

CASSON-LIN'S INVARIANT OF A KNOT AND FLOER HOMOLOGY

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ABSTRACT. A. Casson defined an intersection number invariant which can be roughly thought of as the number of conjugacy classes of irreducible representations of $\pi_1(Y)$ into $SU(2)$ counted with signs, where Y is an oriented integral homology 3-sphere. X.S. Lin defined an similar invariant (signature of a knot) to a braid representative of a knot in S^3 . In this paper, we give a natural generalization of the Casson-Lin's invariant to be (instead of using the instanton Floer homology) the symplectic Floer homology for the representation space (one singular point) of $\pi_1(S^3 \setminus K)$ into $SU(2)$ with trace-free along all meridians. The symplectic Floer homology of braids is a new invariant of knots and its Euler number of such a symplectic Floer homology is the negative of the Casson-Lin's invariant.

1. INTRODUCTION

Around 1986 there appeared two invariants for a closed, oriented, 3-manifold Y with the integral homology of S^3 . The first, introduced by Casson (see [31]), is an integer-valued invariant which roughly counts the number of irreducible $SU(2)$ -representations of the fundamental group $\pi_1(Y)$. The second is a Z_8 -graded homology theory developed by Floer in [9], and is based upon an application of the Morse theory to the Chern-Simons functional on the space \mathcal{B}_Y of equivalent classes of $SU(2)$ connections on Y . The two invariants are closely related because irreducible $SU(2)$ -representations can be interpreted as the critical points of the Chern-Simons functional. The Euler characteristic of the instanton Floer homology is twice Casson's invariant of the integral homology 3-sphere Y .

Analogous to Casson invariant, Lin [22] constructed an invariant of representation spaces corresponding to a braid representative of a knot K in S^3 . It turns out that the Lin's invariant of a knot is the signature of the knot. In [22], Lin used the special representations of $\pi_1(S^3 \setminus K)$ into $SU(2)$ such that all meridians of K are represented by trace zero matrices in $SU(2)$. It naturally arises two questions to understand more on the Casson-Lin's invariant of a knot. (1) Does there exists a similar Floer homology generalization for the Casson-Lin's invariant of a knot? (2) What kind of invariant one can get by replacing the trace zero to a fixed trace representations of knot groups? Recently, Cappell, Lee and Miller have studied the second question from the symplectic theory point of view in [6] and defined a so-called equivariant Casson invariant for 3-manifold with cyclic group action. Independently, Herald studied the same problem from the gauge theory point of view

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in [15]. The present paper is to give a Floer homology type of generalization for the Casson-Lin's invariant of a knot for the first question.

In this paper, we derive the symplectic Floer homology, rather than the instanton Floer homology, as the generalization of the Casson-Lin's invariant of a knot. It is not clear how to make the gauge theory point of view to realize that the critical points of certain functional (Chern-Simons functional) will correspond to the representation of $S^3 \setminus K$ with trace free along all meridians. Instead of the instanton Floer homology, the symplectic Floer homology also provides the generalization of the Casson invariant which help us to shift the point of view. From the Atiyah conjecture as described in [20], [21], we have seen that there is really no difference between the instanton Floer homology and the symplectic Floer homology for integral homology 3-spheres. So we can construct a generalization for the Casson-Lin's invariant of a knot by constructing the symplectic Floer homology.

The paper is organized as follows. In §2, we recall the Casson-Lin's invariant of a knot from the symplectic topology point of view which also provides some basic ingredients to construct the symplectic Floer homology. In §3, we mainly concentrate on the reducible representation of $\pi_1(S^3 \setminus K)$ and discuss the Walker-type correction term for this isolated reducible $U(1)$ -representation. We briefly review the symplectic Floer homology in §4.1 and show that the symplectic Floer homology is an invariant of a knot in §4.2. Some simple calculations of the new invariant are given in §5 in order to fix the sign between the Euler number and the invariant λ_{CL} .

2. CASSON-LIN'S INVARIANT OF A KNOT

In this section, we recall Lin's construction in [22] for the intersection number of the representation spaces corresponding to a braid representative of a knot K in S^3 . The presentation of the Casson-Lin's invariant here is from the symplectic geometry point of view in order to extend the Casson-Lin's invariant to the symplectic Floer homology in §4.

Let (S^3, D_+^3, D_-^3, S^2) be a Heegaard decomposition of S^3 with genus 0, where

$$S^3 = D_+^3 \cup_{S^2} D_-^3, \quad \partial D_+^3 = \partial D_-^3 = D_+^3 \cap D_-^3 = S^2.$$

Suppose that a knot $K \subset S^3$ is in general position with respect to this Heegaard decomposition. So $K \cap S^2 = \{x_1, \dots, x_n, y_1, \dots, y_n\}$, $K \cap D_\pm^3$ is a collection of unknotted, unlinked arcs $\{\gamma_1^\pm, \dots, \gamma_n^\pm\} \subset D_\pm^3$, where $\partial\gamma_i^- = \{x_i, y_i\}$ and $\{\gamma_1^+, \dots, \gamma_n^+\} = K \cap D_+^3$ becomes a braid of n strands inside D_+^3 . Denote a word β in the braid group B_n . For the top end point x_i of γ_i^+ , the bottom end points of $\{\gamma_1^+, \dots, \gamma_n^+\}$ gives a permutation of $\{y_1, \dots, y_n\}$ which generates a map

$$\pi : B_n \rightarrow S_n,$$

where $\pi(\beta)$ is the permutation of $\{y_1, \dots, y_n\}$ in the symmetric group of n letters. Let $K = \overline{\beta}$ be the closure of β . It is well-known that there is a correspondence between a knot and a braid β with $\pi(\beta)$ is a complete cycle of the n letters (see [3]).

There is a corresponding Heegaard decomposition for the complement of a knot K ,

$$\begin{aligned} S^3 \setminus K &= (D_+^3 \setminus K) \cup_{(S^2 \setminus K)} (D_-^3 \setminus K), \\ D_\pm^3 \setminus K &= D_\pm^3 \setminus (D_\pm^3 \cap K), \quad S^2 \setminus K = S^2 \setminus (S^2 \cap K). \end{aligned}$$

Thus from Seifert-Van Kampen theorem,

$$\begin{array}{ccc} \pi_1(S^2 \setminus K) & \rightarrow & \pi_1(D_+^3 \setminus K) \\ \downarrow & & \downarrow \\ \pi_1(D_-^3 \setminus K) & \rightarrow & \pi_1(S^3 \setminus K), \end{array}$$

we obtain a pull-back of representation spaces

$$\begin{array}{ccc} \mathcal{R}(S^2 \setminus K) & \leftarrow & \mathcal{R}(D_+^3 \setminus K) \\ \uparrow & & \uparrow \\ \mathcal{R}(D_-^3 \setminus K) & \leftarrow & \mathcal{R}(S^3 \setminus K), \end{array} \quad (2.1)$$

where $\mathcal{R}(X) = \text{Hom}(\pi_1(X), SU(2))/SU(2)$ for $X = S^2 \setminus K, D_\pm^3 \setminus K, S^3 \setminus K$. The $SU(2)$ admits a natural Riemannian metric which arises by translation from the inner product on the Lie algebra of S^3 . The natural identification of S^3 with $SU(2)$ is an isometry of the standard metric on S^3 and the above Riemannian metric on $SU(2)$. Also note that $S^3 \setminus K$ is an Eilenberg-MacLane space $K(\pi_1(S^3 \setminus K), 1)$ for classical knot K due to the asphericity theorem of Papakyriakopoulos.

In [24], Magnus used the trace free matrices to represent the generators of a free group to show that the faithfulness of a representation of braid groups in the automorphism groups of the rings generated by the character functions on free groups. This is original idea to have representations with trace free along all meridians which Lin worked in [22] to define the knot invariant. It has been carried out by Cappell, Lee and Miller in [6] for the representation of knot groups with the trace of the meridian fixed (not necessary zero). We only give the generation to symplectic Floer homology of the Casson-Lin's invariant for trace free representations of knot groups. For general case as in [6], we will discuss the corresponding Floer homology elsewhere.

Let (M, ω) be a $2n$ -dimensional symplectic manifold and $\phi : M \rightarrow M$ be a symplectic diffeomorphism, i.e. ω is a nondegenerate closed 2-form and $\phi^*\omega = \omega$. By choosing an almost complex structure J on (M, ω) such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric, we have an integer valued second cohomology class $c_1(M) \in H^2(M, \mathbb{Z})$ (the first Chern class). Note that the space of all almost complex structures which are compatible with ω is connected, so $c_1(M)$ is uniquely determined by ω . There are two homomorphisms

$$I_\omega : \pi_2(M) \rightarrow \mathbb{R}; \quad I_{c_1} : \pi_2(M) \rightarrow \mathbb{Z}.$$

Definition 2.1. The symplectic manifold (M, ω) is called monotone if there exists a nonnegative $\alpha \geq 0$ such that $I_\omega = \alpha I_{c_1}$ on $\pi_2(M)$ or $\pi_2(M) = 0$. See also [11].

Let $\mathcal{R}(S^3 \setminus K)^{[i]}$ be the space of $SU(2)$ representations $\rho : \pi_1(S^3 \setminus K) \rightarrow SU(2)$ such that

$$\rho([m_{x_i}]) \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho([m_{y_i}]) \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (2.2)$$

where $m_{x_i}, m_{y_i}, i = 1, 2, \dots, n$ are the meridian circles around x_i, y_i respectively. Note that $\pi_1(S^3 \setminus K)$ is generated by $m_{x_i}, m_{y_i}, i = 1, 2, \dots, n$ and one relation $\prod_{i=1}^n m_{x_i} = \prod_{i=1}^n m_{y_i}$. Corresponding to (2.1), we have

$$\begin{array}{ccc} \mathcal{R}(S^2 \setminus K)^{[i]} & \leftarrow & \mathcal{R}(D_+^3 \setminus K)^{[i]} \\ \uparrow & & \uparrow \\ \mathcal{R}(D_-^3 \setminus K)^{[i]} & \leftarrow & \mathcal{R}(S^3 \setminus K)^{[i]}. \end{array} \quad (2.3)$$

The conjugacy class in $SU(2)$ is completely determined by its trace. So the condition (2.2) can be reformulated for $\rho \in \mathcal{R}(X)^{[i]}$,

$$\text{trace} \rho([m_{x_i}]) = \text{trace} \rho([m_{y_i}]) = 0. \quad (2.4)$$

The space $\mathcal{R}(S^2 \setminus K)^{[i]}$ can be identified with the space of $2n$ matrices $X_1, \dots, X_n, Y_1, \dots, Y_n$ in $SU(2)$ satisfying

$$\text{trace}(X_i) = \text{trace}(Y_i) = 0, \quad \text{for } i = 1, \dots, n, \quad (2.5)$$

$$X_1 \cdot X_2 \cdots X_n = Y_1 \cdot Y_2 \cdots Y_n. \quad (2.6)$$

Lemma 2.2. *Let Q_n be the space $\{(X_1, \dots, X_n) \in SU(2)^n \mid \text{trace}(X_i) = 0, i = 1, \dots, n\}$. Thus Q_n is a monotone symplectic manifold of dimension $2n$.*

