INERTIA GROUPS OF MANIFOLDS AND DIFFEOMORPHISMS OF SPHERES.

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The inertia group of a closed smooth manifold M consists of those topological spheres which do not change the diffeomorphism class of M by connected sum. It is often non-zero; examples have been constructed by Tamura [27] and Brown-Steer [10]. On the other hand, limitations on the size of this group have been given by Wall [30], Browder [7], Kosinski [17] and Novikov [24].

Another inertia group can be defined as those diffeomorphisms of a disk, the identity on the boundary, which, when used to change a diffeomorphism of M, don't change its isotopy class. It is technically more practical to replace isotopy by *concordance* (see § 1)—according to a result of Cerf [11], these concepts coincide if M is simply-connected and of large enough dimension. In case M is a topological sphere, this inertia group determines the group of concordance classes of diffeomorphisms of M.

Our study will be based upon a general method of constructing elements of inertia groups—using a generalization of a construction of Milnor [10]. A special case of this result has been previously obtained by Munkres [21]. In some cases this will enable us to completely determine inertia groups; also, most existing examples of non-zero inertia groups—and many more—will emerge.

Some of these results have been obtained independently by A. Kosinski (unpublished) and R. de Sapio [33].

Two Inertia Groups.

1. All manifolds are smooth and oriented; diffeomorphisms and embeddings with codimension zero are orientation preserving. Γ^n is the group of diffeomorphism classes of smooth topological n-spheres under connected sum (see [28]). If $\sigma \in \Gamma^n$, then Σ_σ will be used to denote a representative manifold. Two other interpretations of Γ^n will be used. They are: (1) the

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group of concordance classes of diffeomorphisms of S^{n-1} (two diffeomorphisms of M are concordant if they extend to a diffeomorphism of $I \times M$—see [31], where the term quasi-diffeotopy is used), under composition. (2) the group of concordance classes $\mathrm{rel}\,\partial D^{n-1}$ of diffeomorphisms of D^{n-1} which are 1 on ∂D^{n-1} (a concordance $\mathrm{rel}\,\partial D^{n-1}$ is one which is 1 on $I \times \partial D^{n-1}$).

In either case, if $\sigma \in \Gamma^n$, let h_σ be used to denote a representative diffeomorphism. The correspondence between the interpretations is given as follows. Given h_σ, a diffeomorphism of S^{n-1}, which can be taken to be 1 on a hemisphere D_0^{n-1}, then $h_\sigma \mid D^{n-1}$ (the opposite hemisphere) is a corresponding diffeomorphism of D^{n-1}, and Σ_σ can be defined as the union of two copies of D^n with boundaries identified by h_σ. See [28], [31] for more details.

2. We will use M^n to denote a closed manifold of dimension n. We consider two subgroups $I_0(M) \subset \Gamma^n$, $I_1(M) \subset \Gamma^{n+1}$ called the *inertia groups* of M. $I_0(M)$ consists of all $\sigma \in \Gamma^n$ such that the connected sum $M \# \Sigma_\sigma$ is diffeomorphic to M (see [17]). $I_1(M)$ consists of all $\sigma \in \Gamma^{n+1}$ such that the diffeomorphism of M which differs from 1 only on an n-disk $D \subset M$, and there coincides with h_σ, is concordant to 1. These groups are obviously of importance in the classification of diffeomorphism classes of manifolds homeomorphic to M and concordance classes of diffeomorphisms of M.

We also define *reduced* inertia groups $I_0(M)$, $I_1(M)$. Let $bP^{n+1} \subset \Gamma^n$ be the subgroup of those σ such that Σ_σ bounds a parallelizable manifold (see [16]). Then we define:

$$\tilde{I}_0(M) = I_0(M)/_{I_0(M) \cap bP^{n+1}}, \qquad \tilde{I}_1(M) = I_1(M)/_{I_1(M) \cap bP^{n+2}}$$

—subgroups of $\Gamma^n/bP^{n+1} = \tilde{\Gamma}^n$, and $\tilde{\Gamma}^{n+1}$, respectively.

3. We relate the two inertia groups by:

PROPOSITION 1. $I_1(M) = I_0(M \times S^1)$.

Recall the *mapping torus* M_h of a diffeomorphism h of M. This is the manifold obtained from $M \times I$ by identifying $(x, 0)$ with $(h(x), 1)$, for every $x \in M$ (see [8]).

LEMMA 1. *If $\sigma \in \Gamma^{n+1}$ and h is a diffeomorphism of M, let h' be obtained from h by changing it on an n-disk $D \subset M$ by h_σ. Then $M_{h'}$ is diffeomorphic to $M_h \# \Sigma_\sigma$.*

See [8] for a proof.

LEMMA 2. *If h is a diffeomorphism of M and $n \geqq 4$ then h is concordant* *to* 1 *if and only if* M_h *is diffeomorphic to* $M_1 = M \times S^1$, *by a diffeomorphism yielding a homotopy-commutative diagram:*

$$\begin{array}{ccc} M_h & \longrightarrow & M \times S^1 \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle p^1} & \\ & S^1 & \end{array}$$

where p, p^1 *are the natural fibrations.*

A more general fact is proved in [8] when M is 1-connected, $n \geqq 5$. But the proof actually shows that if M_h is diffeomorphic to $M \times S^1$, then there exists an h-cobordism V of M with itself and a diffeomorphism g of V which is h on one end and 1 on the other. In the case $n \geqq 4$, it is proved in [26] that V is *invertible,* i. e., there exists another h-cobordism W from M to M such that $V \cup W$—identified along the end of V where $g = 1$—is diffeomorphic to $I \times M$. If we extend g to a diffeomorphism of $V \cup W$ which is 1 on W, we get a concordance from h to 1.

