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KOJIMA'S ETA-FUNCTION FOR MANIFOLD LINKS IN HIGHER DIMENSIONS

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ABSTRACT. Kojima's η-function is generalized to give a new concordance invariant for certain two-component manifold links in higher dimensions. Examples are given of manifold links successfully distinguished by this generalized η-function but not by their Cochran derived invariants.

1. INTRODUCTION

Kojima's η-function[3] is a TOP-concordance invariant defined for two-component classical links with vanishing linking number, which generalizes Goldsmith's invariants [2] in one direction and has an abstruse connection to some of Laufer's invariants [4]. Examples are known of classical links which are distinguishable by Kojima's η-function but not by their Milnor $\bar{\mu}$-invariants [6].

In this paper, Kojima's η-function is generalized to give a new TOP-concordance invariant for a class of manifold links called admissible links. This generalized η-function may be viewed as an ordered collection of rational functions over $\mathbf{Z}$, which vanishes identically for both boundary links and manifold links with vanishing first integral homology group and detects both non-amphicheirality and non-invertibility of admissible links in higher dimensions.

For the same class of manifold links, Cochran [1] has defined, by generalizing a construction of Sato [11], a sequence of derived invariants of link concordance which vanish for boundary links and manifold links with vanishing first integral homology group. In the classical dimension setting, these derived invariants are equivalent, via a non-trivial change of variable, to Kojima's η-function [1]. However, in higher dimensions, our generalized η-function often gives more information. Indeed, in each higher dimension setting, there exist infinitely many different TOP-concordance classes of admissible links whose Cochran derived invariants vanish but which are distinguishable by our generalized η-function.

2. BASIC DEFINITIONS

A *manifold link* in S^{n+2} is an ordered pair $L = (M, K)$ of disjoint, codimension-two, closed, connected, oriented, smooth submanifolds of S^{n+2}. Two manifold links $L_0 = (M_0, K_0)$ and $L_1 = (M_1, K_1)$ are *CAT-concordant*, CAT = DIFF or TOP, if

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there exists an ordered pair $C = (P, Q)$ of proper, oriented CAT-submanifolds of $S^{n+2} \times I$ which is CAT-homeomorphic to $L_0 \times I$ and so that $\partial C = L_0 \times 0 \cup (-L_1) \times 1$.

Let M be a codimension-two, closed, connected, oriented, smooth submanifold of S^{n+2}. Then there exists a compact, connected, oriented, smooth submanifold V of S^{n+2}, called a *Seifert surface* of M, so that $\partial V = M$. A *boundary link* is a manifold link whose components bound disjoint Seifert surfaces. A *semi-boundary link* is a manifold link such that each component bounds a Seifert surface in the complement of the other component. An *admissible link* is a manifold link $L = (M, K)$ in S^{n+2} so that $H_1(K) \cong 0$ unless $n = 1$ and so that K bounds a Seifert surface in the complement of M; see Cochran [1]. Here and throughout, all homology and cohomology groups are assumed to have untwisted integral coefficients. Notice that if $L = (M, K)$ is an admissible link, then M automatically bounds a Seifert surface in the complement of K by Sato [11], and that in the classical dimension setting, a link is admissible if and only if it has a vanishing linking number.

3. THE η-FUNCTION

Let K be a codimension-two, closed, connected, oriented, smooth submanifold of S^{n+2}, and let E be the exterior of K which, by definition, is the closure of $S^{n+2} - N(K)$, where $N(K)$ is a tubular neighborhood of K. By Alexander duality, E has the first integral homology group isomorphic to $\mathbf{Z}$, and as a consequence, the universal abelian cover $\widetilde{E}$ admits infinite cyclic covering transformations with t the generator determined in a canonical way by the given orientation of K. This defines a unique module structure for each homology group of $\widetilde{E}$ over the Laurent polynomial ring $\Lambda = \mathbf{Z}[t, t^{-1}]$.