Proof: An element in $SU(2)$ can be viewed as $\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$, with $a^2 + b^2 + c^2 + d^2 = 1$. A matrix X_j in Q_n is a matrix with $\text{trace}(X_j) = 2a_j = 0$. I.e. the set of all such matrices equal to $\{(b_j, c_j, d_j) \mid b_j^2 + c_j^2 + d_j^2 = 1\}$. Hence Q_n is the product $S^2 \times S^2 \cdots \times S^2$ of n copies of 2-sphere with radius 1. So all S^2 's have the same area 4π . It is known that $S^2 \times S^2 \cdots \times S^2$ is a monotone symplectic manifold if and only if all S^2 has the same area (see [11] page 577). Thus the result follows. \square

It is well known that there is a natural symplectic structure on the coadjoint orbits by Kostant-Souriau theorem. If $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$ for Lie algebra \mathfrak{g} of a Lie group G , every $\omega \in Z^2(\mathfrak{g})$ is exact, $\omega = d\alpha$. The α is unique due to $H^1(\mathfrak{g}) = 0$. There is one-to-one correspondence between G -orbits $\text{orb}_G(\omega) = \{ad_a^* \omega : a \in G\}$ in $Z^2(\mathfrak{g})$ and G -orbits $\text{orb}_G(\alpha) = \{ad_a^* \alpha : a \in G\}$ in \mathfrak{g}^* , where ad^* is the coadjoint action and \mathfrak{g}^* is the dual space of \mathfrak{g} . The symplectic form $\bar{\omega}$ on $\text{orb}_G(\alpha) \cong G/H_\omega$ can be expressed by

$$\bar{\omega}(X_\alpha, Y_\alpha) = -\alpha([X, Y]),$$

for $\pi : G \rightarrow G/H_\omega$ and $X, Y \in \mathfrak{g}, X_\alpha = \pi_* X, Y_\alpha = \pi_* Y$. For $G = SU(2)$,

$$\text{orb}_G(B) = \{ad_a^* B : a \in SU(2)\} \quad \text{for } B = \begin{pmatrix} i & \\ & -i \end{pmatrix}.$$

is an orbit $\{ia^{-1}Ba : a \in SU(2)\}$ in $\mathfrak{su}(2)^*$. Note that $a^{-1}Ba = B$ if and only if $a = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$ with $t \in U(1)$. So

$$\text{orb}_G(B) = SU(2)/U(1) \cong \mathbf{CP}^1.$$

A symplectic form on \mathbf{CP}^1 is given by $\omega(X, Y) = (-B, [A^X, A^Y])$ for $\pi_* A^X = X, \pi_* A^Y = Y$. The trace free condition ensures to lie in the orbit $orb_G(B)$. Hence Q_n can be naturally identified with $\mathbf{CP}^1 \times \cdots \times \mathbf{CP}^1$ as a product of n copies of \mathbf{CP}^1 , again a monotone symplectic manifold.

Denote \tilde{S}^2 be the branched double covering of S^2 with $K \cap S^2$ as branched set. Then Z_2 operates on $\mathcal{R}(\tilde{S}^2)$ and the component of the fixed point set $\mathcal{R}^*(\tilde{S}^2)$ (the irreducible $SU(2)$ representations) with traceless condition coincides with $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ the irreducible $SU(2)$ representations with traceless condition. By [13], the irreducible components carry the natural symplectic structure and dimension $4n - 6$ has been verified in [22] p342- 344.

Lemma 2.3. *The space $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ is a monotone symplectic manifold of dimension $4n - 6$.*

Proof: Note that $\mathcal{R}^*(S^2 \setminus K)^{[i]} = (H_n \setminus S_n)/SU(2)$ in Lin's notation [22], where

$$H_n = \{(X_1, \dots, X_n, Y_1, \dots, Y_n) \in Q_n \times Q_n \mid X_1 \cdots X_n = Y_1 \cdots Y_n\},$$

S_n is the subspace of H_n consisting of all the reducible points. Here $H_n \setminus S_n$ is the total space of a $SU(2)$ -fiber bundle over $\mathcal{R}^*(S^2 \setminus K)^{[i]}$. It is remarkable that the points in H_n where $SU(2)$ -action is not locally free are precisely the set S_n . Note that any product of monotone symplectic manifolds is again monotone. So we have $I_\omega = \alpha I_{c_1}$ for $\alpha \geq 0$ over $(Q_n \times Q_n, \omega)$. (H_n, ω) is pre-symplectic space which is degenerated along the $SU(2)$ -fiber. For $n \geq 2$, S_n has codimension bigger than 2. The tangent space of $H_n \setminus S_n$ splits into a direct sum of tangent spaces along vertical and horizontal directions,

$$T(H_n \setminus S_n) = T\mathcal{R}^*(S^2 \setminus K)^{[i]} \oplus TSU(2). \quad (2.7)$$

Here $TSU(2) \cong T^*SU(2)$ carries a canonical symplectic form $\omega_0 = -d\lambda$, where λ is the Liouville 1-form. Thus $\omega - \omega_0$ is nondegenerate closed 2-form on $\mathcal{R}^*(S^2 \setminus K)^{[i]}$. Let $\pi : H_n \setminus S_n \rightarrow \mathcal{R}^*(S^2 \setminus K)^{[i]}$ be the $SU(2)$ -orbit projection, so $\omega - \omega_0 = \pi_*(\omega)$. From the fibration and (2.7), the Levi-Civita connection A on $T(H_n \setminus S_n)$ is essentially the Levi-Civita connection a on $T\mathcal{R}^*(S^2 \setminus K)^{[i]}$ since $SU(2)$ is parallelizable, $A = a + \theta$, where θ is the trivial connection. Thus by Chern-Weil theorem, $2\pi i c_1 = F_A$, we have $\pi^* c_1 = c_1|_{H_n \setminus S_n} \in H^2(H_n \setminus S_n; \mathbf{Z})$, i.e. $c_1|_{H_n \setminus S_n}$ is the natural pullback of the first Chern class c_1 in $H^2(\mathcal{R}^*(S^2 \setminus K)^{[i]}, \mathbf{Z})$. By the homotopy exact sequence of the fibration $\pi : H_n \setminus S_n \rightarrow \mathcal{R}^*(S^2 \setminus K)^{[i]}$ with fiber $SU(2)$,

$$0 = \pi_2(SU(2)) \rightarrow \pi_2(H_n \setminus S_n) \xrightarrow{\pi_*} \pi_2(\mathcal{R}^*(S^2 \setminus K)^{[i]}) \rightarrow \pi_1(SU(2)) = 0,$$

we obtain the natural isomorphism induced by π_* $\pi_2(H_n \setminus S_n) \cong \pi_2(\mathcal{R}^*(S^2 \setminus K)^{[i]})$. By Stokes theorem, $I_{\omega_0} = 0$. Hence for any $v \in \pi_2(\mathcal{R}^*(S^2 \setminus K)^{[i]}) = \pi_2(H_n \setminus S_n)$ (after identification by π_*),

$$I_\omega(v) = I_{\pi_*(\omega)}(v) + I_{\omega_0}(v) = I_{\pi_*(\omega)}(v).$$

$$I_{c_1}(v) = I_{\pi^*(c_1)}(v).$$

Thus $I_{\pi_*\omega} = \alpha I_{c_1}$ for $\alpha \geq 0$ on $\mathcal{R}^*(S^2 \setminus K)^{[i]}$. Thus we get the monotonicity for $\mathcal{R}^*(S^2 \setminus K)^{[i]}$. \square

Given $\beta \in B_n$, we denote by Γ_β the graph of β in $Q_n \times Q_n$, i.e.

$$\Gamma_\beta = \{(X_1, \dots, X_n, \beta(X_1), \dots, \beta(X_n)) \in Q_n \times Q_n\}.$$

As an automorphism of the free group $Z[m_1] * Z[m_2] * \cdots * Z[m_n]$, this element $\beta \in B_n$ preserves the word $[m_1] \cdots [m_n]$. Thus we have

$$X_1 \cdots X_n = \beta(X_1) \cdots \beta(X_n),$$

or in other words Γ_β is a subspace of H_n . In fact, for $\bar{\beta} = K$, this subspace Γ_β coincides with the subspace of representations $\rho : \pi_1(S^2 \setminus K) \rightarrow SU(2)$ in H_n which can be extended to $\pi_1(D_+^3 \setminus K)$, $\Gamma_\beta = \text{Hom}(\pi_1(D_+^3 \setminus K), SU(2))^{[i]}$. Hence the space $\mathcal{R}^*(D_+^3 \setminus K)^{[i]} = \Gamma_{\beta, irre}/SU(2)$ is the irreducible $SU(2)$ representations with traceless condition over $D_+^3 \setminus K$.

In the special case $\beta = id$, then Γ_{id} represents the diagonal in $Q_n \times Q_n$,

$$\Gamma_{id} = \{(X_1, \dots, X_n, X_1, \dots, X_n) \in Q_n \times Q_n\}.$$

Since $K \cap D_-^3$ represents the trivial braid, this space $\Gamma_{id} \subset H_n$ can be identified with the subspace of representations in $\text{Hom}(\pi_1(S^2 \setminus K), SU(2))^{[i]}$ can be extended to $\pi_1(D_-^3 \setminus K)$, i.e.

$$\Gamma_{id} = \text{Hom}(\pi_1(D_-^3 \setminus K), SU(2))^{[i]}.$$

By Seifert, Van-Kampen Theorem, the intersection $\Gamma_\beta \cap \Gamma_{id}$ is the same as the space of representations of $\pi_1(S^3 \setminus K)$ satisfying the monodromy condition $[i]$ (see (2.1)),

$$\Gamma_\beta \cap \Gamma_{id} = \text{Hom}(\pi_1(S^3 \setminus K), SU(2))^{[i]}.$$

Given $\beta \in B_n$ with $\bar{\beta} = K$, there is an induced diffeomorphism (still denoted by β) from Q_n to itself. Such a diffeomorphism also induces a diffeomorphism $\phi_\beta : \mathcal{R}^*(S^2 \setminus K)^{[i]} \rightarrow \mathcal{R}^*(S^2 \setminus K)^{[i]}$.

