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LECTURES ON GROUPS OF HOMOTOPY SPHERES

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Kervaire and Milnor's germinal paper [15], in which they used the newly-discovered techniques of surgery to begin the classification of smooth closed manifolds homotopy equivalent to a sphere (homotopy-spheres), was intended to be the first of two papers in which this classification would be essentially completed (in dimensions ≥ 5). Unfortunately, the second part never appeared. As a result, in order to extract this classification from the published literature it is necessary to submerge oneself in more far-ranging and complicated works (e.g. [7], [16], [30]), which cannot help but obscure the beautiful ideas contained in the more direct earlier work of Kervaire and Milnor. This is especially true for the student who is encountering the subject for the first time.

In Fall, 1969, I gave several lectures to a graduate seminar at Brandeis University, in which I covered the material which I believe would have appeared in Groups of Homotopy Spheres, II. Two students, Allan Gottlieb and Clint McCrory, prepared mimeographed notes from these lectures, with some extra background material, which have been available from Brandeis University. The present article is almost identical with these notes. I hope it will serve to fill a pedagogical gap in the literature.

The reader is assumed to be familiar with [15], [20]. In these papers, Kervaire-Milnor define the group θ^n of h-cobordism classes of homotopy n-spheres and the subgroup bP^{n+1} defined by homotopy spheres which bound parallelizable manifolds. The goal is to compute bP^{n+1} and θ^n/bP^{n+1}.

Section 1 reviews some well known results on vector bundles over spheres and the homotopy of the classical groups, as well as some theorems of Whitney on embeddings and immersions. Since a homotopy n-sphere Σ^n is h-cobordant to S^n (the n-sphere with its standard differential structure) iff Σ^n bounds a contractible manifold, in order to calculate bP^{n+1} we are interested in finding and realizing "obstructions" to surgering parallelizable manifolds into contractible ones. Section 2 contains some general theorems for framed surgery and describes which "obstructions" exist for each n. In [15] it is shown that bP^{n+1} is zero for $n+1$ odd. Sections 3 and 4 perform the corresponding calculations for $n+1 = 4k$ and $n+1 = 4k+2$ respectively. In section 5, by use of the Thom-Pontryagin construction, the calculation of θ^n/bP^{n+1} is reduced to a question of framed cobordism which is answered by using results from sections 3 and 4. Many results of these notes are summarized in a long exact sequence

$$\cdots \to P^{n+1} \xrightarrow{b} \theta^n \to A^n \to P^n \xrightarrow{b} \theta^{n-1} \to \cdots$$

which is discussed in the appendix.

Throughout these notes all manifolds are assumed to be smooth, oriented, and of dimension greater than 4. In addition all manifolds with boundary are assumed to have dimension greater than 5 (so that the boundary manifold will have dimension greater than 4).

§1. Preliminaries

A) Oriented vector bundles over spheres.

In [28] Steenrod gives the following method for viewing oriented k-plane bundles over S^n as elements of $\pi_{n-1}(SO_k)$. Let ξ be such a bundle. By section 12.9 of [28] the group of ξ may be reduced from $GL(k,\mathbb{R})$ to O_k. Since ξ is oriented O_k may be further reduced to SO_k. Cover S^n by two overlapping "hemispheres". Since the bundle is trivial over each hemisphere, it is determined by the transition function at each point of the equator. This function, $\alpha: S^{n-1} \to SO_k$, is well defined up to homotopy class by the equivalence class of ξ and is the obstruction to framing ξ. In addition the map $[\xi] \rightsquigarrow [\alpha]$ sets up a one-to-one correspondence between (oriented isomorphism) equivalence classes of oriented k-plane bundles and elements of $\pi_{n-1}(SO_k)$. For details the reader should see section 18 [28]. By abuse of notation we refer to $[\xi] \in \pi_{n-1}(SO_k)$.

<u>Lemma 1.1</u>. Let $[\xi] \in \pi_{n-1}(SO_k)$ be an oriented k-plane bundle $/ S^k$. Then $[\xi \oplus \varepsilon] = i_*[\xi] \in \pi_{n-1}(SO_{k+1})$ where we view $SO_k \xrightarrow{i} SO_{k+1}$ as acting trivially on the last component of $\mathbb{R}^{k+1}$ (i.e. the matrix M goes to

$$\begin{pmatrix} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

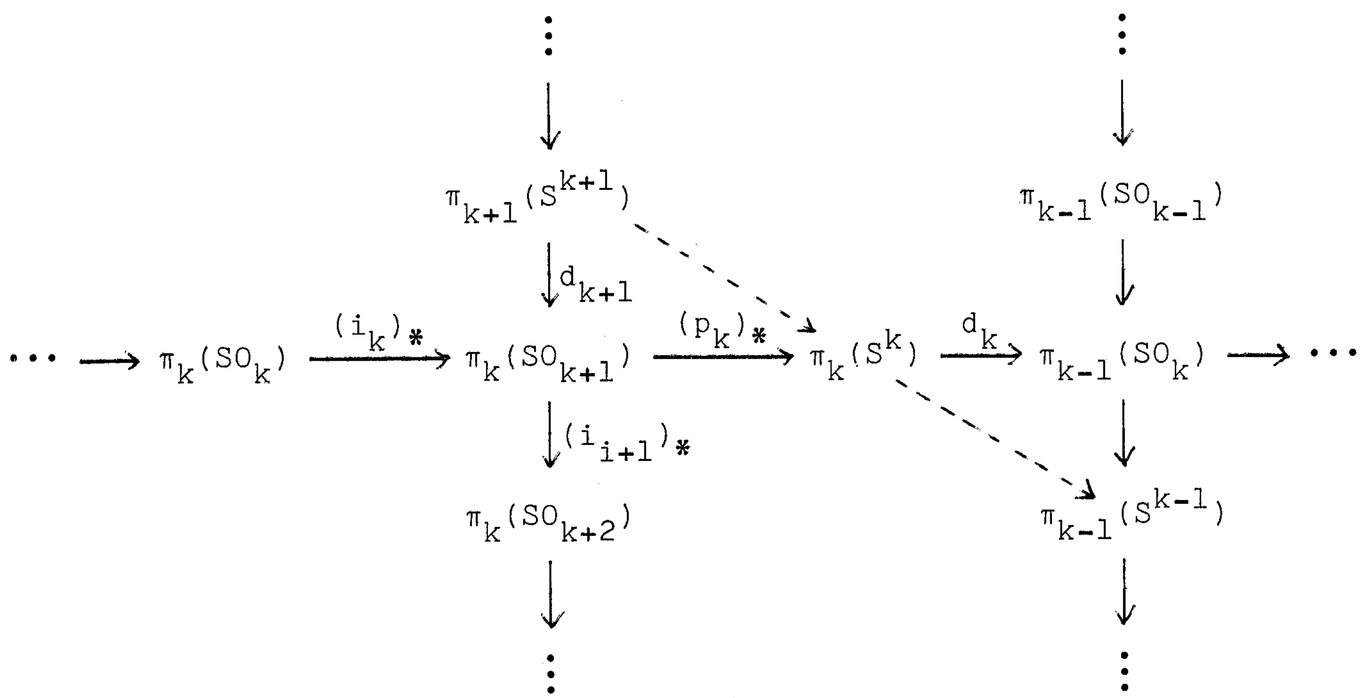
<u>Proof</u>. Cover S^n by two hemispheres as above. At a point x_o on the equation, the transition function for $\xi \oplus \varepsilon^1$ is $T \times \mathrm{id}: \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ where T is the transition function for ξ at x_o. But this characterizes the element i_* as well.

<u>Corollary 1.2</u>. Oriented stable bundles over S^n are in 1-1 correspondence with elements of $\pi_{n-1}(SO)$.

B) <u>Homotopy of the Classical Groups.</u>

Let $(0,\ldots,0,1) = e_k \in S^k \subset \mathbb{R}^{k+1}$. Then the projection $SO_{k+1} \xrightarrow{p_k} S^k$ given by $\sigma \mapsto \sigma(e_k)$ gives a fibre bundle $SO_k \xrightarrow{i_k} SO_{k+1} \xrightarrow{p_k} S^k$. If M is a manifold, let $\tau(M)$ denote the tangent bundle of M.

By weaving together the resulting exact sequences one obtains:



<u>Diagram 1</u>

where $d_k: \pi_k(S^k) \to \pi_{k-1}(SO_k)$ is the induced boundary map. By direct computation one checks that under d_k the generator is taken to $\tau(S^k) \in \pi_{k-1}(SO_k)$ and that, under $(p_{k-1})_*: \pi_{n-1}(SO_k) \to \pi_{n-1}(S^{k-1})$, a k-plane bundle ξ^k over S^n is taken to $O(\xi^k)$, the obstruction to finding a section (c.f. [28] §34.4). When $n = k$, $O(\xi^k) = \chi(\xi^k)$, the Euler class [28]. Since $\chi(\tau(S^k)) = \chi(S^k)$ generator where $\chi(S^k) = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$ is the Euler number, we have that the dashed maps are multiplication by 2 or 0 as indicated. This allows us to calculate the order of $\tau(S^k) \in \pi_{k-1}(SO_k)$. When k is even $(p_k)_*$ takes $\tau(S^k)$ to twice the generator and thus $\tau(S^k)$ has infinite order. When k is odd, twice the generator of $\pi_k S^k$ is in im p_{k*} so that $\tau(S^k)$ has order at most 2. Since S^k is parallelizable iff $k = 1, 3$ or 7 [3] we have

<u>Lemma 1.3</u>. $$\text{order } \tau(S^k) = \begin{cases} \infty & k \text{ even} \\ 1 & k = 1,3,7 \\ 2 & \text{otherwise} \end{cases}.$$

From the bundle exact sequence, we know that $(i_k)_*\colon \pi_j SO_k \to \pi_j SO_{k+1}$ is mono (resp. epi) unless $j = k-1$ (resp. $j = k$). Thus $\ker(\pi_{k-1}(SO_k) \to \pi_{k-1}(SO)) = \ker((i_k)_I\colon \pi_{k-1}(SO_k) \to \pi_{k-1}(SO_{k+1})) = \operatorname{im}(d_k\colon \pi_k S^k \to \pi_{k-1}SO_k)$. Applying Lemma 1.3 we obtain the first part of

<u>Theorem 1.4</u>. (1) $$\ker(\pi_{k-1}(SO_k) \to \pi_{k-1}(SO)) \cong \begin{cases} Z & k \text{ even} \\ 0 & k = 1,3,7 \\ Z_2 & \text{otherwise} \end{cases}$$

(2) $$\operatorname{coker}(\pi_k(SO_k) \to \pi_k(SO)) \cong \begin{cases} Z_2 & k = 1,3,7 \\ 0 & \text{otherwise} \end{cases}$$

(3) Let $V_{N,N-k}$ be the Steifel manifold of $N-k$ frames in N space. We have a bundle $SO_k \xrightarrow{i} SO_N \xrightarrow{p} V_{N,N-k}$. If N is large and $k = 3,7$, $\pi_k(SO_N) \xrightarrow{p_*} \pi_k(V_{N,N-k})$ is onto.

<u>Proof</u>. To prove (2) we need only investigate $\pi_k SO_k \xrightarrow{i_{k*}} \pi_k SO_{k+1} \xrightarrow{(i_{k+1})_*} \pi_k SO_{k+2}$ as the last group is also $\pi_k SO$. $(i_{k+1})_*$ is always epi. If k is even we see, from Diagram 1, that $d_k\colon \pi_k S^k \to \pi_{k-1}(SO_k)$ is mono and thus that i_{k*} is epi. If k is odd but unequal to 1, 3, or 7 the relevant part of Diagram 1 is

$$\begin{array}{ccccc} & & Z & & \\ & & \downarrow & \overset{\cong}{\searrow} & \\ \pi_k SO_k & \xrightarrow{i_{k*}} & \pi_k SO_{k+1} & \longrightarrow & 2Z \\ & & \downarrow {\scriptstyle (i_{k+1})_*} & & \\ & & \pi_k SO_{k+2} & & \end{array}$$

and a trivial diagram chase shows that $(i_{k+1} i_k)_*$ is epi. If $k = 1$, 3 or 7 we have

$$
\begin{array}{ccccc}
 & & \mathbb{Z} & & \\
 & & \big\downarrow d_{k+1} & \searrow^{\times 2} & \\
\pi_k SO_k & \xrightarrow{i_{k*}} & \pi_k SO_{k+1} & \longrightarrow & Z \\
 & & \big\downarrow (i_{k+1})_* & & \\
 & & \pi_k SO_{k+2} & &
\end{array}
$$

which concludes the proof.

