

A NOTE ON HERMITIAN AND QUADRATIC FORMS

D. W. LEWIS

Let K be a field (characteristic $\neq 2$), and let L be a quadratic extension of K . Then $L = K(\sqrt{a})$ for some $a \in K$. Let D be the quaternion algebra $(a, b/K)$ generated by elements i, j satisfying $i^2 = a, j^2 = b, ij = -ji$. We will assume that D is a division algebra, i.e. that the quadratic form $\langle 1, -a, -b, ab \rangle$ is anisotropic. Let $-$ denote the standard involution on L and on D so that $\sqrt{a} = -\sqrt{a}$ on L and $\bar{i} = -i, \bar{j} = -j$ on D .

We will consider hermitian forms ϕ over L and D with respect to the standard involution. Given such a hermitian form ϕ there is an underlying symmetric bilinear form over K given by $\frac{1}{2}(\phi + \bar{\phi})$. (Equivalently we have an underlying quadratic form over K given by taking $\phi(x, x)$ for all x).

Jacobson [1] proved that two hermitian forms over L , or over D , are isometric if and only if their underlying quadratic forms are isometric. We may ask when is a quadratic form over K the underlying form of some hermitian form over L or over D . This question is answered in the case of L by Milnor-Husemoller [2, Appendix 2], where they construct an exact sequence of $W(K)$ -modules

$$0 \rightarrow W(L, -) \rightarrow W(K) \rightarrow W(L),$$

$W(L, -)$, (resp. $W(K), W(L)$), denoting the Witt group of hermitian forms over $(L, -)$, (resp. quadratic forms over K, L).

The first map in the sequence comes by taking the underlying quadratic form and the second map by tensoring by L , i.e. for the symmetric bilinear form $\phi : V \times V \rightarrow K$ we get the extended form

$$\phi \otimes L : V \otimes L \times V \otimes L \rightarrow L, \quad (\phi \otimes L)(x \otimes \lambda, y \otimes \mu) = \phi(x, y)\lambda\mu.$$

In this note we give a different proof of exactness at $W(K)$ which is illuminating in that it tells us how to construct a similar exact sequence in the quaternion algebra case. We also extend by one term the Milnor-Husemoller sequence.

I am grateful to C. T. C. Wall for some helpful comments.

LEMMA 1. *Let the quadratic form ϕ over K be such that $\phi \otimes L$ is hyperbolic over L . Then the Witt class of ϕ is in the ideal of $W(K)$ generated by the two-dimensional form $\langle 1, -a \rangle$.*

Proof. Let V be the K -space on which ϕ is defined. Then there exists $z \in V \otimes L$ such that $(\phi \otimes L)(z, z) = 0$. Writing $z = v \otimes 1 + w \otimes \sqrt{a}$ for some $v, w \in V$ we get

$$(\phi \otimes L)(z, z) = \phi(v, v) + a\phi(w, w) + 2\phi(v, w)\sqrt{a} = 0.$$

Thus $\phi(v, w) = 0$ and $\phi(v, v) = -a\phi(w, w)$. Our interest being in the Witt class of ϕ , we may assume ϕ anisotropic, so $\phi(v, v) \neq 0$. Hence the two-dimensional subspace spanned by v and w splits off as a direct summand of V . The result follows by induction.

Received 26 September, 1978.

[BULL. LONDON MATH. SOC., 11 (1979), 265-267]

PROPOSITION 1. *There is an exact sequence of $W(K)$ -modules*

$$0 \rightarrow W(L, -) \rightarrow W(K) \rightarrow W(L) \rightarrow W(K),$$

the last map being induced by $p: L \rightarrow K$, $p(r+s\sqrt{a}) = s$, for $r, s \in K$.

Proof. Exactness at $W(L, -)$ follows from Jacobson's theorem. At $W(K)$ an easy calculation shows that the image of $W(L, -) \rightarrow W(K)$ is the ideal generated by $\langle 1, -a \rangle$ (i.e. the ideal generated by the norm form of $(L, -)$). By the lemma this ideal is the kernel of the next map.

Finally we must prove exactness at $W(L)$. It is clear that $\phi \otimes L$ must be in the kernel of the map $p_*: W(L) \rightarrow W(K)$ for a form ϕ over K , (consider a one-dimensional form ϕ !). Conversely given ϕ over L such that $p_*(\phi)$ is hyperbolic then for some x we must have $p(\phi(x, x)) = 0$. Hence $\phi(x, x) = c \in K$ and so we can write ϕ as an orthogonal sum $\langle c \rangle \perp \phi'$ where ϕ' , by the Witt cancellation theorem, must be such that $p_*(\phi')$ is hyperbolic. An induction now completes the proof.

