# An Algebraic Classification of Some Links of Codimension Two

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# SOME LINKS OF CODIMENSION TWO

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ABSTRACT. For $q \geqslant 2$, J. Levine proved that two simple $(2q - 1)$-knots are isotopic if and only if their Seifert matrices are equivalent. In this paper, we will prove the analogue of Levine's result for simple boundary $(2q - 1)$-links; we will show that: "For $q \geqslant 3$, two simple boundary $(2q - 1)$-links are isotopic if and only if their Seifert matrices are $l$-equivalent (defined by some algebraic moves)."

An $n$-link of multiplicity $m$, denoted by $L = K_1 \cup \cdots \cup K_m$ is an embedding of $m$ disjoint copies of the $n$-sphere (or homotopy spheres) $K_i$ into the $(n + 2)$-sphere $S^{n+2}$. $L$ is called boundary if it extends to an embedding of $m$ disjoint orientable compact $(n + 1)$-manifolds $M_i$, called the Seifert manifolds, with $\partial M_i = K_i$. Let $X$ denote the link complement. Gutiérrez [1] showed that an $n$-link of multiplicity $m$ is boundary if and only if there is an epimorphism from $\pi_1(X)$ onto $F_m$, the free group in $m$ generators, sending meridians to generators. An $(2q - 1)$-link $L$ is called simple if $\pi_i(X) = \pi_i(\bigvee_m S^1)$ for $i < q$; in case $L$ is a boundary link, we require that the meridians be sent to generators.

For $q \geqslant 2$, Levine [5] proved that two simple $(2q - 1)$-knots are isotopic if and only if their Seifert matrices are "equivalent" (defined by certain algebraic "moves" in [5], also called $S$-equivalent in [7]). In this paper, we will prove the analogue of Levine's Theorems 1–3 for simple boundary $(2q - 1)$-links, $q \geqslant 3$: two simple boundary $(2q - 1)$-links are isotopic if and only if their "Seifert matrices" are related by certain algebraic "moves".

Since our proofs are almost the same as those of [4] and [5], we will only give the outlines here.

1. For simplicity, we will consider only the $(2q - 1)$-link of multiplicity 2. Everything considered here is in the smooth category.

Let $L = K_1 \cup K_2$ be a boundary $(2q - 1)$-link. According to [1], there exist two disjoint $2q$-dimensional Seifert manifolds $M_1$ and $M_2$ for $L$, that is, $\partial M_1 = K_1$ and $\partial M_2 = K_2$. Let $A_1$ be the corresponding Seifert matrix for the

knot $K_1$ (in $S^{2q+1}$) with respect to the basis $\{a_1, \ldots, a_n\}$ of the torsion-free part of $H_q(M_1)$, and $A_2$ the Seifert matrix for $K_2$ with respect to the basis $\{b_1, \ldots, b_m\}$ of $H_q(M_2)/\text{Torsion}$.

A linking from $\theta: (H_q(M_1) \oplus H_q(M_2)) \otimes (H_q(M_1) \oplus H_q(M_2)) \to Z$ is defined by letting $\theta(\alpha \otimes \beta)$ be the linking number $L(z_1, z_2)$ (in $S^{2q+1}$), where $z_1$, a cycle in $M_1$ (or $M_2$), represents $\alpha$ and $z_2$ represents $i_+ \beta$, the translate in the positive normal direction off $M_1$ (or $M_2$) into $S^{2q+1} - M_1 - M_2$ of a cycle representing $\beta$. With respect to the basis $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ of the torsion-free part of $H_q(M_1) \oplus H_q(M_2)$, the matrix $\theta$ has the following form:

$$D = \begin{pmatrix} A_1 & P \\ -\varepsilon P' & A_2 \end{pmatrix},$$

also written as $D = [A_1, A_2, P]$, where $\varepsilon = (-1)^q$ and $P'$ denotes the transpose of $P$. We call $D$ a Seifert matrix for the boundary link $L$. It is obvious that $D + \varepsilon D'$ is unimodular. Algebraically, we will call $D = [A_1, A_2, P]$ a Seifert matrix of type 2 if $A_1 + \varepsilon A_1'$, $A_2 + \varepsilon A_2'$ and $D + \varepsilon D'$ are unimodular. Here $A'$ denotes the transpose of $A$.

