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TRANSVERSALITY STRUCTURES AND P.L. STRUCTURES ON SPHERICAL FIBRATIONS

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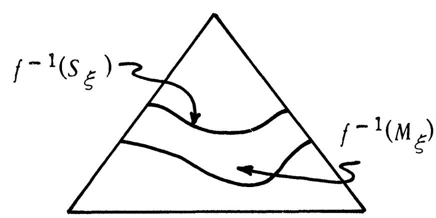
The first author in [L] introduced the notion of "Poincaré transversality" for an N-dimensional spherical fiber space, $\pi : \xi^N \to X$. If $T(\xi^N)$ is the Thom space of ξ^N, then we consider $T(\xi^N) = M_\xi \cup c(S(\xi))$ where M_ξ is the mapping cylinder of π and $S(\xi)$ is the total space of ξ^N. A map $f: \Delta^{N+i} \to T(\xi^N)$ is Poincaré transversal if f is p.l. transversal to $S(\xi) \subset T(\xi)$ with $(f^{-1}(M_\xi), f^{-1}(S(\xi)))$ a codimension 0 submanifold of Δ^{N+i} with the inclusion $f^{-1}(S(\xi)) \subset f^{-1}(M_\xi)$ the spherical fibration induced by f over $f^{-1}(M_\xi)$. This implies $f^{-1}(M_\xi)$ is a Poincaré duality space, (P.D. space), of dimension i with boundary $f^{-1}(M_\xi) \cap \partial\Delta^{i+N}$, and that $(f^{-1}(M_\xi), f^{-1}(S_\xi))$ is its normal tube.



A p.l. manifold M^j mapping by $f: M^j \to T(\xi^N)$ is Poincaré transversal to ξ^N if and only if $f|$(any simplex) is. If f is Poincaré transversal then $f^{-1}(M_\xi)$ is a P.D. space with boundary $f^{-1}(M_\xi) \cap \partial M$ and of dimension $j - N$. One of the main results of [L] is to develop a theory to study the problem of when a map $f: M^j \to T(\xi^N)$ may be shifted to be Poincaré transversal. To do this one introduces the space $W(\xi^N)$, of Poincaré transversal maps of $\Delta^i \to T(\xi^N)$ for all i. In [L] and [J] it is proved that if F_{ξ^N} denotes the homotopy theoretic fiber of $W(\xi^N) \to T(\xi^N)$, then $\pi_i(F_{\xi^N}) \cong \pi_{i-N}(G/PL)$ for $i - N \neq 1, 2,$ or 3. In fact a map of fiber spaces $\xi^N \to \zeta^N$ induces $F_{\xi^N} \to F_{\zeta^N}$ and this map is an isomorphism on π_i for $i - N \neq 1, 2,$ or 3. Also if $f: M^j \to T(\xi^N)$, then homotopying f until it is Poincaré transversal is equivalent to lifting f up to homotopy to $W(\xi^N)$. In this announcement we shall describe further results in this theory.

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By a theory of transversality, τ, for a spherical fiber space, $\pi : \xi^N \to X$, we mean an assignment to each $f : \Delta^i \to T(\xi)$ a deformation of f until it is Poincaré transverse on Δ^i and all its faces. These assignments are required to be compatible with inclusions of faces $\Delta^{i-1} \subset \Delta^i$. Two theories of transversality τ_0 and τ_1 are equivalent (concordant) if there is a theory of transversality for $\xi^N \times I \to X \times I$ which, when restricted to $X \times i$, is τ_i for $i = 0, 1$. In the language of [L], a theory of transversality is exactly a section of $W(\xi^N) \to T(\xi^N)$.

A p.l. structure for ξ^N is a p.l. block bundle (see [**R-S**]), $\pi : E^N \to X$, and a fiber homotopy equivalence of $S(E^N)$ to ξ^N.

Let $PL(\xi^N)$ denote the concordance classes of p.l. structures on ξ^N, Clearly p.l. transversality in E provides a theory of transversality for ξ^N. This defines a function from $PL(\xi^N)$ to concordance classes of theories of transversalities for ξ^N.

THEOREM A. *Let $\pi : \xi^N \to X$ be a spherical fiber space with $N \geqq 3$ and X four connected, then the function $PL(\xi^N) \to \{$concordance classes of theories of transversalities for $\xi^N\}$ is a bijection. In particular ξ^N has a p.l. structure if and only if it has a theory of transversality.*

This is the bundle analogue of Winkelnkemper's philosophy that "transversality unlocks the secret of a manifold." The homotopy theoretic formulation of Theorem A states that given X four connected and $\xi^N : X \to BSG(N)$ then ξ^N lifts to $E : X \to BSPL(N)$ if and only if $W(\xi^N) \to T(\xi^N)$ has a section. The latter in turn is equivalent to the existence of a lift in the following diagram:

$$\begin{tikzcd} & WSG(N) \arrow[d] \\ T(\xi^N) \arrow[r] \arrow[ur, dashed] & MSG(N) \end{tikzcd}$$