Now Proposition 1 follows easily. If $n \leqq 5$, both groups are zero, since $\Gamma^{n+1} = 0$. When $n \geqq 4$, it follows from Lemmas 1 and 2.

Diffeomorphisms of Spheres.

4. When M is a topological sphere, $I_1(M)$ assumes added significance.

Proposition 2. *If* M *is a topological sphere,* $I_1(M)$ *contains at most two elements and* $\Gamma^{n+1}/I_1(M)$ *is naturally isomorphic to* $\Gamma(M)$, *the group of concordance classes of diffeomorphisms of* M.

Define a homomorphism $\phi\colon \Gamma^{n+1} \to \Gamma(M)$ by changing 1 on a disk $D \subset M$, as described in § 2. The kernel is clearly $I_1(M)$. Since the closure of the complement of D is a disk D_0 and any diffeomorphism of M is isotopic to one which is 1 on D_0, ϕ is onto.

We introduce the group $\Gamma(M \operatorname{rel} D_0)$ of concordance classes $\operatorname{rel} D_0$ of diffeomorphisms of M which are 1 on D_0 ([31]), and the obvious homomorphism $\psi\colon \Gamma(M \operatorname{rel} D_0) \to \Gamma(M)$. If $n \geqq 3$, it is proved in [31] that ψ is onto and the kernel has order at most two. Moreover, a diffeomorphism of M represents the generator of Kernel ψ if and only if it is concordant to 1 by a concordance which restricts to the non-trivial bundle map $I \times D_0 \to I \times D_0$ (bundles over I) which is 1 over ∂I—i. e., the one corresponding to the non-trivial homotopy class $(I, \partial I) \to (SO_n, e)$.

There is a natural isomorphism $\Gamma^{n+1} \leftrightarrow \Gamma(M \operatorname{rel} D_0)$ obtained by associating to any diffeomorphism of M, which is 1 on D_0, its restriction to a diffeomorphism of D. Clearly ϕ corresponds to ψ under this isomorphism.

This completes the proof of Proposition 2.

5. $I_1(M)$ also is related to a question of "rotational symmetry" of M, when it is a topological sphere. It follows from § 4 that $I_1(M) = \text{Kernel}\,\psi = 0$ if and only if the non-trivial isotopy from 1 to 1 on D_0 extends to a concordance—and therefore an isotopy, when $n \geqq 6$, according to [11]—from 1 to 1 on M. This can be restated.

PROPOSITION 3. *If M is a topological sphere, then $I_1(M) = 0$ if and only if a non-trivial orthogonal action of S^1 on any disk in M extends to an action of S^1 on M.*[1]

6. Define a function $\gamma\colon \Gamma^n \to \Gamma^{n+1}$ by:

$$\gamma(\sigma) = \text{generator of } I_1(\Sigma_\sigma).$$

PROPOSITION 4. *γ is a homomorphism.*

We use the following characterization of $\gamma(\sigma)$. Let $\{f_t\}$ be the non-trivial linear isotopy from 1 to 1 on D^n. Then a diffeomorphism $h_{\gamma(\sigma)}$ of D^n represents $\gamma(\sigma)$ if and only if the isotopy $\{h_\sigma \circ f_t \circ h_\sigma^{-1}\}$ of S^{n-1} extends to a concordance from 1 to $h_{\gamma(\sigma)}$. This follows readily from § 4. Suppose $\tau \in \Gamma^n$ also. We may assume $h_\sigma \mid D_+^{n-1} = 1$ and $h_\tau \mid D_-^{n-1} = 1$; then $h_{\sigma+\tau} = h_\sigma \circ h_\tau$ agrees with h_τ on D_+^{n-1} and h_σ on D_+^{n-1}. Since D_+^{n-1} and D_-^{n-1} are invariant under f_t, $h_{\gamma(\sigma)}$ may be chosen to be 1 on $D_+^n \subset D^n$ (D_+^n is the "half-moon" defined by a coordinate being non-negative); also $h_{\gamma(\tau)} = 1$ on D_-^n. To construct $h_{\gamma(\sigma+\tau)}$ we need an isotopy from 1 on D^n which extends $h_{\sigma+\tau} \circ f_t \circ h_{\sigma+\tau}^{-1}$ on S^{n-1}. But this can be done by piecing together the isotopy from 1 to $h_{\gamma(\sigma)}$ on D_-^n, and from 1 to $h_{\gamma(\tau)}$ on D_+^n. Then we see that $h_{\gamma(\sigma+\tau)} = h_{\gamma(\sigma)} \circ h_{\gamma(\tau)}$, which says $\gamma(\sigma + \tau) = \gamma(\sigma) + \gamma(\tau)$.