(3) The bundle structure is given in [28] §7. This gives the sequences:

$$\pi_k(SO_N) \xrightarrow{p_*} \pi_k(V_{N,N-k}) \to \pi_{k-1}(SO_k) \to \pi_{k-1}(SO_N)$$

and the result now follows from (1).

We conclude this section by giving some results of Bott and Kervaire.

Theorem 1.5.

(1) $\pi_*(U)$ is periodic with period 2, $\pi_0 U = 0$, and $\pi_1 U = Z$

(2) $\pi_* O$ is periodic with period 8 and the actual homotopy groups are

i mod 8	0	1	2	3	4	5	6	7
$\pi_i O$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$

(3) For all j, $\pi_j(U/SO) \cong \pi_{j-2}(SO)$

(4) For all j, $\pi_{2j}(U_j) \cong Z_{j!}$.

Proof. (1) is proved in complete detail in [21] where a proof of (2) is also indicated. Both (2) and (3) can be found in [4] and (4) occurs in [5].

C) Some theorems of Whitney

Definition. An embedding $M \subset N$ of manifolds is proper if $\partial N \cap M = \partial M$ and M is transverse to ∂N.

Theorem 1.6. Let L^ℓ and M^m be compact proper submanifolds of N^n, $\ell + m = n$, such that L and M intersect transversely and the intersection number of L and M is zero. (The intersection number is an integer if L, M and N are oriented, and changing orientations changes its sign. If L, M or N is nonorientable, the intersection number is in Z_2.) If $\ell, m > 2$ and N is simply connected, then there is an ambient isotopy h_t of N such that $h_1(L) \cap M = \emptyset$.

Proof. Whitney's intersection removal technique is in [31]. See also Milnor [20].

The same technique yields

Theorem 1.7. Let $f: M^m \to N^{2m}$ be an immersion, M closed, with self-intersection number zero. (If m is even and M and N are oriented, the self intersection number is an integer. If M is odd, of if M or N is nonorientable, it is in Z_2.) If $m > 2$ and N is simply connected, then f is regularly homotopic to an embedding.

As a corollary of this theorem (and the approximation of continuous maps by immersions, and the fact that the self intersection number of an immersion can be changed arbitrarily without changing its homotopy type) we have:

Theorem 1.8. If N^{2k} is simply connected, $k > 2$, then any $\alpha \in \pi_k(N)$ can be represented by an embedded sphere.

Theorem 1.9. Let $f: (M^m, \partial M) \to (N^{2m-1}, \partial N)$ be a continuous map such that $f|\partial M$ is an embedding. Then f is homotopic to an immersion keeping $f|\partial M$ fixed.

Proof. See [32].

Definition. Let M and N be closed manifolds. Immersions $f_i: M \to N$, $i = 0,1$, are concordant if there is an immersion $f: M \times I \to N \times I$ such that $F^{-1}(N \times \{i\}) = M \times \{i\}$ and $F|M \times \{i\} = F_i$, $i = 0,1$.

Corollary 1.10. Let M^m and N^{2m} be closed manifolds. Two embeddings $f_i: M \to N$, $i = 0,1$, are homotopic if and only if they are concordant as immersions.

Proof. If $F: M \times I \to N \times I$ is a concordance, then $\pi \circ F: M \times I \to N$ is a homotopy, where $\pi: N \times I \to N$ is projection onto N. If $h: M \times I \to N$ is a homotopy from f_0 to f_1, let $H: M \times I \to N \times I$ be given by $H(x,t) = (h(t),t)$. Applying Theorem 1.9 to H, we obtain a concordance from f_0 to f_1.

§2. Some theorems on framed surgery

Let M be an oriented smooth manifold. Suppose that surgery is performed via the embedding $f\colon S^k \times D^{n-k} \to M^n$ to obtain a manifold $M' = (M - f(S^k \times D^{n-k})) \cup D^{k+1} \times S^{n-k-1}$. (We will always assume $f(S^k \times D^{n-k}) \subset \operatorname{Int} M$.) The "trace" W of the surgery is obtained by attaching the "handle" $D^{k+1} \times D^{n-k}$ to $M \times I$ by identifying $(\partial D^{k+1}) \times D^{n-k}$ with $f(S^k \times D^{n-k}) \times \{1\}$. Thus $\partial W = M' - M$. Let $\varepsilon^N(X)$ denote the trivial N-plane bundle over X. (We will write ε^N when the base space is clear from the context.)

Definition. A framed manifold (M,F) is a smooth manifold M together with a framing F of $\tau(M) \oplus \varepsilon^N(M)$ for some $N > 0$. A framed surgery of (M,F) is a surgery of M (as above) together with a framing G of $\tau(W) \oplus \varepsilon^k(W)$ $(k \geq N-1)$, where W is the trace of the surgery, satisfying $G|M = F \oplus t^{k-N+1}$, where t^{k-N-1} is the standard framing of ε^{k-N+1}. (Here M is identified with $M \times 0 \subset W$, and $\tau(W)|M$ is identified with $\tau(M) \oplus \varepsilon^1$ by using the inward normal vector field on $M \subset \partial W$.) Restricting G to $\partial W - M = M'$ we obtain a framed manifold (M',F'), the result of the framed surgery on (M,F). ($\tau(W)|M' = \tau(M') \oplus \varepsilon^1$ via outward normal field on M'.)

Remarks.

1) There is a corresponding definition of framed cobordism. Two closed framed manifolds (M,F) and (M',F') are framed cobordant if there is a compact framed manifold (W,G) such that $\partial W = M - M'$, $G|M = F$, and $G|M' = F'$. (More precisely, this means there exist integers $i,j,k \geq 0$ such that $G \oplus t^i|M = F \oplus t^j$ and $G \oplus t^i|M' = F' \oplus t^k$. Again we identify $\tau(W)|M$ with $\tau(M) \oplus \varepsilon^1$, and $\tau(W)|M'$ with $\tau(M') \oplus \varepsilon^1$.) It is easy to check that framed cobordism is an equivalence relation. Clearly if (M',F') is obtained from (M,F) by a finite sequence of framed surgeries, then (M',F') is framed cobordant to (M,F). Conversely (M',F') is framed cobordant to (M,F) implies that (M',F') can be obtained from (M,F) by a finite sequence of framed surgeries (compare Milnor [2]).

If (M_1,F_1) and (M_2,F_2) are framed manifolds, $(M_1,F_1) \# (M_2,F_2)$ denotes their framed connected sum. (See [10].) The set of framed cobordism classes of framed closed manifolds forms an abelian group under $\#$.

2) If F is homotopic to F', then clearly (M,F) is framed cobordant to (M,F'). By an easy obstruction argument, homotopy classes of framings of $\tau(M) \oplus \varepsilon^N$ for any fixed N are in one-to-one corre-

spondence with homotopy classes of framings of $\tau(M) \oplus \varepsilon^1$. Thus our definition of framed cobordism gives the same equivalence classes as the definition in Kervaire-Milnor [15].

3) The following conditions are equivalent:

i) M is s-parallelizable ("framable")--i.e. the bundle $\tau(M) \oplus \varepsilon^N$ is trivial for some N. (M is <u>parallelizable</u> if $\tau(M)$ is trivial: "s-parallelizable" means "stably parallelizable".)

ii) $\tau(M) \oplus \varepsilon^1$ is trivial. (This is the definition Kervaire-Milnor [15] gives for s-parallelizable.)

iii) M is a π-manifold (i.e. there is an N such that M embeds in R^N with trivial normal bundle). (See [15] and [20].)

(i) $\Leftrightarrow$ (iii) can be strengthened as follows: Let $i: M^n \to R^{n+k}$ be an embedding, k large. Then

$$\tau(M) \oplus \nu(i) = \tau(R^{n+k})|M = \varepsilon^{n+k} \qquad (\nu = \text{normal bundle})$$

so

$$\varepsilon^N \oplus \tau(M) \oplus \nu(i) = \varepsilon^{N+n+k} .$$

4) A manifold with boundary is s-parallelizable and only if it is parallelizable. (See [15].)

<u>Lemma</u>. Suppose N is large. Then if F is a framing of $\varepsilon^N \oplus \tau(M)$, there exists a framing F' of $\nu(i)$ such that $F \oplus F' \simeq t^{N+n+k}$, and any two such F' are homotopic. Conversely, if F' is a framing of $\nu(i)$, there exists a framing F of $\varepsilon^N \oplus \tau(M)$ such that $F \oplus F' \simeq t^{N+n+k}$, and any two such F are homotopic.

<u>Proof</u>. We will show that if ξ^k and ξ^ℓ are vector bundles over the manifold M^n with $\ell > n+1$, such that $\xi^k \oplus \eta^\ell \cong \varepsilon^{k+\ell}$, and F is a framing of ξ^k, then there exists a framing F' of η^ℓ, unique up to homotopy, such that $F \oplus F' = t^{k+\ell}$. F defines a map $\phi: M \to V_{k+\ell,k}$. Since $V_{k+\ell,\ell}$ is $\ell-1$ connected, $n < \ell$ implies that ϕ is null homotopy (by obstruction theory). Thus by the homotopy lifting property of $V_{k+\ell} \to V_{k+\ell,\ell}$, ϕ extends to a map $M \to V_{k+\ell,k+\ell}$. Thus F' exists. Suppose F'' is another framing of η^ℓ such that $F \oplus F'' \simeq t^{k+\ell}$. Then F' and F'' differ by a map $\alpha: M \to SO_\ell$, and if $i: SO_\ell \to SO_{k+\ell}$, $i \circ \alpha \simeq 0$. But $i_*: \pi_i SO_\ell \cong \pi_i SO_{k+\ell}$ for $i < \ell-1$, so since $n < \ell-1$, $i_*[\alpha] = 0 \to [\alpha] = 0$ (by obstruction theory). Thus $F'' \simeq F$.

<u>Definition</u>. Suppose that (M_1,F_1), (M_2,F_2) are normally framed manifolds

(i.e. F_i is a framing of an embedding $f_i: M_i \subset R^N$, N large). (M_1,F_1) and (M_2,F_2) are normally framed cobordant if there is a manifold W with $\partial W = M_1 \cup M_2$ and an embedding $g: W \to R^N \times I$ such that int $W \cap \partial(R^n \times I) = \emptyset$ and $g|M_i = f_i$, with a framing G of $\nu(g)$ such that $G|M_i = F_i$.

The set of normally framed cobordism classes of closed normally framed manifolds forms a group Ω_f^n under connected sum. By the lemma and remark 2) above, Ω_f^n is canonically isomorphic to the group of (tangentially) framed cobordims classes of (tangentially) framed manifolds. Pontryagin proved that Ω_f^n is isomorphic to the n-stem $\pi_n(S)$, the correspondence being the Thom-Pontryagin construction. For a proof, see [22]. In these notes, a "framed manifold" will usually mean a manifold with a framing of its stable tangent bundle. Normal framings are used only when the Thom-Pontryagin construction is needed.

Theorem 2.1. Let M be a compact framed manifold of dimension $n \geq 4$ such that ∂M is a homology sphere. By a finite sequence of framed surgeries M can be made $[\frac{n-1}{2}]$ connected.

Proof. This is 5.5 and 6.6 of Kervaire-Milnor [15].

This theorem says that for a compact framed manifold, surgery can be done to kill all homotopy groups "below the middle dimension." Therefore, by Poincaré duality, we have:

Corollary 2.2. Suppose that M^n is compact, framed, n odd ≥ 5, and ∂M is a homotopy sphere (resp. $\partial M = \emptyset$). By a finite sequence of framed surgeries M can be made contractible (resp. a homotopy sphere). Thus $bP^n = 0$ for n odd.

