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Forms Over Real Algebras and the Multisignature of a Manifold

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INTRODUCTION

This article gives a detailed account of the classification (up to equivalence), of forms over simple real algebras with involution.

Let R be a ring with an identity. An *involution* J on R is an anti-automorphism of period two.

$$\text{i.e., writing } x^J \quad \text{for} \quad J(x),$$
$$(x+y)^J = x^J + y^J,$$
$$(xy)^J = y^J x^J,$$
$$(x^J)^J = x,$$
$$\text{for all } x, y \in R$$

Let (R, J) denote ring R with involution J. This will be an object in a category whose morphisms are homomorphisms preserving involution.

A form over (R, J) is a map

$$\phi\colon M \times M \to R, \quad M \text{ a right } R\text{-module}$$

such that (i) $\phi(x, y) = \phi(y, x)^J$

(ii) $\phi(xr, y) = r^J \phi(x, y)$

(iii) $\phi(x_1 + x_2, y) = \phi(x_1, y) + \phi(x_2, y)$

(i.e. ϕ is J-symmetric, J-linear in first variable and linear in second variable).

Forms $\phi_1 : M_1 \times M_1 \to R$, $\phi_2 : M_2 \times M_2 \to R$ are said to be *equivalent* if there is an R-module isomorphism $\gamma\colon M_1 \to M_2$ such that $\phi_1 = \phi_2 \circ (\gamma \times \gamma)$.

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There is a category whose objects are forms over (R, J) and whose morphisms are commutative diagrams

$$\begin{array}{ccc} M_1 \times M_1 & \xrightarrow{\phi_1} & R \\ {\scriptstyle f\times f}\downarrow & \nearrow{\scriptstyle \phi_2} & \\ M_2 \times M_2 & & \end{array}$$

, f being an R-homomorphism.

Equivalence of forms is just equivalence of objects in this category. We use the symbol $\sim$ to denote equivalence.

The *sum* of two forms ϕ_1, ϕ_2 on M_1, M_2 is the form denoted $\phi_1 \oplus \phi_2$ on $M_1 \oplus M_2$ given by $(\phi_1 \oplus \phi_2)(x \oplus x', \ y \oplus y') = \phi_1(x, y) + \phi_2(x', y')$. The operation $\oplus$ will make the set $S(R, J)$ of equivalence classes of forms over (R, J) into a semigroup. In the usual way we can define a Grothendieck group $G(R, J)$ for the category of forms. There is a natural homomorphism from $S(R, J)$ to $G(R, J)$ which will be injective whenever a 'Witt cancellation theorem' can be proved for forms over (R, J), (i.e. whenever $\phi_1 \oplus \phi_2 \sim \phi_1 \oplus \phi_3$ implies $\phi_2 \sim \phi_3$). Such a theorem, proved originally by Witt for quadratic forms over fields, holds in particular when R is a division ring [6, Chapter 5].

A form $\phi: M \times M \to R$ is *nonsingular* if the map from M to $\mathrm{Hom}_R(M, R)$, $m \to \phi(m, -)$, is an isomorphism.

Note. A product of forms can also be defined and the concepts of hyperbolic form and Witt ring can be developed. See [4] for details.

In Section 1 we describe how the category of R-modules and the category of M_nR-modules are equivalent (M_nR being the ring of $n \times n$ matrices with entries in R).

In Section 2 we show how forms over (R, J) correspond bijectively to forms over (M_nR, J') where $J' = J$ on restriction to $R \subset M_nR$. In Section 3 we list all the possible involutions on a semisimple real algebra and then in Section 4 we show how forms over simple real algebras can be classified up to equivalence. Section 5 gives a topological application when we define the multisignature of an even dimensional nonsimply connected manifold. This is a topological invariant which is useful in the theory of surgery of manifolds. See Wall [8], who originated the idea of the multisignature.

1. Morita Equivalence

Let C_R denote the category of right R-modules. Rings A and B are said to be *Morita equivalent* if C_A and C_B are equivalent as categories. If

M is an object in C_R then M^n is an object in C_{M_nR} in an obvious way. Conversely if P is an object in C_{M_nR} then $\mathrm{Hom}(R^n, P)$, the set of all M_nR-module homomorphisms R^n to P, is an object in C_R.