Lemma 2.4. *For $\beta \in B_n$ with $\bar{\beta} = K$, the induced diffeomorphism $\phi_\beta : \mathcal{R}^*(S^2 \setminus K)^{[i]} \rightarrow \mathcal{R}^*(S^2 \setminus K)^{[i]}$ is symplectic, and the fixed point set of ϕ_β is $\mathcal{R}^*(S^3 \setminus K)^{[i]}$.*

Proof: For $\beta \in B_n$ with $\bar{\beta} = K$, there is an induced diffeomorphism (see [22]). Note that β maps each X_j to a conjugate of some $X_{j'}$, it leaves Q_n invariant, so it gives rise an area-preserving diffeomorphism of Q_n . I.e.

$$\text{Area}(S_j^2) = \text{Area}(S_{j'}^2) = \text{Area}(S_{\beta(j)}^2),$$

where $S_j^2 = \{X_j \in SU(2) | \text{tr} X_j = 0\}$. Let $\omega = \sum_{j=1}^n \omega_j$ be the symplectic form on Q_n with the symplectic form ω_j on the j -th 2-sphere by Lemma 2.2. Since $\beta \in B_n$ is a complete cycle of n -letters, so

$$\beta^*(\omega) = \sum_{j=1}^n \omega_{\beta(j)} = \omega.$$

From the identification between $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ and $(H_n \setminus S_n)/SU(2)$ in Lemma 2.3, $\beta^*\omega_0 = \omega_0$ on the $SU(2)$ -orbit, the braid β also induces a diffeomorphism ϕ_β which preserves the induced symplectic structure on $(H_n \setminus S_n)/SU(2)$:

$$\begin{aligned} \phi_\beta^*(\pi_*\omega) &= \beta^*(\omega - \omega_0) \\ &= \beta^*(\omega) - \beta^*(\omega_0) \\ &= \omega - \omega_0 = \pi_*\omega. \end{aligned}$$

Note that $\bar{\Gamma}_\beta = (\Gamma_\beta \setminus (\Gamma_\beta \cap S_n))/SU(2)$ is the graph of ϕ_β . Similarly, $\bar{\Gamma}_{id}$ can be thought as a “diagonal”. By Seifert, Van-Kempen theorem (2.3), it is clear that

$$Fix(\phi_\beta|_{\mathcal{R}^*(S^2 \setminus K)^{[i]}}) = \bar{\Gamma}_\beta \cap \bar{\Gamma}_{id} = \mathcal{R}^*(S^3 \setminus K)^{[i]}.$$

Thus we obtain the desired result. \square

Let \tilde{D}_\pm^3 be the double branched covering of D_\pm^3 with $D_\pm^3 \cap K$ as branched set. There is an induced Z_2 -action on the representation space $\mathcal{R}(\tilde{D}_\pm^3)$, and $\bar{\Gamma}_\beta = \mathcal{R}^*(D_+^3 \setminus K)^{[i]}$, $\bar{\Gamma}_{id} = \mathcal{R}^*(D_-^3 \setminus K)^{[i]}$ can be regarded as components of the fixed point set $\mathcal{R}(\tilde{D}_\pm^3)^{Z_2}$. It follows that $\mathcal{R}^*(D_+^3 \setminus K)^{[i]}$, $\mathcal{R}^*(D_-^3 \setminus K)^{[i]}$ have orientations inherited from the orientation on $\mathcal{R}(D_\pm^3)^{Z_2}$. These oriented submanifolds $\mathcal{R}^*(D_+^3 \setminus K)^{[i]}$, $\mathcal{R}^*(D_-^3 \setminus K)^{[i]}$ intersects each other in a compact subspace of $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ from Lemma 1.6 in [22]. Hence we can perturb ϕ_β (via Hamiltonian vector field if necessary), or in another words perturb $\mathcal{R}^*(D_+^3 \setminus K)^{[i]}$ to $\hat{\mathcal{R}}^*(D_+^3 \setminus K)^{[i]}$ by a compactly support isotopy so that $\hat{\mathcal{R}}^*(D_+^3 \setminus K)^{[i]}$ intersects $\mathcal{R}^*(D_-^3 \setminus K)^{[i]}$ transversally at a finite number of intersection points. Denote the perturbed symplectic diffeomorphism by $\tilde{\phi}_\beta$. So its fixed points are all nondegenerated.

Definition 2.5. The Casson-Lin invariant of a knot $K = \bar{\beta}$ is given by counting the algebraic intersection number of $\hat{\mathcal{R}}^*(D_+^3 \setminus K)^{[i]}$ and $\mathcal{R}^*(D_-^3 \setminus K)^{[i]}$, or the algebraic number of $Fix(\tilde{\phi}_\beta)$,

$$\lambda_{CL}(K) = \lambda_{CL}(\beta) = \#Fix(\tilde{\phi}_\beta) = \#(\hat{\mathcal{R}}^*(D_+^3 \setminus K)^{[i]} \cap \mathcal{R}^*(D_-^3 \setminus K)^{[i]}).$$

The results proved in [22] show that the Casson-Lin invariant $\lambda_{CL}(K) = \lambda_{CL}(\beta)$ is independent of its braid representatives, i.e. $\lambda_{CL}(\beta)$ is invariant under the Markov moves of type I and type II on β .

3. LAGRANGIAN PERTURBATIONS ON THE SINGULAR REPRESENTATION

In this section, we study the Walker type correction term around an isolated $U(1)$ reducible representation of the knot group and discuss that the compact supported perturbations used in [22] can be achieved by Lagrangian perturbations from symplectic topology point of view. Hence we get that the Casson-Lin's invariant is Casson-Walker type invariant with non correction terms.

The only singular strata with trace zero condition consist of representations with image in the $U(1)$ subgroup and trace free. Let $\mathcal{S}^*(X)^{[i]} = Hom(\pi_1(X), U(1))^{[i]}/Z_2$ be the $U(1)$ strata of X . Similar to (2.3), we have the following diagram,

$$\begin{array}{ccc} \mathcal{S}^*(S^3 \setminus K)^{[i]} & \rightarrow & \mathcal{S}^*(D_+^3 \setminus K)^{[i]} \\ \downarrow & & \downarrow \\ \mathcal{S}^*(D_-^3 \setminus K)^{[i]} & \rightarrow & \mathcal{S}^*(S^2 \setminus K)^{[i]}. \end{array} \quad (3.1)$$

From Lin's notation $S_n = \mathcal{S}^*(S^2 \setminus K)^{[i]}$ in [22], the subset of $(X_1, \dots, X_n, Y_1, \dots, Y_n) \in H_n$ such that there is a matrix $A \in SU(2)$ with property

$$A^{-1}X_iA, \quad A^{-1}Y_iA \in U(1), \quad \text{for all } i = 1, 2, \dots, n.$$

Note that $Hom(\pi_1(S^3 \setminus K), U(1)) \cong U(1)$, $\mathcal{S}^*(S^3 \setminus K)^{[i]} = U(1)/Z_2$ the upper half unit circle in the complex plane. There is only one reducible conjugacy class of representations $\rho : \pi_1(S^3 \setminus K) \rightarrow$

$U(1) \hookrightarrow SU(2)$ which satisfies the trace condition $\text{tr}(\rho([m_{x_i}])) = \text{tr}(\rho([m_{y_i}])) = 0$. Without loss of generality, we may assume that $\rho : \pi_1(S^2 \setminus K) \rightarrow U(1)$ is the diagonal matrix

$$\rho([m_{x_i}]) = \rho([m_{y_i}]) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Write $\rho = \sigma \oplus \sigma^{-1}$ where $\sigma : \pi_1(S^2 \setminus K) \rightarrow U(1)$ is given by $\sigma([m_{x_i}]) = \sigma([m_{y_i}]) = i$. Then

$$\text{ad}\rho = \text{End}^0(\sigma \oplus \sigma^{-1}) = \mathbf{R} \oplus \sigma^{\otimes 2}.$$

From deformation theory, any deformation in $H^1(\cdot, \mathbf{R})$ direction changes the trace condition. So only $H^1(\cdot, \sigma^{\otimes 2})$ fit into the Mayer-Vietoris sequence:

$$H^1(S^3 \setminus K, \sigma^{\otimes 2}) \rightarrow H^1(D_-^3 \setminus K, \sigma^{\otimes 2}) \oplus H^1(D_+^3 \setminus K, \sigma^{\otimes 2}) \rightarrow H^1(S^2 \setminus K, \sigma^{\otimes 2}) \rightarrow \dots$$

That $\mathcal{S}^*(D_+^3 \setminus K)^{[i]}$ and $\mathcal{S}^*(D_-^3 \setminus K)^{[i]}$ intersect transversally in $T_\rho \mathcal{S}^*(S^2 \setminus K)^{[i]}$ is equivalent to $H^1(S^3 \setminus K, \sigma^{\otimes 2}) = 0$. According to Milnor [27], $H^1(S^3 \setminus K, \sigma^{\otimes 2})$ can be computed by considering the cohomology of the infinite cyclic covering space $\widehat{S^3 \setminus K}$. This last cohomology $H^1(\widehat{S^3 \setminus K})$ is a module of the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$.

$$H^1(\widehat{S^3 \setminus K}) \otimes_{\mathbf{Z}[t, t^{-1}]} C = H^1(S^3 \setminus K, \sigma^{\otimes 2}).$$

Therefore $0 = H^1(\widehat{S^3 \setminus K}) = \mathbf{Z}[t, t^{-1}]/\Delta_K(t)$ if and only if $\Delta_K(-1) \neq 0$ which is true for any knot K .

Lemma 3.1 (Lin [22] Lemma 1.4). *For any $K = \overline{\beta}$, $\mathcal{S}^*(D_+^3 \setminus K)^{[i]}$ and $\mathcal{S}^*(D_-^3 \setminus K)^{[i]}$ intersect transversally in $T_s \mathcal{S}^*(S^2 \setminus K)^{[i]}$. $\mathcal{S}^*(S^3 \setminus K)^{[i]} = \{s\}$ is an isolated point of the intersection between $\mathcal{S}^*(D_+^3 \setminus K)^{[i]}$ and $\mathcal{S}^*(D_-^3 \setminus K)^{[i]}$ in $\mathcal{S}^*(S^2 \setminus K)^{[i]}$.*

Remark: The isolated point $\{s\} = \mathcal{S}^*(S^3 \setminus K)^{[i]}$ plays the same role as the trivial connection in the Casson's invariant for integral homology 3-spheres. But this $U(1)$ reducible may have special isotopy around s which reflects global twisting in the normal bundle of $\mathcal{S}^*(S^2 \setminus K)^{[i]}$ as in rational homology 3-spheres situation [31].

The vector spaces $\{H^1(S^2 \setminus K, \sigma_p^{\otimes 2})|p \in \mathcal{S}^*(S^2 \setminus K)^{[i]}\}$ form a symplectic vector bundle ν over $\mathcal{S}^*(S^2 \setminus K)^{[i]}$. There is a Hermitian structure on ν compatible with its symplectic structure ω . We call an isotopy $\{h_t\}_{0 \leq t \leq 1}$ of $\mathcal{R}(S^2 \setminus K)^{[i]}$ to be a **special** if $h_t|_{\mathcal{S}(S^2 \setminus K)^{[i]}} = \text{Id}$ for all t , and in a neighborhood of $\mathcal{S}^*(S^2 \setminus K)^{[i]}$, h_t is induced by a complex symplectic bundle automorphism of ν . Proposition 1.20 in [31] shows that there exists a special isotopies $\{h_t\}_{0 \leq t \leq 1}$ of $\mathcal{R}(S^2 \setminus K)^{[i]}$ such that $h_1(\mathcal{R}^*(D_+^3 \setminus K)^{[i]})$ is transverse to $\mathcal{R}^*(D_-^3 \setminus K)^{[i]}$.

