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Link concordance and algebraic closure, II

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Introduction

In the study of knot concordance one of the more successful approaches makes use of Seifert "surfaces" and the associated Seifert pairing (see e.g. [L1]). Similarly, these methods apply to the study of link concordance, provided the link components bound disjoint Seifert surfaces ([Ko]). Such links are called *boundary links.* Alternative methods, using the Blanchfield pairing ([Ke]) or homology surgery ([CS]), also require boundary links when applied to link concordance ([D], [CS1]).

This restriction raises the question of whether every link may be concordant to a boundary link. For one-dimensional links, there is the necessary condition that the $\bar{\mu}$-invariants of Milnor [M] must all vanish. In [G1] a proof that every higher-dimensional link is concordant to a boundary link was announced, but was found to contain errors. Since then, this problem has received some attention. In [Sa] a new concordance invariant for n-dimensional links ($n>1$), which vanishes for boundary links, was defined. It turned out, however, that this invariant *always* vanishes. The proof of this was tied to the construction, in [O], of a sequence of related concordance invariants θ_k ($2 \leqq k \leqq \infty$) which live in the homotopy groups of spaces defined from the lower central series quotients F/F_k, when $k<\infty$, or the nilpotent completion of F, when $k=\infty$ (F is the free group whose rank is the number of link components). When $n=1$, θ_k is defined only when all the $\bar{\mu}$-invariants of degree $\leqq k$ vanish. But, as it turned out (see [C]), $\theta_k=0$ for all $k<\infty$ if $n>1$, and exactly when all the $\bar{\mu}$-invariants vanish if $n=1$. Only θ_∞ survives as a new and possibly non-trivial invariant (but living in an uncomputed homotopy group). In [L] θ_∞ was redefined as a (possibly) stronger invariant – living in another uncomputed homotopy group – but it was shown that any element of this group could be realized by a link, at least for $n=1$.

An important step in a different direction was taken in [C] where it was pointed out that certain classes of links, containing the boundary links, could be useful. For example, *homology boundary links* or, even more generally, sublinks

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of homology boundary (SHB) links, have vanishing θ_∞ and, when $n=1$, vanishing $\bar{\mu}$-invariants. Thus the problem breaks into three pieces:

(i) is every link with vanishing θ_∞ (and, if $n=1$, vanishing $\bar{\mu}$-invariants) concordant to an SHB link?
(ii) can θ_∞ be non-zero?
(iii) is every homology boundary link concordant to a boundary link?

Another concept, that of a *finite E-link*, was also defined in [C]. It turns out that every SHB link is a finite E-link and every finite E-link satisfying a condition similar to, but ostensibly stronger than, $\theta_\infty=0$, is concordant to an SHB link (see [L]).

In the present paper we construct a further refinement θ of the Orr invariant θ_∞. The definition makes use of a group-theoretic construction which we call *algebraic closure*, used already in [L] and first considered in [G] in a somewhat different manner. Then $\theta(L)$ is defined in $H_3(\hat{F})$ for 1-dimensional links L satisfying a condition (ostensibly) stronger than that of having vanishing $\bar{\mu}$-invariants. $\hat{F}$ is the algebraic closure of F. Our main results are:

(a) $\theta(L)=0$ if and only if L is concordant to an SHB link.
(b) If $\alpha\in H_3(\hat{F})$ then there exists some L such that $\theta(L)=\alpha$.

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I. Algebra

1. We begin by recalling some notions from [L]. Let G be a group.

Definition. (i) An element $w=w(x_1, \dots, x_m)\in G * F$ will be called a *monomial over* G (with indeterminates $x_1, \dots, x_m$), where $F=F(x_1, \dots, x_m)$ is the free group with basis $x_1, \dots, x_m$. A monomial w over G is *contractible* if w lies in the kernel of the obvious projection $G * F \to F$.

Obviously a monomial is contractible when it is a product of conjugates of elements of G.

(ii) A system S of equations over G:

$$x_i=w_i(x_1, \dots, x_m), \qquad i=1, \dots, m$$

is *contractible* if each w_i is a contractible monomial over G. G is *algebraically closed* (AC) if every contractible system of equations over G has a *unique* solution in G. There are already several notions of *algebraically closed* occurring in the literature (see [Ho], [N], [BDH]). These more classical notions demand solutions of a larger set of equations, but omit any requirement of uniqueness. Thus there is no associated notion of *algebraic closure*.

We gather some facts about AC groups, using the approach of [B].

Proposition 1. (a) *If $\{G_\alpha\}$ is a family of AC subgroups of an AC group G, then $\bigcap_\alpha G_\alpha$ is AC.*

(b) *If $\{G_\alpha\}$ is any family of AC groups, then $\pi_\alpha G_\alpha$ is AC.*

(c) *If G is a central extension of H, then G is AC if and only if H is AC.*

Remark. In general, inverse limits and direct limits of AC groups are AC.

Proof. (a) and (b) are immediate. We prove (c). Suppose H is AC and let $x_i = w_i(x_1, \ldots, x_m)$ be a contractible system S over G. It projects to a contractible system $\bar{S}$ over H with (unique) solution set $x_i = h_i$. Let $g_i \in G$ be any lift of h_i.

We will show that $x_i = w_i(g_1, \ldots, g_m)$ is the unique solution set of S. Uniqueness follows because any solution set $x_i = g_i' = g_i \tau_i$, where $\tau_i \in$ center of G, and, since w_i are contractible, this implies

$$w_i(g_1, \ldots, g_m) = w_i(g_i', \ldots, g_m') = g_i'.$$

On the other hand, if we set $g_i' = w_i(g_1, \ldots, g_m)$, the same argument shows

$$w_i(g_1', \ldots, g_m') = w_i(g_1, \ldots, g_m) = g_i'.$$

Suppose G is AC and $x_i = w_i(x_1, \ldots, x_m)$ is a contractible system S over H. We can lift S to a contractible system $\tilde{S}$ over G: $x_i = \tilde{w}_i(x_1, \ldots, x_m)$ and a solution of $\tilde{S}$ projects to a solution of S. On the other hand, if $g_1, \ldots, g_m$ is any lift of a solution $h_1, \ldots, h_m$ of S, then, by the argument of the previous paragraph, $g_i' = \tilde{w}_i(g_1, \ldots, g_m)$ is a solution set of $\tilde{S}$. But g_i' is a lift of $w_i(h_1, \ldots, h_m) = h_i$ and it follows that any solution set of S lifts to the (unique) solution set of $\tilde{S}$.

It follows from Proposition 1(c), since the trivial group is obviously AC, that every nilpotent group is AC. Using the Remark, we can also conclude that the nilpotent completion of any group is AC.

2.

Definition. A group homomorphism $\Phi: G \to A$ where A is AC, is called a *closing* of G. A closing Φ is *small* if ΦG is contained in no AC proper subgroup of A.

Proposition 2. *Any closing $\Phi: G \to A$ "contains" a unique small closing $\Phi: G \to A'$* i.e. *$\Phi G \subseteq A' \subseteq A$. Every element of A' belongs to the solution set of a contractible system over G. (More precisely, the image system over ΦG.)*

Proof. A' is just the intersection of all AC subgroups containing ΦG. The second statement is proved by exactly the same argument as Lemma 4 of [L].

Definition. Let G be a subgroup of H. Then H is *finitely normally generated* by G if:

(i) H is the normal closure of G and
(ii) H is generated by G and a finite number of additional elements.

Note that if $G_1 \subseteq G_2 \subseteq G_3$ are subgroups and G_{i+1} is finitely normally generated by G_i $(i=1, 2)$, then G_3 is finitely normally generated by G_1.

Lemma 1. *If H is finitely normally generated by G and A is any AC group then the natural map* $\mathrm{Hom}(H, A) \to \mathrm{Hom}(G, A)$ *is an injection.*

Proof. Suppose $h_1, \ldots, h_m$, together with G, generate H. Furthermore each h_i is a product of conjugates of elements of G. By substituting x_i for each appearance of h_i in these words, we obtain a set of contractible monomials $w_i(x_1, \ldots, x_m)$ over G and $\{h_i\}$ is a solution set of the system S: $x_i = w_i(x_1, \ldots, x_m)$. If $\Phi: H \to A$ is any homomorphism, then $\Phi | G$ sends S to a contractible system S' over A. Clearly $\{\Phi(h_i)\}$ is a solution set for S' and so, by uniqueness, $\{\Phi(h_i)\}$ are determined by $\Phi | G$.

Note that we can add the following easy *addendum* to Proposition 2.

A' is the union of all subgroups of A finitely normally generated by ΦG.

It is clear that the solution set of any contractible system over ΦG generates, together with ΦG, a subgroup finitely normally generated by ΦG. Conversely, the argument of Lemma 1 shows that any subgroup finitely normally generated by ϕG is generated (together with ϕG) by the solution set of some contractible system over ϕG.

3.

Definition. For any group G, an *algebraic closure* of G is a closing $\phi: G \to A$ which satisfies the following universal property: for any closing $\psi: G \to B$, there exists a unique homomorphism $\rho: A \to B$ such that $\rho \circ \phi = \psi$.

Proposition 3. *For any group G there exists an algebraic closure which is unique, up to isomorphism, and is a small closing of G.* (*Two closings $\phi: G \to A$, $\psi: G \to B$ are isomorphioc if there exists an isomorphism $\alpha: A \to B$ with $\alpha \circ \phi = \psi$*).

