An Algebraic Classification of Some Knots of Codimension Two[1])

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An n-knot will denote a smooth oriented submanifold K of the $(n+2)$-sphere S^{n+2}, where K is *homeomorphic* to S^n. If n is odd, one can associate to K a square integral matrix A, called a *Seifert matrix* of K, using a submanifold of S^{n+2} bounded by K (see [13] for $n=1$, and [4] or [8] in general). When $n=1$, it is known that two Seifert matrices of isotopic knots are related by certain algebraic "moves" (see [11], [16]). In this paper we will generalize this fact to all n. We then consider, for n odd, n-knots (referred to as simple) whose complements are of the same $((n-1)/2)$ type as a circle, i.e. $\Pi_q(S^{n+2}-K)\approx\Pi_q(S^1)$ for $q\leqslant(n-1)/2$. This is the most that can be asked without making K unknotted (see [7]). We will show that two simple n-knots $(n\geqslant 3)$ are isotopic if and only if their Seifert matrices are related by such "moves". Thus it will follow that the semi-group of isotopy classes of simple n-knots depends only on the residue class, mod 4, of n for $n\geqslant 4$.

By contrast, Lashof and Shaneson [6] (and, independently, Browder) have shown that the isotopy class of an n-knot $(n\geqslant 3)$ K, whose complement is of the same 1-type as a circle is determined by the homotopy type of its exterior pair $(X, \partial X)$, where X is the complement of an open tubular neighborhood of K in S^{n+2} – *except* for one other possible knot $\tau(K)$, obtained from K by removing a tubular neighborhood twisting, and reinserting in S^{n+2}. It is not known whether $\tau(K)$ is ever different from K. As a straightforward application, we will show that $\tau(K)$ is isotopic to K if K is simple.

We conclude with some remarks on the algebraic problems which arise.

1. Let K be a $(2q-1)$-knot in S^{2q+1}. We recall the definition of a Seifert matrix of K. Let M be a smooth oriented submanifold of S^{2q+1} bounded by K. The *l-pairing* of M:

$$\theta: H_q(M)\otimes H_q(M)\to \mathbf{Z}$$

is defined by letting $\theta(\alpha\otimes\beta)$ be the linking number $L(z_1, z_2)$, where z_1 is a cycle in M representing α and z_2 is the translate in the positive normal direction off M of a cycle in M representing β. A Seifert matrix A of K is then a representative matrix of θ with respect to a basis of the torsion-free part of $H_q(M)$ – see e.g. [8].

We recall also the formula [8]:

$$\theta(\alpha\otimes\beta)+(-1)^q\,\theta(\beta\otimes\alpha)=\alpha\cdot\beta$$

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where $\alpha\cdot\beta$ is the intersection number in M. Thus $A+(-1)^q A^T$ is unimodular (A^T is the transpose of A) and, if $q=2$, $A+A^T$ has signature a multiple of 16 (see [9]).

2. Let A be a square integral matrix. Any matrix of the form:

$$\left(\begin{array}{c|cc} A & 0 & \\ \hline \alpha & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|cc} A & \beta & 0 \\ \hline \multirow{2}{*}{0} & 0 & 1 \\ & 0 & 0 \end{array}\right)$$

where α is a row vector, β a column vector, will be called an *elementary enlargement* of A. A is an *elementary reduction* of any of its elementary enlargements. Two matrices (or their associated pairings) are equivalent if they can be connected by a chain of elementary enlargements, reductions and unimodular congruences. It is proved in [11] that Seifert matrices of isotopic 1-knots are equivalent. We shall prove:

THEOREM 1: *Seifert matrices of isotopic knots of any (odd) dimension are equivalent.*

THEOREM 2: *Let q be a positive integer and A a square integral matrix such that $A+(-1)^q A^T$ is unimodular and, if $q=2$, $A+A^T$ has signature a multiple of 16. If $q\neq 2$, there is a simple $(2q-1)$-knot with Seifert matrix A; if $q=2$, there is a simple 3-knot with Seifert matrix equivalent to A.*

THEOREM 3: *Let $q\geqslant 2$ and K_1, K_2 simple $(2q-1)$-knots with equivalent Seifert matrices. Then K_1 is isotopic to K_2.*

3. *Proof of Theorem* 1: Suppose K_1, K_2 are isotopic $(2q-1)$-knots bounding manifolds M_1, M_2, respectively, of S^{2q+1}. We first construct a submanifold V (with corners) of $I\times S^{2q+1}$ meeting $0\times S^{2q+1}$ along $0\times M_1$ and $1\times S^{2q+1}$ along $1\times M_2$ with boundary the union of $0\times M_1$; $1\times M_2$ and the trace X of an isotopy from K_1 to K_2. We use the Pontriagin-Thom construction as follows. First construct a normal vector field to $(0\times M_1)\cup X\cup(1\times M_2)=Y$ in $\dot{I}\times S^{2q+1}$, which is tangent to $\dot{I}\times S^{2q+1}$ along $0\times M_1\cup 1\times M_2$. If $q\neq 1$, there is no obstruction. If $q=1$, the obstruction to extending such a vector field from $0\times M_1\cup 1\times M_2$ over Y is the difference in its winding numbers about K_1 and K_2. But since the field is defined over M_1 and M_2, these winding numbers are zero.