COROLLARY. *$I_1(\Sigma_\sigma) = 0$ if $\sigma = 2\sigma'$ for some $\sigma' \in \Gamma^n$.*

The homomorphism γ can be shown to coincide with the special case of the Λ_2 of Munkres [20]:

$$\Lambda_2\colon H^{n-1}(X;\Gamma^n) \to H^{n+1}(X;\Gamma^{n+1}),$$

where X is the non-trivial $(n-1)$ sphere bundle over S^2.[2]

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[1] The actions referred to are *not* group actions i. e. do not satisfy the formula $gh(x) = g(h(x))$.

[2] Using the Hirsch-Mazur isomorphism $\Gamma^i = \Pi_i(Pl/0)$, γ corresponds to composition with the generator $\eta \in \Pi_{n+1}(S^n)$.

Construction of Some Inertial Spheres.

7. Let n and k be positive integers. Choose elements

$$\sigma \in \Gamma^{n+1}, \quad \tau \in \Gamma^{k+1}, \quad \alpha \in \pi_n(SO_k), \quad \beta \in \pi_k(SO_n).$$

By a slight generalization of a construction of Milnor [19], we define an element $\delta = \delta(\sigma, \alpha; \tau, \beta) \in \Gamma^{n+k+1}$.

Let h_σ, h_τ be representative diffeomorphisms of S^n and D^k—we may assume $h_\tau = 1$ in a neighborhood N of S^{k-1} and $h_\sigma = 1$ on a hemisphere $D \subset S^n$. Let $f\colon (S^n, D) \to (SO_k, e)$ and $g\colon (D^k, S^{k-1}) \to (SO_{n,e})$ represent α, β respectively—we may assume g maps all of N onto e. Now define diffeomorphisms d_1, d_2 of $S^n \times D^k$ by:

$$d_1(x, y) = (h_\sigma(x), f(x) \cdot y)$$
$$d_2(x, y) = (g(y) \cdot x, h_\tau(y)),$$

using the (suspended) action of SO_n on S^n and the usual action of SO_k on D^k.

LEMMA 3. (a) $d_1 \mid D \times D^k = 1$
(b) $d_2 \mid S^n \times N = 1$
(c) $d_2(D \times D^k) = D \times D^k$
(d) d_2 *extends to a diffeomorphism of* $D^{n+1} \times D^k$ *which is* 1 *on* $D^{n+1} \times N$.

One checks (a) and (b) immediately; (c) and (d) follow from the fact that the action of SO_n on S^n preserves D and extends to an action on D^{n+1}.

Now define $d = d_1^{-1} d_2^{-1} d_1 d_2$. It follows directly from Lemma 3 that $d = 1$ on a neighborhood of $S^n \times S^{k-1} \cup D \times D^k$. Thus $d = 1$ outside of an interior disk $D_0 \subset S^n \times D^k$. Let $\delta \in \Gamma^{n+k+1}$ be the element represented by $d \mid D_0$; it clearly depends only upon σ, τ, α and β.

When $\sigma = \tau = 0$, this agrees with Milnor's construction. When $\alpha = 0$ and $\tau = 0$, for example, it is related to a construction of Novikov [25], the twist-spinning operation of Hsiang-Sanderson [13], and a pairing of Bredon [32].

8. The following theorem is basic.

THEOREM 1. *Let M be a closed, smooth $(n+k+1)$-manifold and suppose Σ_σ is embedded in M with normal bundle associated to $\alpha \in \pi_n(SO_k)$. Then, for any $\tau \in \Gamma^{k+1}$, $\beta \in \pi_k(SO_n)$, we have:*

$$\delta(\sigma, \alpha; \tau, \beta) \in I_0(M).$$

An immediate consequence of Theorem 1 and Proposition 1 is:

THEOREM 2. *Let M be a closed, smooth $(n+k)$-manifold, and suppose Σ_σ is embedded in M with normal bundle associated to $\alpha \in \pi_n(SO_{k-1})$. Then, if $S: \pi_n(SO_{k-1}) \to \pi_n(SO_k)$ is suspension, for any $\tau \in \Gamma^{k+1}$, $\beta \in \pi_k(SO_n)$, we have:*

$$\delta(\sigma, S(\alpha); \tau, \beta) \in I_1(M).$$

For example, in both theorems, M can be taken to be the sphere-bundle over Σ_σ associated with $S(\alpha)$. See [33] for a similar result.

Let T be a tubular neighborhood of Σ_σ in M; then T is diffeomorphic to the disk bundle over Σ_σ associated with α. We will show that, if the connected sum $M \# \Sigma_\delta$ is formed along a disk interior to T, then it is diffeomorphic to M by a diffeomorphism which reduces to 1 on $\overline{M - T}$. Equivalently, we simply show that T is diffeomorphic to $T \# \Sigma_\delta$ (along an interior disk) by a diffeomorphism which is 1 near ∂T.

Let d_1 be as in § 7; then T can be described as the union of two copies of $D^{n+1} \times D^k$ identified along $S^n \times D^k$ by d_1. We denote this by $X(d_1)$. Theorem 1 will now follow from the two facts:

(1) $X(d_1)$ is diffeomorphic to $X(d_2^{-1}d_1d_2)$ by a diffeomorphism which is 1 near the boundary—this makes sense since, by Lemma 1-(b), $d_1 = d_2^{-1}d_1d_2$ near $S^n \times S^{k-1}$.