Surgery can be completed in the middle dimension of an even dimensional framed manifold if the middle homology group can be represented in a special way:

Theorem 2.3. Let M^{2k}, $k \geq 3$, be a compact framed (k-1)-connected manifold, ∂M a homotopy sphere (resp. $\partial M = \emptyset$). Suppose there is a basis $\alpha_1,\ldots,\alpha_r, \beta_1,\ldots,\beta_r$ of $H_k(M)$ such that

(1) $\alpha_i \cdot \alpha_j = 0$, $\beta_i \cdot \beta_j = \delta_{ij}$ for all i,j ("$\cdot$" is intersection number. Such a basis is called (weakly) symplectic).

(2) The α_i can be represented by disjoint embedded spheres with trivial normal bundles. (Note that the α_i are spherical by the Hurewicz theorem.)

Then M can be made contractible (resp. a homotopy sphere) by a finite sequence of unframed surgeries. The surgeries can be framed unless $k = 3$ or 7.

<u>Proof</u>. All but the last statement is included in the proof of Theorem 4 of Milnor [20]. As shown in §6 of Kervaire-Milnor [15] (see also the proof of 4.2b below), the obstruction to framing a surgery performed via an embedding $f: S^k \times D^k \to M^{2k}$ lies in $\pi_k(SO_{2k+N}) = \pi_k(SO)$, and this obstruction can be altered by any element in the image of the map $i_*: \pi_k(SO_k) \to \pi_k(SO)$. But i_* is surjective for $k \neq 1,3,7$ (1.4), so any surgery can be framed.

When can the hypotheses of this theorem be satisfied? If k is even (i.e. $n \equiv 0(4)$), we will see that $H_k(M)$ has a symplectic basis if and only if the <u>signature</u> (index) of M is zero. However, (2) always holds for k even, assuming (1) (see §3). If k is odd, $k \neq 3,7$, $H_k(M)$ has a symplectic basis, and the normal bundles of embedded spheres representing this basis are trivial if and only if the <u>Kervaire (Arf) invariant</u> of M is zero (§4). If $k = 3$ or 7, (1) and (2) both hold, but there is an obstruction to framing the surgery. In §4 this obstruction and the Kervaire invariant are shown to be manifestations of a single invariant which can be defined for all odd k.

<u>Corollary 2.4</u>. $bP^6 = bP^{14} = 0$.

§3. <u>Computation of</u> bP^{4k}

In this section we compute bP^{4k} by defining a surjective map from Z to bP^{4k}, and determining its kernel.

Let $\Sigma \in bP^{4k}$ say $\Sigma = \partial M^{4k}$ with M parallelizable. If Σ also bounds a contractible manifold, $\Sigma = 0$ in bP^{4k}, thus $\Sigma = 0$ if we can kill the homotopy of M by framed surgery. Theorem 2.1 allows us to assume that M is $(2k-1)$-connected, which places us in the situation described by Theorem 2.3, the hypotheses of which are satisfied iff the signature of M is zero.

<u>Definition</u>. Let M^{4k} be a compact oriented manifold with $H_{2k}M$ free (e.g. M $(2k-1)$-connected). The <u>signature</u> (index) of M $\sigma(M)$ is the situature of the quadratic (i.e. symmetric bilinear) form $\langle\ ,\ \rangle$: $H_{2k}M \otimes H_{2k}M \to Z$ given by the intersection pairing $\langle\alpha,\beta\rangle = \alpha\cdot\beta$.

<u>Remark</u>. $\sigma(M \# M') = \sigma(M) + \sigma(M')$ where $\#$ is connected sum.

<u>Proof</u>. $\langle\ ,\ \rangle$ is dual to cup product, i.e.

$$\begin{array}{ccc} H_{2k}M \otimes H_{2k}M & \xrightarrow{\langle\ ,\ \rangle} & \mathbb{Z} \\ \cong \downarrow \text{PD} \otimes \text{PD} & & \downarrow \cong \quad \text{commutes} \\ H^{2k}(M,\ M) \otimes H^{2k}(M,\partial M) & \xrightarrow{\ \cup\ } & H^{4k}(M,\partial M) \end{array}$$

and (c.f. Milnor [19]) the signature of the cup product is additive with respect to connected sums.

<u>Theorem 3.1</u>. Let (M^{4k},F) be a compact framed $(2k-1)$-connected manifold with ∂M a homotopy sphere (resp. $\partial M = \emptyset$). Then (M,F) can be framed surgered into a contractible manifold (resp. a homotopy sphere) iff $\sigma(M) = 0$.

<u>Corollary 3.2</u>. The Hirzebruch index theorem (below) implies that, if M^{4k} is framed and $\partial M = \emptyset$, then $\sigma(M) = 0$ and hence M is framed null cobordant.

<u>Proof</u>. ($\Rightarrow$) If a closed manifold N^{4k} bounds a compact manifold, then $\sigma(N) = 0$ (c.f. [17]). By the above remark, σ is thus an invariant of oriented cobordism. Therefore, if $\partial M = \emptyset$ and M can be surgered into a homotopy sphere Σ, $\sigma(M) = \sigma(\Sigma) = 0$.

Now suppose that N^{4k} is compact and $\partial N = \Sigma = \partial D$ with D contractible. We claim that $\sigma(N) = \sigma(N \cup_{\Sigma} D)$. Let $V = N \cup_{\Sigma} D$ and let $i\colon N \to V$ be the inclusion. Then we have the commuting diagram

$$\begin{array}{ccc} H^{2k}V \otimes H^{2k}V & \xrightarrow{\ \cup\ } & H^{4k}(V) \cong Z \\ \cong \downarrow i^* \otimes i^* & & \cong \downarrow i^* \\ H^{2k}(N,\partial N) \otimes H^{2k}(N,\partial N) & \xrightarrow{\ \cup\ } & H^{4k}(N,\partial N) \cong Z \end{array} .$$

As $\langle\ ,\ \rangle$ is dual to $\cup$, the claim follows. If $\partial M = \Sigma$ and M can be surgered to D, let W be the union of the traces of the surgeries. Then $\partial W \cong M \cup (D \cup \Sigma \times I)$. Thus, by our claim, $\sigma(M) = \sigma(\partial W) = 0$.

($\Leftarrow$) We will verify (1) and (2) of Theorem 2.3. Since $\sigma(M) = 0$, $\exists$ a symplectic basis, $\alpha_1,\ldots,\alpha_r$, $\beta_1,\ldots,\beta_r$ for $H_{2k}(M)$ (c.f. [26]). By the Hurewicz theorem, each α_i is spherical and can be represented by an embedding $f_i\colon S^{2k} \to M^{4k}$ (Theorem 1.8). Since $\alpha_i \cdot \alpha_j = 0$, the $f_i(S^{2k})$ can be isotoped so as to be disjoint (Theorem 1.6). Let $\nu(f_1)$ be the normal bundle. $[\nu(f_1)] \in \pi_{2k-1}(SO_{2k})$, and we have the commutative diagram (c.f. §1B)

$$\begin{array}{ccccc} \pi_{2k}(S^{2k}) & \xrightarrow{d_{2k}} & \pi_{2k-1}(SO_{2k}) & \xrightarrow{(i_{2k-1})_*} & \pi_{2k-1}(SO_{2k+1}) = \pi_{2k-1}(SO) \\ & \searrow^{\times 2} & \downarrow P_{2k-1} & & \\ & & \pi_{2k-1}(S^{2k-1}) & & \end{array}$$

Now $\tau(S^{2k}) \oplus \nu(f_i) = f_i^*(\tau(M))$, so since $\tau(M)$ and $\tau(S^{2k})$ are stably trivial, so is $\nu(f_i)$, i.e. $i_*[\nu(f_i)] = 0 \in \pi_{2k-1}(SO)$. Thus $[\nu(f_i)] \in \mathrm{Im}\ \alpha$. But $p_{2k-1}[\nu(f_i)] = X(\nu(f_i))\cdot\mathrm{gen} = (\alpha_i\cdot\alpha_i)\mathrm{gen} = 0$, and $\mathrm{Im}\ d_{2k} \cap \mathrm{Ker}\ p_{2k-1} = 0$, since $p_{2k-1}d_{2k}$ is multiplication by 2, so $[\nu(f_i)] = 0$, i.e. $\nu(f_i)$ is trivial.

Theorem 3.3. Let M^{4k} be a framed $(2k-1)$ connected manifold whose boundary is empty or a homotopy sphere. Then $\sigma(M)$ is a multiple of 8.

Proof. Pick $\alpha \in H_{2k}M$ and let $\alpha' \in H^{2k}(M,\partial M)$ be its Poincaré dual. The mod 2 computation $\alpha' \cup \alpha' = Sq^{2k}\alpha' = V^{2k} \cup \alpha' = 0$ (V^{2k}, the $2k^{\text{th}}$ Wu class, is zero since $\tau(M)$ is stably trivial) shows that $\alpha \cup \alpha$ (and hence its dual $\langle\ ,\ \rangle$) is always even i.e. $\langle\ ,\ \rangle$ is an even quadratic form (c.f. [20] for a more geometric proof). Since the signature of an even unimodular integral quadratic form is a multiple of 8 (c.f. [26]), we need only show the:

Assertion. $\langle\ ,\ \rangle$ is unimodular.

Proof. We have the commuting diagram

$$\begin{array}{ccccccc} H^{2k}(M,\partial M) & \xrightarrow{i^* \cong} & H^{2k}(M) & \xrightarrow{PD \cong} & H_{2k}(M,\partial M) & \xrightarrow{i_* \cong} & H_{2k}(M) \\ \alpha & \rightsquigarrow & \alpha & \rightsquigarrow & \alpha \cup \mu_M = \alpha & \leftsquigarrow & \alpha \\ & & \searrow & \swarrow & \swarrow & & \\ & & \mathrm{Hom}(H_{2k}M, Z) & & \langle\alpha,\cdot\rangle & & \end{array}$$

Where we have abused notation by not distinguishing between elements in isomorphic absolute and relative groups (μ_M is the fundamental class of M). $\langle\ ,\ \rangle$ is unimodular iff the map

$$\begin{array}{ccc} H_{2k}M & \longrightarrow & \mathrm{Hom}(H_{2k}M, Z) \\ \alpha & \rightsquigarrow & \langle\alpha,\cdot\rangle \end{array}$$

is an isomorphism. But the above diagram factors this map into the composition of three isomorphisms.

Theorem 3.4. Let $k > 1$ and $t \in Z$. Then $\exists$ a framed $4k$ manifold (M,F) with ∂M a homotopy sphere $\sigma(M) = 8t$.

A very complete proof can be found in [7] (see also [23]). The manifolds are constructed by plumbing disc bundles over spheres.

We now describe the map mentioned in the first paragraph of §3.

Definition. $b_k: Z \to bP^{4k}$ is defined as follows. Let $b_k(t) = [\partial M^{4k}]$ where M^{4k} is a framed manifold with signature $8t$ having boundary a homotopy sphere.

In the Appendix we will see that b_k can be thought of as a "boundary" map.

Lemma 3.5. (1) b_k is well defined, i.e. if M_1 and M_2 are as above, ∂M_1 is cobordant to ∂M_2.
(2) b_k is surjective.

Proof. For (1), it suffices to show that the connected sum $\partial M \# \partial M'$, a homotopy sphere, is cobordant to zero. From the boundary connected sum $W = M \# -M'$ (c.f. [1]). $\partial W = \partial M \# \partial M'$. But $\sigma(W) = 0$ so, by Theorem 3.1, W can be (interior) surgered into a contractible manifold. (1) follows from (2) is immediate from Theorem 3.4.

Corollary 3.6. $bp^{4k} \cong Z/\ker b_k$.

We now try to determine $\ker b_k$.

Suppose $t \in \ker b_k$. Then we have a framed manifold (M,F) with signature $8t$ whose boundary, Σ, is a homotopy sphere that bounds a contractible manifold D. Attaching D to M by identifying ∂M with ∂D gives an almost framed closed manifold N of dimension $4k$ with $\sigma(N) = 8t$. (An almost framed manifold is a pair (N,G) where G frames $\tau(N)|_{N-\{x\}}$ for some $x \in N$.) Conversely, given an almost framed closed manifold N^{4k} with $\sigma(N) = 8t$, let $D \subset N$ be any embedded disc. Then $N - \text{int}\, D$ is framed and has signature $8t$ and boundary S^{4k-1}. This gives:

Theorem 3.7. $t \in \ker b_k$ if $\exists$ an almost framed closed 4k-manifold with signature 8t.

