

Periodicity of Clifford algebras and exact octagons of Witt groups

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(Received 30 May 1984)

Exact octagons, i.e. circular eight-term exact sequences, have cropped up recently in a few places in the literature. Papers of the author [11], and implicitly [10], the book of M. Warshauer [14], and the notes [5], all contain exact octagons. The first three references involve octagons of Witt groups of quadratic and other kinds of forms, the last reference extending the octagons to the setting of L -groups, i.e. surgery obstruction groups.

It is the purpose of this paper to show that the symmetry pattern displayed by the octagons of [10], [11], and indeed the fact that they are octagons, arises out of the eightfold periodicity of Clifford algebras viewed as algebras with involution.

While our octagons may be viewed as special cases of the L -group octagons of [5], our construction uses only elementary algebra without any of the machinery of algebraic topology as used in [5].

In § 1 we discuss Clifford algebras and give a periodicity theorem for such algebras viewed as algebras with involution. In § 2 we construct an infinite sequence of Clifford algebras out of which arises a long exact sequence of Witt groups, these being the Witt groups of hermitian forms over Clifford algebras with involution. The eightfold periodicity causes the long exact sequence to reduce to an exact octagon. The octagons of [10], [11] arise from special cases of this.

Acknowledgements are due to C. T. C. Wall, who first suggested that the shape of the sequences of [10] was reminiscent of Bott periodicity of Clifford algebras, and to C. Riehm for helpful discussions on some of this work.

1. Clifford algebras and periodicity

Let K be a field of characteristic not equal to 2. Let $q: V \rightarrow K$ be a non-singular quadratic form defined on a finite-dimensional K -space V . Let $C(q)$ be the Clifford algebra of q . See [8], chapter 5, or [6] for basic definitions and results on Clifford algebras. For the standard results on periodicity see [1] and [8], chapter 5. Further results on periodicity may be found in [2], [7], [12] but this paper requires no specific knowledge of these.

The algebra $C(q)$ has two involutions naturally defined on it. (The word involution is used in this paper to mean an anti-automorphism of period 2.) These two involutions, denoted J_ϵ for $\epsilon = 1$ or $\epsilon = -1$, are defined by $J_\epsilon(x) = \epsilon x$ for all $x \in V$. Note that the composite map $J_1 \circ J_{-1}$, which will henceforth be denoted by α , is an automorphism of period 2. The standard \mathbb{Z}_2 -grading of $C(q)$ is completely determined by α .

The notation (A, J) will be used to denote an algebra A equipped with an involution J . The algebras with involution $(C(q), J_\epsilon)$ are those of interest in this paper.

The following is a standard result on Clifford algebras.

PROPOSITION 1. *Let q_1 and q_0 be non-singular quadratic forms over K . Let q_0 be of rank 2 and have determinant δ . Let $q_1 \perp q_0$ denote the orthogonal sum of the forms q_1 and q_0 . Then $C(q_1 \perp q_0)$ is isomorphic to $C(q_0) \otimes_K C(-\delta q_1)$. (This is the usual tensor product, not the graded product!)*

Proof. For a detailed proof see [6], p. 233. Briefly it goes as follows:

Let q_0 be the form $\langle a, b \rangle$, $a, b \in K$, and let $\{e, f\}$ be an orthogonal basis of V_0 such that $q_0(e) = a$, $q_0(f) = b$, where $q_i: V_i \rightarrow K$ ($i = 0, 1$). Identify V_0 , resp. V_1 , with its image in $C(q_0)$, resp. $C(-\delta q_1)$. The universal property of Clifford algebras and also that of the tensor product yields a natural map $C(q_0) \otimes_K C(-\delta q_1) \rightarrow C(q_0 \perp q_1)$ such that $y \rightarrow y$ for all $y \in V_0$, $z \rightarrow dz$ for all $z \in V_1$, where $d = ef$. (Here each V_i has been identified with its image in $V_0 \oplus V_1$ and thence its image in $C(q_0 \perp q_1)$.) This map is an isomorphism, its inverse being the map $C(q_0 \perp q_1) \rightarrow C(q_0) \otimes_K C(-\delta q_1)$ specified by

$$y + z \rightarrow y \otimes 1 + d^{-1} \otimes z \quad \text{for } y \in V_0, z \in V_1.$$

The following is the corresponding result for algebras with involution.