Let T be a tubular neighborhood of X. We can "translate" $(X, v \mid X)$ to a framed submanifold of ∂T which agress with the framed submanifold $(0\times M_1\cup 1\times M_2, v)$ on $\partial T\cap(\dot{I}\times S^{2q+1})$. Let $W=\overline{I\times S^{2q+1}-T}$; the Pontriagin-Thom construction on the above framed submanifolds of ∂W determines a map $\partial W\to S^1$. An extension of this map over W will determine the desired V. The obstruction lies in

$$H^2(W, \partial W) \approx H^2(I \times S^{2q+1}, X \cup \dot{I} \times S^{2q+1}) \approx H^1(X \cup \dot{I} \times S^{2q+1}) = 0.$$

4. Now let $\Phi': V \to I$ be the "height" function defined by the restriction of the projection $I \times S^{2q+1} \to I$. We may assume Φ' has no critical points in a neighborhood of ∂V (omitting corners). Let Φ be a C^2-approximation to Φ' which agrees with Φ' in a neighborhood of ∂V and has only non-degenerate critical points (except at corners) which are mapped one-one into I (see e.g. [10]). We can move V so that Φ becomes the new height function. In fact if $p: V \to S^{2q+1}$ is defined by the projection $I \times S^{2q+1} \to S^{2q+1}$ and Φ is a close enough approximation to Φ', then $x \mapsto (\Phi(x), p(x))$ defines a new imbedding $V \to I \times S^{2q+1}$ which agrees with the original inclusion near ∂V and has Φ as its new height function.

5. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a partition of I satisfying

(i) each t_i is a regular value of Φ,

(ii) at most one critical value of Φ lies in each interval (t_i, t_{i+1}).

Let $\Phi^{-1}(t_i) = t_i \times M'_i$; then each M'_i is bounded by a knot isotopic to K_0 and K_1, and $M'_0 = M_1$, $M'_k = M_2$. This shows that it suffices to consider the case where Φ has only one critical point.

LEMMA 1: *Let $\alpha, \alpha' \in H_q(M_1)$ and $\beta, \beta' \in H_q(M_2)$ and suppose that α is homologous to β and α' homologous to β' in V. Then $\theta_1(\alpha, \alpha') = \theta_2(\beta, \beta')$, where θ_i is the l-pairing of M_i.*

Proof: Let C, C' be $(q+1)$-chains in V such that $\partial C = \alpha - \beta$, $\partial C' = \alpha' - \beta'$. Then it follows from the definition of θ_1, θ_2 that $\theta_1(\alpha, \alpha') - \theta_2(\beta, \beta')$ is the intersection number of C and the translate of C' off V in the positive normal direction – but this is obviously zero.

6. Now consider the following diagram:

$$\begin{array}{ccccccc} & & & & H_{q+1}(V, M_2) & & \\ & & & & \downarrow & & \\ & & & & H_q(M_2) & & \\ & & & & \downarrow & & \\ H_{q+1}(V, M_1) & \to & H_q(M_1) & \to & H_q(V) & \to & H_q(V, M_1) \\ & & & & \downarrow & & \\ & & & & H_q(V, M_2) & & \end{array}$$

consisting of the exact homology sequences of (V, M_1) and (V, M_2). If the index of the critical point of Φ is not q or $q+1$, then

$$H_q(V, M_1) = H_{q+1}(V, M_1) = H_q(V, M_2) = H_{q+1}(V, M_2) = 0$$

and we have

$$H_q(M_1) \approx H_q(V) \approx H_q(M_2).$$

It follows from Lemma 1 that θ_1 and θ_2 are congruent.

If the index of the critical point of Φ is q, then

$$H_q(V, M_1) \approx H_{q+1}(V, M_2) \approx Z$$

and

$$H_q(V, M_2) = H_{q+1}(V, M_1) = 0.$$

If $\alpha \in H_q(M_2)$ is the image of a generator of $H_{q+1}(V, M_2)$, then the composite

$$H_q(M_2) \to H_q(V) \to H_q(V, M_1) \approx \mathbf{Z}$$

can be defined by $\beta \to \alpha \cdot \beta =$ intersection number in M_2 (see [5]). If α has finite order, then it follows that $H_q(M_1) \approx H_q(V) \approx H_q(M_2)$, *modulo torsion*, and, therefore, θ_1 and θ_2 are congruent modulo torsion.

7. Suppose α has infinite order; then α is a multiple of a primitive element α_0 and there exists $\beta_0 \in H_q(M_2)$ with $\alpha_0 \cdot \beta_0 = 1$. Suppose $\gamma'_1, \ldots, \gamma'_s \in H_q(M_2)$ such that:

(i) γ'_i is homologous to γ_i in V, and

(ii) $\alpha_0, \beta_0, \gamma'_1, \ldots, \gamma'_s$ is a basis of $H_q(M_2)$, modulo torsion.