Lemma 3.1. *Suppose that $H_1(K) \cong 0$ unless $n = 1$. Then the homology intersection pair $H_{n+1}(\widetilde{E}) \times H_1(\widetilde{E}) \to \mathbf{Z}$ vanishes identically, and there exists, for each $k \in \{1, n, n+1\}$, a non-trivial conjugate symmetric Laurent polynomial $\lambda(t)$ with $\lambda(1) = 1$ so that $\lambda(t_*)u = 0$ for each $u \in H_k(\widetilde{E})$.*

Proof. By Milnor [7] we have an exact sequence of homology groups

$$\longrightarrow H_*(\widetilde{E}) \overset{t_* - 1}{\longrightarrow} H_*(\widetilde{E}) \overset{p_*}{\longrightarrow} H_*(E) \longrightarrow H_{*-1}(\widetilde{E}) \longrightarrow$$

where $p : \widetilde{E} \to E$ is the projection. It follows by Alexander duality that

$$t_* - 1 : H_*(\widetilde{E}) \to H_*(\widetilde{E})$$

is an epimorphism in dimensions 1, n and $n + 1$, hence an automorphism of Λ-modules in these dimensions since $H_*(\widetilde{E})$ is finitely generated and Λ is noetherian. The second assertion now follows by the proofs of Proposition 1.2 and Corollary 1.3 of Levine [5]. Notice that p_* is the zero-homomorphism in dimensions 1 and $n + 1$. The first assertion follows by the proof of Lemma 5 of Kojima and Yamasaki [3]; see also Milnor [7]. $\square$

Let $\widetilde{E}$ be oriented via the pull-back of the standard orientation of S^{n+2} and let b be an n-boundary of $\widetilde{E}$. Then for each 1-cycle $\tilde{l}$ of $\widetilde{E}$ which is disjoint from b, one may define the *linking number* $lk(\tilde{l}, b)$ to be the usual intersection number of $\tilde{l}$ with a $(n + 1)$-chain bounded by b. It follows by the above lemma that this linking number is well-defined. Consider now a lift u to $\widetilde{E}$ of a codimension-two, closed, connected, oriented, smooth submanifold M of E so that u is disjoint from

b. Then the linking number gives rise to a homomorphism $H_1(M) \to \mathbf{Z}$ defined by $[l] \to lk(\tilde{l}, b)$, where $\tilde{l}$ is the lift of l to u. This determines a cohomology class in $H^1(M)$, which we denote by $\Phi_b(u)$.

Definition 3.2. Let $L = (M, K)$ be an admissible link in S^{n+2} and let $\widetilde{E}$ be the universal abelian cover of the exterior E of K with t the canonical generator of the infinite cyclic covering transformations. By a *zero-push-off* of M, we mean a smooth submanifold of S^{n+2} disjoint from M which is obtained by pushing M slightly along a normal vector field of a Seifert surface of M. Fix now a lift u of M to $\widetilde{E}$, a nearby lift u' of a zero-push-off of M, and a non-trivial Laurent polynomial $\lambda(t)$ so that $b = \lambda(t_*)u'$ is an n-boundary. Then the η-*function* of L is defined to be

$$\eta_L(t) = \frac{1}{\lambda(t)} \sum_{k=-\infty}^{+\infty} \Phi_b(t_*^k u) t^k.$$

Notice that in the classical dimension setting, the η-function defined here coincides with the one defined by Kojima and Yamasaki [3] by identifying $H^1(S^1)$ with $\mathbf{Z}$. Notice also that since $H^1(M)$ is a finitely generated free abelian group, our generalized η-function may be viewed as an ordered collection of r rational functions over $\mathbf{Z}$, where r is the rank of $H^1(M)$. The following theorem generalizes the corresponding result of Kojima and Yamasaki [3] concerning the η-function for classical links and will be proved along the same line as that used in [3].

Theorem 3.3. *The η-function is a well-defined TOP-concordance invariant which satisfies $\eta_L(1) = 0$ and vanishes identically for both boundary links and manifold links with vanishing first integral homology group. More precisely, if two admissible links $L_0 = (M_0, K_0)$ and $L_1 = (M_1, K_1)$ are TOP-concordant, then there exists an orientation preserving homeomorphism from M_0 onto M_1 so that the induced isomorphism on the first integral cohomology groups sends the η-function of L_1 to that of L_0.*

Proof. We shall use the same notations as in the above definition. First notice that the η-function is independent of one's choices of the lift and the zero-push-off of M. The independence of the annihilator $\lambda(t)$ follows because

$$\sum_{k=-\infty}^{+\infty} \Phi_{\mu(t_*)b}(t_*^k u) t^k = \mu(t) \sum_{k=-\infty}^{+\infty} \Phi_b(t_*^k u) t^k$$

for each Laurent polynomial $\mu(t)$. This shows that the η-function is well-defined. The vanishing of the η-function for boundary links and manifold links with vanishing first integral homology group is an immediate consequence of the definition.

