Math. Ann. 215, 47—58 (1975)

A Concordance Classification of p.l. Homeomorphisms of Real Projective Space*

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0. Two p.l. homeomorphisms f and g of a p.l. manifold M are called concordant if there is a p.l. homeomorphism $H: M\times I\to M\times I$, where $I=[0,1]$, such that $H(y,0)=(f(y),0)$ and $H(y,1)=(g(y),1)$. The set of concordance classes of p.l. homeomorphisms forms a group $\bar{D}(M)$ under the composition of maps. Let $D(M)$ denote the subgroup of $\bar{D}(M)$ consisting of the classes of those elements which are homotopic to id, the identity map (our homotopy is free of base point).

Let P^n denote the real projective space, which is the quotient space of the sphere S^n under the antipodal map A. We know that any p.l. homeomorphism of P^{2n} is homotopic to id, and any orientation-preserving p.l. homeomorphism of P^{2n+1} is homotopic to id [10]. In this paper, we will compute $D(P^n)$ for $n\geqq 4$. The main result is the following

Theorem. $D(P^4)=D(P^5)=1$. For $k>0$, $D(P^{4k+2})=D(P^{4k+3})=D(P^{4k+4})$ $=D(P^{4k+5})=kZ_2$, *the direct sum of* k *copies of* Z_2.

The paper is organized as follows: in Section 1, we use some basic facts about the surgery exact sequence to compute $hT(P^n\times I,\partial)$, and consider an onto homomorphism $\Psi: hT(P^n\times I,\partial)\to D(P^n)$. In Section 2, we use "equivariant coning" (2.1) to show that there exists a homomorphism from $D(P^n)$ to $D(P^{n+1})$. In Section 3, we show that except in one case $n=4k+1$, this homomorphism is onto. In Sections 4 and 5, we prove that $D(P^4)=D(P^5)=1$. An interesting byproduct is **Theorem 5.2**: *any h-cobordism of* P^4 *to itself is diffeomorphic to* $P^4\times I$; which is proved by using an argument similar to the one in the proof of [12, Theorem 1.4.]. In Section 6, we see that what makes the case $4k+1$ different from the others is the existence of two non-concordant embeddings of P^{4k+1} in P^{4k+2}, [1] or [8]. In the final section, we use the argument in [1] to prove the main theorem, and hence show that for $n\geqq 5$, the kernel of Ψ is Z_2 which is generated by equivariant suspending the element in (5.1).

The author wishes to thank Professors G. R. Livesay and I. Berstein for their constant help and encouragement.

1. Let $hT(P^n\times I,\partial)$ denote the set of equivalence classes of homotopy triangulations of $P^n\times I$ relative boundary [14]. An element y of $hT(P^n\times I,\partial)$ has a representative of the form (M^{n+1},g), where M is a p.l. manifold, $g: M\to P^n\times I$ is a homotopy equivalence such that $g: M\to\partial(P^n\times I)$ is a p.l. homeomorphism. (M,g) and (M',g') determine the same element if and only if there exists a p.l.

* Partially supported by NSF grant MPS 72-05055 A02.

homeomorphism $h: M \to M'$ such that g is homotopic to $g' \circ h$ rel boundary. If $n \geqq 5$, then M is p.l. homeomorphic to $P^n \times I$ by a p.l. homeomorphism G by the s-cobordism theorem. Thus $(M, g) = (P^n \times I, g \circ G)$. Let $f = g \circ G | P^n \times 0$, and $F = g \quad G \circ (f \times \mathrm{id})$. Hence $y = (M, g) = (P^n \times I, F)$. We define a map Ψ from $hT(P^n \times I, \partial)$ to $D(P^n)$, $n \geqq 5$, by $\Psi(y) = F | P^n \times 1$. Ψ can be shown to be well-defined and onto as in [15].

For $n \geqq 5$, we have the following surgery exact sequence [14]:

$$[\Sigma^2 P^n_+, G/PL] \xrightarrow{\sigma} L_{n+2}(Z_2, a) \xrightarrow{\omega} hT(P^n \times I, \partial) \xrightarrow{\eta} [\Sigma P^n_+, G/PL]$$
$$\xrightarrow{\sigma'} L_{n+1}(Z_2, a) \xrightarrow{\omega'} hT(P^n) \xrightarrow{\eta'} [P^n, G/PL],$$

where $a = +(-)$, if n is odd (even). For $n = 4$, the part of the above sequence from $[\Sigma^2 P^n_+, G/PL]$ to $L_5(Z_2, -)$ is exact also.

Ignoring odd torsions, we know that G/PL has the homotopy type of $Y \times \prod_{j \geqq 2} (K(Z_2, 4j-2) \times K(Z, 4j))$, where $Y = K(Z_2, 2) \times_{\delta Sq^2} K(Z, 4)$, [8] or [14]. For $n \geqq 4$, $[\Sigma P^n_+, K(Z, 4)] = 0$. Hence $[\Sigma P^n_+, Y] = [\Sigma P^n_+, K(Z_2, 2)] = Z_2$ as in [8, IV.2]. For $n \geqq 4k+1$, $[\Sigma P^n_+, K(Z_2, 4k+2)] = H^{4k+1}(P^n; Z_2) = Z_2$. $[\Sigma P^n_+, K(Z, 4k)] = H^{4k-1}(P^n) = 0$, if $n \neq 4k-1$; $= Z$ for $n = 4k-1$. Since $[\Sigma P^n_+, G/PL] = [\Sigma P^n_+, Y] \times \prod_{j \geqq 2} ([\Sigma P^n_+, K(Z, 4j-2)] \times [\Sigma P^n_+, K(Z, 4j)])$, we have the following

Proposition 1.1. *For* $k \geqq 0$, $[\Sigma P^{4k}_+, G/PL] = kZ_2$, $[\Sigma P^{4k+1}_+, G/PL] = [\Sigma P^{4k+2}_+, G/PL] = (k+1) Z_2$, $[Z P^{4k+3}_+, G/PL] = (k+1) Z_2 + Z$.

$L_n(Z_2, a)$, $hT(P^n)$ and $[P^n, G/PL]$ have been computed in [8, 14]. Substituting these into the surgery exact sequence, we have the following four exact sequences for $k \geqq 1$.