Actually, $D$ is a Seifert matrix for the link $L$ corresponding to the manifold $M_1 \# M_2$ with $\partial(M_1 \# M_2) = K_1 \cup K_2$ in the sense of [6, Theorem 3.2]. The $(n \times m)$-matrix $P = (p_{ij})$ in $D$ can be obtained as follows: let $\{c_1, \ldots, c_n\}$ be a basis for $H_q(S^{2q+1} - M_1)/\text{Torsion}$, which is the Alexander dual of $\{a_i\}$, that is, $L(a_i, c_j) = \delta_{ij}$. In $S^{2q+1} - M_1$, we have $b_j = \sum p_{kj} c_k$, hence

$$L(a_i, i_+ b_j) = L(a_i, b_j) = \sum_j L(a_i, c_k) p_{kj} = p_{ij}.$$

Following [5], we now define certain algebraic "moves" for Seifert matrices of type 2. Let $D = [A_1, A_2, P]$ be one. Then any matrix of the form (which is again a Seifert matrix of type 2):

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x \\ 0 & \lambda & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & y & x \\ 0 & 0 & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix},$$

$$\begin{bmatrix} A_1 & P & x' & 0 \\ -\varepsilon P' & A_2 & \tau & 0 \\ -\varepsilon x & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A_1 & P & x' & 0 \\ -\varepsilon P' & A_2 & 0 & 0 \\ -\varepsilon x & y & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $x, y$ are row vectors, and $\lambda, \tau$ are column vectors, will be called an elementary $l$-enlargement of $D$; $D$ is an elementary $l$-reduction. Let $C$ be a unimodular matrix having the same dimension as $A_1$, and $E$ a unimodular matrix having the same dimension as $A_2$. Then each of the operations below will be called an $l$-congruence:

$$D \rightarrow \begin{pmatrix} C & 0 \\ 0 & I_m \end{pmatrix} D \begin{pmatrix} C' & 0 \\ 0 & I_m \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_n & 0 \\ 0 & E \end{pmatrix} D \begin{pmatrix} I_n & 0 \\ 0 & E' \end{pmatrix}.$$

Two Seifert matrices of type 2 are called *l*-equivalent if they can be connected by a chain of elementary *l*-enlargements, *l*-reductions, and *l*-congruences (with $C$ or $E$ having the appropriate dimension).

2. We first prove the analogue of [5, Theorem 1].

THEOREM 1. *Seifert matrices of isotopic boundary* $(2q - 1)$-*links are l-equivalent.*

PROOF. Suppose $L_1 = K_1 \cup K_2$ and $L_2 = J_1 \cup J_2$ are isotopic boundary links with Seifert manifolds $M_1$, $M_2$ and $N_1$, $N_2$, respectively. Then the argument in [5, p. 186] gives us two disjoint $(2q + 1)$-dimensional manifolds $V_i$ ($i = 1$ or $2$) in $S^{2q+1} \times I$ meeting $S^{2q+1} \times 0$ along $M_i$ and $S^{2q+1} \times 1$ along $N_i$, with $\partial V_i = M_i \cup X_i \cup N_i = Y_i$.

After rearranging the level of the critical points for the "height" functions $\Phi_i: V_i \rightarrow I$ as in [5, p. 187], we need only consider the case where $\Phi_1$ has only one critical point and $\Phi_2$ has none. Then we use the argument in [5, pp. 187–188] to conclude that the Seifert matrix $D$ for $L_2 = J_1 \cup J_2$ with respect to an appropriate basis has the following form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & u & v & x \\ 0 & \lambda & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & u & v & x \\ 0 & \lambda & A_1 & P \\ 0 & -\varepsilon x' & -\varepsilon P' & A_2 \end{bmatrix}$$

where $[A_1, A_2, P]$ is the Seifert matrix for $L_1 = K_1 \cup K_2$ associated with $M_1$ and $M_2$. $D$ is *l*-congruent to an elementary *l*-enlargement of $[A_1, A_2, P]$ as in [5, Theorem 1]. Q.E.D.