The problem of lifting $X \to BSG(N)$ to $X \to BSPL(N)$ involves the space $G(N)/PL(N) = G/PL$. The problem of lifting $T(\xi^N) \to MSG(N)$ to $T(\xi^N) \to WSG(N)$ involves the space $F_{SG(N)}$. We now consider the relation of these two spaces. The results in [L] show that they have the same homotopy groups shifted by N dimensions (except for low dimensions). To actually compare these two spaces we need to extend G/PL to a connected spectrum. There are several ways of doing this. One is to use Sullivan's calculation of the homotopy type of G/PL as KO-theory at odd primes and cohomology at 2 (with one low dimensional twist), and then define $G/PL\langle k\rangle$ to be $bo\langle k\rangle$ at odd primes ($bo\langle k\rangle$ is Ω^{4l-k} ($4l$ connected cover of BO)) and cohomology shifted k dimensions (with the twist) at 2; see [S]. Another approach is to define $G/PL\langle k\rangle$ to be Quinn's space $L_k(e)$. This is the semisimplicial complex of surgery problems shifted k-dimensions; see [Q]. These approaches lead to the same results.

THEOREM B. *There is a map $\phi : F_{\xi^N} \to G/PL\langle N\rangle$ which is an isomorphism on π_i for $i \neq N+1$, $N+2$, and $N+3$ provided only that $N \geqq 3$.*

ϕ is a realization on the space level of the isomorphism on homotopy groups given in [L].

II. **Homotopy theoretic reformulations.** If $\pi : \xi^N \to X$ is a spherical fiber space, then we may form a connected spectrum $\boldsymbol{T}(\boldsymbol{\xi})$. The ith space is $T(\xi^N \oplus \varepsilon^{i-N})$ for $i \geqq N$ and the maps are the usual ones $\sum T(\xi^N \oplus \varepsilon^{i-N}) \to T(\xi^N \oplus \varepsilon^{i-N+1})$. In [L] it is shown that the spaces $\{W(\xi^N \oplus \varepsilon^{i-N})\}$ also form a spectrum, $\boldsymbol{W}(\boldsymbol{\xi})$, and that there is a map of spectra

$$\boldsymbol{F}_\xi \to \boldsymbol{W}(\boldsymbol{\xi}) \to \boldsymbol{T}(\boldsymbol{\xi}).$$

The spectrum fiber (or cofiber) $\boldsymbol{F}_\xi$ is made up of the spaces $F_{\xi^N \oplus \varepsilon^{i-N}}$. Let $\boldsymbol{G}/\boldsymbol{PL}$ be the spectrum whose ith space is $G/PL\langle i\rangle$. Thus we have $\phi : \boldsymbol{F}_\xi \to \boldsymbol{G}/\boldsymbol{PL}$ with $\phi_\#$ an isomorphism on π_i for $i \geqq 4$.

We can give another description of the spectrum $\boldsymbol{F}_\xi$. Let $\gamma^i \to BSG(i)$ be the universal i-dimensional spherical fiber space. Then $\boldsymbol{T}(\gamma^i) = \boldsymbol{MSG}$. $p : W(\gamma^i) \to T(\gamma^i)$ with $W(\gamma^i) = p^{-1}(M_{\gamma^i}) \cup p^{-1}(c(S(\gamma^i)))$. Let $E_i = p^{-1}(c(S(\gamma^i)))$. The $\{E_i\}$ form a spectrum $\boldsymbol{\varepsilon}$.

PROPOSITION. $\boldsymbol{\varepsilon} \cong \boldsymbol{F}_\gamma$.

This follows from

THEOREM C.

$$\begin{array}{ccc} p^{-1}(S(\gamma^i)) & \xrightarrow{p} & S(\gamma^i) \\ \cap & & \cap \\ p^{-1}(M_{\gamma^i}) & \xrightarrow{p} & M_{\gamma^i} \end{array}$$

is a homotopy equivalence of pairs in dimensions less than $2i$.

Thus $\boldsymbol{\varepsilon} \to \boldsymbol{W}(\gamma) \to \boldsymbol{T}(\gamma)$ $(= MSG)$ is a cofibration of spectra. This proves $\boldsymbol{\varepsilon} \cong \boldsymbol{F}_\gamma$.

III. **Idea of proofs of Theorems** A **and** B. The proof of Theorem A reduces inductively to the following. Let $\eta = \xi \mid X^{(n)}$ be given a p.l. structure, $n \geqq 5$.

$$\begin{array}{ccc} E & \longrightarrow & \xi \mid X^{(n)} \\ \downarrow & & \downarrow \\ X^{(n)} & = & X^{(n)} \end{array}$$

Thus for any $(n+1)$ cell τ_{n+1} in X we have $\xi \mid \tau_{n+1}$ with a canonical trivialization and $E \mid \partial\tau_{n+1} \to \xi \mid \partial\tau_{n+1}$ a p.l. structure on the boundary. This is equivalent to a p.l. bundle, $E \mid \partial\tau_{n+1}$, and a fiber homotopy trivialization, i.e., an element in $\pi_n(G/PL)$. This defines a cocyle σ whose class in $H^{n+1}(X; \pi_n(G/PL))$ is the obstruction to extending the p.l. structure relative to $X^{(n-1)}$ over $X^{(n+1)}$.