We now examine θ_2 on the elements $\alpha_0, \beta_0, \gamma'_1, \ldots, \gamma'_s$. By Lemma 1 we can conclude from (i) that $\theta_2(\gamma'_i, \gamma'_j) = \theta_1(\gamma_i, \gamma_j)$. Since α is null-homologous in V, $\theta_2(\alpha, \gamma'_i) = \theta_1(0, \gamma_i) = 0$ and $\theta_2(\alpha, \alpha) = \theta_2(0, 0) = 0$. Thus $\theta_2(\alpha_0, \gamma'_i) = \theta_2(\alpha_0, \alpha_0) = 0$; similarly $\theta_2(\gamma'_i, \alpha_0) = 0$. We also recall that (§ 1):

$$\theta_2(\alpha_0, \beta_0) + (-1)^q \theta_2(\beta_0, \alpha_0) = -\alpha_0 \cdot \beta_0 = -1$$

8. We may summarize this as follows. Let A be the matrix representative of θ_1 with respect to the basis $\gamma_1, \ldots, \gamma_s$. The the matrix representative of θ_2 with respect to the basis $\gamma'_1, \ldots, \gamma'_s, \alpha_0, \beta_0$ has the form:

$$B = \left(\begin{array}{c|cc} \multirow{3}{*}{$\mathbf{A}$} & 0 & \\ & \vdots & \eta \\ & 0 & \\ \hline 0 \ldots 0 & 0 & x \\ \xi & x' & y \end{array}\right)$$

where x, y are integers, $x + (-1)^q x' = -1$, ξ is a row vector and η is a column vector.

Recall from e.g. [8] that the polynomial $\Delta_A(t) = \det(tA + (-1)^q A^T)$, where A is a Seifert matrix for a knot K, is an invariant of the isotopy class of K (up to multiplication by a unit in $\mathbf{Z}[t, t^{-1}]$). But it is easily verified that:

$$\Delta_B(t) = \left(tx + (-1)^q x'\right)\left(tx' + (-1)^q x\right) \Delta_A(t)$$

Thus x (or x') is zero, since $x \pm x' = -1$, then x' (or x) is ± 1. It now is easily checked that B is congruent to an elementary enlargement of A.

9. If the index of the critical point of Φ is $q+1$; then its index as a critical point of $-\Phi$ is q. The preceding arguments apply to show that θ_2 is congruent to θ_1, or has, as representative matrix, an elementary reduction of a representative matrix of θ_1.

This completes the proof of Theorem 1.

10. *Proof of Theorem* 2: For $q \neq 2$, this is proved in [4] (see also [9]). For $q=2$, we must show that A is equivalent to a matrix B, where $B+B^T$ is a matrix representative of the intersection pairing of some simply-connected closed 4-manifold. By an argument in [9], such a B can be obtained by adding enough blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to A; but this is a sequence of elementary enlargements of A, so B is equivalent to A.

11. *Proof of Theorem* 3: We reduce Theorem 3 to two lemmas. Recall (see [7]) that a simple $(2q-1)$-knot bounds a $(q-1)$-connected submanifold of S^{2q+1}. A Seifert matrix obtained from the l-pairing of such a submanifold will be called *special.*

LEMMA 2: *Let K be a simple $(2q-1)$-knot with a special Seifert matrix A. If B is an elementary enlargement of A, then B is also a special Seifert matrix of K.*

LEMMA 3: *If $q \geqslant 2$, then simple $(2q-1)$-knots admitting identical special Seifert matrices are isotopic.*

12. We first show that Theorem 3 follows from Lemmas 2 and 3. Let K, K' be simple $(2q-1)$-knots with equivalent Seifert matrices, $q \geqslant 2$. Let A, A' be special Seifert matrices of K, K', respectively. Thus there exists a sequence: $A = A_1, A_2, \ldots, A_k = A'$, where each A_{i+1} is unimodularly congruent to an elementary enlargement or reduction. of A_i It follows from Theorem 2 that, for $q > 2$, each A_i is a special Seifert matrix of a simple $(2q-1)$-knot K_i (actually the proof of Theorem 2 (see [4]) realizes S as a special Seifert matrix of a simple knot). If $q=2$, we can enlarge each A_i by adding a constant number of blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to obtain a new sequence $A'_1, A'_2, \ldots, A'_k$. Each A'_{i+1} is again congruent to an elementary enlargement or reduction of A'_i – and it now follows from the argument in § 10 that each A_i' is a special Seifert matrix of a simple 3-knot K_i.

We now prove that each K_i is isotopic to K_{i+1}. Suppose A_{i+1} is congruent to an elementary enlargement of A_i. It follows from Lemma 2 that A_{i+1} (or A_{i+1}') is a special Seifert matrix of K_i. Then Lemma 3 implies K_i and K_{i+1}, both of which now admit A_{i+1} (or A_{i+1}') as a special Seifert matrix, are isotopic. If A_{i+1} is congruent to an elementary reduction of A_i, the same argument works, switching the roles of K_i and K_{i+1}.