(1.2)

$$Z_2 \xrightarrow{\omega} hT(P^{4k} \times I, \partial) \xrightarrow{\eta} kZ_2 \to 0$$
$$Z_2 \xrightarrow{\omega} hT(P^{4k+1} \times I, \partial) \xrightarrow{\eta} (k+1) Z_2 \to Z_2 \xrightarrow{\omega'} hT(P^{4k+1})$$
$$Z_2 \xrightarrow{\omega} hT(P^{4k+2} \times I, \partial) \xrightarrow{\eta} (k+1) Z_2 \to 0$$
$$0 \xrightarrow{\omega} hT(P^{4k+3} \times I, \partial) \xrightarrow{\eta} (k+1) Z_2 + Z \to Z + Z \to Z + \text{torsion}.$$

The map $\omega': L_{4k+2}(Z_2, +) \to hT(P^{4k+1})$ is trivial: $L_{4k+2}(1) \to L_{4k+2}(Z_2, a)$ is an isomorphism [14, p. 164]. Hence the action of the Wall group is given by adding a Kervaire manifold to $P^{4k+1} \times I$ along the boundary $P^{4k+1} \times 1$, but the Kervaire sphere is just the ordinary sphere in the p.l. category. For $k = 1$, the surgery obstruction is given by a "wrong" framing [9], and the action is trivial also [12, p. 348]. Similarly, $\omega: L_{4k+2}(Z_2, -) \to hT(P^{4k} \times I, \partial)$ is trivial. Hence $[\Sigma^2 P^{4k}_+, G/PL] \xrightarrow{\sigma} L_{4k+2}(Z_2, -)$ is onto. But for $k \geqq 1$ we have the following commutative diagram as in [8, p. 46]:

$$\begin{tikzcd}
{[\Sigma^2 P^{4k+2}_+, G/PL]} \arrow[d] \arrow[dr, "\sigma"] & \\
{[\Sigma^2 P^{4k+1}_+, G/PL]} \arrow[r, "\sigma"] \arrow[d] & Z_2 . \\
{[\Sigma^2 P^{4k}_+, G/PL]} \arrow[ur, "\sigma"] &
\end{tikzcd}$$

Since the bottom map is onto, the other two maps are onto also. From the surgery exact sequence, we see that $\omega=0$ in (1.2). Thus, we have proved the following

Proposition 1.3. *$\eta: hT(P^n\times I,\partial)\to[\Sigma P_+^n, G/PL]$ is $1-1$ for $n\geqq 4$. For $k\geqq 1$, $hT(P^{4k}\times I,\ \partial)=hT(P^{4k+1}\times I,\ \partial)=kZ_2$, $hT(P^{4k+2}\times I,\ \partial)=hT(P^{4k+3}\times I,\ \partial)=(k+1)\,Z_2$.*

We can define a group structure in $hT(P^n\times I,\partial)$ as follows: let $(P^n\times I, F)$ and $(P^n\times I, G)\in hT(P^n\times I,\partial)$ such that $F|P^n\times 1=\mathrm{id}$ and $G|P^n\times 0=\mathrm{id}$. Write $F=(F_1,F_2)$ and $G=(G_1,G_2)$. Then define $(P^n\times I,F)*(P^n\times I,G)=(P^n\times I,F*G)$, where $F*G(y,t)=(F_1(y,t),\frac{1}{2}F_2(y,t))$ for $0\leqq t\leqq\frac{1}{2}$, $=(G_1(y,t),\frac{1}{2}G_2(y,2t-1))$ for $\frac{1}{2}\leqq t\leqq 1$. We can show that $(hT(P^n\times I,\partial),*)$ is a group and the map $\Psi: hT(P^n\times I,\partial)\to D(P^n)$ is a homomorphism (for details, see [15]).

Proposition 1.4. *For $n\geqq 5$, $D(P^n)$ is a finite abelian group.*

Proof. Since $[\Sigma P_+^n, G/PL]$ is abelian and $\eta: hT(P^n\times I,\partial)\to[\Sigma P_+^n, G/PL]$ is an $1-1$ homomorphism, $hT(P^n\times I,\partial)$ is abelian. Ψ is an onto homomorphism. Hence $D(P^n)$ is finite abelian by (1.3).

2. All the p.l. embeddings considered below are assumed to be locally flat.

Let P_1 and P_2 be the images of two p.l. embeddings of P^n in P^{n+1}; or equivariantly, let S_1 and S_2 be the images of two A-equivariant embeddings of S^n in S^{n+1} such that S_i covers P_i. By the Schöeflies theorem, each component B_{i+} (or B_{i-}) of the complement of S^n in S^{n+1} is p.l. homomorphic to the standard $(n+1)$-disk D^{n+1} by a p.l. homomorphism $h_{i+}(h_{i-})$. Also, $AB_{i+}=B_{i-}$, where A is the antipodal map.

Let f be a p.l. homomorphism from P_1 to P_2. We will construct a p.l. homomorphism g of P^{n+1} to itself as follows:

Definition 2.1. Equivariant coning: Let $\bar{f}: S_1\to S_2$ be the 2-fold covering of f. We define a p.l. homomorphism $k: D^{n+1}\to D^{n+1}$ by coning the p.l. homomorphism $h_{2+}\cdot\bar{f}\circ h_{1+}^{-1}$ on $S^n=\partial D^{n+1}$. Then we define $\bar{g}: S^{n+1}\to S^{n+1}$ by $\bar{g}|B_{1+}=h_{2+}^{-1}\circ k\circ h_{1+}$, and $\bar{g}|B_{1-}=A\circ(\bar{g}|B_{1+})\circ A$. $\bar{g}$ is an A-equivariant p.l. homomorphism. Degree $\bar{g}=\pm 1$. Since degree $(A: S^n\to S^n)=-1$ for n even, we can always make degree $\bar{g}=+1$ by either replacing $\bar{g}$ by $A\bar{g}$ or using $A\bar{f}$ instead of $\bar{f}$ in the above construction. $\bar{g}$ induces a p.l. homomorphism $g: P^{n+1}\to P^{n+1}$, which is said to be obtained from f by equivariant coning.

Suppose we replace h_{1+} by another p.l. homomorphism h'_{1+} in the above construction to get g'. $h'_{1+}\circ h_{1+}^{-1}: D^{n+1}\to D^{n+1}$ is concordant to id [6]. Hence g' is concordant to g by composing the concordance between h'_{1+} and h_{1+} with other maps in (2.1). If we replace f by a concordant map f', then we can apply equivariant coning to the p.l. homomorphism $\bar{F}$ of $W=S^{n+1}\times\{0,1\}\cup S^n\times I$ to itself, defined by $\bar{F}|S^{n+1}\times 0=\bar{g}$, $\bar{F}|S^{n+1}\times 1=\bar{g}'$, and $\bar{F}|S^n\times I$ is the 2-fold covering of the concordance between f and f', to get an equivariant concordance between $\bar{g}$ and $\bar{g}'$; because the complement of W in $S^{n+1}\times I$ consists of $2(n+2)$-disks.