LEMMA 1.1. *If* $P \in C_{M_nR}$ *then* $P \cong (\mathrm{Hom}(R^n, P))^n$.

Proof. Define $\psi: M \to (\mathrm{Hom}(R^n, P))^n$ by

$$\psi(x) = (f_1^x, f_2^x, \ldots, f_n^x)$$

where $f_i^x(a_1, a_2, \ldots, a_n) = x \sum_{j=1}^n a_j e_{ij}$, e_{ij} being the matrix with 1_R in the (i, j) place and zero elsewhere. The inverse $\psi^{-1}: (\mathrm{Hom}(R^n, P))^n \to M$ is given by $\psi^{-1}(f_1, \ldots, f_n) = \sum_{i=1}^n f_i(e_i)$, $e_1, \ldots, e_n$ being the standard basis for R^n as a free R-module.

LEMMA 1.2. *If M is an R-module then* $M \cong \mathrm{Hom}(R^n, M^n)$.

Proof. Define $\xi: M \to \mathrm{Hom}(R^n, M^n)$ by $\xi(x) = f_x$ where $f_x(r_1, r_2, \ldots, r_n) = (xr_1, xr_2, \ldots, xr_n)$. The inverse ξ^{-1} is given by

$$\xi^{-1}(f) = f\left(\sum_{i=1}^n e_i\right)$$

This is meaningful since, although $f(\sum_{i=1}^n e_i) \in M^n$, $f(\sum_{i=1}^n e_i A) = f(\sum_{i=1}^n e_i)A$ for all $A \in M_nR$ and taking A to have 1_R on top row and zero elsewhere we get $f(\sum_{i=1}^n e_i) = (x, x, x, \ldots, x)$ for some $x \in M$.

THEOREM 1.3. *R and M_nR are Morita equivalent (i.e. C_R and C_{M_nR} are equivalent).*

Proof. Define a functor $F: C_R \to C_{M_nR}$ by $F(M) = M^n$ for objects and the obvious definition for morphisms. Define a functor $G: C_{M_nR} \to C_R$ by $G(P) = \mathrm{Hom}(R^n, P)$ for objects and $G(h)(f) = h \circ f$ for morphisms $h: P_1 \to P_2$. To prove the theorem we must show that FG and GF are each naturally equivalent to the identity functor.

$FG(P) \cong P$ by Lemma 1.1

$$\begin{array}{ccc} P_1 & \xrightarrow{h} & P_2 \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ FG(P_1) & \xrightarrow{FG(h)} & FG(P_2) \end{array}$$

This diagram commutes since

$$\begin{aligned} FG(h)\,\psi(x) &= FG(h)(f_1^x, f_2^x, \dots, f_n^x) \\ &= (h \circ f_1^x, h \circ f_2^x, \dots, h \circ f_n^x) \\ &= (f_1^{h(x)}, f_2^{h(x)}, \dots, f_n^{h(x)}) \\ &= \psi h(x) \end{aligned}$$

So FG is naturally equivalent to the identity functor. Similarly using Lemma 1.2 it can be shown that GF is naturally equivalent to the identity functor.

2. Forms over R and M_nR

Given an involution J on R there is an involution J' on M_nR defined by $J'(A) = (A^J)^t$ for each matrix A. $(A^J = (a_{ij}^J))$. In particular if R is commutative and J is the identity map then $J'(A) = A^t$. When R is not commutative, transposition on M_nR is not involution. Let $S(R, J)$ be the set of equivalence classes of forms over (R, J). We will show in this section that there is a natural bijection from $S(R, J)$ to $S(M_nR, J')$ for any ring R, not necessarily commutative.