Remark. It turns out that algebraic closure is the analogue of the (topological) $\bar{\omega}$-localization functor of Vogel (see [Le]). In fact, if X is a CW-complex and $X \to E(X)$ the Vogel localization of X, then $\pi_1(X) \to \pi_1(E(X))$ is an algebraic closure of $\pi_1(X)$. Since we will not need to use $\bar{\omega}$-localization in the present work, we will give (in § 5) a direct proof of Proposition 3.

Proposition 4. *An algebraic closure $\phi: G \to \hat{G}$ is 2-connected i.e. $\phi_*: H_i(G) \to H_i(\hat{G})$ is an isomorphism for $i = 1$ and an epimorphism for $i = 2$.*

Proof. That $H_1(G) \to H_1(\hat{G})$ is onto follows from Proposition 2 (addendum) and one-one follows from the universal property of algebraic closure applied to the closing $G \to G/[G, G]$ (by Prop. 1(c)). To prove $H_2(G) \to H_2(\hat{G})$ is onto we use an argument of Bousfield [B]. If $H_2(\phi) \neq 0$, then consider an element of

$H^2(\hat{G}; H_2(\phi))$ corresponding to the natural epimorphism $H_2(\hat{G}) \to H_2(\phi)$. This determines a non-trivial central extension $E \to \hat{G}$ and a lift of ϕ to $\tilde{\phi}: G \to E$. But this violates the universality of ϕ, since $\tilde{\phi}$ is a closing of G. □

In particular, if F is a free group then $H_2(\hat{F}) = 0$.

4.

Proposition 5. *Suppose $\phi: G \to H$ is a 2-connected homomorphism of groups, G is finitely generated, H is finitely presented and H is the normal closure of ϕG. Then $\hat{\phi}: \hat{G} \to \hat{H}$ is an isomorphism. ($\hat{\phi}$ is defined by the universal property applied to the closing $G \xrightarrow{\phi} H \to \hat{H}$).*

Remark. This proposition is false if G or H is not finitely generated since there exist "super-perfect" groups G (i.e. $H_1(G) = 0 = H_2(G)$) with $\hat{G}$ non-trivial.

Proof. Since H is obviously finitely normally generated by ϕG we see that, for some contractible system $S: x_i = w_i(x_1, \ldots, x_m)$ over G, H is generated by a solution set $\{h_i\}$ of $\phi(S)$ – the system over H obtained from S by applying ϕ to the elements of G in the equations. S is constructed by choosing a set of generators $\{h_i\}$ for H and writing each h_i as a product of conjugates of the form hgh^{-1}, where $g \in \phi G$ and $h \in H$. Each h can then be written as a word in $\{h_j\}$ and by substituting x_j for h_j in this expression we obtain the required $w_i(x_1, \ldots, x_m)$.

Let G_S be the group obtained from G by adjoining generators $x_1, \ldots, x_m$ and relations: $x_i = w_i(x_1, \ldots, x_m)$. There is an obvious commutative diagram:

$$\begin{tikzcd} & G_S \arrow[d, "\psi"] \\ G \arrow[ur, "\tau"] \arrow[r, "\phi"'] & H \end{tikzcd}$$

where $\psi(x_i) = h_i$. Note that ψ is onto. Also τ is 2-connected and, therefore, so is ψ. Furthermore, there is a homomorphism $\rho: G_S \to \hat{G}$ defined using S, and a commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\tau"] \arrow[d] & G_S \arrow[d] \arrow[dl, "\rho"'] \\ \hat{G} \arrow[r, "\hat{\tau}"'] & \hat{G}_S \end{tikzcd}$$

Since ρ is obviously a small closing, we conclude that $\hat{\tau}$ is an isomorphism from:

Lemma 2. *Given a commutative diagram:*

$$\begin{tikzcd} G \arrow[rr] \arrow[dr, "\Phi"'] & & \hat{G} \\ & A \arrow[ur, "\psi"'] & \end{tikzcd}$$

where ϕ is a small closing, then ψ is an isomorphism.

Proof. By the universal property of $\hat{G}$ we obtain a homomorphism $\gamma: \hat{G} \to A$ such that $\gamma \circ \psi \circ \Phi = \Phi$. Since Φ is a small closing, by Lemma 1 and Proposition 2 (addendum) we have $\gamma \circ \psi = 1$. Also $\psi \circ \gamma \circ \psi \circ \Phi = \psi \circ \Phi$ and, since $\psi \circ \Phi$ is a small closing, we have $\psi \circ \gamma = 1$. $\square$

We return to the proof of Proposition 5.

Since, as has been noted, ψ is onto and 2-connected we conclude from the Stallings exact sequence [S] that $[K, G_S] = K$, where $K = \text{Kernel } \psi$. It also follows from the hypotheses on G, H that K is the normal closure of a finite subset (i.e. K has finite weight in G_S). We now need:

Lemma 3. *Suppose K is a normal subgroup of finite weight in G and $[G, K] = K$. If $\Phi: G \to A$ is any homomorphism, where A is AC, then $K \subseteq \text{Kernel } \Phi$.*

Proof. Suppose K is the normal closure of $\{a_1, \ldots, a_m\}$. From $[G, K] = K$ we conclude that each a_i can be expressed as a product of conjugates of commutators of the form $[a_j, g]^\varepsilon$, where $g \in G$ and $\varepsilon = \pm 1$. If we replace each occurrence of a_j in these commutators by the indeterminate x_j, we obtain a contractible monomial $w_i(x_1, \ldots, x_m)$ over G with the extra property $w_i(1, \ldots, 1) = 1$.

Now $\{a_i\}$ is a solution set of the system $S: x_i = w_i(x_1, \ldots, x_m)$ over G and so $\{\phi(a_i)\} \subseteq A$ is a solution set of $\phi(S)$. But in A, $\phi(S)$ has a unique solution set and, because $w_i(1, \ldots, 1) = 1$ we see that $x_i = 1$ is already a solution set of $\phi(S)$. Thus $\phi(a_i) = 1$. $\square$

We now return to the proof of Proposition 5. By Lemma 3, $K \subseteq \text{Kernel}$ $\{G_S \to \hat{G}_S\}$ and so we obtain a homomorphism $H \to \hat{G}_S \approx \hat{G}$ which is clearly a small closing. By Lemma 2, $\hat{\phi}$ is an isomorphism. $\square$

Definition. Subgroups K satisfying the hypotheses of Lemma 3 will be called *invisible*.

5. We now prove Proposition 3. The uniqueness is clear, and smallness follows from Proposition 2. For existence consider the set $\mathscr{C}$ of all contractible systems of equations over G i.e. an element $\alpha \in \mathscr{C}$ consists of a positive integer $m = m_\alpha$, and a sequence of m_α contractible monomials $w_1^\alpha, \ldots, w_m^\alpha$ in $G * F$, where F is free on m indeterminants $x_1, \ldots, x_m$. Now construct the group $\mathscr{G}$ by adjoining to G generators $x_1^\alpha, \ldots, x_m^\alpha$ $(m = m_\alpha)$, for each $\alpha \in \mathscr{C}$, and relations $x_i^\alpha = w_i^\alpha(x_1^\alpha, \ldots, x_m^\alpha)$, $1 \leqq i \leqq m_\alpha$. Clearly every contractible system over G has a (non-unique) solution in $\mathscr{G}$, and these solutions generate $\mathscr{G}$. In fact, every element of $\mathscr{G}$ belongs to a solution set of such a system (see Lemma 4 of [L]).

Now let N be the union of all the invisible subgroups of $\mathscr{G}$. Then N is itself a normal subgroup, since any two invisible subgroups normally generate a third. We will show that $\hat{G}=\mathcal{G}/N$ is algebraically closed by proving the following three assertions:

(a) every contractible system over $\hat{G}$ has a solution in $\hat{G}$.

(b) $\hat{G}$ has no non-trivial invisible subgroups.

(c) If H is any group with no non-trivial invisible subgroups then a contractible system over H has at most one solution in H.

Proof of (a). Let $x_i=w_i(x_1, \ldots, x_m)$, $1\leqq i\leqq m$, be a contractible system S_1 over $\hat{G}$.

The monomials $w_1, \ldots, w_m$ involve a finite, number of elements $h_1, \ldots, h_k$ of $\hat{G}$; Let $h_i'\in\mathcal{G}$ be a lift of h_i. There is a contractible system S_2 over G: $y_i=v_i(y_1, \ldots, y_k)$, $1\leqq i\leqq k$, for which $y_i=h_i'$, $1\leqq i\leqq k$, is a solution set, allowing for enlargement of $\{h_i'\}$. We now define a new contractible system of equations S over G: $x_i=w_i'(x_1, \ldots, x_m, y_1, \ldots, y_k)$; $1\leqq i\leqq m$; $y_i=v_i(y_1, \ldots, y_k)$, $1\leqq i\leqq k$, where w_i' is obtained from w_i by substituting $v_j(y_1, \ldots, y_k)$ for h_j wherever it appears in w_i, for all $1\leqq j\leqq k$. S has a solution in $\mathcal{G}$ and, therefore, in $\hat{G}$. Since S_2 is a subsystem of S, any solution of S includes a solution of S_2. If we assume (b) and (c), then solutions in $\hat{G}$ are unique and so, in any solution of S, $y_i=h_i$. Substituting this into the rest of S shows that the $\{x_i\}$ will be a solution of the original system S_1.