3. Let $q$ denote an integer and recall that $\varepsilon = (-1)^q$.

THEOREM 2. *Let* $q \geq 3$, *and* $D = [A_1, A_2, P]$ *a square integral matrix such that* $A_1 + \varepsilon A_1'$, $A_2 + \varepsilon A_2'$, *and* $D + \varepsilon D'$ *are unimodular. Then there is a simple boundary* $(2q - 1)$-*link* $L = K_1 \cup K_2$ *with* $D, A_1, A_2$ *the Seifert matrices of* $L$, $K_1$, $K_2$, *respectively.*

PROOF. Let $B_1$, $B_2$ denote two disjoint $(2q + 1)$-balls in $S^{2q+1}$. We know from [2, pp. 255–257] that there exist two handlebodies $M_1 = D^{2q} \cup h_1 \cup \cdots \cup h_n$, $M_2 = D^{2q} \cup h_1' \cup \cdots \cup h_m'$, where each $h_i$, $h_i'$ is a handle of index $q$; and two embeddings $g_i: M_i \rightarrow B_i \subseteq S^{2q+1}$ such that $g_i(\partial M_i) = J_i$ represents a simple knot with Seifert matrix $A_i$. Let $\{a_1, \ldots, a_n\}$ be a basis for $H_q(g_1(M_1))$ and $\{b_1, \ldots, b_m\}$ a basis for $H_q(g_2(M_2))$; each represents the core of a handle in $M_1$ or $M_2$. As in [2, p. 257] we may choose $n$ $q$-spheres $S_1, \ldots, S_n$ in $B_2$ such that $L(S_i, b_j) = \delta_{ij}$ and $L(S_i, S_j) = 0$. Since $S_i \subseteq B_2$, $L(a_i, S_j) = 0$. Then we define a new embedding $f$ for $M_1$ such that $f = g_1$ near $D^{2q}$, $f(\gamma_i) = g_1(\gamma_i) \# \Sigma p_{ij} S_j$, where $P = (p_{ij})$ and $\gamma_i$ denotes the core of

the handle $h_i$. Let $c_i \in H_q(f(M_1))$ represent $f(\gamma_i)$. Since $f(M_1)$ and $g_2(M_2)$ are $(q - 1)$-connected, the link $L = K_1 \cup K_2$, where $K_1 = \partial f(M_1)$ and $K_2 = \partial g_2(M_2)$, is a simple boundary link [1]. Furthermore, with respect to the basis $\{c_1, \ldots, c_n, b_1, \ldots, b_m\}$, the Seifert matrices of $L$, $K_1$, $K_2$ are $D$, $A_1$, $A_2$, respectively.   Q.E.D.

4. A Seifert matrix of a simple boundary $(2q - 1)$-link $L$ obtained from two disjoint $(q - 1)$-connected Seifert manifolds will be called special.

LEMMA 1. *Let $L = K_1 \cup K_2$ be a simple boundary $(2q - 1)$-link with a special Seifert matrix $D = [A_1, A_2, P]$. If $E$ is an elementary $l$-enlargement of $D$, then $E$ is also a special Seifert matrix of $L$.*

PROOF. The proof is essentially the same as [5, Lemma 2].

LEMMA 2. *For $q \geqslant 3$, two simple boundary $(2q - 1)$-links admitting identical special Seifert matrices are isotopic.*

PROOF. Let $L_1 = K_1 \cup K_2$ and $L_2 = J_1 \cup J_2$ be two simple boundary $(2q - 1)$-links bounding $(q - 1)$-connected Seifert manifolds $M_1$, $M_2$ and $N_1$, $N_2$, respectively, with $M_1 \cap M_2 = \varnothing = N_1 \cap N_2$. Suppose also that there exists an isomorphism $\Phi$: $H_q(M_1 \cup M_2) \to H_q(N_1 \cup N_2)$ preserving the linking form with $\Phi | H_q(M_i) \to H_q(N_i)$ an isomorphism.

Lemma 3 of [5] showed that $M_1$ and $N_1$ are isotopic submanifolds of $S^{2q+1}$. Hence we may assume that $M_1 = N_1$. According to [8], $M_1$, $M_2$ and $N_2$ have handle decompositions:

$$M_1 = D_0^{2q} \cup \alpha_1 \cup \cdots \cup \alpha_n, \quad M_2 = D^{2q} \cup \beta_1 \cup \cdots \cup \beta_m,$$

$$N_2 = D^{2q} \cup \gamma_1 \cup \cdots \cup \gamma_m,$$

where each $\alpha_i$, $\beta_i$, $\gamma_i$ is a handle of index $q$. By a further isotopy keeping $M_1$ fixed, we may assume that the base disks $D^{2q}$ in the handle decompositions of $M_2$ and $N_2$ coincide as imbedded in $S^{2q+1}$.