This theorem leads us to investigate the signature of almost framed closed manifolds. Our tool is of course the:

Hirzebruch signature (née index) theorem. For any closed manifold M^{4k}, $\sigma(M)$ is the Kroneker index $\langle L_k(P_1(M),\ldots,P_k(M)),\mu_M\rangle$ where L_k is a rational function and the P_i's are the Pontrjagin classes (see [19] or [12]). The only fact that we will use about L_k is that $L_k(x_1,\ldots,x_k) = s_k x_k$ + terms not involving x_k where

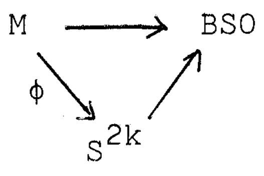
$$s_k = \frac{2^{2k}(2^{k-1}-1)B_k}{(2k)!}$$

(B_k is the k^{th} Bernoulli number.)

Let (M^{4k},F) be an almost framed closed manifold. Since $p_i(M)=0$ $i<k$, $\sigma(M) = s_k p_k(M)$. We will see that the obstruction to extending the almost framing to a stable framing of M (i.e. a framing of $\tau(M)\oplus\varepsilon^N$) actually determines $\sigma(M)$ and is thereby useful in calculating $\ker b_k$ and consequently bP^{4k}.

The obstruction $O(M,F)\in\pi_{4k-1}(SO)\cong Z$ (Theorem 1.5 (2)) can be defined as follows. Let $x\in M$ be the point where F is not defined. Next choose $x\in U\cong D^{4k}$ and let F' be the usual framing of D^{4k} (which orients D^{4k} consistently with M). $O(M,F)\in\pi_{4k-1}(SO)$ is the obstruction to forcing agreement of the stable framings F and F' on $U-\{x\}\cong S^{4k-1}$.

Let $\tau\colon M\to BSO$ be the classifying map of the stable tangent bundle of M. Since $M-\{x\}$ is stably parallelizable, $\tau|M-\{x\}$ is null-homotopy and thus factors (up to homotopy) as



where ϕ collapses to a point the complement of an open disk containing x. Hence $\exists\ \xi$, a stable oriented vector bundle $/S^{4k} \ni \phi^*\xi$ is the stable tangent bundle of M. As usual (c.f. §1A) we view $[\xi]\in\pi_{4k-1}(SO)$. one checks that $[\xi] = \pm O(M,F)$.

The above factorization of τ shows that the Pontryagin classes of almost framed 4k-manifolds can be determined by examining the k^{th} Pontryagin class of stable vector bundles $/S^{4k}$.

Theorem 3.8. (c.f. [18]). If ξ is a stable vector bundle over S^{4k}, then $p_k(\xi) = \pm a_k(2k-1)![\xi]$ where $a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$.

Proof. One task is to make sense of the above equation as a priori the two sides lie in different groups. We will see that each group is isomorphic to Z and hence, up to sign, they are canonically isomorphic to each other.

By definition $p_k(\xi) = \pm C_{2k}(\xi^C)$ (the $2k^{th}$ Chern class of the complexification of ξ) and, just as $[\xi] \in \pi_{4k-1}(SO_N)$, $[\xi^C]$ $\pi_{4k-1}(U_N) \in$ (N large). In fact ξ^C is $i_k(\xi)$ where $i\colon SO_N \to U_N$ is the inclusion.

Let $W_{m,\ell}$ be the space of complex orthonormal ℓ frames in C^m (cf. [28]).

$C_{2k}((\xi^N)^C) \in H^{4k}(S^{3k}, \pi_{4k-1}(W_{N,N-2k+1})) \cong \pi_{4k-1}(W_{N,N-2k+1})$, is the obstruction to extending an $N-2k+1$ dimensional complex framing of ξ^C from the $4k-1$ skeleton to S^{4k} itself. Equivalently it is the obstruction to extending an $N-2k+1$ dimensional framing of ξ^C from the southern hemisphere to S^{4k}. Since ξ^C is the obstruction to extending to complete framing from the southern hemisphere to S^{4k}, we see that $C_{2k}(\xi^C) = P_*(\xi^C)$ where $p\colon U_n \to U_n/U_{2k-1}$ is the usual projection. We have the exact sequence:

$$\pi_{4k-1}(U_N) \xrightarrow{p_*} \pi_{4k-1}(W_{N,N-2k+1}) \xrightarrow{\partial} \pi_{4k-2}(U_{2k-1}) \longrightarrow \pi_{4k-2}(U_N)$$

By (1.5) ($\pi_j(W_{m,\ell})$ is calculated in [3]) the above sequence becomes

$$Z \xrightarrow{p_*} Z \longrightarrow Z_{(2k-1)!} \longrightarrow 0\ .$$

Hence p_* is multiplication by $(2k-1)!$. Since we have

$$\begin{array}{ccccc} Z & & Z & & Z \\ \wr\| & & \wr\| & & \wr\| \\ \pi_{4k-1}(SO_N) & \xrightarrow{i_*} & \pi_{4k-1}(U_N) & \xrightarrow{p_*} & \pi_{4k-1}(W_{N,N-2k+1}) \\ \xi & \rightsquigarrow & \xi^C & \rightsquigarrow & \pm C_{2k}(\xi^C) = \pm p_k(\xi) \end{array}$$

it remains to show that i_* is multiplication by $\pm a$. As N is large we may work with the stable map $i_*\colon \pi_{4k}(SO) \to \pi_{4k}(U)$. But we have the exact sequence

$$\pi_{4k}(U/SO) \longrightarrow \pi_{4k-1}(SO) \xrightarrow{i_*} \pi_{4k-1}(U) \longrightarrow \pi_{4k-1}(U/SO)$$
$$\begin{array}{ccc} \pi_{4k-1}(SO) & & \pi_{4k-1}(U) \\ \wr\| & & \wr\| \\ Z & & Z \end{array}$$

and in §1B we have also shown that $\pi_{4k}(U/SO) \cong \pi_{4k-2}(SO) \cong 0$ and that $\pi_{4k-1}(U/S) = \pi_{4k-3}(SO) = \begin{cases} 0 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$. The result follows.

Let M be an almost framed closed manifold. We have:

Corollary 3.9. $p_k(M) = \pm a_k(2k-1)!O(M,F)$.

Corollary 3.10. $O(M,F)$ is independent of F.

Corollary 3.11. $\sigma(M) = \dfrac{\pm a_k 2^{2k-1}(2^{2k-1}-1)B_k O(M,F)}{k}$.

Corollary 3.12. M is s-parallelizable if $\sigma(M) = 0$.

In order to completely determine bP^{4k} some basic properties of the J-homomorphism are needed (c.f. [13]).

Definition. Given n and ℓ we define $J = J_{n,\ell}: \pi_m(SO_\ell) \to \pi_{m+\ell}(S^\ell)$ as follows: Let $[\alpha] \in \pi_m(SO_\ell)$. $J(\alpha): S^{m+\ell} \to S^\ell$ is constructed in two stages. We view $S^{m+\ell}$ as $(S^m \times D^\ell) \cup (D^{m+1} \times S^{\ell-1})$ and first define $J(\alpha)$ on $S^m \times D^\ell$ as the composition $S^m \times D^\ell \xrightarrow{\psi} D^\ell \xrightarrow{c} S^\ell$ where $\psi(x,y) = \alpha(x)(y)$ and c collapses ∂D^ℓ to a point. The second stage, extending $J(\alpha)$, is trivial as $c \circ \psi(\partial(S^m \times D^\ell))$ is just one point. $J([\alpha])$ is defined as $[J(\alpha)]$. One then verifies the

Lemma 3.13. View S^m as $S^m \times \{0\} \subset S^m \times D^\ell \subset S^{m+\ell}$ with F_0 the standard normal framing $S^m \subset S^m + D^\ell$. Given $[\alpha] \in \pi_m(SO_\ell)$, let F_α be the framing obtained by "twisting F_0 via α" (i.e. at $x \in S^m$, $F_\alpha(x) = \alpha(x)(F_0(x))$). The Thom-Pontryagin construction applied to $(S^m \subset S^\ell, F_\alpha)$ gives $\pm J([\alpha])$.

Since

$$\begin{array}{ccc} \pi_m(SO_\ell) & \xrightarrow{J} & \pi_{m+\ell}(S^\ell) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \Sigma} \\ \pi_m(SO_{\ell+1}) & \xrightarrow{J} & \pi_{m+\ell+1}(S^{\ell+1}) \end{array}$$

commutes (Σ is the suspension homomorphism), we obtain the stable J

homomorphism $J: \pi_m(SO) \to \pi_m(S)$, where $\pi_m(S) = \lim_{\ell}\{\pi_{m+\ell}(S^\ell) \overset{\Sigma}{\to} \pi_{m+\ell+1}(S^{\ell+1})\}$ is the m-stem. The relevance of the J homomorphism to our work to the following:

Theorem 3.14. Given $\alpha \in \pi_{m-1}(SO)$, $\exists$ an almost framed closed manifold (M^m,F) with $O(M,F) = \alpha$ iff $J(\alpha) = 0$.

Proof. $\Rightarrow$ We may assume that F is a framing of $M^m - \text{int}\, D^m$ with D^m a closed disc. Now imbed M^m in $\mathbb{R}^N$ (N large) so that D^m is the northern hemisphere of the standard m-sphere in $\mathbb{R}^N$. Let F_0 be the usual (outward) normal framing of $D^m \subset \mathbb{R}^N$. Let $F_\alpha = F_0|_{S^{m-1}}$ twisted via α. Hence the Thom-Pontryagin construction applied to (S^{m-1},F_α) gives $\pm J(\alpha)$. Since $\alpha = O(M,F), F = F|_{S^{m-1}}$. Thus $(S^{m-1},F_\alpha) = \partial(M^m - \text{int}\, D^m, F)$ so (S^{m-1},F_α) is framed null cobordant. Therefore, the Thom-Pontraygin construction yields $\sigma \in \pi_{m-1}(S)$.

$\Leftarrow$ $S^{m-1} \subset D^m$. Let F_0 be the standard framing of $D^m \subset S^N$ (N large). Since $J(\alpha) = 0$, a framed manifold (N^m,F) such that $\partial(N^m,F) = (S^{m-1},F_\alpha)$. Let $M^n \subset N \underset{S^{m-1}}{} D^m$. Then (M^m,F) is an almost framed closed manifold and $O(M^m,F) = \alpha$.

If we let j_k be the order of the image of the stable J homomorphism $Z \cong \pi_{4k-1}(SO) \overset{J}{\to} \pi_{4k-1}(S)$ we get the following:

Corollary 3.15. The possible values for $O(M,F)$ are precisely the multiples of j_k.

Corollary 3.16. The possible values for $\sigma(M)$ are precisely the multiples of $\dfrac{a_k 2^{2k-1}(2^{2k-1}-1)B_k j_k}{k}$.

In order to (finally) get exact information about bP^{4k} we need a hard

Theorem of Adams 3.17. [1] [33] Let $J: \pi_m(SO) \to \pi_m(S)$.

1) If $m \neq 3(4)$, J is injective.
2) j_k = denominator $(B_k/4k)$.

Although our primary interest in in 2), 1) also has important consequences.

Corollary 3.18. If M is an almost framed closed manifold and $\dim M \neq 0(4)$ then the almost framing of M extends to a complete framing.

Since homotopy $4k$ spheres have signature 0, we get:

Corollary 3.19. Any homotopy sphere is s-parallelizable.

We have already seen that bP^{4k} is a finite factor group of Z. Let t_k be the order of that group, we have (using 3.7 and 3.16) that

$$8t_k = \frac{a_k 2^{2k-1}(2^{2k-1}-1)B_k j_k}{k} .$$

Thus $t_k = a_k 2^{2k-2}(2^{2k-1}-1)(B_k/4k)j_k$ and, applying 3.17, this gives the final

Corollary 3.20. $bP^{4k} = Z_{t_k}$ where $t_k = a_k 2^{2k-2}(2^{2k-1}-1)$ numerator $\cdot$ $(B_k/4k)$.