Proof of (b). Suppose B is an invisible subgroup of $\hat{G}$ with a finite number of normal generators $b_1, \ldots, b_k$. Thus we can write $b_i=w_i$, where w_i is a product of conjugates of commutators $[b_j, g]$, $g\in\hat{G}$. Choose $\bar{b}_j\in\mathcal{G}$, a lift of b_j, and lifts of the other elements of $\hat{G}$ appearing in w_i. This enables us to write $\bar{b}_i=\bar{w}_i n_i$ in $\mathcal{G}$, where $\bar{w}_i$ is a product of conjugates of commutators $[\bar{b}_j, g]$, $g\in\mathcal{G}$, and $n_i\in N$. Since N is the union of the invisible subgroups and any two lie in a third, there is some invisible subgroup K containing all the $\{n_i\}$. Now consider the normal subgroup normally generated by $\{\bar{b}_j\}$ and K. The equations $\bar{b}_i=\bar{w}_i n_i$ show that this is an invisible subgroup of $\mathcal{G}$. Therefore $\{\bar{b}_j\}\subseteq N$ and so every $b_j=1$.

Proof of (c). Suppose S: $x_i=w_i(x_1, \ldots, x_m)$, $1\leqq i\leqq m$, is a contractible system over H with two solution sets $x_i=g_i$ and $x_i=h_i$. Then $x_i=1$ and $x_i=g_i h_i^{-1}$ are solution sets of the contractible system S': $x_i=w_i'(x_1, \ldots, x_m)$, $1\leqq i\leqq m$, where $w_i'=w_i(x_1 h_1, \ldots, x_m h_m)\, h_i^{-1}$. We will now show that w_i' is a product of conjugates of commutators $[x_j, a]$, where a can be any monomial. This will show that any solution set of S' normally generates an invisible subgroup of H, and so $g_i=h_i$, for all i.

Since $w_i'(1, \ldots, 1)=1$, we can first write w_i' as a product of conjugates of $\{x_j\}$: $w_i'=\prod_{r=1}^{s} a_r x_{i_r}^{\varepsilon_r} a_r^{-1}$, where $a_r\in H$, $\varepsilon_r=\pm 1$. This can be rewritten w_i' $=\left(\prod_{r=1}^{s} b_r [a_r, x_{i_r}^{\varepsilon_r}]\, b_r^{-1}\right) b_{s+1}$, where $b_r=x_{i_1}^{\varepsilon_1}, \ldots, x_{i_{r-1}}^{\varepsilon_{r-1}}$ $(b_1=1)$. But w_i' is contractible and so $b_{s+1}=1$, completing the proof.

We now have produced an algebraically closed group $\hat{G}$. Let ϕ: $G\to\hat{G}$ be defined as the composition of the obvious maps $G\to\mathcal{G}\to\hat{G}$. We show that this is an algebraic closure. Suppose f: $G\to A$ is any homomorphism to an algebraically closed group. There is obviously a unique extension to a homomorphism $F: \mathscr{G} \to A$ since the generators and relations that define $\mathscr{G}$ have unique solutions in A. But $N \subseteq \operatorname{Ker} F$, by Lemma 3, and so F induces a unique homomorphism $\hat{G} \to A$.

This completes the proof of Proposition 3.

6.

Example. Suppose L is an m-component *based* link of dimension >1 – i.e. a link with chosen meridian elements $\mu_1, \ldots, \mu_m \in \pi_1(S^{n+2} - L) = \pi$. If F is the free group with basis $x_1, \ldots, x_m$ and $\phi: F \to \pi$ defined by $\phi(x_i) = \mu_i$, then ϕ is 2-connected and so, by Proposition 5, $\hat{\phi}: \hat{F} \to \hat{\pi}$ is an isomorphism. If we perform framed surgery on S^{n+2} along the components of L, we obtain an oriented closed manifold M with $\pi_1(M) \approx \pi$. M represents an element of $H_{n+2}(\pi)$ and so, using $\pi \to \hat{\pi} \approx \hat{F}$, determines an element $\alpha \in H_{n+2}(\hat{F})$ which one sees easily, is a *based* concordance invariant of L.

Alternatively one can use the construction of Orr [O] and define a space $\hat{K}$ to be the mapping cone of the map $K(F, 1) \to K(\hat{F}, 1)$. Then the map $X \to K(\pi, 1) \to K(\hat{F}, 1) \to \hat{K}$, where $X = S^{n+2} - T$ (T a tubular neighborhood of L) extends, in a canonical way, to a map $S^{n+2} \to \hat{K}$. This gives an invariant in $\pi_{n+2}(\hat{K})$ which determines α via the Hurewicz homomorphism.

We will be interested in an analogous construction for 1-dimensional links.

7. It will be useful to have an alternative description of $\hat{G}$, when G is finitely presented.

Proposition 6. *Let G be a finitely presented group. There exists a sequence of groups and homomorphisms:*

$$G = P_0 \to P_1 \to P_2 \to \ldots \to P_n \to P_{n+1} \ldots$$

such that:

(i) $\hat{G} = \lim\limits_{n} P_n$ *and* $G \to \hat{G}$ *is the limit map.*

(ii) *Each* P_n *is finitely presented.*

(iii) $G \to P_n$ *is 2-connected, for every n.*

(iv) P_n *is the normal closure of G.*

Remark. Under the correspondence between algebraic closure and $\bar{\omega}$-localization mentioned in § 3, this proposition follows from Proposition 1.8 of [Le]. But we give a direct proof.

Proof. We refer to the proof of Proposition 3 in § 5. The set $\mathscr{C}$ of contractible systems over G is countable and so we can arrange them in a sequence. Define $\mathscr{G}_n$ by adjoining to G indeterminates for the first n systems in $\mathscr{S}$, and the relations defined by the equations. Clearly $\mathscr{G}_n$ is finitely presented. There are natural homomorphisms:

$$G \longrightarrow \mathscr{G}_1 \xrightarrow{i_1} \mathscr{G}_2 \xrightarrow{i_2} \ldots \longrightarrow \mathscr{G}_n \xrightarrow{i_n} \mathscr{G}_{n+1} \longrightarrow \ldots,$$

and $\mathscr{G} \approx \lim_n \mathscr{G}_n$. Furthermore, $G \to \mathscr{G}_n$ is a 2-connected homomorphism, as one can see e.g. from a consideration of the 2-complexes corresponding to the presentations. We will construct a sequence of invisible subgroups K_n of G_n (possibly passing to a subsequence) such that

(a) $i_n(K_n) \subseteq K_{n+1}$.

(b) For any invisible subgroup K of $\mathscr{G}$ there is some n so that $j_n^{-1}(K) \subseteq K_n$, where $j_n \colon \mathscr{G}_n \to \mathscr{G}$ is the natural homomorphism.

Since the number of invisible subgroups of $\mathscr{G}$ is countable and any two are contained in a third, we can find a nested sequence

$$L_1 \subseteq L_2 \subseteq \ldots \subset L_n \subseteq L_{n+1} \subseteq \ldots$$

of invisible subgroups of $\mathscr{G}$ whose union is N. For each n we can find $k = k_n$ such that $j_k^{-1}(L_n)$ is contained in an invisible subgroup of $\mathscr{G}_k$. In fact, if $b_1, \ldots, b_r$ is a set of normal generators of L_n and we write $b_i = w_i$, where w_i is a product of conjugates of commutators $[b_j, g]$, $g \in \mathscr{G}$, we can find k large enough, by the definition of direct limit, so that every b_j and other elements of $\mathscr{G}$ appearing in $\{w_i\}$ lift to elements of $\mathscr{G}_k$ and the equations $b_i = w_i$ are true in $\mathscr{G}_k$ after substituting in these lifts. We can now define $K(n) \subseteq \mathscr{G}_k$ to be the normal closure of the lifts of $\{b_i\}$. Clearly $K(n)$ is invisible. Since k_n can be chosen arbitrarily large we may assume $k_{n+1} > k_n$. Now for each $k = k_n$ define K_k to be the normal closure in $\mathscr{G}_k$ of the image of all $K(i)$ in $\mathscr{G}_k$, for $i \leqq n$, under the natural homomorphisms $\mathscr{G}_l \to \mathscr{G}_k$ defined for all $l \leqq k$. These obviously satisfy (a), (b).

We will show that $P_n = \mathscr{G}_n / K_n$ satisfy the requirements of Proposition 6. The homomorphisms $i'_n \colon P_n \to P_{n+1}$ are induced by i_n, using (a). Assertion (i) follows directly from (b) and the fact, already noted, that $\mathscr{G} \approx \lim_n \mathscr{G}_n$. Since $\mathscr{G}_n$ is finitely-presented, and K_n is normally generated by a finite set, (ii) follows. Since $G \to \mathscr{G}_n$ is 2-connected and $\mathscr{G}_n \to P_n$ is 2-connected by the Stallings exact sequence, (iii) follows. Finally (iv) follows because $\mathscr{G}_n$ is the normal closure of G – which follows from the contractibility of the systems of equations defining the relations of $\mathscr{G}_n$.

This completes the proof of Proposition 6.

8. We will need to consider certain automorphisms of $\hat{F}$, where F is the free group with basis $x_1, \ldots, x_m$.

Proposition 7. *For any elements $a_1, \ldots, a_m \in \hat{F}$, there exists a unique automorphism ϕ of $\hat{F}$ such that $\phi(x_i) = a_i x_i a_i^{-1}$, $i = 1, \ldots, m$. These automorphisms form a subgroup of the group of all automorphisms of $\hat{F}$.*

Proof. The homomorphism $\Phi \colon F \to \hat{F}$ defined by $\Phi(x_i) = a_i x_i a_i^{-1}$ is a closing and so extends to a unique endomorphism ϕ of $\hat{F}$. We must show that ϕ is bijective.