PROPOSITION 2. *$(C(q_0 \perp q_1), J_\epsilon)$ is isomorphic to $(C(q_0), J_\epsilon) \otimes (C(-\delta q_1), J_{-\epsilon})$ for $\epsilon = 1$ or $\epsilon = -1$.*

Proof. Examine the action of the involutions on basis elements of $V_0 \oplus V_1$.

The above proposition is the key result needed for proving results on periodicity of Clifford algebras viewed as algebras with involution. First some terminology is needed. $C^{r,s}$ will denote the Clifford algebra of the $(r+s)$ -dimensional form $r\langle -1 \rangle \perp s\langle 1 \rangle$. The standard periodicity theorems [1], [8] for Clifford algebras, viewed either as ungraded or as \mathbb{Z}_2 -graded algebras, relate $C^{r,s}$ and $C^{r+s,s}$. The following is the result for algebras with involution.

PROPOSITION 3. *Let the involution J_ϵ on any Clifford algebra be as defined earlier. Then $(C^{r+s,s}, J_\epsilon)$ is isomorphic to $(M_{16} C^{r,s}, *)$, where $M_{16} C^{r,s}$ denotes the ring of 16×16 matrices with entries in $C^{r,s}$ and $*$ is the involution on this matrix ring defined by, for $X \in M_n C^{r,s}$, $X^* = S^{-1}(X^{J_\epsilon})^t S$, S being an element of $M_n C^{r,s}$ which is symmetric with respect to J_ϵ , i.e. $(S^{J_\epsilon})^t = S$. (In fact, S can be taken to be a symmetric matrix in $M_{16} K$.)*

Proof. By repeated application of Proposition 2,

$$(C^{r+s,s}, J_\epsilon) \cong (C^{2,0}, J_\epsilon) \otimes (C^{0,2} J_{-\epsilon}) \otimes (C^{2,0}, J_\epsilon) \otimes (C^{0,2}, J_{-\epsilon}) \otimes (C^{r,s}, J_\epsilon).$$

Also by repeated application of Proposition 2,

$$(C^{8,0}, J_\epsilon) \cong (C^{2,0}, J_\epsilon) \otimes (C^{0,2}, J_{-\epsilon}) \otimes (C^{2,0}, J_\epsilon) \otimes (C^{0,2}, J_{-\epsilon}).$$

Hence $(C^{r+s,s}, J_\epsilon) \cong (C^{8,0}, J_\epsilon) \otimes (C^{r,s}, J_\epsilon)$. But it is easy to check that $C^{8,0} \cong M_{16} K$ for any K . Also, by calculating the dimension of the subspace of $C^{8,0}$ fixed by J_ϵ it is easily seen that J_ϵ is of orthogonal type, $\epsilon = 1$ or -1 , i.e. is the adjoint of a symmetric form. This completes the proof.

Comment 1. There is always a full eightfold periodicity which does not degenerate into a lower-order periodicity for any kind of field. This is in contrast to the ungraded and \mathbb{Z}_2 -graded cases, where the periodicity can reduce to order 4 or 2 for certain kinds of field.

The periodicity of Clifford algebras from the viewpoint of algebras with involution

does not seem to occur explicitly in the literature except for [13], which looks at the special cases of the real and complex numbers.

Comment 2. From the viewpoint of hermitian form theory the above proposition implies that $(C^{r+8,s}, J_e)$ and $(C^{r,s}, J_e)$ are hermitian Morita equivalent. See [4] or [9]. In particular, the Witt groups of non-singular hermitian forms over $(C^{r+8,s}, J_e)$ and $(C^{r,s}, J_e)$ will be isomorphic.