Let P^n also denote the image of the natural inclusion of P^n in P^{n+1}. Given a p.l. homomorphism f of P^n to itself, we may use (2.1) to construct a p.l. homomorphism of P^{n+1} to itself, denoted by Sf. Sf is well-defined up to concordance and induces a map from $D(P^n)$ to $D(P^{n+1})$, which we also denote by S.

Proposition 2.2. *For $n \geqq 5$, $S: D(P^n) \to D(P^{n+1})$ is a homomorphism.*

Proof. We define a map S' from $hT(P^n \times I, \partial)$ to $hT(P^{n+1} \times I, \partial)$ as follows: an element $x = (P^n \times I, F) \in hT(P^n \times I, \partial)$ can be lifted to $(S^n \times I, \bar{F})$ with degree $\bar{F} = +1$. Viewing S^n as the equator of S^{n+1}, we denote the northern (southern) hemisphere by $D_+^{n+1}(D_-^{n+1})$. We define $\bar{G}: S^{n+1} \times I \to S^{n+1} \times I$ by $\bar{G} | S^n \times I = \bar{F}$, $\bar{G}|$ north pole $\times I = \mathrm{id}$, then extend linearly on each $D_+^{n+1} \times t$ for each $t \in I$; and $\bar{G} | D_-^{n+1} \times I = (A \times \mathrm{id}) \circ (\bar{G} | D_+^{n+1} \times I) \circ (A \times \mathrm{id})$. $\bar{G}$ is $(A \times \mathrm{id})$- equivariant, hence induces a homotopy equivalence $G: P^{n+1} \times I \to P^{n+1} \times I$ with $G | \partial(P^{n+1} \times I)$ a p.l. homomorphism. We define $S'x = (P^{n+1} \times I, G) \in hT(P^{n+1} \times I, \partial)$. Applying the same argument again, we can show that S' is well-defined.

Given $x, y \in hT(P^n \times I, \partial)$, we can choose $x = (P^n \times I, F)$ and $y = (P^n \times I, G)$ such $F | P^n \times 1 = G | P^n \times 0 = \mathrm{id}$. By looking at the difinition of $x * y$ in Section 1, we see immediately that $S'(x * y) = (S'x) * (S'y)$. Hence S' is a homomorphism.

Consider the following commutative diagram

$$\begin{tikzcd}
hT(P^n \times I, \partial) \arrow[r, "S'"] \arrow[d, "\Psi"'] & hT(P^{n+1} \times I, \partial) \arrow[d, "\Psi"'] \\
D(P^n) \arrow[r, "S"] & D(P^{n+1}).
\end{tikzcd}$$

Commutativity follows from the definitions. Since S' is a homomorphism and Ψ is an onto homomorphism, we see that S is a homomorphism. q.e.d.

3. In this section, we will show that the homomorphism $S: D(P^n) \to D(P^{n+1})$ defined in the last section is onto for $n \geqq 6$ and $n \neq 4k+1$.

We also write P^n for its image in P^{n+1} under the natural inclusion.

Lemma 3.1. *For $n \geqq 1$. If $x \in D(P^{n+1})$ has a representative f satisfying $f(P^n) = P^n$, then there exists $y \in D(P^n)$ such that $x = Sy$.*

Proof. For n even, let $g = f | P^n$. For n odd, if degree $(f | P^n) = +1$, let $g = f | P^n$; if degree $(f | P^n) = -1$, let $g = C \circ f | P^n$, where $\bar{C}$, the double cover of C, is defined on S^{n+1} by $\bar{C}(x_1, \ldots, x_{n+2}) = (x_1, \ldots, x_n, -x_{n+1}, -x_{n+2})$. $\bar{C}$ is concordant to id by rotations invariant under A. Hence we may replace f by $C \circ f$, and take $g = f | P^n$. We define a concordance $H: P^{n+1} \times I \to P^{n+1} \times I$ between f and Sg as follows: $H | P^{n+1} \times 0 = f$, $H | P^{n+1} \times 1 = Sg$, $H | P^n \times I = g \times \mathrm{id}$; lifting it up, we see that the domain on which H has not been defined is coverd by two disks D_+^{n+1} and D_-^{n+1} such that $(A \times \mathrm{id}) D_+^{n+1} = D_-^{n+1}$, hence we may finish the definition of H by equivariant coning as before. q.e.d.

Lemma 3.2. *For $n \geqq 6$ and $n \neq 4k+1$, any element of $D(P^{n+1})$ has a representative g satisfying $g(P^n) = P^n$.*

Proof. Given a p.l. homeomorphism f of P^{n+1} to itself such that f is homotopic to id, we know that P^n and $f(P^n)$ are concordant in P^{n+1} for $n \geqq 6$ and $n \neq 4k+1$, [1] or [8]: there exists an embedding $F: P^n \times I \to P^{n+1} \times I$ such that $F(P^n \times 1) = f(P^n) \times 1$, $F(P^n \times 0) = P^n \times 0$. We identify P^n with its image in P^{n+1} under the natural inclusion, hence we may assume $F|P^n \times 0 = \mathrm{id}$ by composing with some homeomorphism as in Section 1. Let $H = F^{-1}: F(P^n \times I) \to P^n \times I \subsetneqq P^{n+1} \times I$. We use (2.1) to construct $h: P^{n+1} \times 1 \to P^{n+1} \times 1$ from $H|F(P^n \times 1) \to P^n \times 1$. h is concordant to id by a concordance G as follows: $G|P^{n+1} \times 0 = \mathrm{id}$, $G|P^{n+1} \times 1 = h$, and $G|F(P^n \times I) = H$; we then finish the definition of G by equivariant coning as in (3.1). Since $h(f(P^n)) = P^n$, we take $g = h \circ f$. q.e.d.