By Proposition 2 (addendum) any element $a \in \hat{F}$ belongs to a subgroup G which is finitely normally generated by F. We can also assume $a_1, \ldots, a_m \in G$. Then $\Phi F \subseteq G$ and G is finitely normally generated by ΦF. This implies that

G is generated by the solution set $\{g_i\}$ of some contractible system S over ΦF. Now $S=\phi(S')$ for some contractible system S' over F. If the solution set of S' is $\{g_i'\}\subseteq\hat{F}$, then $\phi(g_i')=g_i$. Thus $a\in G\subseteq\phi(\hat{F})$ and we see that ϕ is an epimorphism.

We can now construct an inverse for ϕ. Choose $b_i\in\hat{F}$ such that $\phi(b_i)=a_i^{-1}$ and define ψ by $\psi(x_i)=b_i x_i b_i^{-1}$. Then $\phi\circ\psi(x_i)=x_i$ and so, by the uniqueness property, $\phi\circ\psi=1$. On the other hand $\psi\circ\phi\circ\psi=\psi$ and so $\psi\circ\phi=1$ on $\psi(\hat{F})$. But ψ is surjective by the previous paragraph.

The proof is completed by the easy observation that the composition of two such automorphisms is another.

Definition. Automorphisms of the type in Proposition 7 will be called *special.*

II. Topology

9. We now turn our attention to one-dimensional links.

Definition. A *based link* of multiplicity m is a collection L of m disjoint imbedded circles (smooth) $L_1, \ldots, L_m$ in $\mathbb{R}^3=S^3-\infty$ with the components numbered and oriented, and a collection of meridian elements $\mu_1, \ldots, \mu_m\in\pi=\pi_1(S^3-L, \infty)$. The *i-th meridian* μ_i, by definition, is represented by a circle in S^3-L which bounds an imbedded (oriented) disk intersecting L at exactly one point, on L_i, with positive sign.

A based link has an associated homomorphism (a *basing*) $\Phi\colon F\to\Pi$ defined by $\Phi(x_i)=\mu_i$, $i=1, \ldots, m$. Since Π is finitely presented and normally generated by $\{\mu_i\}$, Φ induces an epimorphism $\hat{\Phi}\colon\hat{F}\to\hat{\Pi}$.

Definition. An *$\hat{F}$-link* is a based link L such that $\hat{\Phi}$ is an isomorphism with the (perhaps redundant) extra condition that the longitudes of L lie in Kernel $\{\Pi\to\hat{\Pi}\approx\hat{F}\}$.

Recall that an *i-th longitude* of L is an element of Π represented by a translate of L_i into S^3-L along a normal vector field to L_i, which is homologically unlinked with L_i. This latter condition is equivalent to demanding that longitudes lie in the commutator subgroup $[\Pi, \Pi]$.

If $\Phi_1, \Phi_2\colon F\to\Pi$ are different basings of a link L, then $\hat{\Phi}_2=\hat{\Phi}_1\circ\phi$ for some special automorphism ϕ of $\hat{F}$. In fact, we have $\Phi_2(x_i)=g_i\Phi_1(x_i)g_i^{-1}$ for some $g_i\in\Pi$. If we choose $a_1\in\hat{F}$ so that $\hat{\Phi}_1(a_i)=g_i$, then we can define ϕ by $\phi(x_i)=a_i x_i a_i^{-1}$, $i=1, \ldots, m$ (see Proposition 7). In particular, the property of being an $\hat{F}$-link is independent of basing.

Proposition 8. *Suppose L is an $\hat{F}$-link with basing $\Phi\colon F\to\Pi$. Then:*

(a) *Φ induces isomorphisms of lower central series quotients*

$$\Phi_n\colon F/F_n\approx\Pi/\Pi_n, \quad \textit{for all } n<\infty.$$

(b) *Any link concordant to L is also an $\hat{F}$-link.*

Proof. (a) Since $F \to F/F_n$ and $\Pi \to \Pi/\Pi_n$ are closings (Proposition 1), we have homomorphisms $\hat{\Pi} \to \Pi/\Pi_n$, $\hat{F} \to F/F_n$. It is an easy exercise, to see that the composite $\Pi \to \hat{\Pi} \approx \hat{F} \to F/F_n$ defines an inverse for Φ_n.

(b) Let $V \subseteq I \times S^3$ be a concordance from L to L'. The induced homomorphism $\Pi \to G = \Pi_1(I \times S^3 - V)$ is 2-connected and we can apply Proposition 5 to see $\hat{\Pi} \approx \hat{G}$-similarly for L'. Since meridians of L and L' are conjugate in $I \times S^3 - V$, we see that L' is an $\hat{F}$-link if and only if L is. Also notice that longitudes of L and L' are conjugate in G; thus, if one set is trivial in $\hat{G}$ then so is the other.

Remark. A, perhaps slightly stronger, but more concrete notion is what we might call a *strong $\hat{F}$-link.* We define this to be a based link L whose longitudes lie in an invisible subgroup N of Π. It can be shown that $H_2(\Pi/N) = 0$ and, as a consequence $\hat{F} \approx \widehat{(\Pi/N)} \approx \hat{\Pi}$; thus L is an $\hat{F}$-link. Conversely, it can be shown that every $\hat{F}$-link is concordant to a strong $\hat{F}$-link.

For a strong $\hat{F}$-link it is obvious that the longitudes lie in Π_∞ and so all the $\bar{\mu}$-invariants vanish. It is then of great interest to ask whether any link with vanishing $\bar{\mu}$-invariants is a (strong) $\hat{F}$-link.

10. Let L be a link, $\Pi = \Pi_1(S^3 - L)$ and $\tilde{\Pi}$ the quotient of Π by the normal closure of the longitudes. Recall the invariant $\tilde{\theta}(L) \in H_3(\tilde{\Pi})$ from [L]. If $S^3 - L \to K(\Pi, 1) \to K(\tilde{\Pi}, 1)$ is the obvious map, we can extend it uniquely to a map $f: M \to K(\tilde{\Pi}, 1)$, where M is obtained from S^3 by doing surgery along L with the 0-framing. Note that $\tilde{\Pi} = \Pi_1(M)$; we refer to M as the *surgery manifold* of L. Since M is a closed oriented manifold, f defines an element $\tilde{\theta}(L) \in H_3(\tilde{\Pi})$.

Now suppose L is a based $\hat{F}$-link. By the definition, the homomorphism $\Pi \to \hat{\Pi} \approx \hat{F}$ determines a homomorphism $\Psi: \tilde{\Pi} \to \hat{F}$ and we define $\theta(L) = \Psi_* \tilde{\theta}(L) \in H_3(\hat{F})$. Any other basing of L determines an automorphism ϕ of $\hat{F}$ and $\theta(L)$ is transformed into $\phi_* \theta(L)$. So strictly speaking we obtain, as an invariant of an unbased $\hat{F}$-link, an orbit in $H_3(\hat{F})$ of the action of the group $\mathscr{S}$ of special automorphisms of $\hat{F}$. However the statement $\theta(L) = 0$ is independent of basing.

Proposition 9. *If L and L' are concordant $\hat{F}$-links, then $\theta(L) \equiv \theta(L')$ in $H_3(\hat{F})/\mathscr{S}$.*

Proof. Let M, M' be the surgery manifolds of L, L' respectively, and $\tilde{\Pi}$, $\tilde{\Pi}'$ be $\Pi_1(S^3 - L)$, $\Pi_1(S^3 - L')$ modulo the normal closures of the longitudes. If V is a concordance from L to L', let $\tilde{G} = \Pi_1(I \times S^3 - V)$ modulo longitudes. Then one constructs from V a cobordism W between M, M', with $\Pi_1(W) \approx \tilde{G}$. The canonical maps $M \to K(\tilde{\Pi}, 1)$, $M' \to K(\tilde{\Pi}', 1)$ extend to a commutative diagram of maps:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & W \arrow[d] & M' \arrow[l] \arrow[d] \\ K(\tilde{\Pi}, 1) \arrow[r] & K(\tilde{G}, 1) & K(\tilde{\Pi}', 1) \arrow[l] \end{tikzcd}$$

This shows $\tilde{\theta}(L)$ and $\tilde{\theta}(L')$ have the same image in $H_3(\tilde{G})$.

Now choose basings $i: F \to \Pi$, $i': F \to \Pi'$. As usual we have isomorphisms:

$$\hat{F} \approx \hat{\Pi} \approx \hat{G} \approx \hat{\Pi}' \approx \hat{F}.$$

The automorphism of $\hat{F}$ induced by these isomorphisms is special since the images of corresponding meridians of L and L' are conjugate in G. The result follows. $\square$

It is not known whether $H_3(\hat{F})=0$. It is also not known whether $\hat{F}=\bar{F}$ where $\bar{F}$ (see [L]) is the image of the natural homomorphism $\hat{F} \to \lim_n F/F_n$.

11. We recall some special classes of links.

Definition. A link $L=(L_1, \ldots, L_m)$ is a *boundary link* if there exist disjoint oriented surfaces $V_1, \ldots, V_m$ in S^3 such that $\partial V_i = L_i$. These are referred to as *Seifert surfaces.*

It is a theorem of Smythe [Sm] that L is a boundary link if and only if there exists an epimorphism $\Phi: \Pi_1(S^3-L)=\Pi \to F$ such that, for some choice of meridians $\mu_1, \ldots, \mu_m$, $\{\Phi(\mu_i)\}$ generate F. In other words, some basing $F \to \Pi$ of L admits a left inverse.

Definition. L is a *homology boundary link* if there exists an epimorphism $\Pi \to F$.