To see that $\eta_L(1) = 0$, notice that each circle l in M bounds an embedded surface S in the complement of K and the zero-push-off of M since L is admissible [11]. Denote by c the intersection cycle of S with a Seifert surface of K and by $\tilde{l}$ a lift of l to u. Then there exists a lift $\tilde{c}$ of c to $\widetilde{E}$ so that $\tilde{l}$ and $t_*\tilde{c} - \tilde{c}$ are homologous in the complement of u'. It follows that

$$\sum_{k=-\infty}^{+\infty} lk(t_*^k \tilde{l}, b) = \sum_{k=-\infty}^{+\infty} lk(t_*^{k+1}\tilde{c}, b) - \sum_{k=-\infty}^{+\infty} lk(t_*^k \tilde{c}, b) = 0,$$

and hence $\eta_L(1) = 0$ as desired.

It remains to verify that the η-function is a TOP-concordance invariant. To this end, let $C = (P, Q)$ be a TOP-concordance between two admissible links $L_0 = (M_0, K_0)$ and $L_1 = (M_1, K_1)$. Denote by X the exterior of Q and by $\widetilde{X}$ the universal abelian cover of X with t the canonical generator of the infinite cyclic covering transformations. Then X restricts to the exterior E_i of K_i, $\widetilde{X}$ restricts to the universal abelian cover $\widetilde{E}_i$ of E_i, and t restricts to the canonical generator of the infinite cyclic covering transformations of $\widetilde{E}_i$ over $S^{n+2} \times i$ where $i = 0, 1$. Fix now a lift w of P to $\widetilde{X}$ and a nearby lift w' of a proper TOP-submanifold P' of $S^{n+2} \times I$ obtained by pushing a small collar of P off P so that P' restricts to zero-push-offs of M_0 and $-M_1$ over $S^{n+2} \times 0$ and $S^{n+2} \times 1$ respectively. Then we have $\partial w = u_0 - u_1$ and $\partial w' = u'_0 - u'_1$, where u_i is a lift of M_i to $\widetilde{E}_i$ and u'_i is a nearby lift of a zero-push-off of M_i.

Since (P, Q) is a TOP-concordance, there exists an orientation preserving homeomorphism F from $M_0 \times I$ onto P which induces an orientation preserving homeomorphism f from M_0 onto M_1. Suppose we are given a 1-cycle l in M_0. Then $F(l \times I)$ lifts to a 2-chain v in w so that $\partial v = \tilde{l}_0 - \tilde{l}_1$, where $\tilde{l}_0$ and $\tilde{l}_1$ are lifts of l and $f(l)$ to u_0 and u_1 respectively. This v is seen to have a vanishing intersection number with $t^k_* w'$ for each k. Pick now a non-trivial Laurent polynomial $\lambda(t)$ so that $b_i = \lambda(t_*) u'_i$ is an n-boundary of $\widetilde{E}_i$ for $i = 0, 1$ and denote by B_i a $(n + 1)$-chain of $\widetilde{E}_i$ with $\partial B_i = b_i$. We claim that $lk(t^k_* \tilde{l}_0, b_0) - lk(t^k_* \tilde{l}_1, b_1)$, which is equal to the usual intersection number of $t^k_* v$ with $B_0 - \lambda(t_*) w' - B_1$, vanishes identically for each $k \in \mathbf{Z}$. To see this, notice that $H_2(X) \cong 0$ by Alexander duality, and $H_1(\partial X) \cong 0$ by a Mayer-Vietoris sequence, hence $H_2(X, \partial X) \cong 0$, and that $H_{n+1}(X) \cong 0$ by Alexander duality. It follows by the proof of Lemma 5 of Kojima and Yamasaki [3] that the homology intersection pair

$$H_{n+1}(\widetilde{X}) \times H_2(\widetilde{X}, \partial \widetilde{X}) \to \mathbf{Z}$$

vanishes identically. This shows that the isomorphism on the first integral cohomology groups induced by f sends the η-function of L_1 to that of L_0. Hence the η-function is an invariant of TOP-concordance as desired.