From (3.1) and (3.2), we have the following

Proposition 3.3. *For $n \geqq 6$ and $n \neq 4k+1$, $S: D(P^n) \to D(P^{n+1})$ is onto.*

4. In this section, we are going to prove two lemmas which will be needed in Section 5.

We write $S^3 = \left\{(x_1, \ldots, x_4) \,\middle|\, \sum_{1 \leqq i \leqq 4} x_i^2 = 1\right\}$ and $S^1 = S^3 \cap \{x_3 = x_4 = 0\}$. We can also consider S^1 as parametrized by z, $0 \leqq z \leqq 2\pi$. Let $j = (0, 0, 1, 0)$ and $k = (0, 0, 0, 1)$ denote two vectors. Then $j(z) = j$ and $k(z) = k$ together form a framing for the normal bundle of S^1 in S^3. If we apply the Thom-Pontrjagin construction to S^1 and this framing, then we get the trivial element in $\pi_3(S^2) = Z$ [7] or [11].

Let A denote the antipodal map. On S^1, we have $Az = z + \pi \bmod 2\pi$. $P^i = S^i/A$. Since P^1 and P^3 are parallelizable, the normal bundle of P^1 in P^3 is trivial. Given an orthonormal framing (a_1', a_2') of this bundle, we can pull it back by the projection $p: S^1 \to P^1$ to an orthonormal framing (a_1, a_2) of the normal bundle of S^1 in S^3. Let $a_i(z) = a_{i1}(z)\, j + a_{i2}(z)\, k$. Since (a_1, a_2) covers (a_1', a_2'), we have $Aa_i = a_i A$, i.e. $-a_{pq}(z) = a_{pq}(z + \pi)$.

Lemma 4.1. *Let (a_1', a_2') and (a_1, a_2) denote the framings as above. If we apply the Thom-Pontrjagin construction to S^1 and (a_1, a_2) in S^3, then we always obtain an odd integer in $\pi_3(S^2) = Z$.*

Proof. The Thom-Pontrjagin construction gives us a homomorphism from the framings of the normal bundle of S^1 in S^3 to $\pi_3(S^2) = Z$ by mapping (a_1, a_2) to the degree of the map f_a, where $f_a(z) = (a_{pq}(z)) \in SO(2) = S^1$ [7, 11].

Consider the framing (c_1, c_2) on S defined by $c_1(z) = \cos(z)\, j + \sin(z)\, k$, $c_2(z) = -\sin(z)\, j + \cos(z)\, k$. Since $\cos(z + \pi) = -\cos(z)$ and $\sin(z + \pi) = -\sin(z)$, we see that $Ac_i = c_i A$. Thus (c_1, c_2) induces a framing (c_1', c_2') for the normal bundle of P^1 in P^3. Furthermore, the degree of the map f_c, where $f_c(z) = (c_{pq}(z)) \in SO(2) = S^1$, is 1.

Suppose (a_1, a_2) is another framing on S^1 covers a framing (a_1', a_2') on P^1. $Aa_i = a_i A$. We define a new framing (b_1, b_2) on S^1 by $b_{pq}(z) = \sum_k c_{pk}(z)\, a_{kq}(z)$. degree f_b = degree f_c + degree f_a = degree $f_a + 1$. But $b_{pq}(z + \pi) = \Sigma c_{pk}(z + \pi) \cdot a_{kq}(z + \pi) = (-1)^2 b_{pq}(z) = b_{pq}(z)$. Hence $f_b = S^1 \to SO(2)$ factors through P^1. Since $p: S^1 \to P^1 = S^1$ is of degree 2, we see that degree f_b is even. Thus degree f_a is odd. q.e.d.

Let d denote the nontrivial element of $\pi_4(P^3) = Z_2$. We may assume d is transverse regular to P^1. Then $d^{-1}(P^1) = U$ is a 2-dim submanifold of S^4. Let F denote the framing of U induced by an arbitrary framing G for the normal bundle of P^1 in P^3 via d. Pontrjagin defined a cobordism invariant – the Kervaire invariant $c(U, F)$ for such pair (U, F) in S^4 [11, p. 101].

Lemma 4.2. *The above Kervaire $c(U, F)$ is not zero.*

Proof. Lifting d to $b: S^4 \to S^3$, we see that b is transverse to S^1.

$$\begin{tikzcd} S^4 \arrow[r, "b"] \arrow[dr, "d"'] & S^3 \arrow[r, "r"] \arrow[d] & S^2 \\ & P^3 & \end{tikzcd}$$

Let $(S^1, H) = p^{-1}(P^1, G)$, where G is an arbitrary framing on P as above. Then $(U, F) = d^{-1}(P^1, G) = b^{-1}(S^1, H)$. If we apply the Thom-Pontrjagin construction to (S^1, H), then the element $r \in \pi_3(S^2) = Z$ thus obtained is odd by (4.1). Since d is nontrivial, b is nontrivial in $\pi_4(S^3) = Z_2$. Hence b is the suspension of the generator of $\pi_3(S^2)$, and the composition $r \circ b$ in $\pi_4(S^2)$ is non-zero [5]. Let y be the point in S^2 such that $r^{-1}(y) = S^1$. $f = r \circ b$ is transverse regular to y by construction, and $f^{-1}(y$, the standard framing$) = (U, F)$. Since the Kervaire invariant $c(U, F)$ and the Thom-Pontrjagin construction give an isomorphism of $\pi_4(S^2)$ with Z_2 [11], we see that $c(U, F)$ is non-zero. q.e.d.

5. Propositions (1.1) and (1.2) tell us that the map η: $hT(P^4 \times I, \partial) \to [\Sigma P^4_+, G/PL] = Z_2$ is $1-1$. We are going to find a homotopy equivalence $g; P^4 \times I \to P^4 \times I$ with $g | \partial(P^4 \times I) = \mathrm{id}$, and $\eta(g) \neq 0$ in $[\Sigma P^4_+, G/PL]$.

We know that $\pi_4(S^3) = \pi_4(P^3) = Z_2$. Let d denote the nontrivial element of $\pi_4(P^3) = \pi_4(P^3 \times I) = Z_2$ by $d: S^4 \to P^3 \times \frac{1}{2} \subseteq P^3 \times I$.

We choose a 4-disk D in $P^3 \times (1/4, 3/4)$ such that $D \cap P^3 \times \frac{1}{2} = \emptyset$ and $D \cap P^1 \times I = \emptyset$. Then we define a homotopy equivalence h: $P^3 \times I \to P^3 \times I$ such that $h = \mathrm{id}$ outside D and the obstruction of h to id $rel P^3 \times I - \mathrm{int} D$ is $d \in H^4(P^3 \times I, P^3 \times I - \mathrm{int} D; \pi_4 P^3) = Z_2$.