This concept was introduced by Smythe [Sm]. It shares some of the pleasant properties of boundary link (e.g. Seifert matrices) but its geometric interpretation is more obscure. A homology boundary link admits "singular" Seifert surfaces: $V_1, \ldots, V_m$, where $\{V_i\}$ are disjoint oriented proper submanifolds of X, the complement of an open tubular neighborhood of L in S^3, such that $\partial V_i \subseteq \partial X$ is a union of longitudes but homologically equivalent to a single i-th longitude.

Note that any epimorphism $\Phi: \Pi \to F$ contains all the longitudes of L in its kernel. Let $\{\mu_i, \lambda_i\}$ be a system of meridians and longitudes in Π satisfying $[\mu_i, \lambda_i]=1$. Then $\{\Phi(\mu_i)\}$ gives a basis for $H_1(F)$ while $\Phi(\lambda_i) \in [F, F]$. On the other hand $\Phi(\mu_i)$ and $\Phi(\lambda_i)$ must be powers of a common element of F. The only possibility is $\Phi(\lambda_i)=1$.

Definition. An *SHB link* is one which is a sublink of a homology boundary link.

It is shown in [H] that any ribbon link is an SHB link and, more generally, in [C], that an interior band sum of a boundary link is an SHB link. It is an open question whether every slice link is an SHB link and, more generally, Tim Cochran has posed the question: If L is concordant to an SHB link, is L itself an SHB link?

Proposition 10. *If L is an SHB link, then L is an $\hat{F}$-link and $\theta(L)=0$.*

12. To prove this we recall some more definitions from [C].

Definition. (a) A group G is an *E-group* if there exists a 2-complex K satisfying:
(i) $\Pi_1(K) \approx G$.
(ii) $H_1(K)$ is torsion-free.
(iii) $H_2(K)=0$.

If K is finite, then G is a *finite E-group*. These conditions can then be expressed algebraically as follows.

(b) G is a *finite E-group* if $H_1(G)$ is torsion-free and G admits a finite presentation of deficiency $d = \operatorname{rank} H_1(G)$; the deficiency of a presentation is the difference between the number of generators and relations.

(c) A *finite E-link* is one which admits a homomorphism $\Phi: \Pi \to G$, where G is a finite E-group, satisfying:

(i) $\Phi(\Pi)$ normally generates G
(ii) $\operatorname{rank} H_1(\Pi) = \operatorname{rank} H_1(G)$
(iii) the longitudes of L lie in kernel Φ.

We will refer to Φ as an *E-calibration* of L. It is pointed out in [C] that every SHB link is a finite E-link. It is shown in [L] that any finite E-link L with the extra property: $\Phi_* \tilde{\theta}(L) = 0$ in $H_3(G)$ is concordant to an SHB link. Note that $H_3(G) = 0$ if the Whitehead conjecture is true.

Proof of Proposition 10. We first show that any finite E-link is an $\hat{F}$-link. Consider the homomorphisms $F \xrightarrow{i} \Pi \xrightarrow{\Phi} G$ where i is a basing of L and Φ is an E-calibration of L. We can apply Proposition 5 to Φ and $\Phi \circ i$, since $H_2(G) = 0$, to conclude $\hat{F} \approx \hat{\Pi} \approx \hat{G}$. Thus L is an $\hat{F}$-link.

Now suppose L is an SHB-link and $\Phi: \tilde{\Pi} \to G$ the E-calibration of L produced in Proposition 6 of [L]. We will show that $\phi_* \tilde{\theta}(L) = 0$. Let L^0 be a homology boundary link which contains L as a sublink and $f^0: X^0 \to W$ a map, where X^0 is the complement of a tubular neighborhood of L^0, W a one-point union of circles (one for each component of L^0), such that f^0 induces an epimorphism of fundamental groups. Now each component C_i of ∂X^0 is identified with $L_i^0 \times S^1$, where L_i^0 is the corresponding component of L^0, by the tubular neighborhood theorem. We may assume that $f^0 | C_i = f_i \circ p_i$, where $p_i: C_i \to S^1$ is projection, and $f_i: S^1 \to W$ some map. This follows because $f^0 | L_i^0 \times t$ is null-homotopic, by (iii) above, since L^0 is a finite E-link, $(t \in S^1)$ and $\pi_2(W) = 0$. We now construct a 2-complex P from W by using the $\{f_i\}$ corresponding to components of $L^0 - L$ to attach 2-cells. It is clear that we can then extend f^0 to $f: X \to P$. Then $\Phi = f_*$ and, by Lemma 9 of [L] (this implication does not require Φ onto), it follows that $\Phi_* \tilde{\theta}(L) = 0$. Note that f extends to a map $M \to P$, where M is the surgery manifold of L. □

13. Our first main result will be:

Theorem 1. *Let L be an $\hat{F}$-link. Then $\theta(L) = 0$ if and only if L is concordant to an SHB-link.*

Proof. One direction follows from Proposition 9, 10. Before beginning to prove the other direction, we present some more definitions and preliminary results.

Definition. A finitely presented group G is an *F-group* if $H_1(G)$ is torsion-free and $H_2(G) = 0$.

Examples. (i) According to [K], G is the fundamental group of the complement of a link of dimension $\geqq 3$ if and only if G is an F-group with weight $G = \operatorname{rank}$

$H_1(G)$ – the weight is the smallest number of elements which normally generate G.

(ii) If G is an F-group, then so are the $\{P_n\}$ of Proposition 6.

(iii) Any finite E-group is an F-group.

Definition. (a) A *calibration* of an m-component link L is a homomorphism $\Phi\colon \pi \to G$ satisfying:

(i) G is an F-group and rank $H_1(G)=m$.

(ii) G is normally generated by $\Phi(\pi)$.

(iii) The longitudes of L lie in kernel Φ.

(b) If L is a calibrated link, with calibration Φ, then set $\theta(L, \Phi) = \Phi_* \tilde{\theta}(L) \in H_3(G)$.

Lemma 4. *A link L admits a calibration if and only if it is an $\hat{F}$-link. Furthermore $\theta(L)=0$ if and only if $\theta(L, \Phi)=0$ for some calibration Φ.*

Proof. If L is calibrated, consider $F \xrightarrow{i} \Pi \xrightarrow{\Phi} G$ where i is a basing, Φ a calibration. We apply Proposition 5 to i and $\Phi \circ i$ to conclude $\hat{F} \approx \hat{\Pi} \approx \hat{G}$. Then $\theta(L)$ can be identified with the image of $\tilde{\theta}(L)$ under the composition $\tilde{\Pi} \to G \to \hat{G} \approx \hat{F}$. Since $\theta(L, \Phi)$ occurs during this passage we see that $\theta(L)=0$ if $\theta(L, \Phi)=0$.

To prove the converse we need Proposition 6. Consider the composition $\tilde{\Pi} \to \hat{\Pi} \approx \hat{F}$. Since $\tilde{\Pi}$ is finitely presented this homomorphism lifts to a homomorphism $\phi\colon \tilde{\Pi} \to P_n$ (we use the notation of Proposition 6 with $G=F$). To see this, first lift $\{\Phi(g_i)\}$ to some P_n, where $\{g_i\}$ is a finite set of generators of $\tilde{\Pi}$. If $r(g_1, \ldots, g_k)=1$ is any relation in $\tilde{\Pi}$, then $\Phi(r) \in P_n$ goes to 1 in the limit. Therefore we can assume $\Phi(r)=1$ by increasing n. Applying this argument to a finite presentation of $\tilde{\Pi}$ yields Φ. We can, in fact, include some choice of meridians among the generators and specify Φ on these elements so that $\Phi \circ i$ agrees with the given homomorphism $F \to P_n$, where i is a basing. Clearly Φ gives the desired calibration.

Now $\theta(L, \Phi) \to \theta(L)$ in the limit $P_n \to \hat{F}$. Since $H_*(\hat{F}) = \lim_n H_*(P_n)$, we can arrange that $\theta(L, \Phi)=0$, if $\theta(L)=0$, by increasing n sufficiently. □

14. We will also need to use the geometric lemmas of [L] which we recall here with a slightly more general formulation.

Suppose $L=\{L_i\}$ is any collection of disjoint m-dimensional closed submanifolds of an $(n+2)$-manifold M and $f\colon V \to X$ is any map, where V is the complement in M of a tubular neighborhood T of L. We will say f is *vertical on* L (via Φ) if $f \mid \partial T$ is the composition of fiber trivializations $\Phi\colon \partial T \approx L \times S^1$, followed by projections $L_i \times S^1 \to S^1$ and maps $S^1 \to X$, one for each L_i.