4. Examples and Remarks

We recall first the definition of Cochran's derived invariants β_k and refer to [1] for more details. Let $L = (M, K)$ be a semi-boundary link in S^{n+2}. Then there exists a pair (V, Z) of Seifert surfaces for (M, K) so that $V \cap K = M \cap Z = \emptyset$ and that V meets Z transversely. The Sato-Levine invariant $\beta(L)$ of L is the unstable homotopy class in $\pi_{n+2}(S^2)$ determined, via the Pontrjagin-Thom construction, by the framed submanifold $F = V \cap Z \subset S^{n+2}$, whose framing, as well as orientation, is inherited from that of V and Z under the convention described in Cochran [1]; see also Sato [11]. Suppose now that $L = (M, K)$ is an admissible link. Then Cochran showed that the Seifert pair (V, Z) may be chosen so that F is connected, which is called a *characteristic intersection* of L, and that the *derived link* $\mathcal{D}(L) = (F, K)$ is again an admissible link. Define further derivatives recursively by $\mathcal{D}^{k+1}(L) = \mathcal{D}(\mathcal{D}^k(L))$. Then the kth *Cochran derived invariant* $\beta_k(L)$ of L is defined to be the Sato-Levine invariant of a $(k - 1)$th derivative $\mathcal{D}^{k-1}(L)$ of L. These derived invariants are obstructions to admissible links being DIFF-concordant to boundary links and were later shown to be zero for manifold links with vanishing first integral homology group by Orr [8].

We now construct examples showing that in higher dimensions, our generalized η-function often gives more information than Cochran's derived invariants.

Example 4.1. Let $L = (M, K)$ be a manifold link in S^{n+2} and identify S^{n+3} with $D^{n+2} \times S^1 \cup S^{n+1} \times D^2$. Remove a small open ball around a point of K to obtain a pair $(M, K - B)$ of proper submanifolds of $D^{n+2} \subset S^{n+2}$ where B is an open n-ball in K, and then extend it to an embedded pair $(M \times S^1, (K - B) \times S^1)$ in $D^{n+2} \times S^1$ in an obvious way. Gluing up with $\partial \bar{B} \times D^2 \subset S^{n+1} \times D^2$ along the common boundary, one obtains a manifold link $\Sigma(L) = (M^*, K^*)$ in S^{n+3}, where

$$M^* = M \times S^1 \ \text{ and } \ K^* = (K - B) \times S^1 \cup \partial \bar{B} \times D^2.$$

Notice that if L is an admissible link, so is the *untwisted spun link* $\Sigma(L)$. If we consider $H^1(M)$ as a subgroup of $H^1(M^*) \cong H^1(M) \oplus \mathbf{Z}$ in a natural way, then *the η-function of the untwisted spun link $\Sigma(L)$ is same as that of L.* To see this, let E be the exterior of K in S^{n+2}, and let $\widetilde{E}$ be the universal abelian cover of E. Then $E \times S^1$ and $\widetilde{E} \times S^1$ may be viewed in a natural way as subspaces of the exterior of K^* in S^{n+3} and its universal abelian cover respectively. Fix a lift u of M to $\widetilde{E}$ and a nearby lift u' of a zero-push-off of M. Then $u \times S^1$ and $u' \times S^1$ are nearby lifts of M^* and its zero-push-off respectively. The assertion now follows readily, noticing that the 1-cycle $* \times S^1$ in $u \times S^1$, where $* \in u$ is a point, bounds a 2-disk disjoint from all liftings of a given zero-push-off of M^*.

Now beginning with a two-component link L in S^3 with vanishing linking number and performing the above construction recursively, one obtains a sequence of untwisted spun links $\Sigma^{n-1}(L)$ in S^{n+2} where $n \geq 2$. We claim that *the Cochran derived invariants of these untwisted spun links vanish identically.* To see this, notice that if F is a framed characteristic intersection of $\mathcal{D}^k(L)$ where $k \geq 0$, then $F_m = F \times (S^1)^m$ is a characteristic intersection of $\mathcal{D}^k(\Sigma^m(L)) - \Sigma^m(\mathcal{D}^k(L))$ where $m \geq 1$, whose framing is induced from that of F by the natural projection $F_m \to F$. Since the framing of $F_1 = F \times S^1$, considered as an element of $H^1(F_1)$, is zero on the generator of $H_1(F_1)$ represented by S^1, the framed torus F_1 in S^4 has a vanishing Arf invariant, hence represents the zero class of $\pi_4(S^2)$; see Ruberman [9]. It follows that the framed submanifold F_{n-1} represents the zero class of $\pi_{n+2}(S^2)$ for each $k \geq 0$ and each $n \geq 2$. Hence the Cochran derived invariants of $\Sigma^{n-1}(L)$, $n \geq 2$, vanish identically as claimed.