(2) $X(d_1d)$ is diffeomorphic to $X(d_1) \# \Sigma_\delta$ by a diffeomorphism which is 1 near the boundary ($d = 1$ near $S^n \times S^{k-1}$).
Since $d_1d = d_2^{-1}d_1d_2$, Theorem 1 follows.

To prove (1), we use the extension of d_2 whose existence is asserted by Lemma 3-(d) to construct the required diffeomorphism on each copy of $D^{n+1} \times D^k$.

Fact (2) is proven by an argument similar to that which proves Lemma 1.

Some Invariants of $\mathfrak{S}$.

9. We now investigate various techniques for proving non-triviality of $\delta(\sigma, \alpha; \tau, \beta)$.

We will need the following alternative description of δ. Let X_1 be the disk-bundle over Σ_σ associated with $S(\alpha)$ and X_2 the disk-bundle over Σ_τ associated with $S(\beta)$. We then form X_δ by the operation of "plumbing" X_1 and X_2: X_δ is just the union of X_1 and X_2 with an identification of the sub-bundle in X_1 over a disk in Σ_σ with a similar sub-bundle in X_2—both sub-bundles admit obvious diffeomorphisms with $D^{n+1} \times D^{k+1}$. Now Σ_δ can be taken to be ∂X_δ. See [19] for more details in the case $\sigma = \tau = 0$; the argument is precisely the same for general σ, τ.

In the case of $n=k$ even, $\sigma=\tau=0$ and $\alpha=\beta$ a desuspension of the tangent bundle of S^{n+1}, Σ_δ is just the Kervaire sphere [15]. In fact, even if σ, τ are unrestricted X_δ is an n-connected parallelizable $(2n+2)$-manifold with Arf invariant 1. By [16], δ is the generator of bP^{2n+2}, which is zero, if $n=2$ or 6, Z_2 if $n\equiv 0 \bmod 4$, and 0 or Z_2 otherwise (see [9]).

As a consequence of Theorem 1 we, therefore, have:

Example 1 (Brown-Steer [10]). $I_0(V_{n+1,2}) \supset bP^{2n+2}$ if n is even, where $V_{n+1,2}$ is the Stiefel-manifold of 2-frames in $(n+1)$-space.

10. We now use the Eells-Kuiper invariant [12] to study $\delta(\sigma,\alpha;\tau,\beta)$. Suppose $r,s \geqq 1$ are integers. We define:

$$\mu_{r,s}=\frac{a_r a_s B_r B_s (2^{2r}-1)(2^{2s}-1)}{16 a_{r+s} rs (2^{2r+2s-1}-1)} \bmod 1$$

where B_r is the r-th Bernoulli number and $a_r=1$ or 2 as r is even or odd. For example $\mu_{1,1}=1/112$; $\mu_{1,2}=\mu_{2,1}=1/3968$; $\mu_{2,2}=1/32{,}512$.

Let $\Gamma^n{}_{\text{spin}}$ be the subgroup of σ such that Σ_σ bounds a spin-manifold. It follows from [2], [3] that $\Gamma^n{}_{\text{spin}}=\Gamma^n$ unless $n\equiv 1$ or 2 mod 8, in which case it is a subgroup of index 2.

Suppose $n=4r-1$, $k=4s-1$. The μ invariant of Eells-Kuiper [12] defines a homomorphism:

$$\mu\colon \Gamma^{n+k+1}\to Q/Z$$

since $\Gamma^{n+k+1}=\Gamma^{n+k+1}{}_{\text{spin}}$.

If $\alpha\in\pi_n(SO_k)$, then the suspension of α into the stable group $\pi_n(SO)\approx Z$ determines a unique non-negative integer, denoted $|\alpha|$. If $n\geqq 2k+1$, $|\alpha|=0$ ([19, Lemma 5]).

PROPOSITION 5. *If* $\delta=\delta(\sigma,\alpha;\tau,\beta)$, *then:*

$$\mu(\delta)=\mu_{r,s}\,|\alpha|\,|\beta|.$$

This is proved in [12] for $\sigma=\tau=0$, using the relation

$$p_r(\alpha)=\pm a_r(2r-1)!\,|\alpha|$$

(see also [19]), where $p_r(\alpha)$ is the Pontragin class of α. The more general case is proved identically.

Example 2. Suppose $s<2r$, $n=4r-1$, $k=4s-1$. If

$$\lambda\in H_{n+1}(M^{n+k+1};Z)$$

is represented by an imbedded sphere, then $I_0(M)$ has order a multiple of the denominator of the fractions: $\epsilon_{r,s}\mu_{r,s}(p_r(M)\cdot\lambda)/a_r(2r-1)!$ where $\epsilon_{r,s}=2$, if $r=s=1$ or 2 or $r=3$, $s=4$, and 1 otherwise.

This follows from Theorsm 1 and the fact that M contains a copy of a disk bundle associated to $\alpha\in\pi_n(SO_k)$, where $|\alpha|=p_r(M)\cdot\lambda/a_r(2r-1)!$ and β can be chosen to satisfy $|\beta|=\epsilon_{r,s}$ (see [6]). For any given $\alpha\in\pi_n(SO_k)$, we can choose M as the sphere bundle over S^{n+1} associated with $S(\alpha)$ to satisfy the hypotheses of Example 2.