Let ϕ: $M \times M \to R$ be a form over (R, J). Define ϕ^*: $M^n \times M^n \to M_nR$ by

$$\phi^*((x_1, \dots, x_n), \quad (y_1, \dots, y_n)) = (\phi(x_i, y_j))$$

(i.e. the matrix with $\phi(x_i, y_j)$ in (i, j) place)

It is easily checked that ϕ^* is a form over (M_nR, J'). Conversely let η: $P \times P \to M_nR$ be a form over (M_nR, J'). We define a map denoted η_* : $\mathrm{Hom}(R^n, P) \times \mathrm{Hom}(R^n, P) \to R$ as follows:

$$\text{for } f, g \in \mathrm{Hom}(R^n, P), \eta\left(f \sum_{i=1} e_i, g \sum_{i=1}^{n} e_i\right) = \text{matrix } (x_{ij})$$

We will show $x_{ij} = x_{11}$ for all i, j. Let A be the matrix with entries 1_R on the top row and zero elsewhere (as in Lemma 1.2). Then $(\sum_{i=1}^n e_i)A = \sum_{i=1}^n e_i$, and $A^{J'} = A^t$.

$$\therefore \eta\left(f \sum_{i=1}^{n} e_i A, g \sum_{i=1}^{n} e_i\right) = A^t \eta\left(f \sum_{i=1}^{n} e_i, g \sum_{i=1}^{n} e_i\right)$$

Hence the columns of (x_{ij}) are identical. Similarly

$$\eta\left(f\sum_{i=1}^{n} e_i\,,\,g\sum_{i=1}^{n} e_iA\right) = \eta\left(f\sum_{i=1}^{n} e_i\,,\,g\sum_{i=1}^{n} e_i\right)A$$

implies that the rows of (x_{ij}) are identical. So $x_{ij} = x_{11}$ for all i, j. We define $\eta_*(f, g) = x_{11}$. It can be checked that η_* is a form over (R, J). Further $\phi \to \phi^*$ and $\eta \to \eta_*$ each preserve equivalences (i.e. $\phi_1 \sim \phi_2 \Rightarrow \phi_1{}^* \sim \phi_2{}^*$ and $\eta_1 \sim \eta_2 \Rightarrow (\eta_1)_* \sim (\eta_2)_*$). This can be seen by using the functors F and G of Theorem 1.3.

THEOREM 2.1. *$S(R, J)$ and $S(M_nR, J')$ are in one-one correspondence.*

Proof.

$$\begin{array}{c} S(R, J) \longrightarrow S(M_nR, J) \\ \phi \longrightarrow \phi^* \end{array}$$

is well defined, as is

$$\begin{array}{c} S(M_nR, J) \longrightarrow S(R, J) \\ \eta \longrightarrow \eta^* \end{array}$$

We show that (1), $(\phi^*)_* \sim \phi$, and (2), $(\eta_*)^* \sim \eta$

$$\begin{array}{ccc} M^n \times M^n & \xrightarrow{\quad\phi\quad} & R \\ \Big\downarrow{\scriptstyle \xi\times\xi} & \nearrow{\scriptstyle (\phi^*)_*} & \\ \mathrm{Hom}(R^n, M^n) \times \mathrm{Hom}(R^n, M^n) & & \end{array} \qquad (1)$$

The map ξ of Lemma 1.2 gives the equivalence since

$$\begin{aligned} (\phi^*)_*\, \xi \times \xi(x, y) &= (\phi^*)_*(f^x, f^y) \\ &= (\phi^*(f^x, f^y))_* \\ &= \phi(x, y) \end{aligned}$$

$$\begin{array}{ccc} P \times P & \xrightarrow{\quad\eta\quad} & M_nR \\ \Big\downarrow{\scriptstyle \psi\times\psi} & \nearrow{\scriptstyle (\eta_*)^*} & \\ (\mathrm{Hom}(R^n, P))^n \times (\mathrm{Hom}(R^n, P))^n & & \end{array} \qquad (2)$$

The map $\psi \times \psi$ of Lemma 1.1 gives the equivalence since

$$\begin{aligned}(\eta_*)^*(\psi \times \psi)(x, y) &= (\eta_*)^*((f_1^x, \ldots, f_n^x), (f_1^y, \ldots, f_n^y)) \\ &= \text{matrix with entries } \eta_*(f_i^x, f_j^y) \\ &= \eta(x, y)\end{aligned}$$