We are going to discuss the proper perturbation around the reducible s in order to obtain the computation of $\lambda_{CL}(\beta)$. To obtain the surgery formula for $h(\beta)$, one can consider the expression

$$\lambda_{CL}(\alpha^{-1}\beta) - \lambda_{CL}(\beta) = \#\tilde{\Gamma}_{\alpha^{-1}\beta} \cap \overline{\Gamma}_{id} - \#\tilde{\Gamma}_\beta \cap \overline{\Gamma}_{id}, \quad (3.2)$$

where $\alpha^{-1} = \sigma_1^2$ a full twist on the first two strands. More generally, we can consider $\lambda_{CL}(\alpha^{-1}\beta) - \lambda_{CL}(\beta)$ without $\alpha = \sigma_1^{-2}$. However, for $\overline{\alpha^{-1}\beta}$ to represent a knot, we must have α lying in the pure braid group $\ker(\pi : B_n \rightarrow S_n)$. Since $\ker(\pi : B_n \rightarrow S_n)$ is generated by σ_1^2 and its conjugate, it

is enough to get a formula for $\lambda_{CL}(\sigma_1^2\beta) - \lambda_{CL}(\beta)$ where $\sigma_1^2\beta$ is obtained from β by doing surgery along a loop circulating the first two strands.

Given $\alpha \in B_n$, we have an induced automorphism on \overline{H}_n defined by $\alpha_* : \overline{H}_n \rightarrow \overline{H}_n$:

$$\alpha_*(X_1 \cdots X_n, Y_1 \cdots Y_n) = (X_1 \cdots X_n, \alpha(Y_1) \cdots \alpha(Y_n)).$$

Under this automorphism, we have $\alpha_*(\overline{\Gamma}_{id} \cap \overline{\Gamma}_{\alpha^{-1}\beta}) = \overline{\Gamma}_\alpha \cap \overline{\Gamma}_\beta$. As we perturb $\overline{\Gamma}_{\alpha^{-1}\beta}$ to a transverse position $\tilde{\Gamma}_{\alpha^{-1}\beta}$, its image $\alpha_*(\tilde{\Gamma}_{\alpha^{-1}\beta})$ becomes transverse to $\overline{\Gamma}_\alpha$. In addition, near the neighborhood of reducible point s , the counter clockwise motion in perturbing $\tilde{\Gamma}_{\alpha^{-1}\beta}$ is also preserved by α_* . Hence

$$\#(\overline{\Gamma}_{id} \cap \tilde{\Gamma}_{\alpha^{-1}\beta}) = \#(\overline{\Gamma}_\alpha \cap \tilde{\Gamma}_\beta). \quad (3.3)$$

Using (3.3), we can rewrite (3.2) as

$$\begin{aligned} \lambda_{CL}(\alpha^{-1}\beta) - \lambda_{CL}(\beta) &= \#(\overline{\Gamma}_{id} \cap \tilde{\Gamma}_{\alpha^{-1}\beta}) - \#(\overline{\Gamma}_{id} \cap \tilde{\Gamma}_\beta) \\ &= \#(\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}) \cap \tilde{\Gamma}_\beta. \end{aligned}$$

In some sense, $\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}$ can be thought of as a cycle except it contains the reducible point ρ and so is $\tilde{\Gamma}_\beta$. To remedy this situation, we consider the difference:

$$\begin{aligned} [\lambda_{CL}(\alpha^{-1}\beta) - \lambda_{CL}(\beta)] - [\lambda_{CL}(\alpha^{-1}\beta') - \lambda_{CL}(\beta')] &= \#(\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}) \cap \tilde{\Gamma}_\beta - \#(\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}) \cap \tilde{\Gamma}_{\beta'} \\ &= \#(\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}) \cap (\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'}). \end{aligned}$$

At a neighborhood of the reducible point s , we choose a 1-parameter family of $\tilde{\Gamma}_\beta(t)$ which brings $(\tilde{\Gamma}_{\beta'})_\rho$ to $(\tilde{\Gamma}_\beta)_\rho$. Using this isotopy, we can construct a cycle $(\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'})$ globally by taking $\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'}$ outside a neighborhood of s and in the neighborhood connecting up $\partial\tilde{\Gamma}_\beta$ to $\partial\tilde{\Gamma}_{\beta'}$, by the isotopy $\partial\tilde{\Gamma}_\beta(t)$.

As $(\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'})_{global}$ is a cycle in the nonsingular part \overline{H}_n , the intersection number

$$I(s) = \#(\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}) \cap (\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'})_{global},$$

makes sense as the intersection as a relative cohomology class with an absolute class. $I(s)$ also counts irreducibles via the global twisting around the reducible s which is essentially the Walker's correction term. Although the Walker's geometric construction of his correction term is not defined for this situation (the trace zero representations contains non-typical element such as trivial representation for rational homology 3-spheres), we can use the analytic definition defined in [6] for the Walker's correction term $I(s)$.

$$[\lambda_{CL}(\alpha^{-1}\beta) - \lambda_{CL}(\beta)] - [\lambda_{CL}(\alpha^{-1}\beta') - \lambda_{CL}(\beta')] = \#(\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}) \cap (\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'})_0 + I(s). \quad (3.4)$$

1. $\#(\overline{\Gamma}_\alpha - \overline{\Gamma}_{id}) \cap (\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'})_0$ can be expressed in turn of triple Maslov index

$$\#(\overline{\Gamma}_{id} - \overline{\Gamma}_\alpha) \cap (\tilde{\Gamma}_\beta - \tilde{\Gamma}_{\beta'})_0 = \frac{1}{2}Mas(\overline{\Gamma}_\alpha, \tilde{\Gamma}_\beta, \overline{\Gamma}_{id})_s - \frac{1}{2}Mas(\overline{\Gamma}_\alpha, \tilde{\Gamma}_{\beta'}, \overline{\Gamma}_{id})_s. \quad (3.5)$$

This is because, from the definition, $\tilde{\Gamma}_\beta, \tilde{\Gamma}_{\beta'}$, are transverse to $\overline{\Gamma}_\alpha, \overline{\Gamma}_{id}$. Hence the dimension correction terms all disappear (see [6] and [7]).

2. $\lambda_{CL}(\alpha^{-1}\beta') - \lambda_{CL}(\beta')$ in (3.4) can be eliminated by choosing β' as follows. There exists a braid β' such that both $\overline{\beta'}$ and $\overline{\sigma_1^2\beta'}$ represent the trivial knot. In the case $n = 2$, we have choose $\overline{\beta'}$ and $\overline{\sigma_1^2\beta'}$ to be $\overline{\sigma_1^{-1}}, \sigma_1$ respectively. For $n > 2$, we have $\beta' = \sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_n^{-1}$, $\overline{\sigma_1^2\beta'} = \sigma_1\sigma_2^{-1}\cdots\sigma_n^{-1}$. Thus $\lambda_{CL}(\alpha^{-1}\beta') - \lambda_{CL}(\beta') = 0$ and $Mas(\overline{\Gamma}_\alpha, \tilde{\Gamma}_{\beta'}, \overline{\Gamma}_{id})_s = 0$. (3.4) is reduced to

$$\lambda_{CL}(\alpha^{-1}\beta) - \lambda_{CL}(\beta) = \frac{1}{2}Mas(\overline{\Gamma}_\alpha, \tilde{\Gamma}_\beta, \overline{\Gamma}_{id})_s + I(s).$$

3. Using Wall's nonadditivity of signature, Cappell, Lee and Miller in [6] identified the triple Maslov index with the (twisted) signature. So (3.4) becomes the following.

$$\lambda_{CL}(\alpha^{-1}\beta) - \lambda_{CL}(\beta) = \frac{1}{2}\{sign(\overline{\sigma_1^2\beta}) - sign(\overline{\beta})\} + I(s). \quad (3.6)$$

4. $I(s) = 0$ and the signature of $\overline{\sigma_1^2}$ is clear zero.

$$\lambda_{CL}(\beta) = \frac{1}{2}sign(\overline{\beta}; [i^2]) = \frac{1}{2}sign(\overline{\beta}).$$

This is Corollary 2.10 in [22]. In the following, we show that the Walker-type correction term $I(s)$ vanishes in two different methods.

Lemma 3.2. *For $\alpha = \sigma_1^{-2}$, $I(s) = 0$. (see also [6])*

Proof: For $\alpha = \sigma_1^{-2}$, we have

$$\begin{aligned} \alpha(X_1) &= (X_1X_2)^{-1}X_1(X_1X_2) \\ \alpha(X_2) &= (X_1X_2)^{-1}X_2(X_1X_2) \\ \alpha(X_j) &= X_j, \quad j \geq 3. \end{aligned}$$

There is a Casson-deformation X^t from $SU(2) \setminus \{-I\}$ to itself such that X^t commutes with X ,

$$\begin{aligned} \phi(t)(X_1) &= [(X_1X_2)^t]^{-1}X_1[(X_1X_2)^t] \\ \phi(t)(X_2) &= [(X_1X_2)^t]^{-1}X_2[(X_1X_2)^t] \\ \phi(t)(X_j) &= X_j, \quad j \geq 3. \end{aligned}$$

Since trace $tr(\phi(t)(X_j)) = tr(X_j)$ for all j , the formula

$$\overline{\Gamma}_{\phi(t)} = \{(X_1, \dots, X_n, \phi(t)(X_1), \dots, \phi(t)(X_n)) \in Q_n \times Q_n\} / SU(2),$$

gives us a 1-parameter family of subspaces which connects up $\Gamma_{id}/SU(2)$ at $t = 0$ to $\Gamma_\alpha/SU(2)$ at $t = 1$. As $\phi(t)(X_1X_2) = [(X_1X_2)^t]^{-1}(X_1X_2)[(X_1X_2)^t] = X_1X_2$, $\phi(t)(X_j) = X_j, j \geq 3$ we have

$$\phi(t)(X_1X_2 \cdots X_n) = (X_1X_2 \cdots X_n),$$

and so $\Gamma_{\phi(t)}$ is a subspace in $\mathcal{R}(S^2 \setminus K)^{[i]}$. By definition of the Casson-deformation,

$$\phi(t)|_{S^*(S^2 \setminus K)^{[i]}} = id,$$

so $\phi(t)$ is a special isotopy. It follows that $\{\Gamma_{\phi(t)} : 0 \leq t \leq 1\}$ gives us a cycle in $\mathcal{R}(S^2 \setminus K)^{[i]}$ whose boundary is $\Gamma_{id}/SU(2) - \Gamma_{\alpha}/SU(2) = \mathcal{R}(D_-^3 \setminus K)^{[i]} - \Gamma_{\alpha}/SU(2)$. Thus

$$\begin{aligned} -I(s) &= \#(\mathcal{R}(D_-^3 \setminus K)^{[i]} - \Gamma_{\alpha}/SU(2)) \cap (\tilde{\Gamma}_{\beta} - \tilde{\Gamma}_{\beta'})_{global} \\ &= \#(\partial\{\Gamma_{\phi(t)} : 0 \leq t \leq 1\} \cap (\tilde{\Gamma}_{\beta} - \tilde{\Gamma}_{\beta'})_{global}) \\ &= 0. \end{aligned}$$

□

Using spectral flows in [7], the Walker correction term $I(s)$ for $s = \sigma \oplus \sigma^{-1}$, can be computed as the rho invariant $\rho(M, \sigma^{\otimes 2})$ (see Proposition 7.1 in [7] Part III).