Another version of periodicity, which is important for the construction of exact octagons, arises again from Proposition 2. Let $a \in K$, $a \neq 0$ and $q_0 = \langle -a, a \rangle$, a two-dimensional hyperbolic form. Let q be any non-singular quadratic form over K . Consider the following infinite sequence of quadratic forms q_n ($n = 1, 2, 3, \dots$), given by $q_1 = q$, $q_2 = q_1 \perp \langle -a \rangle$, $q_{n+2} = q_n \perp q_0$ for all $n \geq 1$, n being a positive integer, i.e. start with q and alternately add one-dimensional forms $\langle -a \rangle$ and $\langle a \rangle$ to obtain an infinite sequence.

PROPOSITION 4. *Let the involution J_e on a Clifford algebra be as defined earlier. Then $(C(q_{r+8}), J_e)$ is isomorphic to $(M_{16}C(q_r), *)$, where $*$ is the involution on $M_{16}C(q_r)$ given by $X^* = S^{-1}(X^{J_e})^t S$ for an element $S \in M_{16}C(q_r)$ such that $(S^{J_e})^t = S$. (S may be taken to be a symmetric matrix in $M_{16}K$.)*

Proof. Use Proposition 2 in the same manner as in the proof of Proposition 3 to get that $(C(q_{r+8}), J_e) \cong (C(4q_0), J_e) \otimes (C(q_r), J_e)$, where $4q_0 = \langle -a, a, -a, a, -a, a, -a, a \rangle$. It is easily checked that $C(4q_0) \cong M_{16}K$ and that J_e is an orthogonal type involution on $C(4q_0)$.

Comment. As earlier, it follows that $(C(q_{r+8}), J_e)$ and $(C(q_r), J_e)$ are hermitian Morita equivalent. This implies that the infinite sequence of Witt groups of hermitian forms over $(C(q_r), J_e)$ ($r = 1, 2, 3, \dots$) is a periodic sequence of period 8.

2. Exact octagons

Let $C(q_r)$ be as in § 1, and let $W(C(q_r), J_e)$ be the Witt group of non-singular hermitian forms over $C(q_r)$. (Briefly, the Witt group is defined by first taking the Grothendieck group of isometry classes of non-singular hermitian forms and then defining a Witt equivalence relation in such a way that the forms having Witt class zero are precisely those which contain a submodule equal to its own orthogonal complement.) The Witt group of non-singular skew-hermitian forms over $C(q_r)$ will also be considered and this will be denoted by $W_{-1}(C(q_r), J_e)$.

There is, for each r , a natural homomorphism $U_r: W(C(q_r), J_e) \rightarrow W(C(q_{r+1}), J_e)$ defined as follows:

Given $\phi: M \times M \rightarrow C(q_r)$, a non-singular hermitian form over $(C(q_r), J_e)$, M being a right $C(q_r)$ -module, the form

$$U_r(\phi): M \otimes_{C(q_r)} C(q_{r+1}) \times M \otimes_{C(q_r)} C(q_{r+1}) \rightarrow C(q_{r+1})$$

is defined by

$$U_r(\phi)(x \otimes \lambda, y \otimes \mu) = \lambda^{J_e} \phi(x, y) \mu \quad \text{for } x, y \in M, \quad \lambda, \mu \in C(q_{r+1}).$$

Note that because of the way the sequence (q_r) is constructed it follows that $C(q_r)$ embeds in $C(q_{r+1})$ in a natural way and $\phi(x, y) \in C(q_r)$ can thus be identified with an element of $C(q_{r+1})$.