Lemma A. *Let $L=(L_1, \ldots, L_m)$ be a collection of disjoint closed n-dimensional submanifolds of a connected $(n+2)$-dimensional manifold M. Let X be any space and $\rho\colon M-L \to X$ a map satisfying, where $H=\pi_1(X)$ and $K=$*Kernel *$\{\pi_1(M-L) \to \pi_1(M)\}$:*

(i) *H is finitely generated.*

(ii) *H is generated by Image ρ_* together with the normal closure of $\rho_*(K)$.*

Then there is a concordance $V \subseteq I \times M$ from L to L' and an extension of ρ to $\bar{\rho}\colon (I \times M) - V \to X$ satisfying:

(a) *$\bar{\rho}_*$ is onto.*

(b) *$\pi_1(M-L')\to\pi_1((I\times M)-V)$ is onto.*

(c) *If ρ is vertical on L via Φ, then $\bar{\rho}$ is vertical on V via $\bar{\Phi}$ extending Φ.*

(d) *If $\{\mu_i\}$ are meridians of L, there exist meridians $\{\mu_i'\}$ of L' so that $\mu_i=\mu_i'$ in $\pi_1((I\times M)-V)$.*

Lemma B. *Let L, M, X, ρ be as in Lemma A and assume ρ_* is onto. Suppose μ_1, μ_2 are meridians of L_1, L_2 and $\rho_*(\mu_1)$ is conjugate to $\rho_*(\mu_2)$ in H. Then there exists a cobordism $V\subseteq I\times M$ from L to L', where $L'=(L_2', L_3', \dots, L_m')$, $L_i'=L_i$ for $i\geqq 3$, L_2' is a connected sum $L_1 \# L_2$, and $V=(V_2', \dots, V_m')$ where $V_i'=I\times L_i$ for $i\geqq 3$ and V_2' is a boundary connected sum $I\times L_1 \perp\!\!\!\perp I\times L_2$. Furthermore ρ extends to a map $\bar{\rho}\colon (I\times M)-V\to X$ such that, if $\rho'=\bar{\rho}\,|\,M-L'$, then ρ_*' is onto. If ρ is vertical on L, then $\bar{\rho}$ is vertical on V.*

The proofs of these lemmas are identical to those in [L]. It is only necessary to remark that the concordances in the proofs are constructed by adding handles of index 1 and 2 and so extending maps in the present formulation is possible exactly when homomorphisms of the fundamental group extend. By taking X to be an Eilenberg-MacLane space, we see that the present lemmas generalize those in [L].

15. We now start the proof of the theorem. By Lemma 4 we may assume L has a calibration Φ with $\theta(L, \Phi)=0$. By Lemma A, L is concordant to a calibrated link (L', Φ') where Φ' is onto. Since the calibration extends over the complement of the concordance $\theta(L', \Phi')=0$. We now apply Lemma 9 of [L]. Changing notation, we have a link L with calibration $\Phi\colon \tilde{\Pi}\to G$ such that $\theta(L, \Phi)=0$ and Φ is onto. Let K be any 2-complex with $\Pi_1(K)=G$. If M is the surgery manifold of L, then, because $\theta(L, \Phi)=0$ we conclude from Lemma 9 that Φ is induced by a map $f\colon M\to K$. Let X be the complement in S^3 of a tubular neighborhood of L. Then $X\subseteq M$ and $M-X$ is a tubular neighborhood of a collection of circles in M (translates of meridian curves of L). Therefore if we consider $g=f\,|\,X$, we may assume g is vertical on L and $g(\partial X)$ lies in the 1-skeleton of K.

We will take K to be the 2-complex associated to a certain type of "pre-abelian" presentation of G. Let $\{\mu_i\}$ be a choice of meridians of L and $x_i=\Phi(\mu_i)\in G$. Let $y_1, \dots, y_k$ be a finite set of generators for G. Since $\{\mu_i\}$ normally generate Π and image Φ normally generates G, we may write: $y_i=w_i(x_1, \dots, x_m, y_1, \dots, y_k)$, $i=1, \dots, k$, where w_i is a formal product of conjugates of $\{x_j^{\pm 1}\}$. In other words, if w_i is viewed as a monomial over F, the free group on $\{x_i\}$, with indeterminates $\{y_i\}$, then w_i is contractible over F. Define $\bar{G}$ to be the group with generators $\bar{x}_1, \dots, \bar{x}_m$, $\bar{y}_1, \dots, \bar{y}_k$ and relations $\bar{y}_i=w_i(\bar{x}_1, \dots, \bar{x}_m, \bar{y}_1, \dots, \bar{y}_k)$, $i=1, \dots, k$. Note that $\bar{G}$ is a finite E group. The obvious epimorphism $\rho\colon \bar{G}\to G$, defined by $\rho(\bar{x}_i)=x_i$, $\rho(\bar{y}_i)=y_i$ has kernel N and we can choose elements $a_1, \dots, a_r$ which normally generate N in $\bar{G}$. If we choose a representation $a_i=v_i(\bar{x}_1, \dots, \bar{x}_m, \bar{y}_1, \dots, \bar{y}_k)$, then the desired presentation of G is:

$$\left\{\begin{aligned} x_1, \dots, x_m, y_1, \dots, y_k\colon y_i&=w_i(x_1, \dots, x_m, y_1, \dots, y_k),\ i=1, \dots, k\\ 1&=v_i(x_1, \dots, x_m, y_1, \dots, y_k),\ i=1, \dots, r\end{aligned}\right\}.$$

If K is the 2-complex corresponding to this presentation, let $P \subseteq K$ be the subcomplex defined by omitting the 2-cells corresponding to $\{v_i = 1\}$. Clearly $\Pi_1(P) \approx \bar{G}$ so that ρ is induced by the inclusion $P \subseteq K$.

Returning to our map $g: X \to K$, choose interior points $\{p_i\}$, one in each 2-cell of $K - P$; we may assume $\{p_i\}$ are regular values of g. Let $L'_i = g^{-1}(p_i)$, a collection of framed circles in $S^3 - L$, and set $L' = \{L'_i\}$, $L^0 = L \cup L'$. Let X^0 denote the complement of a tubular neighborhood of L^0, so that $X^0 \subseteq X$ and we have commutative diagrams:

$$\begin{array}{ccc} X^0 & \xrightarrow{g^0} & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & K \end{array} \qquad \begin{array}{ccc} \tilde{\Pi}^0 & \xrightarrow{\Phi^0} & \bar{G} \\ \downarrow & & \downarrow \\ \tilde{\Pi} & \xrightarrow{\Phi} & G \end{array}$$

where the vertical maps are inclusions and their induced homomorphisms, g^0 is the restriction of g, $\tilde{\Pi}^0$ is $\Pi_1(S^3 - L^0)$ modulo the parallels of L^0 determined by g^0, and Φ^0 is induced by g^0. Note that g^0 is vertical on L^0 and we may arrange that $\phi^0(\bar{\mu}_i) = \bar{x}_i$, where $\bar{\mu}_i$ is a meridian of L_i.

We now apply Lemma A to change L' by a concordance in $S^3 - L$ so that Φ^0 is onto. Hypothesis (ii) of Lemma A is satisfied because Φ is already onto – see the proof of Theorem 4 of [L] for more details. L is unchanged by this step and g^0 is still vertical on L^0. Now we can apply Lemma B to make each L'_i connected, as in the proof of Theorem 4 in [L]. Note that a meridian of L'_i maps to a conjugate of a_i in $\bar{G}$.

16. We now construct a collection of disjoint circles $L'' = \{L''_1, \ldots, L''_r\}$, one for each L'_i, with the following properties:

(i) In $S^3 - L'$, an isotopy of L'' will make each L''_i a small meridian circle of L'_i,

(ii) In $S^3 - L$, the components of L'' bound disjoint disks.

(iii) The isotopy of (i) extends to an isotopy in S^3 of the meridian disks bounded by L'' to the disks in (ii).

(iv) Each L''_i represents an element of Kernel Φ^0.

Since the quotient homomorphism $\bar{G} \to G$ is 2-connected, it follows from the Stallings exact sequence [S] that $[\bar{G}, N] = N$. Thus we may write a_i as a product of conjugates of commutators of the form $[g, a_j]^\varepsilon$, $\varepsilon = \pm 1$, $g \in \bar{G}$. Since G is normally generated by $\{\bar{x}_i\}$, we may therefore write:

$$a_i = \prod_s [\bar{\alpha}_{is}, \bar{\beta}_{is}]^{\varepsilon} s; \qquad \varepsilon_s = \pm 1; \qquad i = 1, \ldots, r$$

where $\bar{\alpha}_{is}$ is a conjugate of some $\bar{x}_j$, and $\bar{\beta}_{is}$ is a conjugate of some a_j. Since Φ^0 is onto we can choose meridians $\{\alpha_{is}\}$ of L and $\{\beta_{is}\}$ of L' so that $\bar{\alpha}_{is} = \Phi^0(\alpha_{is})$, $\bar{\beta}_{is} = \Phi^0(\beta_{is})$. Finally choose a meridian α_i of L'_i so that $\Phi^0(\alpha_i) = \mathrm{a}_i$ and define:

$$\eta_i = \prod_s [\alpha_{is}, \beta_{is}]^{\varepsilon_s} \qquad i = 1, \ldots, r$$

$$\xi_i = \alpha_i^{-1} \eta_i.$$

We will choose L''_i to represent ξ_i.

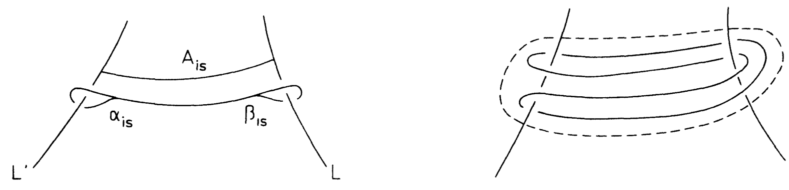

Fig. 1

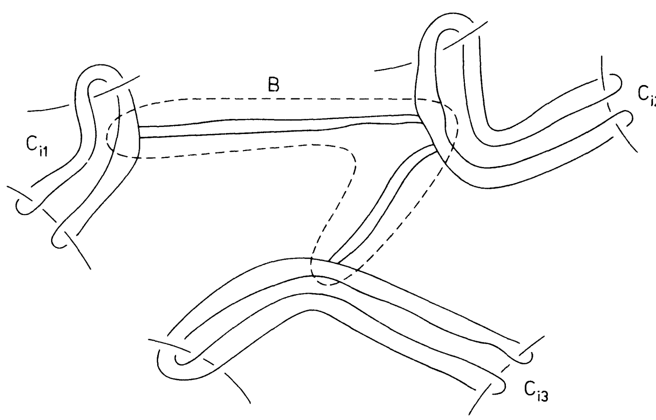

Fig. 2

First note that each $[\alpha_{is}, \beta_{is}]$ can be represented by a circle C_{is} which lies in a regular neighborhood N_{is} of some arc A_{is} connecting a component of L with a component of L'. Furthermore C_{is} bounds an imbedded disk in $N_{is}-L$ and in $N_{is}-L'$. These disks are isotopic, rel C_{is}, in N_{is}. In fact, there is a 3-ball B_{is} inside N_{is} such that $C_{is} \subseteq \partial B_{is}$ and separates ∂B_{is} into the desired disks. See Fig. 1.