Theorem 3.3 (Cappell, Lee and Miller [7]). *Let $s = \sigma \oplus \sigma^{-1}$ be a $U(1)$ representation of $\pi_1(M)$ for the rational homology 3-sphere M . Then the Walker's correction term $I(s)$ is given by*

$$I(s) = -\rho(M, \sigma^{\otimes 2}),$$

where $\rho(M, \sigma^{\otimes 2})$ is the rho invariant of M associated to the representation $\sigma^{\otimes 2}$.

We take the above as analytic definition for the Walker-type invariant. Applying Theorem 3.3 to $M = \hat{S}^3$ the double branched cover of S^3 with branched set K and $s = \sigma \oplus \sigma^{-1}$ which is the fixed point of Z_2 operation on $\mathcal{S}(\hat{S}^3)^{Z_2}$, we have that $ads = \mathbf{R} \oplus \sigma^{\otimes 2}$ acts by 1 on the real part of $\mathbf{su}(2)$ and acts by $\sigma^{\otimes 2} = -1$ on the complex part of $\mathbf{su}(2)$. Thus for $M = \partial X^4$, $\sigma^{\otimes 2} = -1$ is trivially extended to a unitary representation of $\pi_1(X)$. By Theorem 2.4 [2],

$$\begin{aligned} I(s) &= -\rho(M, \sigma^2) \\ &= -(sign_{Id}(X) - sign_{\sigma^{\otimes 2}}(X)) \\ &= 0. \end{aligned}$$

Note that the signature of Hermitian form $x^* \Gamma_z y$ with $\Gamma_z = \frac{1}{2}\{(1 - \bar{z})\Gamma + (1 - z)\Gamma\}$ has the relation $sign_{\xi} = sign(1 - Re\xi)sign_z$ for $z = \frac{1-\xi}{1-\bar{\xi}}$, the only interesting case is when $|z| = 1$, where Γ_z reduces to $\frac{1}{2}(1 - \bar{z})\Gamma$. In particular for $z = -1$, the signature is the same as $sign_{Id}$. See [25] for example.

Let $\rho \in \mathcal{S}^*(S^2 \setminus K)^{[i]}$ be the $U(1)$ reducible representation of $\pi_1(S^2 \setminus K)$ with trace zero condition. A normal neighborhood of ρ can be isomorphic to the cone of $H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^{\perp})/U(1)$, (\mathbf{h} is the Lie algebra of the fixed oriented maximal torus of $SU(2)$, \mathbf{h}^{\perp} is the orthogonal complement of \mathbf{h} with respect to the Killing form of $SU(2)$), the cone bundle $E(\mathcal{S}^*(S^2 \setminus K)^{[i]}) \rightarrow \mathcal{S}^*(S^2 \setminus K)^{[i]}$ is isomorphic to a neighborhood of $\mathcal{S}^*(S^2 \setminus K)^{[i]}$ in $\mathcal{R}(S^2 \setminus K)^{[i]}$ via an exponential map $\exp : \mathcal{N}(\mathcal{S}^*(S^2 \setminus K)^{[i]}) \rightarrow Q_n \times Q_n$. The exponential map allows us to identify a germ of functions $\phi \in C^{\infty}(Q_n \times Q_n, \mathbf{R})$ near $\mathcal{S}^*(S^2 \setminus K)^{[i]}$ with a function on $\mathcal{N}(\mathcal{S}^*(S^2 \setminus K)^{[i]})$. For example, choose a partition of unity $\chi : Q_n \times Q_n \rightarrow \mathbf{R}$ which is 0 outside $\mathcal{N}(\mathcal{S}^*(S^2 \setminus K)^{[i]})$ and 1 near the 0-section and consider a function $g : \mathcal{N}(\mathcal{S}^*(S^2 \setminus K)^{[i]}) \rightarrow \mathbf{R}$ which is induced by a Hermitian pairing on each fiber $H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^{\perp})/U(1)$. Locally, $g(\rho, z) = \sum a_{ij}(\rho) z_i \bar{z}_j$, $z = (z_1, \dots, z_d) \in H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^{\perp})/U(1)$,

$$H_g(\exp(\rho, z)) = \chi(\rho, z)g(\rho, z) \quad (3.7)$$

gives us a function $H_g \in C^\infty(Q_n \times Q_n, \mathbf{R})$. The vector field $\text{grad}H_g$ associated to this perturbation has the property that $g(\rho, z)$ is quadratic in the normal z -direction and so $\text{grad}H_g = 0$ when it is restricted to the zero section, the Hessian $\text{Hess}(H_g)$ of H_g at each fiber $H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^\perp)/U(1)$ is given by the Hermitian pairing $\sum a_{ij}(\rho) z_i \bar{z}_j$.

Let $\mathcal{Lag}(W)$ be the space of complex Lagrangians in $W = H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^\perp)$. $\mathcal{Lag}(W)$ is a homogeneous manifold whose tangent space can be identified with the space of Hermitian pairings on $H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^\perp)$ (c.f. [31] p13-14). We consider the open subspace Π_0 of perturbation data in Π such that

1. $\text{grad}H_g(\rho) = 0$ for $\rho \in \mathcal{S}^*(S^2 \setminus K)^{[i]}$,
2. $(\phi_\beta + H_g)(\bar{\Gamma}_{id})$ is mapped injectively into a complex Lagrangian subspace in $H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^\perp)$.

Let $\text{Lag}(\mathcal{W})$ denote the smooth fiber bundle over $\mathcal{S}^*(S^2 \setminus K)^{[i]}$ whose fiber at a point $\rho \in \mathcal{S}^*(S^2 \setminus K)^{[i]}$ is the homogeneous space $\mathcal{Lag}(W)$, and $\Gamma(\text{Lag}(\mathcal{W}))$ denote the space of smooth sections of this bundle completed with respect to an appropriate Sobolev norm. Then by assigning to π the section of Lagrangian $\rho \mapsto (\phi_\beta + H_g)(\bar{\Gamma}_{id})$ in $H^1(S^2 \setminus K, \mathbf{h}_{Ad\rho}^\perp)$ there exists a mapping $p : \Pi_0 \rightarrow \Gamma(\text{Lag}(\mathcal{W}))$.

Proposition 3.4. (i) *The mapping $p : \Pi_0 \rightarrow \Gamma(\text{Lag}(\mathcal{W}))$ is a submersion whose image is the entire space $\Gamma(\text{Lag}(\mathcal{W}))$.*

(ii) *Π_0 is path connected, i.e. any two different perturbations can be homotopically connected to each other through a 1-parameter family in Π_0 .*

Proof: The argument follows exactly as in the proof of Theorem B and Proposition 3.3 in [20]. Since the correction term $I(s)$ vanishes, there is non obstruction to connect any two different perturbations. \square

Now the Walker-type correction term vanishing implies that the compact support perturbation H_g does not effect the invariant $\lambda_{CL}(\beta)$. Also the Floer homology defined in the next section will be independent of the Lagrangian perturbation around the reducible s , unlike the situation studied in [20]. So we can choose the counter clock orientation (pick H_g the quadratic form in the positive direction) to perturb $\mathcal{R}^*(D_+^3 \setminus K)^{[i]}$ (as graph of ϕ_β) into $\hat{\mathcal{R}}^*(D_+^3 \setminus K)^{[i]}$ (the graph of $\tilde{\phi}_\beta$) which meets $\mathcal{R}^*(D_-^3 \setminus K)^{[i]}$ transversely. Hence the Casson-Lin's invariant $\lambda_{CL}(\beta)$ is also the Casson-Walker type invariant with trivial correction term $I(s)$.

4. THE SYMPLECTIC FLOER HOMOLOGY OF A BRAID

4.1. Floer homology for symplectic fixed points. Let $H : \mathbf{R} \times M \rightarrow \mathbf{R}$, $H(s, x) = H_s(x)$ be a smooth time-dependent Hamiltonian function such that $H_s = H_{s+1} \circ \phi$. The symplectic diffeomorphism $\psi_s : M \rightarrow M$ generated by H are defined by

$$\begin{aligned} \frac{d\psi_s}{ds} &= X_{H,s} \circ \psi_s, \\ \psi_0 &= Id, \quad \omega(X_{H,s}, \cdot) = dH_s. \end{aligned}$$

They satisfy $\psi_{s+1} \circ \phi_H = \phi \circ \psi_s$, where $\phi_H = \psi_1^{-1} \circ \phi$. For a generic Hamiltonian H_s , the fixed points of ϕ_H are all nondegenerate ([11], [16]), denoted by $Fix(\phi_H)$. They can be identified with the critical points of the perturbed symplectic action functional on the space of smooth paths

$$\Omega_\phi = \{\gamma : \mathbf{R} \rightarrow M \mid \gamma(s+1) = \phi(\gamma(s))\}.$$

If M is simply connected, then $\pi_1(\Omega_\phi) = \pi_2(M)$; otherwise, we will apply the construction to components of contractible loops of M . The perturbed symplectic action functional $a_H : \Omega_\phi \rightarrow \mathbf{R}/\alpha\mathbf{Z}$ is defined as a function whose differential is

$$da_H(\gamma)\xi = \int_0^1 \omega(\dot{\gamma} - X_s(\gamma), \xi) ds.$$

So the critical points of a_H are the paths of the form $x(s) = \psi_s(x_0)$ such that $x(s+1) = \phi(x(s))$ which are in one-to-one correspondence with the fixed point set $Fix(\phi_H)$.