Considering $P^1 \times I \subseteq P^3 \times I$, $P^1 \times I \cap P^3 \times \frac{1}{2} = P^1 \times \frac{1}{2}$, we may assume $d: S^4 \to P^3 \times \frac{1}{2}$ is transverse regular to $P^1 \times \frac{1}{2}$. Then $d^{-1}(P^1 \times \frac{1}{2}) = U$ is a 2-dim submanifold of S^4. If F is the framing of U induced by an arbitrary framing for the normal bundle of P^1 in P^3 via d, then the Kervaire invariant $c(U, F)$ is non-zero by (4.2).

The rest of the argument is almost the same as the one used in the proof of [12, Theorem 1.4]. Without changing $d^{-1}(P^1 \times \frac{1}{2}) = U$, we may alter d to make $d | D_1 \to D$ a diffeomorphism on a small disk $D_1 \subseteq S^4$. Identifying D with the complement $S^4 - \mathrm{int} D_1$, we may choose the above $h: P^3 \times I \to P^3 \times I$ such that $(h|D) \cup (d|D_1) = d$. Then h will be transverse regular to $P^1 \times I$, and $h^{-1}(P^1 \times I) = P^1 \times I \cup U = W$. Let $f = h | W: W \to P^1 \times I$, a map of degree 1 on $P^1 \times I$ and degree 0 on U, because f factors through $P^1 \times \frac{1}{2}$.

Now, we construct a homotopy equivalence $g: P^4 \times I \to P^4 \times I$ as follows: let $\bar{h}: S^3 \times I \to S^3 \times I$ denote the double cover of h, we define $\bar{g} = \bar{h}$ on $S^3 \times I$ and $g = \mathrm{id}$ on $S^4 \times \partial I$. Viewing h as a p.l. map, we may complete the definition of $\bar{g}$ by equivariant coning as in the proof of (3.1). Hence $\bar{g}$ covers a homotopy equivalence $g: P^4 \times I \to P^4 \times I$ with $g | \partial(P^4 \times I) = \mathrm{id}$ and $g^{-1}(P^1 \times I) = h^{-1}(P^1 \times I) = P^1 \times I \cup U = W f = h | W = g | W$ is a degree 1 map. Arguing as in p. 350 of [12], we have a induced surgery problem (W, f, H) and the Kervaire surgery obstruction $c(W, f, H) = c(U, F) = 1$.

Suppose $\eta(g)=0$. Since η is $1-1$, we see that g is homotopic rel boundary to a p.l. homeomorphism. Making the homotopy transverse regular to $P^1\times I$ rel boundary, we get a cobordism of (W,f,H) to (W',f',H') such that $f: W'\to P^1\times I$ is a homotopy equivalence, indeed a p.l. homeomorphism. But the Kervaire invariant is a cobordism invariant [2], so this is impossible. Hence $\eta(g)\neq 0$. Thus we have proved the following

Proposition 5.1. *There exists a homotopy equivalence $g: P^4\times I\to P^4\times I$ with $g=\mathrm{id}$ on the boundary, and $\eta(g)\neq 0$ in $[\Sigma P^4_+, G/PL]$.*

We can deduce the following theorem from (5.1).

Theorem 5.2. *Any h-cobordism of P^4 to itself is* p.l. *homeomorphic to $P^4\times I$.*

Proof. Let W be an h-cobordism with $\partial W=P^4_0\cup P^4_1$ and $f: W\to P^4\times I$ a homotopy equivalence with $f^{-1}(P^4\times i)=P^4_i$. Since every homotopy equivalence of P^n is homotopic to a homeomorphism, we have a homotopy equivalence $f': W'=P^4_0\times I\cup W\cup P^4_1\times I\to P^4\times I$ such that $f'|\partial W'$ is a p.l. homeomorphism. Thus $(W',f')\in hT(P^4\times I,\partial)$, a set consists of two elements: $(P^4\times I,\mathrm{id})$ and $(P^4\times I,g)$ constructed in (5.1). In either case, W', hence W, is homeomorphic to $P^4\times I$. q.e.d.

Since $\pi_i(PL/0)=\Gamma_i=0$ for $i\leqq 6$, we have the following

Theorem 5.2′. *Any h-cobordism between P^4 to itself is diffeomorphic to $P^4\times I$.*

Proposition 5.3. $D(P^4)=1$.

Proof. Given a p.l. homeomorphism f of P^4, we have a homotopy F between f and id. As in (5.2), $(P^4\times I,F)$ is equivalent to either the identity or $(P^4\times I,g)$. If it is equivalent to the latter, then we may replace F by $F*g$. Hence there always exists a p.l. homeomorphism $K: P^4\times I\to P^4\times I$ such that $\mathrm{id}\circ K$ is homotopic to F rel boundary. Thus K is a concordance between $K_0=\mathrm{id}$ and $K_1=f$. q.e.d.

Similarly, we have the following

Proposition 5.3′. *Any diffeomorphism of P^4 is concordant to the identity.*

Proposition 5.4. *Let $T: S^5\to S^5$ be a differentiable involution with two fixed points, then T is equivalent to an orthogonal action.*

Proof. Around each fixed point, the action is orthogonal. Cut out two small invariant neighborhood of the fixed points from S^5. The orbit space of the region left is an h-cobordism of P^4 to itself, hence diffeomorphic to $P^4\times I$ by (5.2′). Thus the action (T,S^5) is equivalent to $(A,D^5)\bigcup_f(A,D^5)$, where f is an A-equivariant diffeomorphism of S^4 to itself. Hence (T,S^5) is equivalent to $(A,D^5)\bigcup_{\mathrm{id}}(A,D^5)$, the standard action $(x_1,\ldots,x_6)\to(x_1,-x_2,\ldots,-x_6)$, by (5.3′). q.e.d.

Proposition 5.5. *$S: D(P^4) \to D(P^5)$ is onto, hence $D(P^5) = 1$.*

Proof. All we have to show is Lemma 3.2. is true for $n=4$ also. Let f be a p.l. homeomorphism of P^5 homotopic to id. We know from Theorem 6.1. of [3] that P^4 and $f(P^4)$ are concordant. The concordance between them is an h-cobordism, which is a product $P^4 \times I$ by (5.2). Then we can carry out our proof as in (3.2). By (3.1), we see that $S: D(P^4) \to D(P^5)$ is onto. q.e.d.

6. We always identify P^n with its image in P^{n+1} under the natural inclusion.

Proposition 6.1. *For each $4k+2$, $k \geqq 1$, there exists a* p.l. *homeomorphism g_k of P^{4k+2} to itself such that g_k is non-concordant to* id.