PROPOSITION 5. *There is a long exact sequence of Witt groups and group homomorphisms*

$$\dots \longrightarrow W(C(q_{r-1}), J_{-1}) \xrightarrow{U_{r-1}} W(C(q_r), J_{-1}) \xrightarrow{U_r} W(C(q_{r+1}), J_{-1}) \xrightarrow{U_{r+1}} \dots$$

Proof. For convenience write $C = C(q_{r-1})$, $C' = C(q_r)$, $C'' = C(q_{r+1})$. Firstly, because $q_{r+1} = q_{r-1} \perp q_0$ and $C(q_0) = M_2 K$, it follows, via Proposition 1, that

$$C'' \cong M_2 K \otimes_K C \cong M_2 C.$$

The involution J_{-1} on C'' becomes $J_{-1} \otimes J_1$ on $C(q_0) \otimes C$ and J_{-1} on $C(q_0) \cong M_2 K$ is of symplectic type, i.e. of the form $X \rightarrow S^{-1}X^t S$ for $X \in M_2 K$, where $S^t = -S$. Thus, by hermitian Morita theory, $W(C'', J_{-1})$ is isomorphic to $W_{-1}(C, J_1)$. (Similarly, it can be shown that $W(C'', J_1)$ is isomorphic to $W(C, J_{-1})$ since J_1 on $C(q_0)$ is of orthogonal type.)

Now $C' = C(q_r)$, $C = C(q_{r-1})$ and C' is a kind of quadratic extension of C obtained by the adjunction of an element e such that $e^2 = (-1)^r a$. (e will be an extra basis element adjoined to an orthogonal basis for q_r in order to get a diagonalization of q_{r+1} .) Each $x \in C'$ can be uniquely written in the form $x = c_1 + ec_2$ for $c_1, c_2 \in C$. Define the map $T: C' \rightarrow C$ by $T(c_1 + ec_2) = c_2$. It is easy to verify that $(x^{J_{-1}}) = -T(x)^{J_1}$ for all $x \in C'$. It then can be seen that if ϕ is a hermitian form over (C', J_{-1}) then $T \cdot \phi$ gives a skew-hermitian form over (C, J_1) .

Hence there is a map $T: W(C', J_{-1}) \rightarrow W_{-1}(C, J_1)$. It will be proved, in a lemma below, that there exists an isomorphism $\theta: W(C'', J_{-1}) \rightarrow W_{-1}(C, J_1)$ such that $\theta \circ U_r = T$, i.e. the following diagram commutes:

$$\begin{array}{ccc} W(C', J_{-1}) & \xrightarrow{U_r} & W(C'', J_{-1}) \\ & \searrow T & \downarrow \theta \\ & & W_{-1}(C, J_1) \end{array}$$

Thus U_r and T have the same kernel, and hence to prove exactness of the sequence at the point $W(C', J_{-1})$, it suffices to show that the kernel of T coincides with the image of U_{r-1} . This can be shown as follows. Let $\psi: M \times M \rightarrow C$ represent an element of $W(C, J_{-1})$. Then $T \circ U_{r-1}(\psi)$ is a form $M \otimes_C C' \times M \otimes_C C' \rightarrow C$, given by

$$x \otimes \lambda, y \otimes \mu \rightarrow T(\lambda^{J_{-1}} \phi(x, y) \mu)$$

for $x, y \in M$, $\lambda, \mu \in C'$.

Let $N = M \otimes 1$, a C -submodule of $M \otimes_C C'$ viewed as a C -module. Then the reader may easily check that N is self-orthogonal with respect to the form $T \circ U_{r-1}(\psi)$, i.e. that $TU_{r-1}(\psi)(n_1, n_2) = 0$ for all $n_1, n_2 \in N$ and that $TU_{r-1}(\psi)(n, x) = 0$ for all $n \in N$ implies that $x \in N$. Thus the image of U_{r-1} is contained in the kernel of T .

Now suppose $\phi: M \times M \rightarrow C'$ represents an element of $W(C', J_{-1})$ which lies in the kernel of T . Then M , viewed as a C -module, must have a C -submodule N which is self-orthogonal with respect to the form $T\phi$. Define a form $\psi: N \times N \rightarrow C$ by

$$\psi(n, n') = \phi(n, n').$$

Then it is easily checked that $N \otimes_C C'$ is isomorphic to M and that $U_{r-1}\psi$ is isometric to ϕ . All that is now left to complete the proof of Proposition 5 is the following lemma.