Note 1. The correspondences $\phi \rightarrow \phi^*$ and $\eta \rightarrow \eta^*$ can be viewed as functors between the category of forms over (R, J) and the category of forms over (M_nR, J'). The morphisms in the category of forms over (R, J) being commutative diagrams

$$\begin{array}{ccc} M_1 \times M_1 & \xrightarrow{\phi_1} & R \\ {\scriptstyle f \times f}\big\downarrow & \nearrow{\scriptstyle \phi_2} & \\ M_2 \times M_2 & & \end{array}$$

where f is an R-module homomorphism. The theorem gives an isomorphism of Grothendieck groups. (as the correspondences preserve sums)

Note 2. The previous theorem will follow from the more general theory of Fröhlich and McEvett [4], though they do not get the correspondence explicitly in this way.

We can generalize theorem 2.1 slightly in the following way. Choose a nonsingular form ϕ on R^n over (R, J). ϕ is nonsingular provided $R^n \rightarrow \text{Hom}(R^n, R)$, $x \rightarrow \phi(x, -)$, is an isomorphism. For $A \in M_nR$, define the adjoint $A^* \in M_nR$ be requiring $\phi(xA, y) = \phi(x, yA^*)$ for all x, y in R^n. The mapping $A \rightarrow A^*$ is an involution on M_nR which we will denote by $*$. If C denotes the $n \times n$ matrix with entries $\phi(e_i, e_j)$ then $C^{J'} = C$ and C is invertible.

LEMMA 2.2.

$$A^* = C^{-1}A^{J'}C$$

Proof. $\phi(xA, y) = \phi(x, yA^*)$. Putting $x = e_i$, $y = e_j$ we will get

$$A^{J'}C = CA^*$$
$$A^* = C^{-1}A^{J'}C$$

LEMMA 2.3. *If η is a form over $(M_nR, *)$ then $C\eta$ is a form over (M_nR, J').*

Proof.

$$\begin{aligned} C\eta(x, y) &= C(\eta(y, x))^* \\ &= CC^{-1}\eta(y, x)^{J'}C \\ &= \eta(y, x)^{J'}C \\ &= (C\eta(y, x))^{J'} \text{ since } C = C^{J'} \\ C\eta(xB, y) &= CB^*\eta(x, y) \\ &= B^{J'}C\eta(x, y) \\ C\eta(x + x', y) &= C\eta(x, y) + C\eta(x', y) \end{aligned}$$

$C\eta$ is a form over (M_nR, J')

LEMMA 2.4. *If η is a form over (M_nR, J') then $C^{-1}\eta$ is a form over $(M_nR, *)$.*

Proof. Similar to previous one.

THEOREM 2.5. *$S(R, J)$ and $S(M_nR, *)$ are in one-one correspondence.*

Proof. Follows from Theorem 2.1 and the preceding lemmas. We finish this section with some examples.

EXAMPLE 1. The only kind of form $\eta: R^n \times R^n \to M_nR$ is a 'direct sum' of n copies of the form

$$\begin{aligned} R \times R &\longrightarrow R \\ (x, y) &\longrightarrow x^Jcy \end{aligned} \qquad \text{for some } c \in R$$

i.e. $\eta((x_1, \ldots, x_n), (y_1, \ldots, y_n))$ is the diagonal matrix with entries $x_i^Jcy_i$, η_* turns out to be the form

$$\begin{aligned} R \times R &\longrightarrow R \\ x, y &\longrightarrow x^Jcy \end{aligned}$$

EXAMPLE 2. Choose $C \in M_nR$, $C = C^{J'}$. Define $\eta: M_nR \times M_nR \to M_nR$ by $\eta(A, B) = A^{J'}CB$. This is a form over (M_nR, J') and η_* is the form on R^n with matrix C.