Proof. For $k \geqq 1$, there exists an embedding f of P^{4k+1} in P^{4k+2} such that $Q^{4k+1} = f(P^{4k+1})$ is not concordant to P^{4k+1}, [1, 8]. Let h be any p.l. homeomorphism from P^{4k+1} to Q^{4k+1}. We then apply equivariant coning (2.1), to h and obtain a p.l. homeomorphism g_k of P^{4k+2} to itself. g_k is not concordant to id: if it were, then the image of the concordance between g_k and id, when restricted to $P^{4k+1} \times I$, would be a concordance between P^{4k+1} and Q^{4k+1}, a contradiction. q.e.d.

We will write $\{g_k\}$ for the concordance class which contains g_k.

Proposition 6.2. *To every $x \in D(P^{4k+2})$, there exists $y \in D(P^{4k+1})$ such that that $x = Sy$ or $\{g_k\} \circ Sy$.*

Proof. Let h be a representation for x. $h(P^{4k+1})$ is concordant to either P^{4k+1} or Q^{4k+1} in (6.1), [1] or [8]. In the first case, we can proceed as we did in Section 3 to show that $x = Sy$. We also can reduce the second case to the first by considering $g_k^{-1} \circ h$. q.e.d.

From (3.3), (5.5), and (6.2); we know that any element of $D(P^n)$, $n \geqq 6$, has a representative of the following form: $S^{a_t}(g_{b_t} \circ S^{a_{t-1}}(g_{b_{t-1}} \circ \cdots S^{a_1} g_{b_1}) \ldots)$, with positive intergers a_j and b_j such that $4 | a_j$ for $j < t$, $a_t + 4b_t + 2 = n$, and $b_i = (a_{i-1}/4) + b_{i-1}$.

Given an element $(P^n \times I, F) \in hT(P^n \times I, \partial)$, we can make F transverse regular rel boundary on $P^m \times I$, $4 \leqq m \leqq n$. For $m \neq 4k+1$, F is homotopic rel boundary to H such that H induces a homotopy equivalence $H^{-1}(P^m \times I) \to P^m \times I$, which can be seen as follows: if n is even, then we can use theorems (10.5) and (8.1) of [4] to show inductively that the above assertion is true for $m = n-2, n-4, \ldots$; and $m = 4k-1$ follows from this, the proof of [1, Theorem 1], and the Theorem on [8, p. 16]. The case n is odd can be reduced to the above case by Theorem 6.1. of [3] and the theorem on [8, p. 16].

From (1.1) and (1.3), we know that the map c:

$$hT(P^n \times I, \partial) \to \prod_k [\Sigma P_+^n, K(Z_2, 4k+2)]$$

induced by η, $2 \leqq 4k+2 \leqq n$, is $1-1$. Let c_{4k+2} denote the component of c to each summand. These values can be detected by the surgery obstructions: given an element $y = (P^n \times I, F)$ as in the last paragraph, $c_{4k+2}(y)$ is the surgery obstruction (Arf invariant) of the induced normal map $H^{-1}(P^{4k+1} \times I) \to P^{4k+1} \times I$.

c_{4k+2} can be related to the Browder-Livesay invariant for the uniqueness of desuspension, [1] or [3] as follows: Let G_k be a homotopy from id to the map g_k in (6.1) for $k \geqq 1$; and let G_0 be the map g in (5.1). Using S' defined in (2.2), we see that the element $y = S'^{a_t}(G_{b_t} \circ S'^{a_{t-1}}(\ldots G_{b_0})\ldots)$ satisfying $c_{4k+2}(y) = 1$ for $k = b_i$; $= 0$, otherwise. But $c_{4b+2}(y)$, $b \geqq 1$, is the Browder-Livesay invariant for P^{4b+1} and $g_b(P^{4b+1})$ in P^{4b+2}.

7. We will compute $D(P^n)$ for $n \geqq 6$ in this section. As in the previous sections, we identify P^n with its image in P^{n+1} under the natural inclusion.

Lemma 7.1. $n \geqq 6$, $1 < 4k+1 < n$. *There exists a homotopy equivalence* $F: P^n \times I \to P^n \times I$ *with* $F = \mathrm{id}$ *on the boundary and* $F^{-1}(P^{4r+1} \times I) \to P^{4r+1} \times I$ *is of Arf invariant* 1 *for some r; if and only if there exists submanifolds* V^{m+1} *in* $P^n \times I$, $1 < 4k+1 \leqq m < n$ *for some k, with* $\partial V^{m+1} = P^m \times 0 \cup P^m \times 1$ *such that* $V^t \subsetneqq V^{t+1}$, V^{m+1} *is* p.l. *homeomorphic to* $P^m \times I$ *for* $m \neq 4k+1$, *but* $V^{4k+2} = P^{4k+1} \times I \# K^{4k+2}$, *where* K^{4k+2} *is the Kervaire manifold.*

Proof. Let k be the largest number among such r's. The only if part has been proved in Section 6 and in [1].

Conversely, consider $U^{m+1} = P^m \times I$ in $P^n \times I$, $4k+1 \leqq m < n$, and N^{4k+2} is p.l. homeomorphic to $P^{4k+1} \times I \# K^{4k+2}$ with $\partial U = P^{4k+1} \times 1 \cup Q^{4k+1} \times 0$, where P^{4k+1} and Q^{4k+1} are two non-concordant embeddings in P^{4k+2} [1]. Gluing two copies of of $P^n \times I$ together to get another copy of $P^n \times I$. Let $Y^{m+1} = V^{m+1} \cup U^{m+1}$. For $m > 4k+1$, Y^{m+1} is p.l. homeomorphic to $P^m \times I$, with $\partial Y = P^m \times \partial I$: and we may apply equivariant handle exchanges in the interior of Y^{4k+3} to make Y^{4k+2} p.l. homeomorphic to $P^{4k+1} \times I$ with boundary $= P^{4k+1} \times 1 \cup Q^{4k+1} \times 0$ [3].

Then we use the equivariant coning (2.1) repeatedly to construct a p.l. homeomorphism $H: P^n \times I \to P^n \times I$: we map Y^{4k+2} to $P^{4k+1} \times I \subsetneqq P^n \times I$, and use (2.1) to extend to $P^{4k+2} \times \partial I$; then we extend to a p.l. homeomorphism from Y^{4k+3} to $P^{4k+2} \times I$ as in (3.2), and repeat the procedure. Consider the homotopy equivalence $G_k: P^{4k+2} \times I \to P^{4k+2} \times I$ between g and id in Section 6. Let $G = S'^t G_k$, where $t = m - 4k - 2$. We define $F = H * G$ as in Section 1. q.e.d.