Now choose a smooth map $\mathbf{R} \rightarrow \mathcal{J}(M, \omega) : s \mapsto J_s$ such that $J_s = \phi^* J_{s+1}$. Such a structure determines a L^2 -metric on Ω_ϕ and the gradient of a_H . In particular, the gradient flow of a_H can be identified with the solution $u : \mathbf{R}^2 \rightarrow M$ of the pseudoholomorphic curves

$$\bar{\partial}_{J,H}(u) = \partial_t u + J_s(u)(\partial_s u - X_s(u)) = 0, \quad (4.1)$$

$$u(s+1, t) = \phi(u(s, t)), \quad (4.2)$$

in the sense of Gromov (see [14] for $X_s = 0$ case and [11] for general case). If all fixed points in $Fix(\phi_H)$ are nondegenerate, then any solution of (4.1) and (4.2) with finite energy

$$E(u) = \frac{1}{2} \int_{\mathbf{R}} \int_0^1 (|\partial_s u - X_s(u)|^2 + |\partial_t u|^2) ds dt < \infty \quad (4.3)$$

has limit, by Gromov's compactness (see [11] and [14]),

$$\lim_{t \rightarrow \pm\infty} u(s, t) = \psi_s(x^\pm), \quad x^\pm = \phi_H(x^\pm), \quad (4.4)$$

Let $\mathcal{M}(x, y)$ be the space of solutions u of (4.1), (4.2) and (4.3) with $x, y \in Fix(\phi_H)$. This gives a perturbed Cauchy-Riemann operator $D_u : L_{1,\phi}^p(u^*TM) \rightarrow L_{0,\phi}^p(u^*TM)$ defined by

$$D_u \xi = \nabla_t \xi + J_s(u)(\nabla_s \xi - \nabla_\xi X_s(u)) + \nabla_\xi J_s(u)(\partial_s u - X_s(u)), \quad (4.5)$$

where ∇ denotes the covariant derivative with respect to the s -dependent metric $\langle v_1, v_2 \rangle_s = \omega(v_1, J_s v_2)$, and $L_{0,\phi}^p(u^*TM)$ (respectively $L_{1,\phi}^p(u^*TM)$) is the completions of the space of smooth vector fields $\xi(s, t) \in T_{u(s,t)}M$ along u with $\xi(s+1, t) = d\phi(u(s, t))\xi(s, t)$ and compact support on $S^1 \times \mathbf{R}$, with respect to the L^p -norm (respectively L_1^p -norm) on $S^1 \times \mathbf{R}$. If $x, y \in Fix(\phi_H)$ are nondegenerate and u satisfy (4.1), (4.2) and (4.3), then D_u is Fredholm and its index is given by the Maslov class of u :

$$index D_u = \mu_u(x, y).$$

The Maslov class μ_u is invariant under homotopy, additive for catenations and satisfies

$$\mu_{u\#v} = \mu_u - 2c_1(v), \quad (4.6)$$

for $v \in \pi_2(M)$ by [11] Proposition 1b. A Hamiltonian function H is called regular if the fixed point of ϕ_H are all nondegenerate and the operator D_u is onto for every $u \in \mathcal{M}(x, y)$ and $x, y \in \text{Fix}(\phi_H)$. Floer in [11] showed that the set $\mathcal{H}^{reg} = \mathcal{H}^{reg}(J)$ of regular Hamiltonian functions is generic in the sense of Baire with respect to a suitable C_ε^∞ -topology. For $H \in \mathcal{H}^{reg}$, $x, y \in \text{Fix}(\phi_H)$, the space $\mathcal{M}(x, y)$ is a manifold whose local dimension near u is the Maslov class $\mu_u(x, y)$. By (4.6),

$$\mu : \text{Fix}(\phi_H) \rightarrow \mathbf{Z}_{2N},$$

defined up to an additive constant, such that

$$\mu_u(x, y) = \mu(x) - \mu(y) \pmod{2N},$$

for every $u \in \mathcal{M}(x, y)$. Here the integer N is the minimal Chern number defined by $I_{c_1} : \pi_2(M) \rightarrow \mathbf{Z}$, $\text{Im} I_{c_1} = \langle N \rangle$ since \mathbf{Z} is PID and every ideal is generated by an element in \mathbf{Z} .

Let C_i be the free generated \mathbf{Z} module over $x \in \text{Fix}(\phi_H)$ and $\mu(x) \equiv i \pmod{2N}$. Then the Floer boundary map

$$\partial : C_i \rightarrow C_{i-1}, \quad x \mapsto \sum_{\mu(x) - \mu(y) = 1} \# \hat{\mathcal{M}}(x, y) y, \quad (4.7)$$

is defined by taking the sum of the algebraic number over all one dimensional components of $\hat{\mathcal{M}}(x, y)$, where $\hat{\mathcal{M}}(x, y)$ is the balanced moduli space of J -holomorphic curves, see also [11] and [20]. These numbers are defined by comparing the flow orientation of u with the coherent orientation of $\mathcal{M}(x, y)$ as in [12]. One can check that ∂ is well-defined and $\partial \circ \partial = 0$ as Floer did in [11] for $\phi = id$ case. The homology of this chain complex (C_i, ∂) is defined to be the Floer homology for symplectic fixed points. I.e.

$$HF_*^{sym}(M, \phi, H, J) = \ker \partial / \text{Im} \partial, \quad * \in \mathbf{Z}_{2N}.$$

The Floer homology groups are independent of the almost complex structures J_s and the perturbation $H \in \mathcal{H}^{reg}$ used in the construction [11]. They depend on ϕ only up to Hamiltonian isotopy. There is a natural isomorphism

$$HF_*^{sym}(M, \phi^0, H^0, J^0) = HF_*^{sym}(M, \phi^1, H^1, J^1),$$

whenever ϕ^0 and ϕ^1 are related by a Hamiltonian isotopy.

For the monotone symplectic manifold (M, ω) , there are $2N$ -graded Floer homology groups $HF_*^{sym}(\phi)$ for every symplectic diffeomorphism ϕ , where N is the minimal Chern number. The Euler characteristic of the Floer homology is the Lefschetz number of ϕ ,

$$\chi(HF_*^{sym}(M, \phi)) = L(\phi).$$

If there are degenerated fixed points, then one can choose a Hamiltonian perturbation. For any two symplectic diffeomorphisms ϕ, ψ , there is a natural isomorphism

$$HF_*^{sym}(\phi) \xrightarrow{\cong} HF_*^{sym}(\psi \circ \phi \circ \psi^{-1}). \quad (4.8)$$

(see [26] for the so called Donaldson quantum category)

Note that the symplectic Floer homology is essentially an infinite dimensional version of Morse-Novikov theory for the symplectic action functional. The unique covering space $\tilde{\Omega}_\phi$ of the space Ω_ϕ of the contractible loops has deck-transformation group $\Gamma = \text{Im}(\pi_2(M) \rightarrow H_2(M))$. The symplectic action functional is well-defined over $\tilde{\Omega}_\phi$, then applying the Novikov's construction, one obtains the symplectic Floer homology as a module over the Novikov ring $\Lambda_\omega(\Gamma)$ (see [26] and [18]). For our case, we do not need to extend the symplectic Floer homology with Novikov ring coefficients.

4.2. Floer homology of braids. In this subsection, we are going to apply the construction in §4.1 to the monotone symplectic manifold $\mathcal{R}^*(S^2 \setminus K)^{[i]}$ and the symplectic diffeomorphism ϕ_β induced from a braid β .

For any knot $K = \overline{\beta}$, $\beta \in B_n$ a braid, we apply the previous section to Ω_{ϕ_β} the space of contractible loops in $\mathcal{R}^*(S^2 \setminus K)^{[i]}$. If there are degenerated fixed points of ϕ_β , then we apply the Lagrangian perturbation around s , as discussed in §3, it gives arise a Hamiltonian vector field perturbation to make all the fixed points of ϕ_β to be nondegenerated. Instead of relative index with shift, we can assign an relative index for each element in $\text{Fix}(\tilde{\phi}_\beta)$ with respect to the $U(1)$ -representation s , $\mu(x) = \mu_u(x, s) \pmod{2N}$. Thus the symplectic Floer chain complex is

$$C_i = \{x \in \text{Fix}(\tilde{\phi}_\beta) \cap \mathcal{R}^*(S^2 \setminus K)^{[i]} : \mu(x) = i \in \mathbb{Z}_{2N}\}.$$

Proposition 4.1. *There is a well-defined symplectic Floer homology $HF_*^{\text{sym}}(\phi_\beta, H, J), * \in \mathbb{Z}_{2N}$ of a braid representing a knot.*

Proof: This basically follows from argument in [11] or [19]. We first show that ∂ is well-defined and does not involved the $U(1)$ reducible representation $s \in \text{Fix}(\tilde{\phi}_\beta)$. Recall the special isotopy does fix the $U(1)$ -strata and $\{s\} = \mathcal{S}^*(S^3 \setminus K)^{[i]}$. Because any trajectories connecting with s always come in at least two dimensional families. Followed the method in [11] p 587, we can ensure that these two dimensional families belong to regular Morse cells and their index is at least 2, so that s does not contribute to ∂ . In order to show $\partial \circ \partial = 0$, we consider the term for a fixed y in the following.

$$\partial \circ \partial(x) = \sum_{z \in C_{i-1}} \sum_{y \in C_{i-2}} \#\hat{\mathcal{M}}(x, z) \cdot \#\hat{\mathcal{M}}(z, y)y.$$

For the pair (x, y) , there is the two dimensional moduli space of J -holomorphic curves $\mathcal{M}^2(x, y)$. The ends of $\hat{\mathcal{M}}(x, y)$ consists of all the components $\hat{\mathcal{M}}(x, z) \times \hat{\mathcal{M}}(z, y)$ for $z \in \text{Fix}(\tilde{\phi}_\beta) \cap \mathcal{R}^*(S^2 \setminus K)^{[i]}$. It is impossible for z to be the reducible s because the $U(1)$ symmetry group would add to the extra one more parameter families inside the moduli space. Hence the standard argument shows that $\partial \circ \partial = 0$. Another way to make sure that the reducible representation s does not contribute is to apply the gluing technique developed in [20] for the $U(1)$ reducible representations. See [20] §4. \square

In order to get an invariant of knots from braids, we have to verify that $HF_*^{\text{sym}}(\phi_\beta, H, J)$ (independent of (H, J)) is invariant under Markov moves. A Markov move of type I changes $\sigma \in B_n$ to $\xi^{-1}\sigma\xi \in B_n$ for any $\xi \in B_n$, and the Markov move of type II changes $\sigma \in B_n$ to $\sigma_n^\pm \sigma \in B_{n+1}$, or

the inverses of these operations. It is well-known that two braids β_1 and β_2 has isotopic closure if and only if β_1 can be changed to β_2 by a sequence of finitely many Markov moves [3].

Theorem 4.2. *For $\overline{\beta_1} = \overline{\beta_2} = K$ as a knot, $\beta_1 \in B_n, \beta_2 \in B_m$, then there is a natural isomorphism*

$$HF_i^{\text{sym}}(\phi_{\beta_1}) \cong HF_i^{\text{sym}}(\phi_{\beta_2}), \quad i \in \mathbb{Z}_{2N}.$$

So the symplectic Floer homology $\{HF_i^{\text{sym}}(\phi_\beta)\}_{i \in \mathbb{Z}_{2N}}$ is a knot invariant.

Proof: We only need to show that for $\beta \in B_n$ with $\overline{\beta}$ being a knot K , the Markov moves of type I and type II on β provide a Hamiltonian isotopy of ϕ_β . Hence from the invariance property of the symplectic Floer homology, we get that $\{HF_i^{\text{sym}}(\phi_\beta)\}_{i \in \mathbb{Z}_{2N}}$ is an invariant of knot $K = \overline{\beta}$.