In the special case $r=s=1$, if we choose $|\alpha|=2$, which is possible, we find that $I_0(M)=\Gamma^7$—a result of Tamura [27]. More generally, $|\alpha|$ can be chosen to be $\epsilon_{s,r}$, if $r<2s$.

Corollary. *If $s<2r<4s$, there exists a k-sphere bundle M over S^{n+1} such that $I_0(M)$ has order a multiple of the denominator of $\epsilon_{r,s}\epsilon_{s,r}\mu_{r,s}$.*

The next non-tricial example is a 7-sphere bundle M over S^8 with $I_0(M)$ a subgroup of Γ^{15} of index $\leqq 2$.

Reduced Inertia Groups.

11. In §§ 9, 10 we studied δ by techniques which are particularly sensitive for distinguishing elements of bP^{n+k+2}. We now examine the reduced inertia groups (see § 2). It is possible to obtain some of these results using the Browder-Novikov theory [23], [24].

Recall the homomorphism:

$$T\colon \Gamma^n\to \text{Cokernel}\,\{J_n\colon \pi_n(SO)\to\pi_n(S)\}$$

defined by the Thom construction, where J_n is the Hopf-Whitehead homomorphism (see [16]). The kernel of T is precisely bP^{n+1}; the associated monomorphism, $\tilde{\Gamma}^n\to \operatorname{Cok} J_n$, will also be denoted by T. Recall that T is onto, unless $n=2$, 6 or 14, when the image is a subgroup of index 2, or $n\equiv 6 \bmod 8$, when it is a subgroup of index $\leqq 2$ (see [9] and [18]).

We determine $T(\delta)$, when α or β is zero. If $\sigma\in\Gamma^n$, denote the corresponding element of $\tilde{\Gamma}^n$ by $\tilde{\sigma}$.

We use the bilinear anti-commutative composition pairing [29]:

$$\pi_i(S)\times\pi_j(S)\to\pi_{i+j}(S),\quad (\nu,\xi)\to\nu\circ\xi.$$

If $\nu\in\pi_{i+j}(S^j)$, $\theta\in\pi_j(SO)$, then the composition $\theta\circ\nu\in\pi_{i+j}(SO)$ is defined. Let $E\colon\pi_{i+j}(S^j)\to\pi_{i+j+1}(S^{j+1})$ be the suspension homomorphism—then E^∞ will denote suspension into the stable stem. The following formula holds [17]:

$$J_{i+j}(\theta\circ\nu)=\pm J_j(\theta)\circ E^\infty(\nu).$$

This implies the existence of an induced bilinear composition pairing:

$$\pi_{i+j}(S^j) \times \operatorname{Cok} J_j \to \operatorname{Cok} J_{i+j}.$$

PROPOSITION 6. *If* $\delta = \delta(\sigma, \alpha; \tau, \beta)$ *and* $J\colon \pi_i(SO_j) \to \pi_{i+j}(S^j)$ *is the (non-stable) Hopf-Whitehead homomorphism:*

$$\begin{aligned} T(\delta) &= \pm EJ(\alpha) \circ T(\tilde{\tau}) && \text{if } \beta = 0, \\ &\ \pm EJ(\beta) \circ T(\tilde{\sigma}) && \text{if } \alpha = 0. \end{aligned}$$

It follows from Proposition 6, the above formula, and consideration of suspension [17], that $T(\tilde{\delta}) = 0$ if $n > k$ and $\beta = 0$, or $k > n$ and $\alpha = 0$.

Proposition 6, and its proof, is closely related to [25, Lemma 6]. A similar fact has been proved by Milnor [21] and Bredon [32].

12. Since $\delta(\sigma, \alpha; \tau, \beta) = -\delta(\tau, \beta; \sigma, \alpha)$, it suffices to consider only $\beta = 0$.

It follows from the description of δ in § 9, that Σ_δ arises from a spherical modification ([16]) on ∂X_2. In case $\beta = 0$, $\partial X_2 = S^n \times \Sigma_\tau$ and the modification is constructed from an imbedding:

$$i\colon S^n \times D^{k+1} \to S^n \times \Sigma_\tau$$

defined by $i(x,y) = (h_\sigma(x), f(x) \cdot y)$, where f represents $S(\alpha) \in \pi_n(SO_{k+1})$ and D^{k+1} is identified with a disk in Σ_τ. Then:

$$\Sigma_\delta = \overline{S^n \times \Sigma_\tau - i(S^n \times D^{k+1})} \cup D^{n+1} \times S^k$$

where the boundaries are identified by $i/S^n \times S^k$.

Σ_δ and $S^n \times \Sigma_\tau$ are connected by the cobordism

$$X = I \times S^n \times \Sigma_\tau \cup D^{n+1} \times D^{k+1}$$

where the pieces are attached by the imbedding $S^n \times D^{k+1} \to 1 \times S^n \times \Sigma_\tau$ corresponding to i.

Suppose $S^n \subset R^N$, $N \gg n$, has a normal frame F_0 obtained from the standard normal frame by a "twist" by a map representing

$$-S^{N-n-k}(\alpha) \in \pi_n(SO_{N-n}).$$

Consider $\Sigma_\tau \subset R^M$, $M \gg k$, with a normal frame F_1. Then the product imbedding $S^n \times \Sigma_\tau \subset R^N \times R^M$, with the product framing $F_0 \times F_1$, defines, by the Thom construction, a representative of $\pm EJ\alpha \circ T(\tilde{\tau})$ (see [14]). The theorem will be proved by extending this to a framed imbedding of X in $R \times R^N \times R^M$.