EXAMPLE 3. If $\eta: M \times M \to M_nR$ then trace η is a form over $(R, J' \mid R)$ and trace η is equivalent to a 'sum' of n copies of η_* (ψ^{-1} of Lemma 1.1 gives the equivalence).

3. Involutions on Semisimple Real Algebras

Any simple real algebra is M_nK up to isomorphism where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (reals, complex numbers or quaternions). First we look at involutions on M_nK.

$M_n\mathbb{R}$ The map $X \to X^t$ is clearly an involution. The composite of two involutions is an automorphism and every automorphism is inner by Skolem-Noether theorem so any involution on $M_n\mathbb{R}$ must be of the form

$$X \to A^{-1}X^tA \qquad \text{and} \qquad A^t = \pm A$$

since the map must have period two.

$M_n\mathbb{C}$ The maps $X \to X^t$ and $X \to \bar{X}^t$ are both involutions. So we get that any involution on $M_n\mathbb{C}$ is either

(i) $X \to A^{-1}X^tA$ where $A^t = \pm A$,

or (ii) $X \to A^{-1}\bar{X}^tA$ where $\bar{A}^t = \pm A$.

In case (ii) if $\bar{A}^t = -A$ then $(\overline{iA})^t = iA$ so we can always take A hermitian symmetric.

$M_n\mathbb{H}$ The map $X \to \bar{X}^t$ is an involution ($-$ being the usual involution on $\mathbb{H}$). (Note that $X \to X^t$ is not an involution on $M_n\mathbb{H}$.) Every involution on $M_n\mathbb{H}$ is of the form $X \to A^{-1}\bar{X}^tA$ where $\bar{A}^t = \pm A$. See Albert [1]. We can not put iA in place of A to remove the skew symmetric case as $iA \neq Ai$ in $M_n\mathbb{H}$, i.e. we have two distinct types of involution.

In all we have, up to equivalence, seven types of involution (2 on $M_n\mathbb{R}$, 3 on $M_n\mathbb{C}$, and 2 on $M_n\mathbb{H}$) and each one of the seven could be interpreted as being of the form * of section 2.

If R is a semi-simple real algebra then $R = \sum_{i=1}^n R_i$ where each R_i is of the form M_nK up to isomorphism. Any involution of R either preserves components R_i or else swaps pairs of them (being of period two). Hence all the possible types of involution on R could be listed. In section 5 we will construct a form over $(\mathbb{R}\pi, J)$, $\mathbb{R}\pi$ being the real group ring of a finite group π, and J being the involution induced by mapping each element of π to its inverse. We can describe how J behaves on simple components of $\mathbb{R}\pi$ as follows. An involution J on a real algebra R is said to be *positive* if for each non-zero $a \in R$ the linear map $R \to R$ which sends x to $aa^J x$ has positive trace. This definition is due to Weil [10].

LEMMA 3.1. *The above mentioned involution J on $\mathbb{R}\pi$ is positive.*

Proof. Let $\pi = \{g_1, g_2, \ldots, g_n\}$ and g_1 be the identity of π.

$$x \in \mathbb{R}\pi \Rightarrow x = \sum_{i=1}^{n} r_i g_i, \ (r_i \in \mathbb{R} \text{ for each } i)$$

We want to find the diagonal entries in the matrix, with respect to the basis $g_1, \ldots, g_n$, of the map $x \to aa^J x$. So we want the coefficient of g_i in $aa^J g_i$. This is, for each i, the same as the coefficient of g_1 in aa^J i.e., if

$$a = \sum_{i=1}^{n} a_i g_i$$

then each diagonal entry of the matrix is $\sum_{i=1}^{n} a_i^2$ i.e. positive trace.

Weil proves that a positive involution preserves simple components and also that on $M_n K$, $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ there is a unique positive involution (namely $X \to X^t$ for $K = \mathbb{R}$ and $X \to \bar{X}^t$ for $K = \mathbb{C}$ or $\mathbb{H}$). So J on $\mathbb{R}\pi$ is precisely determined.