We may certainly choose the $\{A_{is}\}$ to be disjoint and, therefore, also the $\{N_{is}\}$. However, to keep track of the basepoint we choose a ball B in S^3-L^0 meeting each B_{is} in a sub-ball containing the basepoint of C_{is}. We now band sum the $\{C_{is}\}$ together, in the proper order, choosing the bands inside B, missing the interior of every B_{is}, connecting the components of $C_{is} \cap B$ containing the basepoints. See Fig. 2.

The result is a collection of circles $\{C_i\}$ in S^3-L^0 representing $\{\eta_i\}$ which bound disjoint disks D_i in S^3-L and D_i' in S^3-L'. Furthermore there is an isotopy of S^3 rel $\{C_i\}$ moving $\{D_i\}$ onto $\{D_i'\}$. To construct $\{L_i''\}$ we now only need band sum $\{C_i\}$, again inside B, to meridian curves representing $\{\alpha_i^{-1}\}$.

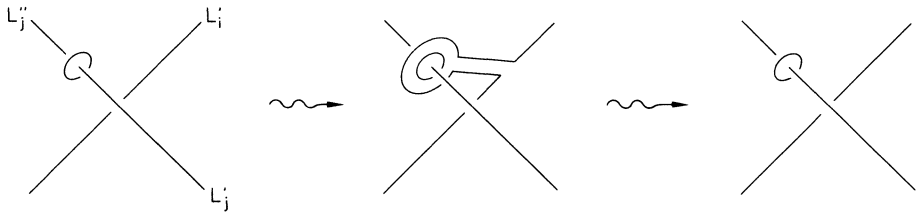

Fig. 3

These latter curves can be chosen to bound disks in $S^3 - L$ which are disjoint from each other, all the $\{B_{is}\}$ and the bands used to construct $\{C_i\}$. The resulting L'' clearly satisfies (i)–(iv).

17. We now perform surgery on S^3 along the components of $L' \cup L''$. We specify the 0-framing on the components of L'', while L' has the framing with respect to which g^0 is vertical. Let Σ be the resulting oriented manifold and consider $L \subseteq \Sigma$. We will prove:

(a) Σ is diffeomorphic to S^3.
(b) $L \subseteq \Sigma$ is concordant to $L \subseteq S^3$.
(c) $L \subseteq \Sigma$ is concordant to an SHB link.

These assertions will, of course, prove the theorem. Assertion (a) follows from (i) with an application of the Kirby calculus [Ki]. If we choose a projection picture of L' in which the components of L'' are small meridian circles, then any crossing of L' can be changed by a handle-slide over an appropriate component of L'' – see Fig. 3.

Therefore L' can be transformed into the trivial link by handle slides and now $\{L_i' \cup L_i''\}$ is a collection of Hopf links lying in disjoint balls. The components of L'' still have the 0-framing. But it is well-known that surgery on this link, whatever the framing on L', produces S^3. This proves (a).

To prove (b) consider the cobordism V between Σ and S^3 produced by adding handles to S^3 along the components of $L' \cup L''$ with the given framings. There is a concordance C between $L \subseteq S^3$ and $L \subseteq \Sigma$ defined by $I \times L \subseteq V$, but of course $V \neq I \times S^3$. We will do framed surgeries on 2-spheres in $V - C$ which will transform V into $I \times S^3$. By (ii) the components of L'' bound disjoint disks $\{D_i\}$ in $S^3 - L$. The desired 2-spheres are obtained by taking the union of $\{D_i\}$ (pushed into the interior of V) with cores of the handles added along L''. Since L'' has the 0-framing these 2-spheres have trivial normal bundles. We denote by W the result of surgery on V along these framed 2-spheres. Note that the diffeomorphism type of W is unchanged if we change the $\{D_i\}$ by an isotopy, rel L'', in S^3. In particular, by (i) and (iii), we may assume that L'' and $\{D_i\}$ are meridian circles and disks of L'. Furthermore the handle slides used above to transform $L' \cup L''$ into a union of Hopf links do not change the diffeomorphism type of V or W.

We now have a representation of V as a boundary connected sum $\coprod_i V_i$, where each V_i is obtained by plumbing together a trivial 2-disk bundle over

S^2 with some other 2-disk bundle over S^2. W is then obtained from V by doing surgery on the 0-sections of the trivial bundle in each V_i. We leave it to the reader to convince himself that this gives $I\times S^3$.

Finally we prove (c) by showing that $L\subseteq\Sigma$ is a finite E-link, as defined in [C], and invoking Theorem 4 of [L]. Recall the map $g^0: X^0\to P$ inducing $\Phi^0: \bar{\Pi}^0\to\bar{G}$. We will show that g^0 induces a map $Y\to P$, where Y is the complement of a tubular neighborhood of L in Σ, which is vertical on L, and the induced homomorphism $\Pi_1(Y)\to\bar{G}$ is onto. Condition (ii) of Theorem 4 of [L] is satisfied by Lemma 9 of [L]. In fact condition (ii) serves only to create a map to a 2-complex realizing the given homomorphism, but in our situation the map is already given.

We show that g^0 extends over the surgeries performs on S^3 to create Σ. First of all, since g^0 is vertical on L', we have an obvious extension of g^0 over the copies of $S^1\times D^2$ attached to ∂X^0 along the components of L', since the framing of L' is the one with respect to which g^0 is vertical.

As for the surgeries along L'', we can invoke property (iv) of L'' to extend g^0 over the handles adjoined along L''. We thus obtain the desired map $g': Y\to P$ which agrees with g^0 on X^0-L''. In particular, since $\Pi_1(X^0-L'')\to\Pi_1(X^0)$ is onto and $\Phi^0=g^0_*$ is onto, we conclude that g'_* is onto.

This completes the proof of Theorem 1. □

18. As companion to Theorem 1 we have the following realization result:

Theorem 2. *For any element $\alpha\in H_3(\hat{F})$, there is a (based) $\hat{F}$-link L such that $\theta(L)=\alpha$.*

Proof. To prove this we must reinterpret $\theta(L)$ as an invariant similar to that defined by Kent Orr [O] and then use an argument similar to that in the proof of Theorem 2 of [L].

Consider the map $i: K(F,1)\to K(\hat{F},1)$, of Eilenberg-MacLane complexes, inducing the canonical inclusion $F\to\hat{F}$. Let $\hat{K}$ be the mapping cone of i. Equivalently, $\hat{K}$ is obtained by attaching 2-cells to $K(\hat{F},1)$ via attaching maps representing the basis elements $\{x_i\}\subseteq F\subseteq\hat{F}$. If L is a based $\hat{F}$-link we are provided with a homomorphism $\Phi: \bar{\Pi}\to\hat{F}$, where $\bar{\Pi}=\Pi_1(S^3-L)$ modulo the normal closure of the longitudes, inducing a map $f: M\to K(\hat{F},1)$ where M is the surgery manifold of L. We can then obtain S^3 from M by surgery along the meridians which define the basing of L. By the definition of $\hat{K}$, there is a canonical map $f': S^3\to\hat{K}$ obtained by extending f over the surgery. Note that the trace V of the surgery defines a cobordism between S^3 and M and f, f' extend to a map $V\to\hat{K}$. Therefore, if $\theta'(L)\in\Pi_3(\hat{K})$ is the homotopy class of f', then we have (*) $h\theta'(L)=j_*\theta(L)$ where $h: \Pi_3(\hat{K})\to H_3(\hat{K})$ is the Hurewicz homomorphism and $j: K(\hat{F},1)\to\hat{K}$ is the canonical cofibration.

Now $\hat{K}$ is 1-connected since F normally generates $\hat{F}$. By Proposition 4, the 2-cells of $\hat{K}$ are attached along a basis of $H_1(\hat{F})$ and so $j_*: H_i(\hat{F})\approx H_i(\hat{K})$ for $i>1$. Since $H_2(\hat{F})=0$, again by Proposition 4, $\hat{K}$ is 2-connected. We conclude that h and j_* are both isomorphisms and so formula (*) identifies $\theta(L)$ and $\theta'(L)$.

19. Suppose $\alpha\in H_3(\hat{F})$ and let $\alpha'\in\Pi_3(\hat{K})$ be defined by $h(\alpha')=j_*(\alpha)$. Choose $f': S^3\to K$ representing α' and interior points $p_1,\ldots,p_m$, one in each 2-cell of

$\hat{K}-K(\hat{F},1)$. We may assume these are regular values of f' and define $L_i=(f')^{-1}(p_i)$; then f' restricts to a map $f\colon S^3-L\to K(\hat{F},1)$ which is vertical on L. We may assume each L_i is non-empty. Let M be the surgery manifold of L-since L_i may be disconnected, this means we do framed surgery on each component of each L_i, using the framing defined by the verticality of f. Note that this is the 0-framing if L_i if connected. Then f has a canonical extension over M, $f''\colon M\to K(\hat{F},1)$ and $\alpha=f''_*[M]$. Also notice that for any meridian μ_i of any component of L_i, $f_*(\mu_i)$ is conjugate in $\hat{F}$ to $x_i\in F\subseteq\hat{F}$. It follows that, if each L_i were connected then L would be a link and $\theta(L)\equiv\alpha$ modulo the action of $\mathscr{S}$, the group of special automorphisms.