We may as well have chosen $K_1=K$ and $K_k=K'$ if $q>2$, but if $q=2$ we need to show that K_1 is isotopic to K and K_k is isotopic to K'. It follows from Lemma 2 that A_1' is a special Seifert matrix of K, since A_1' is obtained from A_1 by a sequence of elementary enlargements. Then Lemma 3 implies K and K_1 are isotopic – similarly for K' and K_k.

13. *Proof of Lemma* 2: Let M be a $(q-1)$-connected submanifold of S^{2q+1} bounded by K, and $\alpha_1, \ldots, \alpha_n$ a basis of $H_q(M)$, modulo torsion, such that A is the corresponding matrix representative of the l-pairing of M. Let $x_1, \ldots, x_n$ be an arbitrary sequence of integers. It follows from Alexander duality that there exists a cycle $z \in H_q(S^{q+1}-M)$ such that the linking numbers $L(z, \alpha_i)=x_i$, for $i=1, \ldots, n$. Now $S^{2q+1}-M$ is $(q-1)$-connected and so z is spherical; by general position, z can be represented by an imbedded q-sphere $\sigma \subset S^{2q+1}-M$. The normal bundle to σ is trivial and so a tubular neighborhood T can be identified with $\sigma \times D^{q+1}$ – we may assume T disjoint from M. Orient ∂T so that the positive normal direction in S^{2q+1} points into T and let M' be the connected sum in S^{2q+1} of M and ∂T. Then $H_q(M')$ has rank two greater than the rank of $H_q(M)$, and $\alpha_1, \ldots, \alpha_n$ may be extended to a basis of $H_q(M')$, modulo torsion, by adjoining the homology classes β_1, β_2 of $\sigma \times y_0$ and $x_0 \times S^q \subset \sigma \times S^q = \partial T$, respectively. The representative matrix of the l-pairing of M' with respect to the basis $\alpha_1, \ldots, \alpha_n, \beta_1, \beta_2$ is

$$\begin{pmatrix} \multicolumn{1}{c|}{} & \pm x_1 & 0 \\ \multicolumn{1}{c|}{A} & \vdots & \vdots \\ \multicolumn{1}{c|}{} & \pm x_n & 0 \\ \hline x_1 \ldots x_n & x & 0 \\ 0 \ldots 0 & \pm 1 & 0 \end{pmatrix}$$

which is congruent to:

$$\begin{pmatrix} \multicolumn{1}{c|}{A} & 0 & 0 \\ \multicolumn{1}{c|}{} & \vdots & \vdots \\ \hline x_1 \ldots x_n & 0 & 0 \\ 0 \ldots 0 & 1 & 0 \end{pmatrix}$$

If z is chosen so that $L(\alpha_i, z)=x_i$ for $i=1,\dots,n$, and ∂T is oriented so that the positive normal direction points out from T, then the representative matrix of the l-pairing of M' with respect to $\alpha_1,\dots,\alpha_n,\beta_1,\beta_2$ is:

$$\left(\begin{array}{c|cc} \mathbf{A} & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \pm x_1 \dots \pm x_n & x & \pm 1 \\ 0 \dots 0 & 0 & 0 \end{array}\right)$$

which is congruent to:

$$\left(\begin{array}{c|cc} \mathbf{A} & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \dots 0 & 0 & 1 \\ 0 \dots 0 & 0 & 0 \end{array}\right)$$

Thus we can realize any elementary enlargement of A as a special Seifert matrix of K.

14. *Proof of Lemma* 3: Suppose K and K' are $(2q-1)$-knots bounding $(q-1)$-connected submanifolds M and M' of S^{2q+1} with l-pairings θ and θ'. Suppose also that there exists an isomorphism $\Phi: H_q(M) \to H_q(M')$ preserving the l-pairings, i.e. $\theta = \theta' \circ (\Phi \otimes \Phi)$.

Let us assume, for now, $q>2$; we will show that M and M' are isotopic submanifolds of S^{2q+1}. According to [15], M and M' have handle decompositions:

$$M = D^{2q} \cup h_1 \cup \cdots \cup h_r$$
$$M' = D^{2q} \cup h'_1 \cup \cdots \cup h'_r$$

where each h_i, h'_i is a handle of index q – diffeomorphic to $D^q \times D^q$. The $h_i(h'_i)$ are attached to D^{2q} by disjoint imbeddings $S^{q-1} \times D^q \to \partial D^{2q}$. Let $C_i(C'_i)$ be the "core" of $h_i(h'_i)$, i.e. the submanifold corresponding to $D^q \times 0$ – then $\partial C_i = C_i \cap D^{2q}$.

The imbedded disks $(C_i, \partial C_i) \subset (M, D^{2q})$ represent a basis $\{\alpha_i\}$ of $H_q(M, D^{2q}) \approx \approx H_q(M)$. According to handle body theory (see [17]), we can choose a handle-decomposition realizing any prescribed basis $\{\alpha_i\}$. Thus if $\{\alpha'_i\}$ is the basis of $H_q(M')$ defined by $(C'_i, \partial C'_i) \subset (M', D^{2q})$, we may, by setting $\alpha'_i = \Phi(\alpha_i)$, assume $\theta(\alpha_i, \alpha_j) = \theta'(\alpha'_i, \alpha'_j)$.