Lemma 7.2. $n \geqq 6$. *Let* $F: P^n \times I \to P^n \times I$ *be a homotopy equivalence with* $F = \mathrm{id}$ *on the boundary. We can make F transverse regular to* $P^{4r+1} \times I$ rel *boundary for* $1 < 4r+1 < n$. *Then the induced normal map* $F^{-1}(P^{4r+1} \times I) \to P^{4r+1} \times I$ rel *boundary is of Arf invariant* 0 *for each r.*

Proof. The proof is by induction. Proposition (6.1) and (7.1) showed that (7.2) is true for $n = 6$. We now assume the lemma is true for all $t < n$, but not n: there exists certain r with $4r+1 < n$ which doesn't satisfy the assertion of (7.2). Let k denote the largest of such r's. There are four cases:

(a) $n = 4h$. By using [1, Theorem 1] and [8, p. 16, Theorem] as in the last section, we may change F through homotopy rel boundary to make $F^{-1}(P^{n-1} \times I)$ p.l. homeomorphic to $P^{n-1} \times I$. V, the double cover of $F^{-1}(P^{n-1} \times I)$, and $S^{n-1} \times I$ are characteristic submanifolds for $(S^n \times I, A \times \mathrm{id})$ in the sense of [3]. By a similar argument as in the proof of [3, Lemma 3.2], we assert that there is a characteristic submanifold W for $(S^n \times I \times I, A \times \mathrm{id})$ with $\partial W \cap S^n \times I \times 1 = V \times 1$, $\partial W \cap S^n \times I \times 0 = S^{n-1} \times I \times 0$, and $\partial W \cap S^n \times \partial I \times I = S^{n-1} \times \partial I \times I$. Since $\dim W = 4h+1$ is odd, we may apply equivariant handle exchanges in the interior of $S^n \times I \times I$ as in [3] to make W equivariantly p.l. homeomorphic to $S^{n-1} \times I \times I$. Hence there exists a p.l. embedding $H: P^{n-1} \times I \times I \to P^n \times I \times I$ with $H(P^{n-1} \times I \times I) = W/(A \times \mathrm{id})$, $H(P^{n-1} \times I \times 0) = P^{n-1} \times I \times 0 \subsetneq P^n \times I \times 0$, $H(P^{n-1} \times I \times 1) = F^{-1}(P^{n-1} \times I) \times 1$. As in the proof of (3.2), we may compose H with a p.l. homeomorphism of $P^{n-1} \times I \times I$ to itself to make $H | P^{n-1} \times I \times 0 =$ the natural inclusion. Consider the homotopy equivalence $G = \mathrm{id} \cup F \cup \mathrm{id}: Y \to Y$, where $Y = P^n \times 1 \times I \cup P^n \times I \times 1 \cup P^n \times 0 \times I$. Y is p.l. homeomorphic to $P^n \times I$. We define a homotopy equivalence $K: P^{n-1} \times I \to P^{n-1} \times I$ by $K = G \circ H$ on $P^{n-1} \times I = P^{n-1} \times 1 \times I \cup P^{n-1} \times I \times 1 \cup P^{n-1} \times 0 \times I$. Since we identify P^{n-1} with its image under the natural inclusion $i: P^{n-1} \to P^n$ and $H | P^{n-1} \times I \times 0 =$ the natural inclusion, $K = \mathrm{id}$ on the boundary. The induced map $K^{-1}(P^{4k+1} \times I) \to P^{4k+1} \times I$ is of Arf invariant 1, which contradicts the induction hypothesis [we also can prove (a) by using (7.1) as in (d) below].

(b) $n = 4h+2$. (6.1) and (7.1) tell us that $k \neq h$. Hence we can make $F^{-1}(P^{n-1} \times I)$ p.l. homeomorphic to $P^{n-1} \times I$. Then we can proceed as we did in the case (a) for the dimensional reason [3].

(c) $n = 4h+1$. We may change F through homotopy rel boundary to make $F^{-1}(P^{n-1} \times I)$ p.l. homeomorphic to $P^{n-1} \times I$ by Theorem 6.1 of [3] and [8, p. 16, Theorem]. As in (a), we have a characteristic submanifold W^{4h+2} in $S^{4h+1} \times I \times I$ such that $\partial(W/T) = P^{4h} \times I \times 0 \cup P^{4h} \times \partial I \times I \cup F^{-1}(P^{4h} \times I) \times 1$; where $T = A \times \mathrm{id}$. We apply equivariant handle exchanges to make W $2h$-connected. A reverses the orientation in P^{4h}, hence T reverses the orientation in W. Since $\dim W = 4h = 2$, the bilinear form $B(x, y) = x \cdot T_* y$ defined $H_{2h+1}(W)$, modulo its torsion, is symmetric. We can use the same argument as in [3] to show that the index of B, denoted by $c(W, T)$, is the Browder-Livesay invariant – the obstruction to get a concordance between $P^{4h} \times I \times 0$ and $F^{-1}(P^{4h} \times I) \times 1$ in $P^{4h+1} \times I \times I$ rel boundary. But by using exactly the same argument as the proof of Theorem 1 of [1, pp. 58—63], we can show that $c(W, T)$ is equal to the half of the difference between c_1 and c_2, where c_1 is the obstruction to get a concordance between $P^{4h-1} \times 0 \times 0$ and $P^{4h-1} \times 1 \times 0$ in $P^{4h} \times I \times 0$ and c_2 is the obstruction between $P^{4h-1} \times 0 \times 1$ and $P^{4h-1} \times 1 \times 1$ in $F^{-1}(P^{4h} \times I) \times 1 \cdot c_1 = 0$ by [1, Theorem 1]. $c_2 = 0$ follows from the proof of [1, Theorem 1]. Thus $c(W, T) = 0$. Hence we may apply equivalent bundle exchanges to make W a product. Then we just use the argument in the second half of (a) to finish the proof.

(d) $n = 4h+3$[1]. By (7.1) above, we may assume there exist submanifold V^{m+1} in $P^n \times I$ satisfying the assertions in (7.1).