Suppose we have the Markov move of type I: change β to $\xi^{-1}\beta\xi$ for some $\xi \in B_n$. The element ξ in B_n induces a diffeomorphism $\xi : Q_n \rightarrow Q_n$ is orientation preserving as observed by Lin in [22]. Note that B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$. For any σ_i^\pm , the induced diffeomorphism $\sigma_i^\pm \times \sigma_i^\pm : Q_n \times Q_n \rightarrow Q_n \times Q_n$ is an orientation preserving and symplectic diffeomorphism. So ξ is also a symplectic orientation preserving diffeomorphism since orientation preserving and symplectic form preserving properties are invariant under the composition operation. Hence there is a symplectic diffeomorphism

$$\xi \times \xi : Q_n \times Q_n \rightarrow Q_n \times Q_n,$$

which commutes with the $SU(2)$ -action and

$$\begin{aligned} \xi \times \xi(\mathcal{R}^*(S^2 \setminus K)^{[i]}) &= \mathcal{R}^*(S^2 \setminus K)^{[i]} \quad (\text{changing variables by } \xi \times \xi), \\ \xi \times \xi(\mathcal{R}^*(D_-^3 \setminus K)^{[i]}) &= \mathcal{R}^*(D_-^3 \setminus K)^{[i]} \quad (\text{in new coordinate } \xi(X_1), \dots, \xi(X_n)), \\ \xi \times \xi(\mathcal{R}^*(D_+^3 \setminus K)^{[i]}) &= \mathcal{R}^*(D_+^3 \setminus K)^{[i]} \quad (\text{in new coordinate } \xi(X_1), \dots, \xi(X_n)), \end{aligned}$$

as oriented manifolds. Let $f_\xi : \mathcal{R}^*(S^2 \setminus K)^{[i]} \rightarrow \mathcal{R}^*(S^2 \setminus K)^{[i]}$ be the induced symplectic diffeomorphism of $\xi \times \xi$. Hence we have

$$\phi_\beta = f_\xi \circ \phi_{\xi^{-1}\beta\xi} \circ f_\xi^{-1},$$

from changing variables via f_ξ . Note that $Fix(\phi_{\xi^{-1}\beta\xi})$ is identified with $Fix(\phi_\beta)$ under f_ξ . So we get $\tilde{\phi}_\beta = f_\xi \circ \tilde{\phi}_{\xi^{-1}\beta\xi} \circ f_\xi^{-1}$ if necessary for Hamiltonian perturbations. Therefore by (4.8),

$$HF_i^{\text{sym}}(\phi_{\xi^{-1}\beta\xi}) \cong HF_i^{\text{sym}}(f_\xi \circ \phi_{\xi^{-1}\beta\xi} \circ f_\xi^{-1}) = HF_i^{\text{sym}}(\phi_\beta). \quad (4.9)$$

It is clear that the argument goes through for the inverse operation of Markov move of type I.

Suppose we have the Markov move of type II: change β to $\sigma_n\beta \in B_{n+1}$. Recall that $\sigma_n(x_i) = x_i, 1 \leq i \leq n-1, \sigma_n(x_n) = x_n x_{n+1} x_n^{-1}$ and $\sigma_n(x_{n+1}) = x_n$. We need to identify the Floer homology from the construction in \hat{H}_n into the one from \hat{H}_{n+1} . Following Lin [22], there is an imbedding $g : Q_N \times Q_n \rightarrow Q_{n+1} \times Q_{n+1}$ given by

$$g(X_1, \dots, X_n, Y_1, \dots, Y_n) = (X_1, \dots, X_n, Y_n, Y_1, \dots, Y_n, Y_n).$$

Such an imbedding commutes with the $SU(2)$ -action and $g(H_n) \subset H_{n+1}$, and induces an imbedding

$$\hat{g} : \hat{H}_n (= \mathcal{R}^*(S^2 \setminus \overline{\beta})^{[i]}) \rightarrow \hat{H}_{n+1} (= \mathcal{R}^*(S^2 \setminus \overline{\sigma_n\beta})^{[i]}).$$

Note that the symplectic structure of \hat{H}_{n+1} restricted on $\hat{g}(\hat{H}_n)$ is the symplectic structure on \hat{H}_n . Hence \hat{g} is a symplectic imbedding which makes the space \hat{H}_n naturally into the symplectic submanifold $\hat{g}(\hat{H}_n)$ of \hat{H}_{n+1} . Under this imbedding, we have $\hat{g}(\phi_\beta) : \hat{H}_{n+1} \rightarrow \hat{H}_{n+1}$ is given by

$$(X_1, \dots, X_n, X_1, \dots, X_n) \mapsto (X_1, \dots, X_n, \beta(X_n), \beta(X_1), \dots, \beta(X_n), \beta(X_n)). \quad (4.10)$$

The image of $\hat{g}(\phi_\beta)$ is invariant under the operation of σ_n . Also the corresponding symplectic diffeomorphism $\phi_{\sigma_n\beta}$ is given by

$$\begin{aligned} & \phi_{\sigma_n\beta}(X_1, \dots, X_n, X_{n+1}, X_1, \dots, X_n, X_{n+1}) \\ &= (X_1, \dots, X_{n+1}, \beta(X_1), \dots, \beta(X_{n-1}), \beta(X_n)X_{n+1}\beta(X_n)^{-1}, \beta(X_n)). \end{aligned} \quad (4.11)$$

Thus we have

$$\hat{g}(\mathcal{R}^*(D_-^3 \setminus \overline{\beta})^{[i]}) \subset \mathcal{R}^*(D_-^3 \setminus \overline{\sigma_n\beta})^{[i]}, \quad \hat{g}(\mathcal{R}^*(D_+^3 \setminus \overline{\beta})^{[i]}) \subset \mathcal{R}^*(D_+^3 \setminus \overline{\sigma_n\beta})^{[i]}.$$

The fixed points of $\phi_{\sigma_n\beta}$ are elements

$$\beta(X_i) = X_i, 1 \leq i \leq n_1; \quad \beta(X_n)X_{n+1}\beta(X_n)^{-1} = X_n, \quad \beta(X_n) = X_{n+1},$$

which is equivalent to $\beta(X_i) = X_i, 1 \leq i \leq n$, i.e.

$$Fix(\phi_{\sigma_n\beta}) = Fix(\hat{g}(\phi_\beta)) = Fix(\phi_\beta).$$

For degenerate fixed point of ϕ_β , we perturb $\mathcal{R}^*(D_+^3 \setminus \overline{\beta})^{[i]}$ via Hamiltonian vector field with compact support in \hat{H}_n so that all elements in $Fix(\tilde{\phi}_\beta)$ are nondegenerated. By the standard isotopy extension argument, we can further perturb $\mathcal{R}^*(D_+^3 \setminus \overline{\sigma_n\beta})^{[i]}$ ($\supset \hat{g}(\mathcal{R}^*(D_+^3 \setminus \overline{\beta})^{[i]})$) with compact support in \hat{H}_{n+1} such that

$$\tilde{\mathcal{R}}^*(D_+^3 \setminus \overline{\sigma_n\beta})^{[i]} \cap \hat{g}(\hat{H}_n) = \hat{g}(\tilde{\mathcal{R}}^*(D_+^3 \setminus \overline{\beta})^{[i]}),$$

and all elements in $Fix(\tilde{\phi}_{\sigma_n\beta})$ are nondegenerated. So we have

$$Fix(\tilde{\phi}_{\sigma_n\beta}) = Fix(\tilde{\phi}_\beta).$$

In [22], Lin verified that orientations involved in the process are preserved. Hence the algebraic number of $Fix(\tilde{\phi}_{\sigma_n\beta})$ equals to one of $Fix(\tilde{\phi}_\beta)$. The corresponding relative Maslov indexes are also unchanged under \hat{g} .

Take the standard 2-sphere $S^2 \subset \mathbf{R}^3$ and the height function $H = x_3$ (determined a vector field X_H by $\omega(X_H, \cdot) = dH$). The level sets are circles at constant height and the Hamiltonian flow ϕ_H^t rotates each circle at constant speed

$$\frac{d\phi_H^t}{dt} = X_H \circ \phi_H^t, \quad \phi_H^0 = id.$$

Thus ϕ_H^t is simply the rotation of the 2-sphere about its axis x_3 through the angle t . For any two elements $\beta(X_n), X_{n+1}$ in S^2 , there is a great circle connecting them with direction l which is the normal direction to the disk spanned by the great circle. Hence the rotation about the axis l

generates a Hamiltonian flow η_t which connects $\beta(X_n), X_{n+1}$ at t_0 and t_1 . Using this Hamiltonian flow, we get

$$\begin{aligned} & \psi_t(X_1, \dots, X_{n+1}, X_1, \dots, X_{n+1}) \\ &= (X_1, \dots, X_n, \eta_t(X_{n+1}), \beta(X_1), \dots, \beta(X_{n-1}), \beta(X_n)\eta_t(X_{n+1})\beta(X_n)^{-1}, \beta(X_n)) \end{aligned}$$

from $H_{n+1} \rightarrow H_{n+1}$, clearly it satisfies (2.6) and commutes with $SU(2)$ action. Note that the image of a reducible point in H_{n+1} under ψ_t is also reducible. So ψ_t maps S_{n+1} to itself. Thus we get a Hamiltonian isotopy $\psi_t : \hat{H}_{n+1} (= (H_{n+1} \setminus S_{n+1})/SU(2)) \rightarrow \hat{H}_{n+1}$ between $\psi_{t_0} = \hat{g}(\phi_\beta)$ by (4.10) and $\psi_{t_1} = \phi_{\sigma_n\beta}$ by (4.11). So there is a natural isomorphism

$$HF_i^{\text{sym}}(\phi_{\sigma_n\beta}) \cong HF_i^{\text{sym}}(\hat{g}(\phi_\beta)) = HF_i^{\text{sym}}(\phi_\beta), \quad i \in \mathbb{Z}_{2N}. \quad (4.12)$$

The first isomorphism is from the invariance property of the symplectic Floer homology under the Hamiltonian isotopy ψ_t and the second from the natural identification. We can similarly prove that

$$HF_i^{\text{sym}}(\phi_{\sigma_n^{-1}\beta}) \cong HF_i^{\text{sym}}(\phi_\beta), \quad i \in \mathbb{Z}_{2N}.$$

Combining (4.9), (4.12) and the above discussion, we obtain the desired result. \square

Corollary 4.3. *As in Theorem 4.2, we have the Euler characteristic $\chi(HF_*^{\text{sym}}(\phi_\beta))$ of the symplectic Floer homology*

$$\chi(HF_*^{\text{sym}}(\phi_\beta)) = -\lambda_{CL}(K) = \frac{1}{2} \text{sign}(K).$$

The sign in Corollary 4.3 will be fixed by an example in §5. For a knot K , its Gordian number $u(K)$ (also called the unknotting number) is the smallest number of times that the string must be allowed to pass through itself if the knot is to be changed to the unknot. Murasugi (Theorem 10.1 in [29]) showed that the absolute value of the signature of a knot is not greater than twice the unknotting number. Hence combining with Corollary 4.3, we have

$$|\chi(HF_*^{\text{sym}}(\phi_\beta))| \leq u(K), \quad \text{for } K = \overline{\beta}. \quad (4.13)$$

The equality holds for the $(2, q)$ torus knot (see §5). We end this section by several remarks which we will discuss them in details elsewhere.