LEMMA. Use the notation of the above proof. There exists an isomorphism

$$\theta: W(C'', J_{-1}) \rightarrow W_{-1}(C, J_1)$$

such that $\theta \circ U_r = T$.

Proof. The isomorphism will be given by a hermitian Morita equivalence [3] from (C'', J_{-1}) to (C, J_1) . To obtain this equivalence it is necessary to have a $C'' - C$ bimodule progenerator B together with a form $h: B \times B \rightarrow C$, h hermitian over (C, J_1) , with h admitting (C'', J_{-1}) in the sense that $h(zu, v) = h(u, z^{J_{-1}}v)$ for all $u, v \in B$, $z \in C''$. Given a hermitian form ϕ over (C'', J_{-1}) , $\phi: M \times M \rightarrow C''$, M a right C'' -module, then the hermitian Morita equivalence gives a form over (C, J_1) defined on the right C -module $M \otimes_{C''} B$ by

$$(M \otimes_{C''} B) \times (M \otimes_{C''} B) \rightarrow C; \quad (x \otimes u, y \otimes v) \rightarrow h(u, \phi(x, y)v)$$

for all $x, y \in M$, $u, v \in B$.

A suitable choice for B is to take $B = C'$, which may be viewed as a free right C -module of rank 2, with basis $\{1, e\}$, and which has a left C'' -module structure defined as follows, Regard C' as being contained in C'' in the obvious way. Denote the C'' -action on C' by $*$ and define $\lambda * c' = \lambda c'$ if $\lambda \in C'$, $c' \in C'$ (i.e. ordinary multiplication in C'), and define $f * c' = e\gamma(c')$ for all $c' \in C'$, where γ is given by $\gamma(c'_1 + ec'_2) = c'_1 - ec'_2$ (writing $c' = c'_1 + ec'_2$ for $c'_1, c'_2 \in C$). Since any $\lambda \in C''$ can be expressed in the form $\lambda = \lambda_1 + f\lambda_2$ for $\lambda_1, \lambda_2 \in C'$, the C'' -action on C' is now defined in the obvious way.

[Comment—the motivation for this definition of the C'' -module structure on C' comes from using Proposition 1 to give an isomorphism of C'' with $C \otimes_K C(\langle -a, a \rangle)$ and the isomorphism $C(\langle -a, a \rangle) \rightarrow M_2 K$ given by

$$e \rightarrow \begin{pmatrix} 0 & -a \\ 1 & 0 \end{pmatrix}, \quad f \rightarrow \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

C'' is thus identified with $M_2 C$ and acts on the free C -module C' taking $\{1, e\}$ as the C -basis of C' . Beware that the component C in $C \otimes_K C(\langle -a, a \rangle)$ does not correspond to $\{\lambda I \in M_2 C: \lambda \in C\}$ but to $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^\alpha \end{pmatrix} \in M_2 C: \lambda \in C \right\}$, α being the grading automorphism of C defined earlier.]

The form $h: C' \times C' \rightarrow C$ is defined by $h(u, v) = T(u^{J_{-1}}v)$ for all $u, v \in C'$. The map θ of this lemma is the isomorphism arising from the hermitian Morita equivalence given by C' and h .

To prove the lemma it must be shown that $\theta \circ U_r = T$. Let $\phi: M \times M \rightarrow C'$ be a form over (C'', J_{-1}) . Then $T\phi: M \times M \rightarrow C$ is given by $(x, y) \rightarrow T(\phi(x, y))$ for $x, y \in M$ and

$$\theta((U_r)\phi): ((M \otimes_{C''} C'') \otimes_{C''} C') \times ((M \otimes_{C''} C'') \otimes_{C''} C') \rightarrow C$$

is given by

$$((x \otimes \lambda) \otimes u, (y \otimes \mu) \otimes v) \rightarrow h(u, \lambda^J \phi(x, y) \mu v)$$

for $x, y \in M$, $\lambda, \mu \in C''$, $u, v \in C'$.