15. Now consider the links $\{\partial C_i\}$ and $\{\partial C'_i\}$ in ∂D^{2q}; by [17] and § 1 we have:

$$L(\partial C_i, \partial C_j) = \alpha_i \cdot \alpha_j = -\theta(\alpha_i, \alpha_j) - (-1)^q \theta(\alpha_j, \alpha_i);$$

similarly for $L(\partial C'_i, \partial C'_j)$. Therefore $L(\partial C_i, \partial C_j) = L(\partial C'_i, \partial C'_j)$, for $i \neq j$, and, since $q>2$, the links $\{\partial C_i\}$ and $\{\partial C'_i\}$ are isotopic in ∂D^q.

Clearly we may assume that the base disks D^{2q} in the handle decompositions of M and M' coincide as imbedded in S^{2q+1}. Thus the cores C_i and C'_i, as imbedded in S^{2q+1}, may be assumed to coincide on their boundaries: $\partial C_i = \partial C_i'$. We next show how to isotopically defrom $\{C_i\}$ onto $\{C_i'\}$, keeping $\{\partial C_i\}$ fixed and avoiding any intersections with D^{2q} (except, of course, along ∂C_i).

Assume inductively that $C_i = C_i'$ for $i < k$. We will isotopically deform C_k to C_k', avoiding intersections with $D^{2q} \cup C_1 \cup \cdots \cup C_{k-1}$. Given such an isotopy, we can extend it to an isotopy of $h_k \cup h_{k+1} \cup \cdots \cup h_i$ in

$$S^{2q+1} - (D^{2q} \cup h_1 \cup \cdots \cup h_{k-1});$$

the result is an isotopy of M to a new imbedding satisfying $C_i = C_i'$ for $i \leqslant k$. We begin with an isotopy of C_k to C_k', $\operatorname{rel} \partial C_k$, avoiding D^{2q}; this exists according to Wu [20] because the imbeddings of C_k and C_k' and $S^{2q+1} - \operatorname{int} D^{2q}$ are homotopic $\operatorname{rel} \partial C_k = \partial C_k'$ and $S^{2q+1} - \operatorname{int} D^{2q}$ is simply-connected. We would then like to use Whitney's procedure, as in [20], to remove the intersections of this isotopy with $I \times C_i$ $(i = 1, \ldots, k-1)$ in $S^{2q+1} - D^{2q}$, since $q \geqslant 2$ and $S^{2q+1} - D^{2q}$ is simply-connected. The only obstruction to this is the intersection number, which is easily seen to be (up to sign) $\theta(\alpha_i, \alpha_k) - \theta'(\alpha_i', \alpha_k') = 0$.

16. We now have achieved $C_i = C_i'$ for $i = 1, \ldots, r$. By the tubular neighborhood theorem we may assume $h_i \cap D^{2q} = h_i' \cap D^{2q}$. Let $v_i(v_i')$ be the positive unit normal field to $h_i(h_i')$ on $C_i = C_i'$.

By the tubular neighborhood theorem, we may assume that $h_i(h_i')$ is the orthogonal complement of $v_i(v_i')$ in a normal disk bundle neighborhood N of $C_i = C_i'$ in S^{2q+1}. Therefore if we can homotopically deform v_i to v_i', $\operatorname{rel} \partial C_i$, we obtain an isotopy, $\operatorname{rel} h_i \cap D^{2q}$, of h_i to h_i' within N. Doing this for all i achieves, finally, an isotopy of M to M'.

Since $v_i = v_i'$ along ∂C_i, v_i differs from v_i' by an element of $\Pi_q(S^q) \approx Z$ (the normal space to C_i in S^{2q+1} has dimension $q+1$). But this element can be identified with $\theta(\alpha_i, \alpha_i) - \theta'(\alpha_i', \alpha_i') = 0$, and so Lemma 3 is proved – *for* $q > 2$.

17. For $q = 2$, more work is required to repair those parts of the preceding argument which are no longer valid. First of all M and M' are not necessarily diffeomorphic. On the other hand, they are simply-connected 4-manifolds with boundaries diffeomorphic to S^3 and isomorphic intersection pairings (since their l-pairings are isomorphic). It then follows from [19] that, after adding on a number of copies of $S^2 \times S^2$, M and M' will be diffeomorphic. Since these enlargements of M and M' can be realized by adding to the Seifert matrices blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, as is demonstrated in § 13, we may as well assume that M and M' are diffeomorphic to start with. In fact, by [18], there is a diffeomorphism $f: M \to M'$ preserving the l-pairings i.e. $\theta = \theta' \circ (f_* \otimes f_*)$, where $f_*: H_2(M) \to H_2(M')$ is the induced homomorphism.