4. FORMS OVER SIMPLE REAL ALGEBRAS

A simple real algebra is, up to isomorphism, a matrix ring $M_n K$ where $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The possible involutions on $M_n K$ have been listed in the previous section. We will show now that forms over $(M_n K, J)$ can be classified up to equivalence in the same way as forms over K (the type of form over K depending on J).

First we define a skew form over $(M_n K, J)$ to satisfy the same conditions as a form over $(M_n K, J)$ except that we require $\phi(x, y) = -\phi(y, x)$ (i.e. ϕ is skew-symmetric with respect to J). Equivalence, sums, etc., are all defined as before, and Theorems 2.1 and 2.5 give a correspondence between skew forms over $(M_n K, J)$ and skew forms over K.

For a form ϕ over $(M_n \mathbb{R}, J)$ where $X^J = A^{-1} X^t A$ and $A^t = -A$ we get that $A\phi$ is a skew form over $(M_n K, J')$ where $X^{J'} = X^t$. Similarly for the involutions on $M_n \mathbb{C}$ and $M_n \mathbb{H}$ when $\bar{A}^t = -A$ we get skew forms. Thus we can classify, up to equivalence, forms over $(M_n K, J)$ by using Theorem 2.5.

Involution		Form over K	Invariants
1. $K = \mathbb{R}, X^J = A^{-1}X^tA,$	$A^t = A$	quadratic	rank and signature
2.	$A^t = -A$	alternating	rank
3. $K = \mathbb{C}, X^J = A^{-1}X^tA,$	$A^t = A$	quadratic	rank
4.	$A^t = -A$	alternating	rank
5. $K = \mathbb{C}, X^J = A^{-1}\bar{X}^tA$		hermitian	rank and signature
6. $K = \mathbb{H}, X^J = A^{-1}\bar{X}^tA,$	$\bar{A}^t = A$	hermitian	rank and signature
7.	$\bar{A}^t = -A$	skew-hermitian	rank

Nonsingular forms over K are determined up to equivalence by rank alone, or by rank and signature, as indicated in the above table. The routine argument for the classification would go in the following way.

Each of the seven types is represented by a square matrix with respect to an ordered basis (if A and B represent the same form but with respect to two different bases then $B = P^tAP$ or $\bar{P}^tAP$, depending on type).

The ***rank*** of a form is defined to be the rank of its matrix. This is easily seen to be independent of choice of basis.

A form ϕ of type 2 or 4 will have a skew-symmetric $n \times n$ matrix representing it. Clearly n must be even and $\phi(x, x) = 0$ for all x. It is easy then to find a symplectic basis for ϕ, i.e., a basis such that the matrix of ϕ is the block matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.) (There exists a 2 dimensional subspace U such that $\phi \mid U$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. U is a direct summand and we can repeat the process till we get a basis for the whole space. After re-ordering this basis we get the above block matrix). Hence rank alone classifies forms of type 2 or 4.

For each of the other five cases $\phi(x, x) = 0 \; \forall x$ implies $\phi = 0$. Hence we can find a basis with respect to which ϕ is given by a diagonal matrix. (For case 1, as $\phi(x, x) \neq 0 \; \forall x$ we can represent ϕ by a matrix of the form $\left(\begin{smallmatrix} a & L^t \\ L & N \end{smallmatrix}\right)$ where $a \in K$, $a \neq 0$, L is an $(n-1) \times 1$ matrix, and N an $(n-1) \times (n-1)$ matrix.)

$$\begin{pmatrix} a & L^t \\ L & N \end{pmatrix} \sim \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{a}LL^t + N \end{pmatrix} \text{ using } P = \begin{pmatrix} 1 & -\frac{1}{a}L^t \\ 0 & I \end{pmatrix}.$$

The result follows by induction. Similarly for the other four cases.