As in (a), we may assume the existence of a characteristic submanifold W^{n+1} for $(S^n \times I \times I, A \times \mathrm{id})$ with $\partial W \cap S^n \times I \times 0 = S^{n-1} \times I \times 0$, $\partial W \cap S^n \times I \times 1 = V^{4h+2} \times 1$, $\partial W \cap S^n \times \partial I \times I = S^{n-1} \times \partial I \times I$. We may make W $(2h+1)$-connected. $T = A \times \mathrm{id}$ reverses the orientation in W. Since $\dim W = n+1 = 4h+4$, the bilinear form $B(x, y) = x \cdot T_* y$ defined on $H_{2h+2}(W)$ is skew-symmetric. Hence we are in the Arf invariant case [3]. Let $c(W, T) \in Z_2$ be the Browder-Livesay invariant defined by this form B as in [3]. If $c(\mathrm{W}, T) = 0$, then we can make W a product, and we may proceed as in (a).

[1] This is essentially the proof of the Corollary in [8, p. 83].

We now suppose $c(W, T)=1$. We have seen in Section 1 that the action of the Wall group $L_{4h+4}(Z_2, -)=Z_2$, which is given by the Arf-Kervaire invariant, [13, 14, p. 162], on $hT(P^{4h+2}\times I, \partial)$ is trivial. Hence we have a map of triad $\phi:(M;\partial_- M, \partial_+ M)\to(P^{4h+2}\times I\times I; P^{4h+2}\times I\times 0\cup P^{4h+2}\times\partial I\times I, P^{4h+2}\times I\times 1)$ of degree 1 satisfying the assertions of [14, Theorem 5.8]. Then we use Lemma 1 of [1] and the argument in the proofs of Lemma 8 and Theorem 3 of [1] to show that $\bar{M}$, the double cover of M, can be embedded as a characteristic submanifold for $(S^{4h+3}\times I\times I, A\times \mathrm{id})$ with the Browder-Livesay invariant $c(\bar{M}, T)=1$ such that $\partial M\cap P^{4h+3}\times I\times 0=P^{4h+2}\times I\times 0$, $\partial M\cap P^{4h+3}\times\partial I\times I=P^{4h+2}\times\partial I\times I$. Also there exist concordance N^{r+1} between $P^r\times 0\times 1$ and $P^r\times 1\times 1$ for all $r<4h+2$, $N^r\subseteqq N^{r+1}$ and $K^{4h+3}=U=\partial M\cap P^{4h+3}\times I\times 1$: which follows from the trivialily of the action of the Wall group above.

Joining two copies of $P^{4h+3}\times I\times I$ together along $P^{4h+3}\times I\times 0$ and reparametrizing the last factor, we obtain a characteristic submanifold $R=\bar{M}\cup W$ for $(S^{4h+3}\times I\times I, T)$ such that $\partial(R/T)\cap P^{4h+3}\times I\times 0=U$, $\partial(R/T)\cap P^{4h+3}\times I\times 1 =F^{-1}(P^{4h+2}\times I)\times 1$, $\partial(R/T)\cap P^{4h+3}\times\partial I\times I=P^{4h+2}\times\partial I\times I$. The Browder-Livesay invariant $c(R, T)=1+1=0\in Z_2$; hence we may make R equivariantly a product; there exists a p.l. embedding $H; P^{4h+2}\times I\times I\to P^{4h+3}\times I\times I$ with the image of $H=R$ as in the case (a), we may assume $H|P^{4h+2}\times 1\times I=$ the natural inclusion. For $4k+1\leqq r\leqq 4h+2$, the submanifolds $H^{-1}(N^{r+1}\cup P^r\times 0\times I\cup V^{r+1})$ in $P^{4h+2}\times I=P^{4h+2}\times I\times 0\cup P^{4h+2}\times 0\times I\cup P^{4h+2}\times I\times 1$ satisfy the conditions in (7.1); hence contradict the induction hypothesis by (7.1). q.e.d.

From (7.1) and (7.2), we see easily that a p.l. homeomorphism of the form $S^{a_t}(g_{b_t}\circ S^{a_{t-1}}(g_{b_{t-1}}\circ\cdots)\ldots)$ is not concordant to id; otherwise the element $G=F*(S'^{a_t}G_{b_t})$, where F is the concordance, would contradict (7.2). From this, (1.4), (2.2), and (5.5); we have the following:

Theorem 7.3. *For* $h\geqq 1$, $D(P^{4h+2})=D(P^{4h+3})=D(P^{4h+4})=D(P^{4h+5})=hZ_2$, *the direct sum of h copies of Z_2.*

References

1. Berstein, I., Livesay, G. R.: Non-unique desuspension of involutions. Inventiones math. **6**, 56—66 (1968)
2. Browder, W.: Surgery on simply-connected manifolds. Band 65 Ergebn. der Math. Berlin-Heidelberg-New York: Springer 1972
3. Browder, W., Livesay, G. R.: Fixed point free involutions on homotopy spheres. Tohôku Math. J. **25**, 69—88 (1973)
4. Cappell, S., Shaneson, J.: The codimension 2 placement problem and homology equivalent manifolds. Annals of Math. 277—348 (1974)
5. Hu, S. T.: Homotopy theory. New York: Academic Press 1959
6. Hudson, J. F. P.: Piecewise linear topology. New York: Benjamin 1969
7. Kervaire, M. A.: An interpretation of G. Whitehead's generalization of H. Hopf's invariant. Ann. of Math. **69**, 345—365 (1959)
8. Lopez de Medrano, S.: Involutions on manifolds. Band 59. Ergebn. der Math. Berlin-Heidelberg-New York: Springer 1971
9. Novikov, S. P.: Homotopy equivalent smooth manifolds. I. Translation A.M.S. **48**, 271—396 (1965)

10. Olum, P.: Mappings of manifolds and the notion of degree. Annals of Math. **58**, 458—480 (1953)
11. Pontrjagin, L. S.: Smooth manifolds and their applications in homotopy theory. Translations A.M.S. **11** (1959)
12. Shaneson, J. L. : Non-simply-connected surgery and some results in low dimensional topology. Common. Math. Helv. **45**, 333—352 (1970)
13. Wall, C. T. C.: Surgery of non-simply-connected manifolds Ann. of Math. **84**, 217—276 (1966)
14. Wall, C. T. C.: Surgery on compact manifolds New York: Academic Press 1970
15. Wang, K.: Free S^1 actions and the group of diffeomorphisms. Transaction A.M.S. **191**, 113—128 (1974)

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(Received September 4, 1974)