We reformulate the situation so far as follows. M is a simply-connected 4-manifold, ∂M diffeomorphic to S^3, and we have imbeddings $g, g': M \to S^5$ such that $g(\partial M) = K$ and $g'(\partial M) = K'$. The l-pairings of $g(M)$ and $g'(M)$ are identical, as pairings on $H_2(M)$.

18. Now let

$$M = D^4 \cup h_1^1 \cup \cdots \cup h_k^1 \cup h_1^2 \cup \cdots \cup h_l^2 \cup h_1^3 \cup \cdots \cup h_m^3 \,{}^*$$

be a handle decomposition of M, where h_j^i is a handle of index i. Since $H_1(M) = 0$ we can choose the handles of index 2 in such a way that the first k of them – $h_1^2, \ldots, h_k^2$ – homologically cancel out the handles of index 1 (see e.g. [10], Theorem 7.6]) i.e. if $V = D^4 \cup h_1^1 \cup \ldots \cup h_k^1$, then the boundary operator $H_2(M, V) \to H_1(V)$ maps the subgroup of $H_2(M, V)$ generated by the "cores" of $h_1^2, \ldots, h_k^2$ isomorphically onto $H_1(V)$. Then

$$\Delta = D^4 \cup h_1^1 \cup \cdots \cup h_k^1 \cup h_1^2 \cup \cdots \cup k_k^2$$

is acyclic. Set

$$M_0 = \Delta \cup h_{k+1}^2 \cup \cdots \cup h_l^2 .$$

We will show that $g \mid M_0$ and $g' \mid M_0$ are isotopic.

First we show that $g|\Delta$ and $g'|\Delta$ are isotopic by extending $g' \circ g^{-1}: g(\Delta) \to g'(\Delta)$ to an orientation preserving diffeomorphism of S^5. Begin by extending it to a tubular neighborhood T of $g(\Delta)$ diffeomorphic to $g(\Delta) \times I$ (with corners rounded). Now ∂T is a homology 4-sphere bounding the contractible 5-manifold $\overline{S^5 - T}$ – similarly for T' a tubular neighborhood of $g'(\Delta)$. That $\overline{S^5 - T}$ is acyclic follows from Alexander duality; that $\overline{S^5 - T}$ is simply-connected follows from the fact that Δ collapses onto a 2-dimensional polyhedron (it has only handles of index one and two) which has codimension >2 in S^5. We now invoke the case $n = 5$ of the following lemma (stated by Kato for the PL case in [3]) since $\Gamma^5 = 0$, to obtain the extension over S^5.

19. LEMMA 4: *If C_1, C_2 are contractible smooth manifolds of dimension $n \geqslant 5$, then any diffeomorphism $d: \partial C_1 \to \partial C_2$ extends to a diffeomorphism of C_1 onto C_2, after perhaps changing d on an $(n-1)$-disks in ∂C_1.*

Proof of Lemma 4: Consider $W = C_1 \cup_d D_2$, a homotopy n-sphere. Since $n \geqslant 5$, W is homeomorphic to S^n ([15]) and, by changing d, we can insure that W is diffeomorphic to S^n. Then W bounds a copy of D^{n+1} which determines an h-cobordism

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*) In fact an argument of A. Wallace, communicated to me by C. T. C. Wall, shows that only handles of index 2 are needed after connected sum with enough copies of $S^2 \times S^2$. This obviates the need for the arguments of § 18, 19 and 21, since $\Delta = D^4$ and $M_0 = M$.

from C_1 to C_2, trivial from ∂C_1 to ∂C_2 (identifying them by d). By the relative version of the h-cobordism theorem ([15, Cor. 3.2]), d extends to a diffeomorphism from C_1 onto C_2.

20. Now M_0 is obtained from Δ by attaching handles of index two. Since we may assume that $g|\Delta = g'|\Delta$, the problem of showing $g|M_0$ isotopic to $g'|M_0$ is similar to (but not exactly the same as) the argument above in § 15, 16. We need first to know that the l-pairings on $H_2(M_0)$, defined from g and g', are identical, but this is because they are induced from the l-pairings on $H_2(M)$ by the inclusion $M_0 \subset M$. The argument in § 15 will then serve to show that g and g' are isotopic on the cores of the handles. To extend the isotopy to the entire handles, it will suffice to show that the normal 2-fields to the imbedded cores in S^5, defined by applying the differentials of g and g' to the standard normal 2-fields to the cores in the handles of M_0, are homotopic. But the obstructions are elements of $\Pi_2(V_{3,2})=0$, where $V_{3,2}$ is the Stiefel manifold of 2-fields in 3-space.

21. We now have shown that $g|M_0$ is isotopic to $g'|M_0$. The proof of Lemma 3 for $q=2$ will be completed by showing that ∂M can be isotopically deformed, in M, inside M_0. This will use the engulfing theorem in its most naive form ([12, Lemma 2.7]). Let N be the closed simply-connected 4-manifold obtained from M by attaching a 4-disk D to ∂M. Given any handle-decomposition of N it follows from the engulfing theorem that the handles of index one are contained in a 4-disk imbedded in N. Applying this to the *dual* handle decomposition of that postulated in § 18, we find that $N-M_0$ is contained in a 4-disk D' in N. Since any two similarly oriented n-disks in an unbounded n-manifold are isotopic, D and D' are isotopic in N. It also may be arranged that any given point in $(\operatorname{int} D) \cap (\operatorname{int} D')$ is fixed during the isotopy. It is then easy to see that $\partial D(=\partial M)$ and $\partial D'$ are isotopic in $M=\overline{N-D}$. This completes the proof of Lemma 3, and so Theorem 3.