Since there exist infinitely many two-component links in S^3 with vanishing linking numbers and pairwisely distinct η-functions by Cochran [1] or Kojima and Yamasaki [3], we are now able to construct, for each $n > 1$, infinitely many different TOP-concordance classes of admissible links in S^{n+2} whose Cochran derived invariants vanish but which are distinguishable by our generalized η-function.

Remark. It should be noticed that the same construction as above has been used by Saito [10] to give infinitely many different TOP-concordance classes of surface links in S^4 with vanishing Cochran derived invariants. The invariants he used were a collection of $\mathbf{Z}_2$-valued numerical invariants obtained by applying Sato's idea of asymmetric linking number to surface links in various finite cyclic branched coverings.

In the classical dimension setting, the η-function is conjugate symmetric [3]. Our next examples show that this is not always the case in higher dimensions.

Example 4.2. We start with a pair (S^1, S^n) of oriented spheres disjointly embedded in S^{n+2}, where $n > 1$, so that the linking number $r = lk(S^1, S^n)$ is non-zero, and then extend the embedded S^1 to an embedded $S^1 \times S^{n-1}$ in a small tubular neighborhood of S^1 in S^{n+2}. Pick another unknotted n-sphere K in S^{n+2} which lies outside a $(n+2)$-ball containing $S^1 \times S^{n-1}$ and S^n, and choose an oriented arc α embedded in S^{n+2} going from S^n to $S^1 \times S^{n-1}$ so that α is disjoint from everything else from its interior and links K algebraically s times where $s \neq 0$. Now adding an embedded 1-handle to $S^1 \times S^{n-1} \cup S^n$ with α being the core and removing its interior, one obtains a manifold link $L = (M, K)$ in S^{n+2}, where M is diffeomorphic to $S^1 \times S^{n-1}$. This manifold link is readily seen to be an admissible link having a non-conjugate symmetric η-function given by $\eta_L(t) = r\iota t^s - r\iota$, where ι is the pull-back of the canonical generator of $H^1(S^1)$ under the natural projection $M \to S^1$.

Remark. Recall that a manifold link $L = (M, K)$ in S^{n+2} is *amphicheiral* if there exists an orientation reversing homeomorphism F of S^{n+2} onto itself so that $F(L) = L$, and is *invertible* if there exists an orientation preserving homeomorphism F of S^{n+2} onto itself so that $F(L) = -L$. Kojima's η-function is conjugate symmetric, hence detects only non-amphicheirality of two-component oriented links in S^3. In contrast to this, the asymmetry of our generalized η-function allows us to detect both non-amphicheirality and non-invertibility of admissible links in higher dimensions. To see this, notice that if an admissible link L is amphicheiral (respectively invertible) up to TOP-concordance, then there exists an orientation preserving (respectively orientation reversing) homeomorphism of M onto itself so that the induced automorphism on $H^1(M)$ sends $\eta_L(t)$ to $-\eta(t^{-1})$. It follows that the admissible links constructed in Example 4.2 are neither amphicheiral nor invertible even up to TOP-concordance.

Remark. It is worth remarking that the generalized η-function can be used to construct certain TOP-concordance invariants which take values in the unstable homotopy groups of spheres. This could be done along the same line as that used by Cochran [1] for Kojima's η-function of classical links. To describe this, suppose we are given an admissible link $L = (M, K)$ in S^{n+2}. Then the invariant $\theta_L(t) = \eta_L(t) + \eta_L(t^{-1})$ may be expanded in positive powers of $x = (t-1)(t^{-1}-1)$ so that $\theta_L(t) = \sum a_k x^k$ where $a_k \in H^1(M)$. This may be verified by noticing that $\theta_L(1) = 0$ and $\theta_L(t) = \theta_L(t^{-1})$, and that $t^k + t^{-k} - 2$ is a polynomial in positive powers of x for $k > 0$. Twisting the zero-framing of M in S^{n+2} by using the homotopy class in $[M, S^1] \cong [M, SO(2)]$ corresponding to a_k, one obtains a framed submanifold of S^{n+2}, hence an unstable homotopy class $\theta_k(L)$ in $\pi_{n+2}(S^2)$ via the Pontrjagin-Thom construction for each $k > 0$. This gives rise to a new sequence $\{\theta_k\}_{k>0}$ of TOP-concordance invariants which vanish identically for both boundary links and manifold links with vanishing first integral homology group. In the classical dimension setting, it follows by Theorem 7.1 of Cochran [1] that $\theta_k = 2\beta_k$ for each $k > 0$. But we do not know whether this is always the case for admissible links in higher dimensions.

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