An imbedding of X is defined by merely extending the composite imbedding:

$$S^n \times D^{k+1} \xrightarrow{i} S^n \times \Sigma_\tau \subset R^N \times R^M$$

to an imbedding of

$$D^{n+1} \times D^{k+1} \subset [0,\infty) \times R^N \times R^M$$

which meets $0 \times R^N \times R^M$ transversely along $i(S^n \times D^{k+1})$.

Now suppose $i(S^n \times 0) = S^n \times a$, $a \in \Sigma_\tau$. The normal frame F_2 to $S^n \times a$ in $S^n \times \Sigma_\tau$ induced by the differential of i, is obtained from the standard normal frame by twisting with $S(\alpha)$. To extend i to an imbedding of $D^{n+1} \times D^{k+1}$, we may first extend $i \mid S^n \times 0$ to an imbedding i^1 of $D^{n+1} \times 0$ (transverse to $0 \times R^N \times R^M$ along $i(S^n \times 0)$) and then extend F_2 to a normal $(k+1)$-frame to $i^1(D^{n+1} \times 0)$ in $R \times R^N \times R^M$. Therefore an extension of $F_0 \times F_1$ to a normal framing of X is equivalent to an extension of $F_0 \times F_1 \times F_2$ (a normal frame to $S^n \times a$ in $R^N \times R^M$) to a normal framing of $i^1(D^{n+1} \times 0)$ in $R \times R^N \times R^M$.

But $F_0 \times F_1 \mid S^n \times a = F_0 \times (F_1 \mid a)$, and $F_1 \mid a$ is trivial. Since F_0, F_2 are obtained from trivial frames by twisting by $-\alpha$ and α, respectively, it follows that $F_0 \times F_1 \times F_2$ is homotopic to a trivial frame on $S^n \times a$, which will extend to a normal frame on an imbedding i^1 of $D^{n+1} \times 0$.

This completes the proof of Proposition 6.

13. Proposition 6, together with Theorems 1 and 2, have obvious consequences about the reduced inertia groups.

Example 4. (see [24, 13.3]) If M contains an imbedded topological $(n+1)$-sphere with normal bundle associated to $\alpha \in \pi_n(SO_k)$, then $T(\tilde{I}_0(M))$ contains, as a subgroup $EJ(\alpha) \circ T(\tilde{\Gamma}^{k+1})$.

Example 5. If M contains an imbedded topological $(n+1)$-sphere with normal bundle associated to $\alpha \in \pi_n(SO_{k-1})$, then $T(\tilde{I}_1(M))$ contains, as a subgroup, $E^2J(\alpha) \circ T(\tilde{\Gamma}^{k+1})$.

Example 6. If M contains an imbedded $(n+1)$-sphere Σ_σ with trivial normal k-plane bundle, then $T(\tilde{I}_0(M))$ contains, as a subgroup

$$J\pi_k(SO_{n+1}) \circ T(\tilde{\sigma}),$$

and $T(\tilde{I}_1(M))$ contains, as a subgroup, $EJ\pi_{k+1}(SO_n) \circ T(\tilde{\sigma})$.

Example 6 follows by noticing that $EJ\pi_i(SO_j) = J\pi_i(SO_{j+1})$, when $j > i$, and, when $j \leqq i$, their compositions with an element of the form $T(\tilde{\sigma})$, $\tilde{\sigma} \in \Gamma^j$, are zero, according to remarks on § 11.

In Examples 4 and 5, a sample M is the sphere bundle over S^{n+1} associated with $S(\alpha)$. In Example 6, we can take for M a manifold of the form $\Sigma_\sigma \times V$, where V is any k-manifold.

If α is the non-zero element of $\pi_1(SO_k)$ $(k > 2)$ then there exist $\tau \in \Gamma^{k+1}$ such that $EJ(\alpha) \circ T(\tilde{\tau})$ is non-zero for $k = 7$, 13, 15 and $k \equiv 0 \bmod 8$. This follows from [5] and [29]. Therefore, we have, as a consequence of Example 4 (see [24, Lemma 13.4]) for similar results):

COROLLARY 1. *Suppose M is a manifold of dimension* 9, 15, 17 *or* $8t+2$ $(t \geqq 1)$ *satisfying:*

(a) *M is not a spin-manifold.*
(b) *$H_2(\pi_1(M); Z_2) = 0$, e.g., $\pi_1(M) = 0$, Z, or finite of odd order.*

Then $\tilde{I}_0(M)$ is non-zero.

Condition (b) implies $H_2(M; Z_2)$ is entirely spherical. Then, (a) implies there is an imbedded 2-sphere with non-trivial normal bundle. A similar fact is proved in [21].

Similarly, we derive from Example 5:

COROLLARY 2. *If M satisfies* (a), (b) *of Corollary* 1 *and has dimension* 8, 14, 16 *or* $8t+1$ $(t \geqq 1)$, *then $\tilde{I}_1(M)$ is non-zero.*

As an application of Example 6 we compute reduced inertia groups in some special cases (see also [33]).