22. Let $f: S^n \to R^{n+2}$ be a smooth imbedding. Since its normal bundle is trivial, f extends to an imbedding $F: S^n \times D^2 \to R^{n+2}$ whose isotopy class is uniquely determined by f, if $n>1$. Let $h: S^n \times S^1 \to S^n \times S^1$ be the diffeomorphism defined by $(x,y) \to \to (\Phi(y)\cdot x, y)$ where $\Phi: S^1 \to SO(n+1)$ represents the non-zero element of $\Pi_1(SO(n+1))$. Define

$$R_0 = S^n \times D^2 \cup_{F \circ h} \overline{R^{n+2} - F(S^n \times D^2)}$$

Representing R^{n+2} as the interior of D^{n+2}, the construction of R_0 represents R_0 as the interior of a compact manifold D_0 with boundary diffeomorphic to S^{n+1}. By [15], D_0 is diffeomorphic to D^{n+2} if $n \geqslant 3$; therefore R_0 is diffeomorphic to R^{n+2}. Consider the knot $S^n \times 0 \subset S^n \times D^2 \subset R_0 \approx R^{n+2}$, which we denote by $\tau(K)$, if K is the knot $f(S^n)$. It follows easily that the isotopy class of $\tau(K)$ depends only on that of K.

The interest of $\tau(K)$ is that its complement is diffeomorphic to the complement of K and, besides K itself, is, up to isotopy, the only knot with this property (see [2] and [6]). It is not known whether $\tau(K)$ is ever *not* isotopic to K. We can prove:

COROLLARY 1: *If K is a simple $(2q-1)$-knot, $q \geqslant 2$, then $\tau(K)$ is isotopic to K.*

Proof: Let M be a submanifold of $R^{n+2} \subset S^{n+2}$ bounded by $K = f(S^n)$ – we may assume that $M \cap F(S^n \times D^2) = F(S^n \times \varrho)$ where ϱ is any ray from the origin in D^2. Since $h(S^n \times x_0) = S^n \times x_0$, for any $x_0 \in S^1$, we may define

$$M' = S^n \times \varrho \cup (M \cap \overline{R^{n+2} - F(S^n \times D^2)}),$$

a submanifold of R_0 bounded by $\tau(K)$. It is obvious that M' is diffeomorphic to M and the l-pairings coincide. Thus, by Theorem 3 (or even Lemma 3), K and $\tau(K)$ are isotopic.

23. Another consequence of Theorem 3 – not surprisingly – is the unknotting theorem of [7] and [14].* For if K is a $(2q-1)$-knot with complement X and universal abelian covering $\tilde{X}$, and A is a Seifert matrix for K, then $tA + (-1)^q A^T$ is a relation matrix for $H_q(\tilde{X}; Q)$, as a module over the rational group ring $Q[Z] = Q[t, t^{-1}]$ (see [8]). Now A is equivalent to a non-singular matrix A' (allowing $A' = 0$) by Proposition 1 in § 24. But if A' has rank r, it follows easily that $H_q(\tilde{X}; Q)$ has dimension r as a Q-module. Thus, if $H_q(\tilde{X}; Q) = 0$, A must be equivalent to 0, which implies, by Theorem 3, that K is unknotted for $q \geqslant 2$.

24. We now turn to the algebraic problem presented by the notion of equivalence of matrices. Results are very incomplete, and most of them are contained implicitly in [16].

PROPOSITION 1: *Any matrix A such that $A + A^T$ is unimodular is equivalent to a non-singular matrix (i.e. with non-zero determinant) or zero.*

Proof: By the argument in [16, p. 484], if A is singular it admits an elementary reduction. Thus by a sequence of elementary reductions (and congruences) we may make A non-singular (or zero).

25. PROPOSITION 2: *Suppose that A and B are equivalent non-singular matrices. Then* $\det A = \det B = d$, *and A is congruent to B over any ring R in which d is a unit.*

*) The argument here applies, of course, only for odd dimensional knots. The case $q = 2$ was also announced by C. T. C. Wall: Proc. Camb. Phil. Soc. *63* (1967), p. 6.

Proof: This is proved implicitly in [16, Theorem 2] more or less as follows. Consider the matrix $tA-A^T$, with entries considered as elements of $Z[t,t^{-1}]=\Lambda$, and the Λ-module H_A with $tA-A^T$ as relation matrix. Consider also the bilinear form $[,]: H\otimes_\Lambda H\to Q(\Lambda)/\Lambda=S(\Lambda)$ ($Q(\Lambda)$ is the quotient field of Λ) defined, with respect to the same generators of H_A as $tA-A^T$ is a relation matrix, by the matrix $(tA-A^T)^{-1}$. Note that $tA-A^T$ is non-singular over $Q(\Lambda)$ because $\Delta_A(t)=\det(tA-A^T)$ is non-zero. In fact the leading coefficient of $\Delta_A(t)$ is $\det A\neq 0$.