§4. Computation of bP^n for $n = 2 \bmod 4$.

We proceed as in §3, computing bP^n by studying the kernel of a surjective map $Z_2 \to bP^n$.

Suppose that $\Sigma \in bP^n$, i.e. $\Sigma = \partial M^{2k}$, where k is odd, and M is a parallelizable manifold. By Theorem 2.1, M can be made $(k-1)$-connected by a finite sequence of framed surgeries. We wish to discuss the "obstruction" to a compact, framed $(k-1)$-connected manifold (M^{2k}, F), k odd, satisfying the hypotheses of Theorem 2.3.

First notice that the intersection pairing $H_k M \otimes H_k M \to Z$ is skew-symmetric (since k is odd) and unimodular (by the proof of Theorem 3.3). Therefore [26] there is a symplectic basis for $H_k(M)$, i.e. there is a basis $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$ for $H_k(M)$ with intersection matrix

$$\begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix}$$

As in §3, each α_i is spherical by the Hurewicz theorem, and so if $k > 2$, the α_i are represented by disjoint embedded spheres (by 1.7 and 1.8). Furthermore any two embeddings $f\colon S^k \to M^{2k}$ representing an element $\alpha \in H_k(M^{2k})$ are concordant as immersions by 1.10. Now

$$f^*\tau(M) \cong \tau(S^k) \oplus \nu(f)$$

so

$$\varepsilon^N(s^k) \oplus f^*\tau(M) \cong \varepsilon^N(S^k) \oplus \tau(S^k) \oplus \nu(f)$$

$\varepsilon^N(S^k) \oplus f^*\tau(M) = f^*(\varepsilon^N \oplus \tau(M))$, so the framing F of $\phi^N \oplus \tau(M)$ pulls back to give a framing f^*F of $\varepsilon^N \oplus f^*\tau(M)$. If F_0 is the usual framing of $\varepsilon^{N-1} \oplus \tau(D^{k+1})$, $F_0|S^k$ gives a framing of $\varepsilon^{N-1} \oplus \tau(D^{k+1})|S^k = \varepsilon^N \quad \tau(S^k)$:

$$(*) \qquad \underbrace{\varepsilon^N \oplus f^*\tau(M)}_{f^*F} \cong \underbrace{\varepsilon^N \oplus \tau(S^k)}_{f_0|S^k} \oplus \nu(f)$$

Thus since f^*F gives a trivialization of $\varepsilon^N \oplus \tau(S^k) \oplus \nu(f)$, the framing $F_0|S^k$ assigns to each point in S^k an element of $V_{2k+N,k+N}$. Thus we get an element

$$\Phi(f) \in \pi_k(V_{2k+n,k+n}) \cong Z_2 \quad (k \text{ odd})$$

depending on M, F, and f. We will show that $\Phi(f)$ does not in fact depend on the choice of the embedding f representing α. Suppose $f_0, f_1 \colon S^k \to M$ are embeddings representing α. Let $H\colon S^k \times I \to M \times I$ be an immersion concordance between them (1.10). Then we have the following bundles and framings over the space $S^k \times I$

$$(**) \qquad \underbrace{\varepsilon^{N-1} \oplus H^*\tau(M \times I)}_{H^*G} \cong \underbrace{\varepsilon^{N-1} \oplus \tau(S^k \times I)}_{G_0|S^k \times I} \oplus \nu(H)$$

where G is the framing of $\varepsilon^{N-1} \oplus \tau(M \times I)$ corresponding to F under the identification $\tau(M \times I) = \varepsilon^1 \oplus \tau(M)$, and G_0 corresponds to F_0 under $\tau(S^k \times I) = \varepsilon^1 + \tau(S^k)$. Thus $G_0|S^k \times I$ determines a map $f\colon S^k \times I \to V_{2k+N,k+N}$, which is a homotopy from $\Phi(f_0)$ to $\Phi(f_1)$ since $(**)$ restricted to $S^k \times \{i\}$ yields $(*)$, $i = 0,1$. Therefore we obtain a well-defined element $\Phi(\alpha) \in \pi_k(V_{2k+N,k+N}) \cong Z_2$.

<u>Remark 1</u>. In fact it is true that if the embeddings $F_0, F_1\colon S^k \to M^{2k}$ are homotopic, then they are regularly homotopic. This is an easy corollary of Smale-Hirsch immersion theory [27] [11]. (In fact for $M = R^{2k}$, Φ is identical with Smale's obstruction to homotoping an immersion of S^k to the standard embedding.) Thus if f_0, f_1 are

embeddings representing $\alpha \in H_k(M^{2k})$, $\nu(f_0) \cong \nu(f_1)$, and so (*) defines $\Phi(\alpha)$ indepdnently of the choice of embedding.

Remark 2. There is an alternate way to define Φ, using Smale-Hirsch theory. Given $\alpha \in H_k(M^{2k})$, M s-parallelizable, there is a certain regular homotopy class of immersions $f\colon S^k \times D^k \to M$ such that $F \circ i$ represents α, where $i\colon S^k \to S^k \times D^k$ by $i(x) = (x,0)$ (see [24]). $\Phi(\alpha)$ is defined to be the self-intersection number of the immersion $f \circ i$. For a presentation of this definition, see [24] and [30].

Theorem 4.2. (a) For $k \neq 3,7$, $\Phi(\alpha) = 0$ if and only if ν is trivial. (b) For $k = 3$ or 7 (i.e. $\dim M = 6$ or 14), $\nu(f)$ is trivial, and $\Phi(\alpha) = 0$ if and only if the surgery on M via $f\colon S^k \times D^k \to M^{2k}$ can be framed.

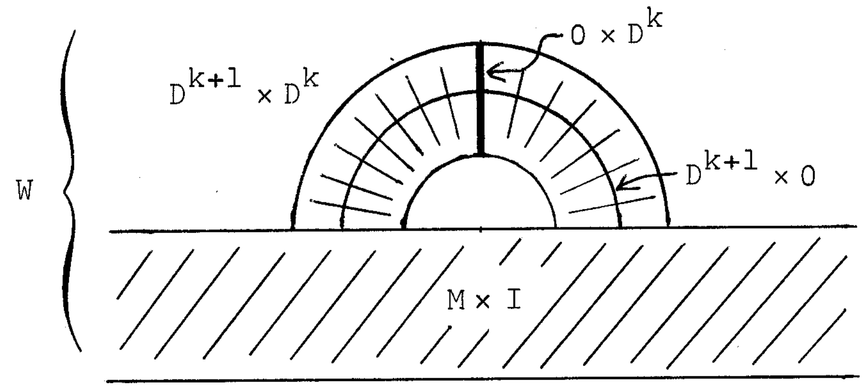
Proof of (a). Consider the long exact homotopy sequence of the bundle $SO_k \to SO_{2k+N} \to V_{2k+N,k+n}$:

$$\cdots \longrightarrow \pi_k SO_k \xrightarrow{i_*} \pi_k(SO_{2k+N}) \xrightarrow{p_*} \pi_k(V_{2k+N,k+N})$$
$$\xrightarrow{\partial_*} \pi_{k-1}(SO_k) \xrightarrow{i_*^!} \pi_{k-1}(SO_{2k+N}) \longrightarrow \cdots .$$

It is clear from the definitions that $\partial_*\Phi(\alpha) = [\nu(f)] \in \pi_{k-1}(SO_k)$. For $k \neq 3,7$, i_* is surjective (1.4), so p_* is 0, so ∂_* is injective. Thus $\Phi(\alpha) = 0 \Leftrightarrow [\nu(f)] = \partial_*\Phi(\alpha) = 0$.

Remark. Thus for $k \neq 3,7$, $\Phi(\alpha)$ can be defined directly as the obstruction to trivializing $\nu(f)$, i.e. $\Phi(\alpha) = [\nu(f)] \in \operatorname{Ker} i_*^! \cong Z_2$, and the two definitions correspond via the isomorphism $\partial_*\colon \pi_k(V_{2k+N,k+N}) \xrightarrow{\cong} \operatorname{Ker} i_*^!$.

Proof of (b). $\nu(f)$ is trivial because $\ker i_*^! = 0$ for $k = 3$ or 7 (1.4). As stated in the proof of Theorem 2.3, the obstruction to framing the surgery lies in $\operatorname{Coker} i_*$. For $k = 3$ or 7, $\operatorname{Im} i_*$ is a subgroup of $\pi_k(SO_{2k+N}) = \pi_k(SO)$ of index 2 (1.4), i.e. $\operatorname{Coker} i_* \cong Z_2$. Furthermore, since $\operatorname{Ker} i_*^! = 0$, p_* is surjective, i.e. $p_*\colon \operatorname{Coker} i_* \cong \pi_k(V_{2k+N,k+N})$. To see that $p_*(\mathcal{O}) = \Phi(\alpha)$, recall the definition of $\mathcal{O}$: A trivialization of $\nu(f)$ gives an embedding $S^k \times D^k \subset M$, and we would like to frame the trace $W = M \times I \bigcup_{S^k \times D^k} D^{k+1} \times D^k$ of the surgery of M via this embedding:



We have a framing of the stable tangent bundle of $M \times I$ which restricts to the given framing F of $\tau(M) \oplus \varepsilon^N \cong \tau(M \times I) | M \times \{1\}$. We also have a canonical framing $F_0 \times F_0'$ of $\tau(D^{k+1} \times D^k)$. Thus comparing $F_0 \times F_0'$ with F on $S^k \times 0$, we get a map $g\colon S^k \to SO_{2k+N}$. Changing the framing of $\nu(f)$ by an element of $\pi_k(SO_k)$ changes the homotopy class of g by an element in the image of $i_*\colon \pi_k(SO_k) \to \pi_k(SO_{2k+N})$. This defines $\mathcal{O} \in \operatorname{Coker} i_*$. Now $p_*(\mathcal{O})$ is the homotopy class of $S^k \xrightarrow{g} V_{2k+N,2k+N} \xrightarrow{p} V_{2k+N,k+N}$. p forgets the last k vectors, so $p \circ g$ compares F_0 with F on $S^k \times 0$ and thus $[p \circ g] = \Phi(\alpha) \in \pi_k(V_{2k+N,k+N}) = Z_2$. This completes the proof of 4.2.

Let $\Phi_2\colon H_k(M;Z_2) \to Z_2$ be the map

$$H_k(M;Z_2 \xrightarrow{\cong} H_k(M;Z) \otimes Z_2 \xrightarrow{\Phi \otimes id} Z_2 .$$

We will show that Φ_2 is a "honsingular quadratic function."

<u>Definition</u>. Let V be a finite dimensional vector space over Z_2, $< , >$ a symmetric bilinear form on V. A <u>quadratic function</u> with associated pairing $< , >$ is a function $\psi\colon V \to Z_2$ such that

$$\psi(\alpha + \beta) = \psi(\alpha) + \psi(\beta) + \langle\alpha,\beta\rangle .$$

ψ is called <u>nonsingular</u> if $< , >$ is nonsingular. Let $\alpha_1,\ldots,\alpha_r$, $\beta_1,\ldots,\beta_r$ be a symplectic basis for $(V,< , >)$. Define the <u>Arf invariant</u> of $(\psi,< , >)$ by

$$A(\psi,< , >) = \sum_i \psi(\alpha_i)\psi(\beta_i) .$$

<u>Remark</u>. It's not hard to show that A is independent of the choice of symplectic basis.

<u>Proposition 4.3</u>. A and rank V are complete invariants of the isomorphism class of $(V,<\ ,\ >,\psi)$. (Isomorphism class means the obvious thing.)

<u>Proof</u>. See [2].

<u>Proposition 4.4</u>. Let M,Φ be as above. Then for $\alpha,\beta \in H_k(M)$ represented by embedding spheres,

$$\Phi(\alpha+\beta) = \Phi(\alpha) + \Phi(\beta) + (\alpha\cdot\beta)_2,$$

where $(\alpha\cdot\beta)_2$ is the intersection number of α and β reduced mod 2.