COROLLARY 3. *If $\sigma \in \Gamma^n$, $\tau \in \Gamma^k$, $n \geqq k$, then*

$$I_0(\Sigma_\sigma \times \Sigma_\tau) = J\pi_k(SO_n) \circ T(\sigma).$$

The inclusion $I_0(\Sigma_\sigma \times \Sigma_\tau) \supset J\pi_k(SO_n) \circ T(\tilde{\sigma})$ follows from Example 6. For the reverse inclusion we examine the subset P of $\operatorname{Cok} J_{n+k}$ determined, from the Thom construction, by all possible normal framing of $\Sigma_\sigma \times \Sigma_\tau$ ([16]). By the additivity of this operation ([16, Lemma 4.4]), every element of $\tilde{I}_0(\Sigma_\sigma \times \Sigma_\tau)$ is the difference of two elements of P.

It follows by obstruction theory that any normal frame to $\Sigma_\sigma \times \Sigma_\tau$ is homotopic to a product framing $F_\sigma \times F_\tau$, on the complement of a point;

where F_σ, F_τ are normal frames to Σ_σ, Σ_τ, respectively. Thus any element of P is represented by a composition $a_\sigma \circ a_\tau$, where $a_\sigma \in T(\tilde{\sigma})$, $a_\tau \in T(\tilde{\tau})$. Now the difference of two such elements is a sum $J_k\alpha_1 \circ a_\sigma + J_n\alpha_2 \circ a_\tau$, where $\alpha_1 \in \pi_k(SO)$, $\alpha_2 \in \pi_n(SO)$. Since $n \geqq k$, $J_n\alpha_2 \circ \alpha_\tau \in \text{Image}\, J_{n+k}$. We only need show that $(\text{Image}\, J_k) \circ \alpha_\sigma \subset J\pi_k(SO_n) \circ T(\tilde{\sigma})$. If $n \geqq k+1$, this is clear. When $n = k$, the composition is zero, according to a remark in § 11.

As an application of Corollary 3, we notice that there exist π-manifolds M with non-zero reduced inertia group $I_0(M)$. This disproves a conjecture of Novikov [24].

Corollary 4. *If* $\sigma \in \Gamma^n$, *then* $I_1(\Sigma_\sigma) = J\pi_1(SO) \circ T(\sigma)$.

This follows from Corollary 3 and Proposition 1.

Diffeomorphisms of Spheres (continued).

14. We now study the homomorphism $\gamma\colon \Gamma^n \to \Gamma^{n+1}$ defined in § 6. It follows from Corollary 4 that γ induces a comutative diagram

$$\begin{array}{ccccc} 0 \to bP^{n+1} & \longrightarrow & \Gamma^n & \longrightarrow & \text{Cok}\, J_n \\ & & \downarrow \gamma & & \downarrow \tilde{\gamma} \\ 0 \to bP^{n+2} & \longrightarrow & \Gamma^{n+1} & \longrightarrow & \text{Cok}\, J_{n+1} \end{array}$$

where $\tilde{\gamma}(\theta) = \eta \circ \theta$, η the generator of $\pi_1(S)$. It follows from the non-zero compositions, mentioned in § 13, that $\tilde{\gamma} \neq 0$ when $n = 8$, 14, 16 or $n = 1$ mod 8, $n > 1$. We point out a few more facts about γ.

Proposition 7. $\gamma(\Gamma^n{}_{\text{spin}}) \subset \Gamma^{n+1}{}_{\text{spin}}$ *and the induced homomorphism* $\Gamma^n/\Gamma^n{}_{\text{spin}} \to \Gamma^{n+1}/\Gamma^{n+1}{}_{\text{spin}}$ *is an isomorphism for* $n \equiv 1 \bmod 8$ *and zero otherwise. Recall* (10) *that* $\Gamma^n/\Gamma^n{}_{\text{spin}} \approx Z_2$ *for* $n \equiv 1$ *or* $2 \bmod 8$ *and zero otherwise.*

If Σ_σ bounds a Spin-manifold M and $(\Sigma_\sigma \times S^1) \# \Sigma_\gamma$ is diffeomorphic to $\Sigma_\sigma \times S^1$ we define a new manifold as follows. Consider the connected sum along the boundary of $I \times \Sigma_\sigma \times S^1$ and $I \times \Sigma_\gamma$. The boundary consists of three components $(0 \times \Sigma_\sigma \times S^1) \# (0 \times \Sigma_\gamma)$, $1 \times \Sigma_\gamma \times S^1$ and $1 \times \Sigma_\gamma$. To the first two components attach copies of $M \times S^1$. The resulting manifold W has boundary Σ_γ.

That W is a Spin-manifold follows from a Mayer-Vietoris argument, as in [7], which proves that:

$$H^2(W) \to H^2(M \times S^1) \oplus H^2(M \times S^1) \quad \text{(coefficients in } Z_2)$$

is injective, while $M \times S^1$ is a Spin-manifold.

Finally, it follows from [2], [3] that, if $n \equiv 1 \bmod 8$, $\sigma \in \Gamma^n$ and Σ_σ does not bound a Spin-manifold then $\Sigma_\sigma \times S^1$ (S^1 has the non-trivial Spin structure) does not bound a Spin-manifold. It follows that $\eta \circ T(\tilde{\sigma})$ cannot be represented by an element of $\Gamma^{n+1}{}_{\text{spin}}$; thus $\gamma(\sigma) \notin \Gamma^{n+1}{}_{\text{spin}}$. This completes the proof of Proposition 7.