Remarks: (1) It is natural to ask if there is an instanton Floer homology of braids which is the original motif of the present paper. In particular, based on the work in [15], one may construct an instanton Floer homology (like in [19]) which is isomorphic to the one we defined in this section. Note that there is a period 4 instanton Floer homology of knots in [10] and [4] Part II.

(2) The set-up in [6] provides a possible relation between the instanton Floer homology of integral homology 3-spheres and the symplectic Floer homology of braids.

(3) In [28], D. Mullins discussed the Casson-Walker invariant for 2-fold branched covers of S^3 . His identification leads to the relations among the Casson-Walker invariant $\lambda_{CW}(\hat{S}^3, K)$ of a knot in the rational homology 3-sphere \hat{S}^3 which is double branched cover of S^3 along K , the signature of the knot $\text{sign}(K)$ and the Jones polynomial $V_K(t)$:

$$2\lambda_{CW}(\hat{S}^3, K) - \lambda_{CL}(K) = \frac{1}{3} \frac{d}{dt} (\log V_K(-t))|_{t=1}.$$

From our approach, it may shed a light on finding out whether any link between the Floer homology and the Jones polynomials, one of the most important problems in the 3-manifold topology.

(4) The computation of $HF_*^{\text{sym}}(\phi_{\beta})$, for $\bar{\beta} = K$, is interesting (and hard) problem in its own right. For symplectic (weakly) monotone manifolds and symplectic diffeomorphisms which are exact, the Floer homology groups are computed by Floer [11], Hofer and Salamon [16]. For symplectic diffeomorphisms which are isotopic to the identity through symplectic diffeomorphisms, the Floer homology is computed by Lê Hồng and Ono [18] associated with Calabi invariant. It is unknown how to compute the Floer homology for general symplectic diffeomorphisms, even the Euler characteristic number, see the Problem 2 in [10]. We will develop some techniques to compute the symplectic Floer homology for the connected sum of knots representing by braids in a future paper.

5. COMPUTATIONS OF THE SYMPLECTIC FLOER HOMOLOGY OF BRAIDS

In this section, we mainly concern a computation of the symplectic Floer homology for the trefoil knot in order to fix the sign in Corollary 4.3.

Let K_n be a subset of B_n which are the representatives of knots. From the natural inclusion $B_n \hookrightarrow B_{n+1}$, we have a direct system of $\{B_n\}_{n \geq 2}$ and a similar direct system $\{K_n\}_{n \geq 2}$. The direct limit of the direct system $\{K_n\}_{n \geq 2}$ exists and

$$\mathcal{K} = \lim_{\rightarrow} K_n = \cup_{n \geq 2} K_n,$$

is the space of all knots in terms of braids by a theorem of Alexander. Note that \mathcal{K} is again a subset (non group structure) of the direct limit of $\{B_n\}_{n \geq 2}$, $\mathcal{B} = \lim_{\rightarrow} B_n$. What we have shown in §4 is that there is a map HF such that

$$HF : \mathcal{K} \rightarrow \mathcal{HF}; \quad \bar{\beta} (= K \in K_n) \mapsto HF_*^{\text{sym}}(\phi_{\beta}), \quad (5.1)$$

provides a new invariant for knots.

For a unknotted knot $K_0 = \overline{\sigma_1^{\pm}}$, we have

$$\mathcal{R}^*(S^3 \setminus K_0)^{[i]} = \emptyset \quad (\text{empty set}), \quad \mathcal{S}^*(S^3 \setminus K_0)^{[i]} = \{s\},$$

by the unknotting theorem. So the Floer chain complex is trivial and $HF_*^{\text{sym}}(\phi_{\sigma_1^{\pm}}) = 0$. By Markov's result, any unknotted knot can be obtained by finite sequence of Markov moves from σ_1^{\pm} , then by Theorem 4.2,

$$HF_*^{\text{sym}}(\phi_{\sigma}) = 0, \quad \text{if } \bar{\sigma} \text{ is unknotted.} \quad (5.2)$$

For the right handed trefoil knot $3_1 = \overline{\sigma_1^3}$ (or $(2, 3)$ torus knot), we have that $\mathcal{R}^*(S^2 \setminus \overline{\sigma_1^3})^{[i]}$ is a 2-sphere with four cone points deleted by [22] Lemma 2.1. Thus

$$\pi_1(\mathcal{R}^*(S^2 \setminus \overline{\sigma_1^3})^{[i]}) = Z^3, \quad \pi_2(\mathcal{R}^*(S^2 \setminus \overline{\sigma_1^3})^{[i]}) = 0,$$

the grading for the symplectic Floer homology is an integral grading ($N \equiv 0$). By §4, we use the special reducible representation $\{s\}$ to fix the grading. Recall in [22], up to conjugate, we can assume

that

$$X_1 = \begin{pmatrix} i \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & -i \cos \theta_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad 0 \leq \theta_1 \leq \pi.$$

Conditions $X_1 X_2 = Y_1 Y_2$ and $\text{tr}(Y_i) = 0, i = 1, 2$ provide

$$Y_1 = \begin{pmatrix} i \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & -i \cos \theta_2 \end{pmatrix}, \quad -\pi \leq \theta_2 \leq \pi.$$

So $\mathcal{R}^*(S^2 \setminus \overline{\sigma_1^3})^{[i]}$ is parameterized by (θ_1, θ_2) modulo the involution $(\theta_1, \theta_2) \rightarrow (-\theta_1, -\theta_2)$. The orientation is given by the counterclockwise orientation of (θ_1, θ_2) -plane.

$$\mathcal{R}^*(D_-^3 \setminus \overline{\sigma_1^3})^{[i]} = \{(\theta_1, \theta_1) \mid 0 < \theta_1 < \pi\},$$

$$\mathcal{R}^*(D_+^3 \setminus \overline{\sigma_1^3})^{[i]} = \text{Graph}(\phi_{\overline{\sigma_1^3}}) = \{(X_1, X_2, \sigma_1^3(X_1), \sigma_1^3(X_2))\},$$

where $Y_1 = \sigma_1^3(X_1) = (X_1 X_2)^2 X_1^{-1} (X_1 X_2)^{-1}$. A straightforward calculation shows that

$$\text{Graph}(\phi_{\overline{\sigma_1^3}}) = \{(\theta_1, 4\theta_1) \mid 0 < \theta_1 < \pi\},$$

$$\text{Fix}(\phi_{\overline{\sigma_1^3}}) = \{\rho = (\frac{2\pi}{3}, \frac{2\pi}{3})\} = \mathcal{R}^*(D_+^3 \setminus \overline{\sigma_1^3})^{[i]} \cap \mathcal{R}^*(D_-^3 \setminus \overline{\sigma_1^3})^{[i]}, \quad (5.3)$$

is a single element and nondegenerate as in Figure 1.

Figure 1. The right handed trefoil knot $\overline{\sigma_1^3}$

The Maslov index of ρ , $\mu(\rho) = \mu(\rho, s)$, is the same Maslov index of two Lagrangian submanifolds $L_0 = \mathcal{R}^*(D_-^3 \setminus \overline{\sigma_1^3})^{[i]}$ and $L_1 = \text{Graph}(\phi_{\overline{\sigma_1^3}})$ by [8]. This index can be computed as the winding number of a path of L_1 about the diagonal L_0 as in [1] (Note that in [8], L_0 is the $\{(\theta_1, 0)\}$). So we have that L_1 clockwise winds the L_0 once. $\mu(\rho) = -1$. Hence

$$HF_i^{\text{sym}}(\phi_{\overline{\sigma_1^3}}) = \begin{cases} Z & \text{if } i = -1 \\ 0 & \text{if } i \neq -1 \end{cases} \quad (5.4)$$

The Euler number $\chi(HF_*^{\text{sym}}(\phi_{\overline{\sigma_1^3}})) = -1 = -\lambda_{CL}(\overline{\sigma_1^3})$ in [22] p 348. Note that the orientation we used is the opposite one in [30] p 220. This example fixes the sign in Corollary 4.3.

Similarly, we can describe the graph $\phi_{\sigma_1^{-3}}$ for the left handed trefoil knot $-3_1 = \overline{\sigma_1^{-3}}$:

$$\text{Graph}(\phi_{\sigma_1^{-3}}) = \{(\theta_1, \frac{\pi}{2} + 4\theta_1) \mid 0 < \theta_1 < \pi\}.$$

The Maslov index of ρ_- is zero since there is a $\frac{\pi}{2}$ -shift.

$$HF_i^{\text{sym}}(\phi_{\sigma_1^{-3}}) = \begin{cases} Z & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases} \quad (5.5)$$

For the $(2, q)$ torus knot $\overline{\sigma_1^q}$ (q must be odd), we have the same method to calculate, by induction,

$$\text{Graph}\phi_{\sigma_1^q} = \{(\theta_1, (q+1)\theta_1) \mid 0 < \theta_1 < \pi\},$$

$$\text{Fix}(\phi_{\sigma_1^q}) = \{\rho_k = (\frac{2\pi k}{q}, \frac{2\pi k}{q}) \mid k = 1, 2, \dots, \frac{q-1}{2}\}.$$

The Maslov indexes are determined by $\mu(\rho_1) = -1$. $\mu(\rho_{k-1}, \rho_k)$ is the Maslov index of two Lagrangian submanifolds intersecting transversally at two smooth points ρ_{k-1}, ρ_k . By [1] §1.4,

$\mu(\rho_{k-1}, \rho_k)$ is the number of rotations of Det^2 , thus $\mu(\rho_{k-1}, \rho_k) = -2$ (orientation). By additivity of the Maslov index, we have the nontrivial Floer chain groups:

$$C_{-2k+1}^{\text{sym}}(\phi_{\sigma_1^q}) = Z < \rho_k >, \quad k = 1, 2, \dots, \frac{q-1}{2}.$$

Therefore we have the trivial Floer boundary for this case again. So

$$HF_i^{\text{sym}}(\phi_{\sigma_1^q}) = \begin{cases} Z & i = -2k + 1, \quad k = 1, 2, \dots, \frac{q-1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (5.6)$$

Hence $\chi(HF_*^{\text{sym}}(\phi_{\sigma_1^q})) = -\frac{q-1}{2}$, the signature of the $(2, q)$ torus knot is $-(q-1)$ in [23]. The number $\frac{q-1}{2}$ is also the unknotting number of the $(2, q)$ torus knot by [29].

In order to compute the Floer homology, one has to study the possible nontrivial boundary map. This is the case for general (p, q) torus knot (a braid representative $(\sigma_1 \sigma_2 \cdots \sigma_{p-1})^q, (p, q) = 1$). There are some methods in determining the representations of knot groups in $SU(2)$ (not using braid representations) as in [5] and [17].

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