In order to make each L_i connected we will apply Lemma's A and B to change (L,f) by a concordance or cobordism in $I\times S^3$. We point out now that the resulting (L',g) with its associated surgery manifold M' and extension $g''\colon M'\to K(\hat{F},1)$ satisfies $g''_*[M']=f''_*[M]=\alpha$ and $g_*(\mu_i)$ is conjugate to x_i for any meridian μ_i of L'_i. To see this first note that f,g extend to a map $F\colon I\times S^3-V\to K(\hat{F},1)$ vertical on V, where V is the concordance or cobordism, and meridians of L_i and L'_i are conjugate in $I\times S^3-V$. Secondly, we will use V to construct a cobordism W between M and M' and extend f'', g'' over W. Attach handles to $I\times S^3$ along $L\subset 0\times S^3$, $L'\subseteq 1\times S^3$ using framings which come from a framing of V. This defines a cobordism W' between M and M' but the maps f'', g'' certainly do not extend over W'. Consider, inside W', the submanifold A obtained by adjoining the cores of the handles to $V\subseteq I\times S^3$. Since the components of V are surfaces of genus 0, A is a collection of 2-spheres. By the choice of framings these 2-spheres have trivial normal bundles. The map F together with f'', g'' defines a map $F'\colon W'-A\to K(\hat{F},1)$ which is vertical on A. We now define W to be obtained from W' by surgery on the components of A. F' extends to $F''\colon W\to K(\hat{F},1)$ giving the desired bordism between f'' and g''.

We now show how to apply Lemmas A and B to make each L_i connected. Suppose some L_i is disconnected and μ, ν are meridians of two components of L_i. Then $f_*(\mu)=gf_*(\nu)g^{-1}$ for some $g\in\hat{F}$. If $g\in f_*\Pi_1(S^3-L)$ we can apply Lemma B immediately to replace L by a link with fewer components. If $g\notin f_*\Pi_1(S^3-L)$ we first apply Lemma A. According to Proposition 3 and the addendum to Proposition 2, there is some finitely generated subgroup $H\subseteq\hat{F}$ which contains $f_*\Pi_1(S^3-L)$ and the element g and is normally generated by F. Since the meridians of L all map to conjugates of $\{x_i\}$, this implies that the images of the meridians normally generate H. (Note that we need each L_i to be non-empty for this conclusion). H now satisfies the hypotheses of Lemma A. We use condition (d) of Lemma A to choose meridians μ', ν' of components of L'_i so that $g_*(\mu')=gg_*(\nu')g^{-1}$, but now $g\in g_*\Pi_1(S^3-L')$ and we can apply Lemma B to L' as above.

20. To complete the proof of Theorem 2 we prove: If L is any based $\hat{F}$-link and ϕ is any special automorphism of $\hat{F}$, then there is a based $\hat{F}$-link L' such that $\theta(L')=\phi_*\theta(L)$.

Let $\Phi\colon\Pi\to\hat{F}$ be the homomorphism associated to the basing of L and suppose $\phi(x_i)=g_i x_i g_i^{-1}$, $g_i\in\hat{F}$. If $\{g_i\}\subseteq\text{Image }\Phi$ then we can rechoose the meridians of L so that the associated homomorphism is $\phi \circ \Phi$. This gives the desired L'. If $\{g_i\} \nsubseteq \text{Image } \Phi$, we apply Lemma A. As above, it follows from Proposition 3 and the addendum to Proposition 2 that there is a finitely generated subgroup $H \subseteq \hat{F}$ which contains Image Φ, $\{g_i\}$ and is normally generated by F. The basing of L consists of meridians $\{\mu_i\}$ such that $\Phi(\mu_i) = x_i$ and so H satisfies the hypotheses of Lemma A. The resulting concordant link L' comes equipped with $\Phi': \Pi' \to \hat{F}(\Pi' = \Pi_1(S^3 - L'))$ and, by (d), meridians $\{\mu_i'\}$ such that $\Phi'(\mu_i') = x_i$. Furthermore, the discussion above shows $\theta(L') = \Phi'_*[M'] = \Phi_*[M] = \theta(L)$. But now $\{g_i\} \subseteq \text{Image } \Phi'$ and we can change the basing of L' so that $\theta(L') = \phi_* \theta(L)$.

This completes the proof of Theorem 2. □

III. Conclusion

21. We point out some remaining questions.

(i) What is $H_3(\hat{F})$?

(ii) Is every link with vanishing $\bar{\mu}$-invariants an $\hat{F}$-link? This is closely related to the question of whether $\hat{F} = \bar{F}$, the algebraic closure of F in its nilpotent completion ($\bar{F}$ is used in [L]). This question, in turn, is related to the question of whether $H_2(\bar{F}) = 0$. There is a transfinite tower construction of the (transfinite) lower central series of $\hat{G}$ similar to that used in [B] for the HZ-localization. Then $H_2(\bar{F}) = 0$ if and only if this tower, for $\hat{F}$, terminates at the first infinite ordinal and so $\bar{F}$ is the transfinitely nilpotent quotient of $\hat{F}$.

(iii) Is every homology boundary link concordant to a boundary link? As some slight positive evidence, it is not hard to show that any elements $\alpha_1, \ldots, \alpha_m \in F$ which normally generate F can be realized as the image of meridians of some ribbon homology boundary link.

(iv) What is the situation in dimensions $n > 1$? Every link is an $\hat{F}$-link and the analogue of $\theta(L)$ is defined, as in § 17, as an element of $\Pi_{n+2}(\hat{K})$. As in Theorem 2 of [L], every element of $\Pi_{n+2}(\hat{K})$ arises from a link L in which the components are connected but may not be spheres. The methods of [Le] seem appropriate – it has already been points out in § 6 that $\hat{F} \approx \Pi_1(EW)$, where EW is the Vogel localization of $W = K(F, 1)$.

References

[B] Bousfield, A.: Homological localization towers for groups and π-modules. Mem. Am. Math. Soc. **186**, (1977)

[C] Cochran, T.D.: Link concordance invariants and homology of nilpotent groups. Invent. Math. **90**, 635–646 (1987)

[CS] Cappell, S., Shaneson, J.: The codimension two placement problem and homology equivalent manifolds. Ann. Math. **99**, 277–348 (1974)

[CS1] Cappell, S.: Link cobordism. Comment. Math. Helv. **55**, 20–49 (1980)

[D] Duval, J.: Forme de Blanchfield et cobordisme d'entrelacs bords. C.R. Acad. Sci. Paris 299, p. 935–938 (1984)

[G] Gutierrez, M.: Concordance and homology I. Fundamental group. Pac. J. of Math. **82**, 75–91 (1979)

[G1] Gutierrez, M.: Unlinking up to cobordism. Bull. Am. Math. Soc. **79**, 1299–1302 (1973)

[H] Hillman, J.: Trivializing ribbon links by Kirby moves. Bull. Aust. Math. Soc. **21**, 21–28 (1980)
[Ke] Kearton, C.: Cobordism of knots and Blanchfield duality. J. London Math. Soc. **10**, 406–408 (1975)
[K] Kervaire, M.: Les noeuds de dimensions superieures. Bull. Soc. Math. France **93**, 225–271 (1965)
[Ki] Kirby, R.: A calculus for framed links in S^3. Invent. Math. **45**, 35–56 (1978)
[Ko] Ko, K.H.: Seifert matrics and boundary links. Trans. Am. Math. Soc. **299**, 657–683 (1987)
[Le] LeDimet, J.: Cobordisme d'enlacements de disques. Preprint Univ. de Nantes
[L] Levine, J.: Link concordance and algebraic closure of groups. Comment. Math. Helv. (to appear)
[L1] Levine, J.: Knot Cobordism Groups in Codimension Two. Comment. Math. Helv. 44, 229–244 (1969)
[M] Milnor, J.: Isotopy of links. Algebraic Geometry and topology: A Symposium in Honor of S. Lefschetz, pp. 280–306. Princeton, N.J.: Princeton Univ. Press 1957
[O] Orr, K.: New link invariants and applications. Comment. Math. Helv. **62**, 542–560 (1987)
[Sa] Sato, N.: Cobordisms of semi-boundary links. Topology Appl. **18**, 225–234 (1984)
[Sm] Smythe, N.: Boundary links. Topology Seminar: Wisconsin 1965. Bing, R.H. (ed.) (Annals of Math. Study Vol. 60, pp. 69–72). Princeton, N.J.: Princeton Univ. Press 1966
[S] Stallings, J.: Homology and central series of groups. J. Algebra **2**, 170–181 (1965)
[Ho] Howie, J.: How to generalize one-relator group theory. Combinatorial group theory and topology. Gersten, S., Stallings, J. (eds.), (Annals of Math. Studies Vol. 111). Princeton, N.J.: Princeton Univ. Press 1987
[N] Neumann, B.H.: A note on algebraically closed groups. J. London Math. Soc. **27**, 227–242 (1952)
[BDH] Baumslag, G., Dyer, E., Heller, A.: The topology of discrete groups. J. Pure Appl. Algebra **16**, 1–47 (1980)

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