<u>Proof</u>. Let $f,g\colon S^k \to M$ be embeddings representing α,β respectively. Joining $f(S^k)$ and $g(S^k)$ by a tube gives an immersion $f\#g$ representing $\alpha+\beta$. Observe that the definition of Φ makes sense for an immersion (it is an invariant of regular homotopy) and it is not hard to see that

$$(*) \qquad \Phi(f\#g) = \Phi(f) + \Phi(g)$$

The self-insersection number of the immersion $f\#g$ is just $(\alpha\cdot\beta)_2$. Thus if $(\alpha\cdot\beta)_2 = 0$, $f\#g$ is an embedding (after isotopy) representing $\alpha+\beta$, so the proposition is true by (*). If $(\alpha\cdot\beta)_2 = 1$, let $h\colon S^k \to M$ be a small null-homotopic immersion with self-intersection number $I(h) = 1$. Then by 1.7 $f\#g\#h$ is regularly homotopic to an embedding representing $\alpha+\beta$, so

$$\Phi(\alpha+\beta) = \Phi(f\#g\#h) = \Phi(f) + \Phi(g) + \Phi(h) = \Phi(\alpha) + \Phi(\beta) + \Phi(h)\ .$$

Thus we must show that $\Phi(h) = 1 = (\alpha\cdot\beta)_2$.

For a given manifold M, h is obtained by composing a fixed immersion $h_0\colon S^k \to R^{2k}$ having self-intersection number 1, with a coordinate embedding $R^{2k} \to M$. (For a definition of h_0, see [6].) Since $\Phi(h) = \Phi(h_0)$, it is enough to check that $\Phi(h) = 1$ for some particular choice of M. Let $M = S^k \times S^k$. $H_k(S^k \times S^k) \cong Z \oplus Z$, with generators α,β represented by the embeddings $a,b\colon S^k \to S^k \times S^k$ given by $a(x) = (x,x_0)$, $b(x) = (x_0,x)$. Clearly $(a,b)_2 = 1$. Let $d\colon S^k \to S^k \times S^k$ be the diagonal map $d(x) = (x,x)$. d is an embedding representing $\alpha+\beta$. Therefore $\Phi(d) = \Phi(a) + \Phi(b) + \Phi(h)$ for any framing F of $\varepsilon^N \oplus \tau(S^k \times S^k)$. Let F be the framing of $\varepsilon' \oplus \tau(S^k \times S^k)$ which is the restriction of the standard framing of

R^{2k+1} to the standard embedding $S^k \times S^k \subset R^{2k+1}$. Then it is clear that $\Phi(a) = \Phi(b) = 0$, so $\Phi(h) = \Phi(d)$. (Or one can produce a framing F such that $\Phi(a) = \Phi(b) = 0$ by the proof of Proposition 4.11 below.) For $k \neq 3,7$, $\Phi(d) = [\nu(d)] = [\tau(S^k)] = 1$. It remains to show that $\Phi(d) = 1$ for $k = 3$ or 7.

It should be possible to give a direct proof that $\Phi(d) = 1$. In lieu of such, here is an alternative proof of Proposition 4.4 for $k = 3$ or 7. It is sufficient to show that if $h: S^k \to R^{2k}$ is an immersion with self-intersection number 1, then $\phi(h) = 1$. But it is easily seen that for any immersion $f: S^\ell \to R^{2\ell}$, $\phi(f)$ is precisely Smale's obstruction to regularly homotoping f to the standard embedding of S^ℓ in $S^{2\ell}$ [27]. (It follows that $\phi(f)$ equals the self-intersection number of f --this is immediate when ℓ is odd, because $\phi(f)$ and the self-intersection number are in Z_2. (See [27].) Therefore $\phi(h) = 1$.

Corollary 4.5. $\Phi_2: H_k(M;Z_2) \to Z_2$ is a nonsingular quadratic function with associated pairing $<\alpha,\beta> = (\alpha\cdot\beta)_2$.

Definition. Let (M^{2k},F), i odd be a compact framed (k-1)-connected manifold such that $H_k(M;Z)$ is free abelian. The Kervaire (Arf) invariant $c(M,F)$ is defined as

$$A(\Phi_2, (\ ,\ )_2) \in Z_2.$$

Remark. By a previous remark, for $k \neq 3,7$, $c(M,F)$ does not depend on F, so for $k \neq 3,7$ we let $c(M) = c(M,F)$.

Theorem 4.6. Let (M^{2k},F), k odd, be a compact framed (k-1)-connected manifold with ∂M a homotopy sphere (resp. empty). (M,F) can be made contractible (resp. a homotopy sphere) by a finite sequence of framed surgeries if and only if $c(M,F) = 0$.

Proof. ($\Leftarrow$) Let $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r$ be a symplectic basis for $H_k(M;Z)$. Suppose $c(M,F) = 0$, i.e. $\sum_i \Phi(\alpha_i)\Phi(\beta_i) = 0 \in Z_2$.

Claim. We can find a new symplectic basis $\alpha_1',\ldots,\alpha_r',\beta_1',\ldots,\beta_r'$ for $H_k(M;Z)$ such that $\Phi(\alpha_i') = 0$ for all i. Assuming this, Theorem 4.2 (a) implies that the α_i' are represented by embedded spheres with trivial normal bundles. By Theorem 2.3, the homotopy groups of M can be killed by surgery. By 4.2 (b) the surgery can be framed even when $k = 3$ or 7.

<u>Proof of Claim</u>. If $\Phi(\alpha_i)\Phi(\beta_i) = 0$, take

$$\alpha_i' = \begin{cases} \alpha_i & \text{if } \Phi(\alpha_i) = 0 \\ \beta_i & \text{if } \Phi(\alpha_i) \neq 0 \quad (\text{and hence } \Phi(\beta_i) = 0) \end{cases}$$

$$\beta_i' = \begin{cases} \beta_i & \text{if } \Phi(\alpha_i) = 0 \\ \alpha_i & \text{if } \Phi(\alpha_i) \neq 0 \end{cases}$$

Since $\sum \Phi(\alpha_i)\Phi(\beta_i) = 0$, $\Phi(\alpha_i)\Phi(\beta_i) \neq 0$ for an even number of values of i. Suppose $\Phi(\alpha_i)\Phi(\beta_i) \neq 0$ and $\Phi(\alpha_2)\Phi(\beta_2) \neq 0$. Let

$$\alpha_1' = \alpha_1 + \alpha_2 \qquad \beta_1' = \beta_1$$

$$\alpha_2' = \beta_2 - \beta_1 \qquad \beta_2' = \alpha_1 \, .$$

It is easy to check that replacing $\alpha_1, \alpha_2, \beta_1, \beta_2$ by $\alpha_1', \alpha_2', \beta_1', \beta_2'$ gives a new symplectic basis with $\Phi(\alpha_1') = \Phi(\alpha_2') = 0$. Thus for each pair of values of i such that $\Phi(\alpha_i)\Phi(\beta_i) = 0$, we can replace the four basic elements involved with new ones such that $\Phi(\alpha_i) = 0$.

($\Longrightarrow$). By an argument completely analogous to the one given in the proof of 3.1, it suffices to show that if (M,F) is as in the theorem and there is a framed manifold (V,G) with $\partial V = M$ and $G|\partial V = F$, then $c(M,F) = 0$. Let $i_*\colon H_k(M) \to H_k(V)$ be induced by inclusion.

<u>Assertion (1)</u>. $i_*(\alpha) = 0 \Longrightarrow \Phi(\alpha) = 0$. Represent α by an embedding $f\colon S^k \to M$. Since V is framed, we can perform surgery to make V $(k-1)$-connected (without touching $M = V$) by Theorem 3.1. Now $i_*(\alpha) = 0 \Longrightarrow i \circ f$ is a null-homologous singular sphere in V, and therefore $i \circ f$ is null-homotopic, since $H_k V \cong \pi_k V$ by the Hurewicz theorem. Therefore $i \circ F$ extends to a continuous map $g\colon D^{k+1} \to V$. By 1.9 g is homotopic rel S^k to an immersion. Consider the following commutative diagram of bundle isomorphisms and framings (where $i \circ f = f$ for simplicity):

$$\begin{array}{ccc}
\overbrace{\varepsilon^{N-1} \oplus h^*\tau(V)|S^k}^{h^*G} & \xrightarrow{\ \cong\ } & \overbrace{\varepsilon^{N-1} \oplus \tau(D^{k+1})|S^k}^{f_0|S^k} \oplus \nu(h)|S^k \\
\Big\uparrow \cong & & \Big\uparrow \cong \\
\underbrace{\varepsilon^{N} \oplus f^*\tau(V)}_{f^*F} & \xrightarrow{\ \cong\ } & \underbrace{\varepsilon^{N} \oplus \tau(S^k)}_{f_0|S^k} \oplus \nu(f)
\end{array}$$

This diagram shows that the map $S^k \to V_{2k+N,k+N}$ representing $\Phi(\alpha)$ lifts to a map $D^{k+1} \to V_{2k+N,k+N}$, i.e. $\Phi(\alpha) = 0$, which proves assertion (1).

Assertion (2). There is a symplectic basis $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r$ for $H_k(M;Z)$ such that $\alpha_i \in \operatorname{Ker} i_*$ for all i. Together with (1), this implies that $c(M,F) = \sum_i \Phi(\alpha_i)\Phi(\beta_i) = 0$, as desired.

For (2), consider the commutative diagram (Z coefficients)

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_{k+1}(V,M) & \xrightarrow{\partial} & H_k(M) & \xrightarrow{i_*} & H_k(V) & \xrightarrow{j_*} & H_k(V,M) & \longrightarrow \cdots \\
& \uparrow{\scriptstyle \cap \mu_V} & & \uparrow{\scriptstyle \cap \mu_m} & & \uparrow{\scriptstyle \cap \mu_V} & & \uparrow{\scriptstyle \cap \mu_m} & \\
\cdots \longrightarrow & H^i(V) & \xrightarrow{i^*} & H^k(M) & \xrightarrow{\delta_*} & H^{k+1}(V,M) & \xrightarrow{j_*} & H^{k+1}(V) & \longrightarrow \cdots
\end{array}$$

Now $(u \cap \mu_M)\cdot(v \cap \mu_M) = (u \cup v) \cap \mu_M$ (intersection is dual to cup product), so it will suffice to find a symplectic basis $u_1,\ldots,u_r$, $v_1,\ldots,v_r$ for $H^k(M;Z)$ such that $u_i \in \operatorname{Ker} \delta_k$, $i = 1,\ldots,r$.

Lemma 4.7. $\operatorname{Ker} \delta_k$ is its own annihilator with respect to the cup product pairing, i.e. $u \cup v = 0$ for all $v \in \operatorname{Ker} \delta_k \Leftrightarrow u \in \operatorname{Ker} S_k$.

Proof. $\delta_{2k}(i^*\alpha \cup \beta) = \alpha \cup \delta_k\beta$ for every $\alpha \in H^k(V)$, $\beta \in H^k(M)$ [29]. Now $\delta_{2k}\colon H^{2k}(M) \overset{\cong}{\to} H^{2k+1}(V,M)$ (both groups are Z, and δ_{2k} is surjective by the diagram), so $u \in \operatorname{Ker} \delta_k = \operatorname{Im} i^* \Rightarrow u \cup v = 0$ for all $v \in \operatorname{Ker} \delta_k$. Conversely, if $u \in H^k(M)$ and $u \cup v = 0$ for all $v \in \operatorname{Ker} \delta_k = \operatorname{Im} i^*$, then $\alpha \cup \delta_k u = \delta_{2k}(i^*\alpha \cup u) = \delta_{2k}(0) = 0$ for all $\alpha \in H^k(M)$. Since the cup product pairing is nonsingular, $\delta_k u = 0$.

Remark. The proof of this lemma used only that V and $M^{2k} = \partial V$ are oriented manifolds and $H^k(M)$ is free.

Corollary 4.8. $\operatorname{Ker} \delta_k$ is a direct summand of half rank of $H^k(M)$.