The diagonal entries we get will be in the set of nonzero real numbers for cases 1, 5, 6, nonzero complex numbers for case 3, and $\{z \in \mathbb{H}: \bar{z} = -z, z \neq 0\}$ for case 7. Multiplying the basis elements by scalars we can reduce the diagonal entries to ± 1 for cases 1, 5, and 6 and to $+1$ for Cases 3 and 7, (all non-zero elements of $\mathbb{C}$ and $\mathbb{H}$ have square roots). Thus two forms of type 3 or 7 are equivalent if and only if they have the same rank. See also [3] for more on type 7. For types 1, 5, 6 we define the *signature* of the form to be $p - q$ where p (resp. q) is the number of appearances of $+1$ (resp. -1) on the diagonal. For each type the fact that the signature is independent of choice of basis follows by the usual argument (i.e. by showing that p is the dimension of the maximal subspace on which the form is positive definite). The two invariants, rank and signature, determine the form up to equivalence.

We are now able to define the rank, (for all types), and the signature, (for types 1, 5, 6 only), of a form over $M_n K$ to be that of the corresponding form over K.

Next let R be a semisimple real algebra and J the unique positive involution on R.

$$R = \sum_{i=1}^{n} R_i \text{ and each } R_i \cong M_{n_i} K_i$$

for some K_i, n_i.

A form ϕ over (R, J) is easily shown to split into a sum of forms ϕ_i over $(R_i, J \mid R_i)$ since $R_i = e_i R$, e_i being the central idempotents of R. Since $J \mid R_i$ is positive it follows that ϕ_i will have a signature and a rank. We could define the multirank (resp. multisignature) of ϕ to be the collection of rank ϕ_i (resp. signature ϕ_i). Non-singular forms ϕ would be classified up to equivalence by multirank and multisignature.

5. Multisignature of a Manifold

Let M^{2k} be a closed (i.e. compact without boundary), connected, oriented topological manifold of dimension $2k$, with finite fundamental group. Let $\pi = \pi_1 M^{2k}$ be the fundamental group. Let $\tilde{M}$ be the universal covering space of M. This will be a closed oriented manifold of dimension $2k$. π acts on $\tilde{M}$ as a group of transformations and thus induces an action on $H^*\tilde{M}$, the cohomology of $\tilde{M}$. (We use cohomology with coefficient group $\mathbb{R}$.) Hence $H^*\tilde{M}$ is an $\mathbb{R}\pi$-module and in particular $H^k\tilde{M}$ is an $\mathbb{R}\pi$-module.

We define a form ϕ by

$$\phi: H^k\tilde{M} \times H^k\tilde{M} \to \mathbb{R}\pi$$

$$\phi(x, y) = \sum_{g \in \pi} (x \cdot yg^{-1})g,$$

the $\cdot$ denotes the cup product in $H^*\tilde{M}$, and $x \cdot yg^{-1}$ may be viewed as a real number since $x \cdot yg^{-1} \in H^{2k}\tilde{M} \cong \mathbb{R}$. (More precisely we get a real number by evaluating $x \cdot yg^{-1}$ on the fundamental homology class of M.) Hence we get that ϕ is a form over $(\mathbb{R}\pi, J)$, J the unique positive involution on $\mathbb{R}\pi$, when k is even, and ϕ is a skew form when k is odd. So ϕ has a multisignature for k even (ϕ also has a multirank but since ϕ is nonsingular this is not of great value). It is a collection of integers indexed by the irreducible real representations of π. For k odd, ϕ is a skew form and so we get a signature for ϕ_i only when the corresponding R_i is of the form $M_n\mathbb{C}$, i.e. we get a multisignature which is a set of integers indexed by the irreducible real representations ρ such that ρ and $\bar{\rho}$ are inequivalent. This index set may of course be empty, depending on π. We define the multisignature of the manifold M^{2k} to be the multisignature of ϕ.

This is a topological, (in fact bordism), invariant and is of use in surgery of manifolds. When $\pi = 1$, it reduces to a single integer which is the signature (or index) defined in [5, p. 84]. The multi-signature can be interpreted in terms of the Atiyah–Singer signature [2, p. 578–579]. See [7] for details. It can be defined for manifolds with boundary (and is of more value here since for closed manifolds it turns out to be almost trivial). Also a multisignature can be defined for non-orientable manifolds (the involution on $\mathbb{R}\pi$ having to be modified to allow for nonorientability). See [8] for all this.

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