Consider now the map $f: M \rightarrow (M \otimes_{C''} C'') \otimes_{C''} C'$, $f(x) = (x \otimes 1) \otimes 1$ for all $x \in M$ and also the map $g: (M \otimes_{C''} C'') \otimes_{C''} C' \rightarrow M$, $g((x \otimes \lambda) \otimes u) = x(\lambda * u)$ for $x \in M$, $\lambda \in C''$, $u \in C'$. It is easily checked that f and g are each right C -module homomorphisms.

Clearly $g \circ f$ is the identity map since $1 * 1 = 1$. Also $f \circ g$ is the identity map because

$$\begin{aligned} (f \circ g)((x \otimes \lambda) \otimes u) &= f(x(\lambda * u)) \\ &= (x(\lambda * u) \otimes 1) \otimes 1 \\ &= (x \otimes (\lambda * u)) \otimes 1 \\ &= (x \otimes 1) \otimes (\lambda * u) \\ &= (x \otimes \lambda) \otimes u. \end{aligned}$$

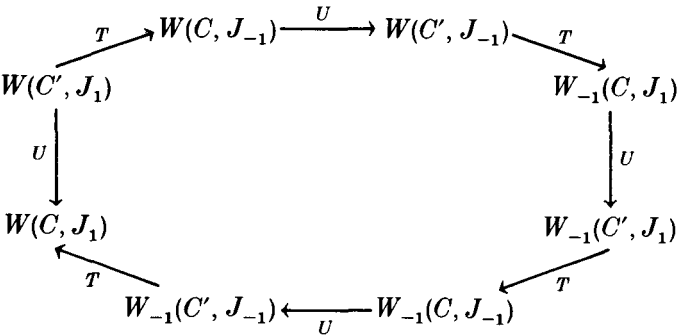
Thus f is a C -module isomorphism. Furthermore

$$\theta(U_r(\phi))(f(x), f(y)) = h(1, \phi(x, y)) = T(\phi(x, y))$$

so that f is an isometry of $\theta(U_r(\phi))$ and $T\phi$.

This proves the lemma.

COROLLARY. Write $C = C(q)$, $C' = C(q \perp \langle -a \rangle)$, q a non-singular quadratic form over K . Then there is an exact octagon as follows, the mappings T and U being the appropriate ones arising out of Proposition 5.



Proof. Because of the comment at the end of § 1 the long exact sequence of Proposition 5 reduces to an exact octagon. By Morita theory the Witt groups in the long sequence can be rewritten as those in the above octagon and the appropriate mappings T, U are the natural ones described above.

Comment 1. Taking $q = \langle a \rangle$ yields $C = K(\sqrt{a})$, $C' = M_2 K$ and, using a Morita equivalence of $M_2 K$ with K , the above octagon becomes the first exact sequence of [10] which is really a ‘degenerate octagon’, in fact an octagon with three zero terms. Taking $q = \langle b \rangle$, where $(a, b/K)$ is a quaternion division algebra so that $C = K(\sqrt{b})$, $C' = (a, b/K)$, yields the second exact sequence of [10] for $(a, b/K)$ with maximal subfield $K(\sqrt{b})$. This, again, is a degenerate octagon, having one zero term.

The symmetry pattern of the sequences of [10] is thus seen to be that of Clifford algebra periodicity for algebras with involution.

It should also be noted that if J_1 is used on each C_r in Proposition 5, instead of J_{-1} , then a similar long exact sequence arises which reduces to precisely the same octagon as in the corollary.

Comment 2. In [11] exact octagons of Witt groups of equivariant forms are constructed, i.e. forms invariant under a finite group action. Such forms possess an equivariant Clifford algebra, as in [3], which is like the usual Clifford algebra with a group action defined on it in the obvious way. All that has been done should go through

in the equivariant case so that the octagons of [11] also arise out of Clifford algebra periodicity.

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