The p.l. structure defines a theory of transversality for $\xi \mid X^{(n)}$. Again considering τ_{n+1}, we have $\xi \mid \tau_{n+1}$ with a theory of transversality for $\xi \mid \partial\tau_{n+1}$. $\xi \mid \tau_{n+1}$ is canonically trivial as a spherical fiber space, and thus there is a map $S^N \times S^N \to T(\xi^N \mid \tau_{n+1})$. Using the theory of transversality over the boundary, we get $f^{-1}(M_{\xi|\partial\tau})$, $f^{-1}(S_{\xi|\partial\tau})$ an n-dimensional G-framed P.D. space in $S^n \times S^N$. But G-framed P.D. bordism in dimension n is isomorphic to $\pi_n(G/PL)$ for $n \geqq 4$. The isomorphism assigns to $(Y, Z : \nu_Y \to \boldsymbol{R}^N)$ the surgery obstruction of the problem $Z^{-1}(0) \to Y$ where Z is shifted p.l. transverse to 0. Thus we have a cocyle $\sigma' \in C^{N+n+1}(T(\xi^N)) \otimes \pi_n(G/PL)$. Theorem A is proved by showing $\sigma' = \Phi(\sigma)$ where Φ is the Thom isomorphism.

To prove Theorem B, we need only calculate F_{ε^N} since in **[L]** it is proved that all the fibers F_{ξ^N} are the same homotopy type. Using the characteristic homomorphism (with a shift of N dimensions) definition of $G/PL\langle N \rangle$, we need only assign to $M^{N+i} \to F_{\varepsilon^N}$ a surgery problem between P.D. spaces. To define the homomorphism we take the usual surgery obstruction of the problem. We do not need to know that this is the only obstruction to doing surgery on P.D. spaces, only that it is an obstruction. A map $M^{N+i} \to F_{\varepsilon^N}$ is equivalent to a transversal map $f: M^{N+i} \to T(\varepsilon^N)$ together with a homotopy of f to zero in $T(\varepsilon^N)$, $F: M^{N+i} \times I \to T(\varepsilon^N)$. Shift F slightly until it is p.l. transverse regular to $\mathrm{pt} \in T(\varepsilon^N)$. Let V^{i+1} be the pre-image. $f^{-1}(M_{\varepsilon^N})$ is an i-dimensional P.D. space and if we have not shifted F too much then $V^{i+1} \cap M^{N+i} \times 0 \subset f^{-1}(M_{\varepsilon^N})$ is a degree one normal map.

IV. **Applications.**

COROLLARY 4.1. *A stable spherical fiber space $\xi \to X$ has a p.l. structure if and only if it has an $MSPL$ orientation lifting its MSG orientation.*

PROOF. A p.l. bundle has a natural $MSPL$ orientation. Conversely, $MSPL \to MSG$ factors through $W(\gamma)$. Thus lifting the canonical MSG orientation to an $MSPL$ orientation also lifts it to a $W(\gamma)$ orientation.

COROLLARY 4.2. *Let M^n be a 4-connected P.D. space, then M^n is homotopy equivalent to a closed p.l. manifold if and only if M^n satisfies Poincaré duality for the homology and cohomology theories associated* *to MSPL with a fundamental class which reduces to the MSG fundamental class.*

PROOF. Use S-duality and 4.1.

COROLLARY 4.3. (THE INTRINSIC BROWDER-NOVIKOV THEOREM). *Let M^n be a 4-connected P.D. space. A necessary and sufficient condition for M^n to be the homotopy type of a closed p.l. manifold is that M^n have two regular neighborhoods $R' \subset R \subset S^{n+K}$, for some large K, with R triangulated so that for every simplex σ^j, $(\sigma^j \cap R', \partial\sigma \cap R')$ is a codimension 0 submanifold which is the tube of a P.D. pair of formal dimension $j - K$.*

Thus M^n is homotopy equivalent to a closed p.l. manifold if and only if M^n may be put in "Poincaré general position" with respect to the simplices of some triangulation of its regular neighborhood. See also [J].

COROLLARY 4.4. *Let $f: N^n \to M^n$ be a homotopy equivalence of 4-connected manifolds and $\mathcal{M}_f$ be the mapping cylinder of f. Then $(\mathcal{M}_f, N \cup M)$ is a Poincaré pair. If the section $T(\nu(N \cup M)) \to W(\nu(N \cup M))$ corresponding to the natural manifold structure on $N \cup M$ extends to a section $T(\nu(\mathcal{M}_f)) \to W(\nu(\mathcal{M}_f))$ then f is homotopic to a p.l. homeomorphism.*

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