15. Of special interest is whether $\gamma(\sigma)$ is zero, in view of Propositions 2, 3. This is determined by $\tilde{\gamma}$, when n is odd. For n even, we must consider whether $\gamma(\sigma)$ can be non-zero in bP^{n+2}. This is answered in some cases by:

PROPOSITION 8. *Suppose* $\sigma \in \Gamma^n{}_{\text{spin}}$, $n = 4t - 2$, *and* $\gamma(\sigma) \in bP^{n+2}$. *If* $t \leqq 5$, *or* t *is odd, or, more generally, if:*

$$(*) \qquad \text{order}(\text{Image}\, J_{n+1}) = \text{denominator}\, \frac{B_t}{4t},$$

then $\gamma(\sigma) = 0$.

That (*) holds for t odd is a theorem of Adams [1]. It is conjectured to hold for all t.

If $\gamma = \gamma(\sigma) \in bP^{n+2}$, then Σ_γ bounds a parallelizable manifold V. Suppose $\gamma \neq 0$; then, by Proposition 2, $2\gamma = 0$, and it follows from [16] that one may assume:

$$\text{index}\, V = 2^{2t}(2^{2t-1} - 1)\ \text{numerator}\, \frac{4B_t}{t}, \text{ for the given value of } t.$$

Since $\sigma \in \Gamma^n{}_{\text{spin}}$, we can construct a Spin manifold W, as in the proof of Proposition 7. If we adjoin the manifold V along ∂W, we obtain a closed manifold X. Clearly X is a Spin-manifold, because W and V are.

We now compute the $\hat{A}$-genus of X [4]. Coefficients of cohomology are rational. First notice that all the decomposable Pontragin numbers of X are zero. In fact, we have the isomorphism:

$$H^i(X, \Sigma_\sigma \times S^1) \approx H^i(M \times S^1, \Sigma_\sigma \times S^1) \oplus H^i(M \times S^1, \Sigma_\sigma \times S^1) \oplus H^i(V, \Sigma_\gamma).$$

Any Pontragin class $p_i(X)$ pulls back to a class $\alpha_i \in H^{4i}(X, \Sigma_\sigma \times S^1)$—since $H^{4i}(\Sigma_\sigma \times S^1) = 0$. Under the above isomorphism $\alpha_i \leftrightarrow \alpha_i' + \alpha_i'' + \alpha_i'''$, where α_i', α_i'', α_i''' are pull-backs of the Pontragin classes of $M \times S^1$, $M \times S^1$ and V. Thus a decomposable Pontragin number in $H^{n+2}(X)$ pulls back to $\alpha \in H^{n+2}(X, \Sigma_\sigma \times S^1)$, $\alpha \leftrightarrow \alpha' + \alpha'' + \alpha'''$, where α', α'', α''' are products of the

α_i', α_i'', α_i''', respectively. But $H^*(M \times S^1, \Sigma_\sigma \times S^1) \approx H^*(M, \Sigma_\sigma) \oplus H^*(S^1)$ and α_i', α_i'' are of the form $\beta_i' \otimes 1$, $\beta_i'' \otimes 1$. Thus, their products in H^{n+2} are all zero. Finally $\alpha_i''' = 0$, since V is parallelizable.

Now, it is easily seen that the index of X is equal to the index of V, since the index of the pair $(M \times S^1, \Sigma_\sigma \times S^1)$ is zero and

$$H^{2t}(\Sigma_\sigma \times S^1) = H^{2t-1}(\Sigma_\sigma \times S^1) = 0.$$

Using in addition the index theorem and the vanishing of the decomposable Pontragin classes of X, we have the formula ([12]).

$$\hat{A}(X) = \frac{-\text{index } V}{2^{2t+1}(2^{2t-1}-1)} = -\tfrac{1}{2}\,\text{numerator}\,\frac{4B_t}{t}$$

using the calculation of index V. It is a consequence of a theorem of von Staudt [22] that the 2-primary part of numerator $\frac{4B_t}{t}$ is 1, if t is even, and 2, if t is odd. But this violates the Atiyah-Hirzebruch Theorem [4], which asserts that $\hat{A}(X)$ must be integral and, when t is odd, divisible by 2.

16. In conclusion, we discuss γ for $n \leqq 18$, using the computations in [29], and our preceding results. For $n \leqq 7$ and $n = 11, 12, 13, 15$, $\gamma = 0$. For $n = 8$, 14 and 16, $\tilde{\gamma}$ (and, therefore, γ) is a monomorphism. For $n = 10$ and 18, $\gamma \mid \Gamma^n_{\text{spin}} = 0$; Γ^n_{spin} is a subgroup of index 2 of Γ^n, and I do not know whether $\gamma = 0$. For $n = 9$, $\gamma(\Gamma^9) \approx \tilde{\gamma}(\tilde{\Gamma}^9) \approx Z_2$ and $\operatorname{Ker}\gamma = \Gamma^9_{\text{spin}}$. For $n = 17$, $\gamma(\Gamma^{17}) \approx \gamma(\tilde{\Gamma}^{17}) \approx Z_2 + Z_2$ and $\operatorname{Ker}\gamma \subset \Gamma^{17}_{\text{spin}}$.

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