Proof. Consider the following diagram with exact rows ($A^* = \operatorname{Hom}(A,Z)$):

$$\begin{array}{ccccccccc}
0 \longrightarrow & \operatorname{Ker} \delta_k & \xrightarrow{f} & H^k(M) & \longrightarrow & R & \longrightarrow & 0 & \quad (R = \operatorname{Coker} f) \\
& \wr\| & & \wr\| & & \wr\| & & & \\
0 \longrightarrow & R^* & \longrightarrow & (H^k(M))^* & \longrightarrow & (\operatorname{Ker} \delta)^* & \longrightarrow & 0 &
\end{array}$$

The middle arrow is an isomorphism since $\cup$ is a nonsingular pairing (by Poincaré duality). The left dotted arrow exists because Ker δ_k annihilates itself under $\cup$. It is an isomorphism because Ker δ_k equals its annihilator. A diagram chase then proves that the second dotted arrow is well-defined and injective. Therefore R is free, and so Ker $\delta_k \cong R^* \cong R$. (Thus the right dotted arrow is an isomorphism and both sequences split.)

To complete the proof of assertion (2), let $u_1,\ldots,r_r$ be any basis of Ker δ_k, and let $v_1,\ldots,v_r$ be a "dual basis" of $R \cong (\text{Ker } \delta_k)^*$, i.e. $u_i \cup v_j = \delta_{ij}$. Ker δ_k annihilates itself under $\cup$, so $u_i \cup u_j = 0$. However, $v_i \cup v_i$ may be nonzero. Let $v_i' = v_i - (v_i \cup v_i)u_i$. Then it's easy to check that $u_1,\ldots,u_r,v_1',\ldots,r_r'$ is a symplectic basis for H^kM.

This completes the proof of Theorem 4.6.

Now we apply Theorem 4.6 to the computation of bP^n for $n = 2k$, k odd, $\neq 3$ or 7. (Recall that $bP^6 = bP^{14} = 0$ (Corollary 2.4).) We wish to define a map

$$b_k: Z_2 \to bP^{2k}$$

by $b_k(t) = \partial M$, where M is any compact framed $(k-1)$-connected $2k$-manifold with ∂M a homotopy sphere and $c(M) = t$. To show that b_k is well-defined, we must prove:

Theorem 4.9. (a) $c(M_1) = c(M_2) \Rightarrow \partial M_1$ is h-corbordant to ∂M_2.
(b) For each odd $k \neq 3$ or 7 and each $t \in Z_2$ there is a framed manifold M^{2k} such that ∂M is a homotopy sphere and $c(M) = 5$. Thus b_k is surjective.

Proof. (a) Let F_1 and F_2 be framings for M_1 and M_2, and let $(N,G) = (M_1,F_1) \# (M_2,F_2)$ be the framed boundary connected sum (cf. §6). $\partial N = \partial M_1 \# -\partial M_2$, and $c(N) = c(M_1) + c(M_2) = 0$. Therefore, by Theorem 4.6, N can be made contractible by framed surgeries. Thus $\partial M_1 \# - \partial M_2$ bounds a contractible manifold, i.e. ∂M_1 is h-cobordant to ∂M_2.

(b) If $t = 0$, take $M^{2k} = D^{2k}$. If $t = 1$, M^{2k} is constructed by plumbing two copies of the tangent disc bundle of S^k (see [7] or [23]).

This theorem shows that $bP^{2k} = 0$ if $b_k = 0$ and $bP^{2k} = Z_2$ if

$b_k \neq 0$, so we would like to determine when $b_k = 0$. This happens if and only if there exists a compact framed $(k-1)$-connected $2k$-manifold M with boundary the standard sphere S^{2k-1} such that $c(M) = 1$. Attaching a disc to the boundary of such an M we obtain an almost framed closed $2k$-manifold N with $c(N) = 1$. By Corollary 3.18, N is framed. Thus $b_k = 0 \Leftrightarrow$ there is a closed framed $(k-1)$-connected $2k$-manifold N with $c(N) = 1$. By the proof of Theorem 4.6, c is well-defined on framed cobordism classes. The framed cobordism group Ω_f^n is isomorphic to the n-stem $\pi_n(S) = \pi_{n+k}(S^k)$ for k large (§2). Thus for each odd k, the Kervaire invariant gives a map

$$c_k\colon \pi_{2k}(S) \to Z_2$$

and for $k \neq 3,7$, $b_k = 0 \Leftrightarrow c_k \neq 0$. According to Browder [6], $c_k = 0$ if $k \neq 2^\ell - 1$. (Kervaire [14] originally showed $c_6 = 0$ and $c_9 = 0$; then Brown and Peterson [8] showed $c_{4\ell+1} = 0$.) Mahowald and Tangora have shown that $c_{15} \neq 0$, and Barratt, Mahowald and Jones have shown $c_{31} \neq 0$ (see [6], [33], [34]). Therefore we have

<u>Theorem 4.10</u>. For k odd $bP^{2k} = \begin{cases} Z_2 & k \neq 2^\ell - 1 \\ 0 & k = 3,7,15, \text{ or } 31 \end{cases}$.

The following proposition, which extends our discussion of the existence of closed framed $2k$-manifolds with nonzero Kervaire invariant to the case $k = 3$ or 7, will be needed in §5:

<u>Proposition 4.11</u>. For $k = 3$ or 7 there is a framing F of $S^k \times S^k$ such that $c(S^k \times S^k, F) = 1$.

<u>Proof</u>. $H_k(S^k \times S^k) \cong Z \oplus Z$, with generators α, β represented by the embedded spheres $S^k \times *$, $* \times S^k$ respectively. Let G be any framing of $\tau(S^k \times S^k) \oplus \varepsilon^1$, $c(S^k \times S^k, G) = \Phi(\alpha)\Phi(\beta)$. <u>Claim (1)</u>: G can be altered so as to realize any values of $\Phi(\alpha)$ and $\Phi(\beta)$. This implies the proposition. Let $f\colon S^k \to SO_{2k+1}$. <u>Claim 2</u>: chainging G on $S^k \times *$ by f alters $\Phi(\alpha)$ by the map $S^k \xrightarrow{f} SO_{2k+1} \to V_{2k+1,k+1}$. Assuming this, we prove (1) as follows: For $k = 3$ or 7, $\pi_k(SO_N) \to \pi_k(V_{N,N-k})$ is surjective (1.4). Now $S^k \times S^k = (S^k \vee S^k) \cup D^{2k}$. Thus we can change G on $S^k \vee S^k$ to obtain a framing F on $S^k \vee S^k$ such that $\Phi(\alpha) = \Phi(\beta) = 0$. The obstruction to extending F over the $2k$-cell is an element of $\pi_{2k-1}(SO_N) = 0$ for $k = 3$ or 7 (1.5).

<u>Proof of (2)</u>. This follows from the definition of Φ. Suppose $g\colon S^k \to V_{2k+1,k+1}$ represents $\Phi(\alpha)$. It is clear that changing G by

f changes g to the map $\tilde{g}(x) = f(x)\,g(x)$, where SO_{2k+1} acts on $V_{2k+1,k+1}$ by rotation. But we can assume that $g(x)$ is the standard $(k+1)$-frame for x in the northern hemisphere, and that $f(x)$ is the identity element of SO_{2k+1} for x in the southern hemisphere, so $[\tilde{g}] = [g] + [h]$, where $h(x)$ is the standard $(k+1)$-frame in R^{2k+1} rotated by $f(x)$, i.e. $h(x) = \pi f(x)$, $\pi: SO_{2k+1} \to V_{2k+1,k+1}$. Thus $[\tilde{g}] = [g] = \pi_*[f]$, which proves (2).

Remark. We can define $c(M,F)$ for any compact framed $2k$-manifold, k odd, with ∂M empty or a homotopy sphere, as follows. Convert (M,F) to a $(k-1)$-connected framed manifold (N,G) by a finite sequence of framed surgeries, and let $c(M,F) = c(N,G)$. The proof of Theorem 4.6 shows that $c(M,F)$ is well-defined, and that it is an invariant of framed surgery. For $k = 3$ or 7, c is not an invariant of unframed surgery by Proposition 4.11, since $S^k \times S^k$ is null-corbordant. Theorem 4.10 implies that c is not an invariant of unframed surgery for some other values of k. Since $c_{15} \neq 0$, there is a closed framed 14-connected 30-manifold N with $c(N) = 1$. However, since N is framed it has zero Stiefel-Whitney and Pontryagin numbers, so N is unframed (oriented) null-corbordant. (In contrast, recall that the signature of a $4k$-manifold is an invariant of unframed surgery.)

Recall that if $k \neq 3, 7$ and M^{2k} is $(k-1)$-connected, then $c(M,F)$ does not depend on F. However, it is not known whether $c(M,F)$ depends on F for arbitrary M.

§5. Computation of θ^n/bP^{n+1}.

The results of this section are all in Kervaire-Milnor [15]. Suppose that the homotopy sphere Σ^n is embedded in R^{n+k} (k large) with a framing F of its normal bundle (recall that homotopy spheres are π-manifolds by Corollary 3.19). Then the Thom construction applied to (Σ,F) yields an element $T(\Sigma,F)$ of $\pi_{n+k}(S^k)$, which is an invariant of the normal cobordism class of (Σ,F). $T((\Sigma,F) \# (\Sigma',F')) = T(\Sigma,F) + T(\Sigma',F)$.

Lemma 5.1. Let $f: \Sigma^n \to SO_k$, and let $\alpha = [f] \in \pi_n SO_k$. Then if F is altered to F' via α,

$$T(\Sigma,F') = T(\Sigma,F) \pm J(\alpha) .$$

Proof. Recall (Lemma 3.13) that $T(S^n,F_\alpha) = \pm J(\alpha)$, where F_α is the standard framing F_0 of S^n altered by α. Thus

$$(\Sigma,F') = (\Sigma,F') \# (S^n,F_0) = (\Sigma,F) \# (S^n,F_\alpha) ,$$

and the lemma follows by applying T to both sides.

Corollary 5.2. $T(\Sigma) = \{T(\Sigma,F),$ F a framing of $\Sigma^n \subset R^{n+k}\}$ is a coset of $J(\pi_n SO_k)$ in $\pi_{n+k}S^k$.

Therefore we can define $T: \theta^n \to \text{Coker } J_n$, where $J_n: \pi_n SO \to \pi_n S$ is the J-homomorphism.

Proposition 5.3. $bP^{n+1} = \text{Ker } T$.

Proof. $\Sigma \in bP^{n+1} \Leftrightarrow \Sigma$ bounds a parallelizable manifold. $T\Sigma = 0 \Leftrightarrow$ there exists a normal framing, F, of Σ such that (Σ,F) bounds a normally framed manifold.

Thus we have an exact sequence

$$0 \longrightarrow bP^{n+1} \longrightarrow \theta^n \xrightarrow{T} \text{Coker } J_n .$$

Corollary 5.4. θ^n is a finite group $(n \geq 4)$.

Now $\theta^n/bP^{n+1} \cong \text{Im } T$. Suppose $\tilde{\alpha} \in \text{Coker } J_n$. $\tilde{\alpha} \in \text{Im } T$ if and only if $\tilde{\alpha}$ is represented by $\alpha \in T_{n+k}S^k$ (k large) such that $\alpha = T(\Sigma,F)$ for some (Σ,F). By the inverse Thom construction, and $\alpha \in \pi_{n+k}S^k$ equals $T(M,F')$ for some framed manifold (M,F'). $\alpha = T(\Sigma,F)$ is and only if (M,F') is framed cobordant to a homotopy sphere (Σ,F). Define

$$P^n = \begin{cases} 0 & n \text{ odd} \\ Z & n \equiv 0(4) \\ Z_2 & n \equiv 2(4) \end{cases}$$

(so that $bP^{n+1} = \text{Im}(b)$, $b: P^{n+1} \to \theta^n$ as in §3 and §4, and define $\phi: \Omega_n^f \to P^n$ by

$$\phi(M^n,F) = \begin{cases} 0 & n \text{ odd} \\ \sigma(M) & n \equiv 0(4) \\ c(M,F) & n \equiv 2(4) \end{cases}$$

ϕ is well-defined, since σ and c are invariants of framed cobordism, and $\phi(M,F) = 0 \Leftrightarrow (M,F)$ is framed cobordant to a homotopy sphere (Corollary 2.2 and Theorems 3.1 and 4.6). Let $\phi' = \phi T^{-1}$:

$$\begin{array}{ccc} \pi_n S & & \\ \cong \uparrow T & \searrow \phi' & \\ \Omega_n^f & \xrightarrow{\phi} & P^n \end{array}$$

Clearly $\phi'(\operatorname{Im} J_n) = 0$, so ϕ' induces a map

$$\phi'' : \operatorname{Coker} J_n \to P^n .$$

By the above analysis of $\operatorname{Coker} J_n$ we have:

<u>Theorem 5.4</u>. The sequence

$$P^{n+1} \xrightarrow{b} \theta^n \xrightarrow{T} \operatorname{Coker} J_n \xrightarrow{\phi''} P^n$$

is exact.