COROLLARY. *Let $\phi: V \times V \rightarrow K$ be a quadratic form on the K -space V . Then ϕ is the underlying form of a hermitian form over $(L, -)$ if and only if*

- (1) $\dim V = 2r$,
- (2) $\phi \otimes L$ is hyperbolic,
- (3) $\det \phi = (-a)^r$, modulo square classes in K .

Proof. Hyperbolic forms of dimension $2r$, r odd, are not the underlying forms of hermitian forms. Condition (3) is needed to exclude these. The result now follows from the proposition.

Next we look at the quaternion algebra case. Let \wedge denote the involution on D given by $\hat{i} = -i, \hat{j} = j$. Note that forms over D which are hermitian with respect to the involution \wedge may equivalently be viewed as skew-hermitian with respect to the standard involution $-$. (The equivalence is given by $\phi \rightarrow i\phi$.) We have Witt groups, $W(D, -)$ and $W(D, \wedge)$, of hermitian forms with respect to those involutions. Also given a form $\phi: V \times V \rightarrow L$ which is hermitian with respect to $-$, V being an L -space, we can extend ϕ to a hermitian form $\phi \otimes (D, \wedge)$ over (D, \wedge) as follows:

$$\phi \otimes (D, \wedge): V \otimes_L D \times V \otimes_L D \rightarrow D, \quad x \otimes \lambda, y \otimes \mu \rightarrow \hat{\lambda}\phi(x, y)\mu.$$

(Of course, we can also extend ϕ to a hermitian form over $(D, -)$ but that will not be needed here.)

LEMMA 2. *Let ϕ be a hermitian form over $(L, -)$ such that $\phi \otimes (D, \wedge)$ is hyperbolic. Then the Witt class of ϕ is in the ideal of $W(L, -)$ generated by $\langle 1, -b \rangle$.*

Proof. For ϕ defined on the L -space V there must exist $z \in V \otimes D$ such that $(\phi \otimes (D, \wedge))(z, z) = 0$. Writing $z = x \otimes 1 + w \otimes j$ we get $\phi(v, w) = 0$ and

$$\phi(v, v) + b\phi(w, w) = 0.$$

The result follows as in Lemma 1.

PROPOSITION 2. *There is an exact sequence*

$$0 \rightarrow W(D, -) \rightarrow W(L, -) \rightarrow W(D, \wedge) \rightarrow W(L).$$

The map $W(D, -) \rightarrow W(L, -)$ is the unique injective map through which $W(D, -) \rightarrow W(K)$ factors (i.e. it is induced by $D \rightarrow L, z_1 + z_2 j \rightarrow z_1$). The next map is the extension to (D, \wedge) and the final map is that induced by $D \rightarrow L, z_1 + z_2 j \rightarrow z_2$.

Proof. The image of $W(D, -) \rightarrow W(L, -)$ is easily checked to be the ideal generated by $\langle 1, -b \rangle$, but this, by Lemma 2, is the kernel of the following map. Exactness at $W(D, \wedge)$ follows easily by the same arguments as used in Proposition 1.

COROLLARY. *Let $\psi : V \times V \rightarrow K$ be a quadratic form on the K -space V . Then ψ is the underlying form of a hermitian form over $(D, -)$ if and only if*

- (1) $\dim V = 4r$,
- (2) ψ is the underlying form of a form ϕ over $(L, -)$ where $\phi \otimes (D, \wedge)$ is hyperbolic,
- (3) if $\psi = \psi_0 \perp h$, ψ_0 anisotropic, h hyperbolic, then dimension of h is zero modulo eight.

Proof. Follows from the proposition together with the fact that for r odd the hyperbolic form of dimension $4r$ is not the underlying form of a hermitian form over $(D, -)$.

References

1. N. Jacobson, "A note on hermitian forms", *Bull Amer. Math. Soc.*, 46 (1940), 264–268.
2. J. Milnor and D. Husemoller, *Symmetric bilinear forms* (Springer-Verlag, Berlin, Heidelberg, New York, 1973).

Department of Mathematics,
University College,
Belfield, Dublin 4,
Ireland.