We connect the boundaries of $D_0^{2q}$ and $D^{2q}$ with a path $\tau$ and then thickening $\tau$ to $\tau \times I^{2q-1} = Q$ avoiding all handles, and meeting $D$ and $D_0$ transversely in two $(2q - 1)$-disks. But $M_1 \cup Q \cup M_2$, with appropriate orientation, is just $M_1 \natural M_2$, the boundary connected sum of $M_1$ and $M_2$ [3]. Moreover, $M_1 \natural M_2$ is a Seifert manifold for the $(2q - 1)$-knot $K_1 \# K_2$. Similarly, $M_1 \natural N_2$ is a Seifert manifold for $K_1 \# J_2$. The special Seifert matrix for $L_1$ and $L_2$ is just a special Seifert matrix for both $K_1 \# K_2$ and $K_1 \# J_2$. Let $D_1 = D_0^{2q} \natural D^{2q} = D_0 \cup Q \cup D$. Then $M_1 \natural M_2$ and $M_1 \natural N_2$ have the following handle decompositions:

$$M_1 \natural M_2 = D_1 \cup \alpha_1 \cdots \cup \alpha_n \cup \beta_1 \cdots \cup \beta_m,$$

$$M_1 \natural N_2 = D_1 \cup \alpha_1 \cdots \cup \alpha_n \cup \gamma_1 \cdots \cup \gamma_m.$$

According to [5, p. 192], we can move one handle $\beta_i$ (onto $\gamma_i$) at a time by an isotopy in $S^{2q+1} - (D_1 \cup \alpha_1 \cdots \cup \alpha_n \cup \beta_1 \cup \cdots \cup \beta_{i-1})$. Thus we can map $M_1 \natural M_2$ diffeomorphically onto $M_1 \natural N_2$ by an isotopy in $S^{2q+1} -$

$(D_1 \cup \alpha_1 \cdots \cup \alpha_n)$. Since the thickened path $Q \subseteq D_1$, we see that $L_1 = K_1 \cup K_2$ is isotopic to $L_2 = J_1 \cup J_2$.   Q.E.D.

The next theorem follows from Lemmas 1 and 2 exactly as in [5, p. 189].

THEOREM 3. *Let* $L_1 = K_1 \cup K_2$ *and* $L_2 = J_1 \cup J_2$ *be two simple boundary* $(2q - 1)$-*links,* $q \geqslant 3$, *with l-equivalent Seifert matrices. Then* $L_1$ *is isotopic to* $L_2$.

5. A $(2q - 1)$-link $L = K_1 \cup K_2$ in $S^{2q+1}$ is splittable if there exist two disjoint $(2q + 1)$-balls $B_1$ and $B_2$ in $S^{2q+1}$ such that $K_1 \subseteq B_1$ and $K_2 \subseteq B_2$ [6, p. 110]. The next theorem follows immediately from Theorems 1–3.

THEOREM 4. *A simple boundary* $(2q - 1)$-*link* $L = K_1 \cup K_2$, $q \geqslant 3$, *is splittable if and only if it has a Seifert matrix of the form* $[A_1, A_2, 0]$.

## REFERENCES

1. M. A. Gutiérrez, *Boundary links and unlinking theorem*, Trans. Amer. Math. Soc. **171** (1972), 491–499.

2. M. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France **93** (1965), 225–271.

3. M. Kervaire and J. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537.

4. J. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

5. _____, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. **45** (1970), 185–198.

6. Y. Shinohara and D. W. Sumners, *Homology invariants of cyclic coverings with application to links*, Trans. Amer. Math. Soc. **163** (1972), 101–121.

7. H. F. Trotter, *On S-equivalence of Seifert matrices*, Invent. Math. **20** (1973), 173–207.

8. C. T. C. Wall, *Classification of (n − 1)-connected 2n-manifolds*, Ann. of Math. (2) **75** (1962), 163–189.

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