It is not hard to see that the isomorphism class of $(H_A; [,])$ is an invariant of the equivalence class of A. Furthermore the element $\Delta_A(t)=\det(tA-A^T)$ is an invariant of the equivalence class, up to multiplication by powers of t. From this it follows that $\det A=\det B$.

If A is unimodular over R, then $t-A^{-1}A^T$ is a presentation matrix of $H_A\otimes R$ $(\otimes=\otimes_Z)$, so $H_A\otimes R$ is a free R-module of the same rank as A and $(tA-A^T)^{-1}$ is the matrix for $[,]$ with respect to an R-basis for $H_A\otimes R$. From this it follows that there exists a matrix P with entries in, and unimodular over, R such that

$$P(tA-A^T)^{-1}P^T=(tB-B^T)^{-1} \quad \text{in} \quad S(\Lambda)\otimes R.$$

Let $\overline{(tA-A^T)}$ be the "adjoint" of $tA-A^T$ ([1, p. 305)]. Then

$$\left.\begin{aligned} P\overline{(tA-A^T)}P^T=\overline{tB-B^T} \quad &\text{in} \quad (\Lambda/(\Delta(t))\otimes R, \\ &\text{where} \quad \Delta(t)=\Delta_A(t)=\Delta_B(t). \end{aligned}\right\} \qquad (*)$$

Since A and B are unimodular over R, $\Delta(t)$ has as leading coefficient a unit of R. From this it follows that there exists a unique well-defined R-linear map $\gamma: \Lambda/(\Delta(t))\otimes R\to R$ defined by the properties $\gamma(1)=1$ and $\gamma(t^i)=0$ for $0<i<$ degree $\Delta(t)$. Now every entry of $\overline{tA-A^T}$ and $\overline{tB-B^T}$ has degree $<\operatorname{rank} A=\operatorname{degree}\Delta(t)$. Therefore, applying γ to equation (*) we find that $PAP^T=B$ in R.

26. We cannot strengthen the conclusion of Proposition 2 to conclude that A and B are congruent over K, as the following example shows. Set:

$$A=\begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}, \quad B=\begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix}$$

We first show that A and B are not congruent over $\mathbf{Z}$. Consider the solutions X of $X^TAX=X^TBX=2$; they are $X=\begin{pmatrix}\pm 1\\ 0\end{pmatrix}$. If $P^TAP=B$, it follows that $PX=\pm X\left(\text{say } X=\begin{pmatrix}1\\ 0\end{pmatrix}\right)$. Now $BX=\begin{pmatrix}2\\ 0\end{pmatrix}$ and so Y^TBX is even, for any Y. Choose Y so that $PY=\begin{pmatrix}0\\ 1\end{pmatrix}$; then $Y^TBX=Y^TP^TAPX=\pm(PY)^TAX$ which one can calculate to be ± 1.

To see that A and B are equivalent, we consider the following elementary enlargements, respectively, of A and B:

$$A' = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & 0 & 0 \\ & & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad B' = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{B}} & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

A' and B' are congruent; in fact, $PA'P^T = B'$, where

$$P = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

27. On the other hand, the converse of Proposition 2 is false. Consider the following matrices for $\varepsilon = \pm 1$:

$$A = \begin{bmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & p^2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1+\varepsilon)+1 \\ 0 & 0 & 0 & 0 & 0 & p+1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \varepsilon x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 \\ p^4 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & p(1+\varepsilon)+1 \\ 0 & 0 & 0 & 0 & 0 & p+1 \end{bmatrix}$$

where p is any odd prime, and $x = \frac{1}{4}(p^4 - 1)$. It may be checked directly that $A + \varepsilon A^T$ and $B + \varepsilon B^T$ are unimodular and $\det A = \det B$ is divisible by p. But A and B are congruent over any ring in which p is a unit. In fact $PAP^T = B$ where:

$$P = \begin{bmatrix} p & & & & & \\ & 1/p & & & \multicolumn{2}{c}{0} \\ & & p & & & \\ & & & 1/p & & \\ \multicolumn{2}{c}{0} & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Finally, A and B are not equivalent. To see this consider $A - \varepsilon A^T$ and $B - \varepsilon B^T$ over Z_p. It follows from Witts Theorem (see e.g. [21]) for $\varepsilon = -1$, or the well-known classification of skew-symmetric forms (see e.g. [1]) for $\varepsilon = +1$, that the congruence class of $A - \varepsilon A^T$ over Z_p is an invariant of the equivalence class of A. But $A - \varepsilon A^T$ and $B - \varepsilon B^T$ have ranks 2 and 4, respectively, over Z_p.

28. Finally we remark that the genus of the non-degenerate quadratic form $A+A^T$ is an invariant of the equivalence class of A (see [16, Prop. 5.1]).

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