The new information here is that $\theta^n/bP^{n+1} \cong \operatorname{Ker} \phi''$. If n is odd, of course $\phi'' = 0$, since $P^n = 0$. If $n = 0(4)$, we have seen that $\phi'' = 0$ (by Corollary 3.12). If $n \equiv 2(4)$, $\phi'' = 0$ for $n \neq 2^i - 2$, and $\phi \neq 0$ for $n = 6, 14, 30$, or 62 (by the discussion preceding Theorem 4.10).

In summary, we have computed bP^{n+1} (except for $n+1 = 2^i - 2$, $i > 6$), and we have $\theta^n \cong \operatorname{Coker} J_n$ except when $n = 2^i - 2$. Then we have computed θ^n up to group extension.

<u>Remark</u>. Brumfield and Frank have then proved that for $n \neq 2^k - 1$ or $2^k - 2$

$$0 \longrightarrow bP^{n+1} \longrightarrow \theta^n \longrightarrow T\theta^n \longrightarrow 0$$

splits. (See e.g. [9].)

<u>Appendix</u>. <u>The Kervaire-Milnor long exact sequence</u>.

The results of these notes can be elegantly expressed by means of a long exact sequence

(i) $\cdots \longrightarrow A^{n+1} \xrightarrow{p} P^{n+1} \xrightarrow{b} \theta^n \xrightarrow{i} A^n \xrightarrow{p} P^n \longrightarrow \cdots$

θ^n is the group of homotopy n-spheres [15]. A^n is the group of "almost framed" cobordism classes of almost framed (i.e. framed execpt at a point) closed n-manifolds. (If M_1 is framed except at x_1 and M_2 is framed except at x_2, an almost framed cobordism between M_1 and M_2 is a corbordism W between M_1 and M_2 and a framing of $W - \alpha$, α an arc in int W from x_1 to x_2 which restricts to the given framings on the ends of the corbordism.) P^n is the group of framed cobordism classes of parallelizable n-manifolds with boundary a homotopy sphere. (A framed cobordism between M_1 and M_2 is a framed manifolds W with boundary $M_1 \underset{\partial M}{\cup} N \underset{\partial M_2}{\cup} M_2$, where N is an h-corbordism between ∂M_1 and ∂M_2, and the framing of W restricts to the given framings of M_1 and M_2.)

b is induced by the boundary map, and it is well defined by the definition of P^n. i is induced by "inclusion", i.e. any homotopy sphere is s-parallelizable, and so is almost framed. i is clearly well-defined. p is induced by "punching out a disc" containing the non-framed point to obtain a parallelizable manifold with boundary S^n. p is clearly well defined.

The discussions preceding Theorems 3.5 and 4.10 show that Ker(b) = Im(p). It is clear from the definition of A^n that Ker(i) = Im(b). It is also easy to see that Ker(p) = Im(i).

Corollary 2.2 implies that $P^n = 0$ for n odd. Theorems 3.1, 3.2, and 3.3 imply that $P^n \cong Z$ for $n \equiv 0(4)$. Theorems 4.6 and 4.9 imply that $P^n \cong Z_2$ for $n \equiv 2(4)$.

Now A^n lies in the exact sequence

(ii) $\cdots \longrightarrow \pi_n(SO) \xrightarrow{J} \pi_n(S) \xrightarrow{t} A^n \xrightarrow{0} \pi_{n-1}(S) \xrightarrow{J} \pi_{n-1}(S) \longrightarrow \cdots$

where J is the stable J-homomorphism, t is the inverse Thom construction which takes $\pi_n(S) \cong \Omega_f^n$ (the framed cobordism group), followed by the inclusion of Ω_f^n in A^n. Theorem 3.14 says that Ker(J) = Im(0). Ker(0) = Im(t) is clear. Ker(t) = Im(J) is easy to show (cf. Lemma 3.13).

Corollary 3.16 determines $\text{Im}[A^n \xrightarrow{p} P^n \cong Z]$, $n \equiv 0(4)$. (This map assigns to an almost framed closed manifold its signature divided by 8.) Theorem 4.10 determines $\text{Im}[A^n \xrightarrow{p} P^n \cong Z_2]$ for almost all $n \equiv 2(4)$. (This map assigns to a manifold its Kervaire invariant.)

The results of §5 can be interpreted as follows. By the exact sequence (ii), Coker (J) $\cong$ Im(t) $\subset A^n$. The discussion following 5.3

shows that $Im(i) \subset Im(t)$, so we have the exact sequence

$$P^{n+1} \xrightarrow{b} \theta^n \xrightarrow{T} \text{Coker }(J) \xrightarrow{\phi} P^n$$

where $T(\Sigma) = i(\Sigma)$ and ϕ: Coker $(J) \subset A^n \xrightarrow{p} P^n$.

Remark. Let θ_f^n be the group of framed h-corbordism classes of framed homotopy n-spheres. Then we have the exact sequences

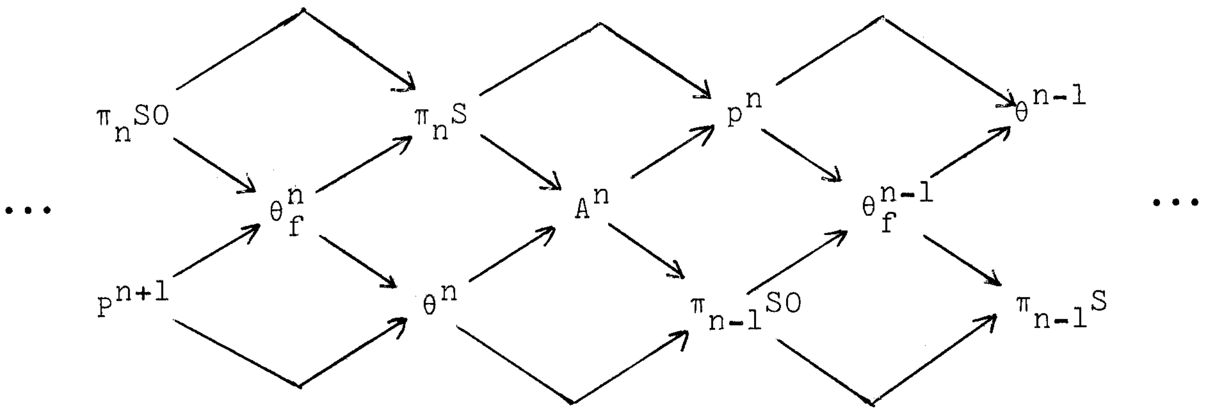
(iii) $$\cdots \longrightarrow P^{n+1} \longrightarrow \Omega_f^n \longrightarrow P^n \longrightarrow \theta_f^{n-1} \longrightarrow \cdots$$

($\Omega_f^n \to P^n$ is "punching out a disc"; $P^n \to \theta_f^{n-1}$ is "taking the boundary"), and

(iv) $$\cdots \longrightarrow \pi_n SO \longrightarrow \theta_f^n \longrightarrow \theta^n \xrightarrow{0} \pi_{n-1} SO \longrightarrow \cdots$$

($\pi_n SO \to \theta_f^n$ sends α to S^n with its standard framing changed by α; $\theta_f^n \to \theta^n$ forgets the framing).

Combining the long exact sequences (i), (ii), (iii), (iv) (replacing Ω_f^n by $\pi_n S$ in (iii)), we obtain the Kervaire-Milnor "braid":



This braid is isomorphic to a braid of the homotopy groups of G, PL, and O (see e.g. [24]). Levine [16] has a nonstable version of the Kervaire-Milnor braid.

References

1. Adams, J.F., On the groups J(X) IV, Topology 5 (1966), 21-71.

2. Arf, Untersuchung über quadratische Formen in Körpern der Characteristik 2, Crelles Math. Journal, 183 (1941), 148-167.

3. Bott, R. and Milnor, J. W., On the parallelizability of the spheres, Bull. Amer. Math. Soc. 64 (1958), 87-89.

4. Bott, R., The stable homotopy of the classical groups, Annals of Math. 70 (1959), 313-337.

5. Bott, R., A report on the unitary group, Proceedings of Symposia in Pure Math. Volume 3, Differential Geometry (1961).

6. Browder, W. The Kervaire invariant of framed manifolds and its generalizations, Annals of Math. 90 (1969), 157-186.

7. Browder, W. Surgery on simply connected manifolds, Ergebnisse, Springer-Verlag, N.Y.

8. Brown, E.M. and Peterson, F.P., The Kervaire invariant of (8k+2)-manifolds, Bull. Amer. Math. Soc. 71 (1965), 190-193.

9. Brumfiel, G., Homotopy equivalences of almost smooth manifolds, Wisconsin Summer Institute on Algebraic Topology, 1970.

10. Haefliger, A., Knotted (4k-1)-spheres in 6k-space, Ann. of Math. 75 (1962), 452-466.

11. Hirsch, M.W., Immersions of manifolds, Trans. Amer. Math. Soc., 93 (1959), 242-276.

12. Hirzebruch, F., Topological Methods in Algebraic Geometry, Springer Verlag (1966).

13. Kervaire, M.A., An interpretation of G. Whitehead's generalization of Hopf's invariant, Annals of Math. 69 (1959), 345-365.

14. Kervaire, M.A., A manifold which does not admit any differentiable structure, Comment. Math. Helv. 34 (1960), 257-270.

15. Kervaire, M.A. and Milnor, J.W., Groups of homotopy spheres, Annals of Math. 77 (1963), 504-537.

16. Levine, J., A classification of differentiable knots, Annals of Math. 82 (1965) 15-50.

17. Milnor, J.W., On simply connected 4-manifolds, Symposium internacional de topologia algebraica, Mexico (1958).

18. Milnor, J.W. and Kervaire, M.A., Bernoulli numbers, homotopy groups, and a theorem of Rochlin, Proc. of the International Congress of Mathematicians, Edinburgh (1958).

19. Milnor, J.W. and Stasheff, J., Characteristic classes, Annals of Math. Studies No. 76, Princeton U. Press, 1975.

20. Milnor, J.W., A procedure for killing homotopy groups of differentiable manifolds, Proceedings of Symposia in Pure Mathematics, Vol. 3, Differential Geometry (1961).

21. Milnor, J.W., Morse Theory, Princeton (1963).

22. Milnor, J.W., Topology from the Differentiable Viewpoint, The University Press of Virginia, (1965).

23. Milnor, J.W., Differential Topology, Lectures in Modern Mathematics, Vol. II, Saaty ed., Wiley (196).

24. Orlik, P., Seminar notes on simply connected surgery, Institute for Advanced Study, 1968.

25. Rourke, C.P. and Sanderson, B.A., Blockbundles III, Annals of Math. 87 (1968), 431-483.

26. Serre, J.P., Formes bilinéaires symétriques entières à discriminant ± 1, Séminaire Cartan, 14 (1961-62).

27. Smale, S., The classification of immersions of spheres in euclidean spaces, Annals of Math. 69 (1959), 327-344.

28. Steenrod, N.A., The Topology of Fibre Bundles, Princeton University Press (1951).

29. Spanier, E.M., Algebraic Topology, McGraw-Hill (1966).

30. Wall, C.T.C., Surgery of non-simply-connected manifolds, Annals of Math. 84 (1966), 217-276.

31. Whitney, H., The self-intersections of a smooth n-manifold in 2n-space, Annals of Math. 45 (1944), 220-246.

32. Whitney, H., Singularities of a smooth n-manifold in (2n-1)-space, Annals of Math. 45 (1944), 247-293.

33. Barratt, M.G., Jones, J.D.S., and Mahowald, M.E., The Kervaire Invariant in dimension 62 (to appear).

34. Mahowald, M.E., and Tangora, M., Some differentials in the Adams spectral sequence, Topology 6 (1967), 349-370.