Vol. 5 No. 2 Systems Science and Mathematical Sciences May, 1992

A SIMPLIFIED APPROACH TO EMBEDDING PROBLEMS IN NORMAL BORDISM FRAMEWORK

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Abstract. The purpose of this paper is to simplify the computations of the normal bordism groups $\Omega_i(W_f, M \times P^\infty; \psi_f)$ and $\Omega_i(\mathcal{C}_f, \partial W; \theta_f)$ which Salomonsen and Dax introduced respectively to study the existence and isotopy classification of differential embeddings of manifolds in manifolds in the metastable range. A simpler space pair $(K_f, M \times P^\infty)$ is constructed to replace $(W_f, M \times P^\infty)$. It is shown that $(K_f, M \times P^\infty)$ is homotopy equivalent to $(W_f, M \times P^\infty)$ and homotopy $(n-1)$-equivalent to $(\mathcal{C}_f, \partial W)$. To demonstrate the efficacy of this simplification, the isotopy groups $[M^n \subset \mathbf{R}P^{n+k}]$, if $n \le 2k-4$ and M^n is a closed $(n-k+2)$-connected manifold, and $[M^n \subset L(p; q_1, \cdots, q_m)]$, if $3n \le 4m-2$, M^n is a closed $(2n-2m+1)$-connected manifold and L is a $(2m+1)$-dimensional lens space, are specifically computed.

Key words. Normal bordism group, differential embedding, isotopy.

§1. Introduction

The existence and isotopy classification of differential embeddings of manifolds in manifolds may be approached in a normal bordism framework by converting the problem into the study of normal bordism groups $\Omega_i(X, A; \psi)$ of a certain elaborate space pair (X, A) ([1] and [2]). Partly owing to the complicated construction of (X, A), the known such settings, however, are not very convenient for the computation of the concerned bordism groups. In this paper, we introduce a simpler space pair $(K_f, M \times P^\infty)$ which simplifies Salomonsen's approach in [1]. We also compare our setting with Dax's construction in [2] so as to unify the approaches in [1] and [2]. Our simplification facilitates the computations and enables us to attack the embedding problems in the metastable range on a larger scale. Applications in this respect with more concrete computations will appear in [3], [4] and other forthcoming papers. The purpose of the present paper is to provide a proper setting for the computations and demonstrate the efficacy by several isotopy classification results.

Throughout the paper we shall follow Salomosen's definition and notation of normal bordism groups in [5]. Manifolds shall always mean differential smooth manifolds in the C^∞ sense and embeddings shall always mean C^∞ embeddings.

In §2 we recall Salomensen's program for embedding problems in the normal bordism framework, namely the construction of $(W_f, M \times P^\infty)$ for a generic map $f : M^n \longrightarrow$

Received December 8, 1990.

U^{n+k} of manifolds and the embedding obstruction $\varepsilon(f)$ in $\Omega_{n-k}(W_f, M \times P^\infty; \psi)$. Our simplified space pair $(K_f, M \times P^\infty)$ is introduced in §3 and is shown to be of the same homotopy type as $(W_f, M \times P^\infty)$. In §4 we recall Dax's construction of the space pair $(\mathcal{C}_f, \partial W)$ and prove that there is an $(n-1)$-equivalence between $(K_f, M \times P^\infty)$ and $(\mathcal{C}_f, \partial W)$, which reveals the essential unity of the different approaches in the metastable range. In §5 we compute, as examples, the group $[M^n \subset U]$ of isotopy classes of embeddings of an $(n-k+2)$-connected compact n-manifold without boundary in $U = P^{n+k}$, the real projective space, or $U = L(p; q_1, \cdots, q_m)$, the $(2m+1)$-dimensional lens space. The main theorems are as follows.

Theorem 1.1. *Suppose that $n \leq 2k-4$ and M^n is an $(n-k+2)$- connected compact manifold without boundary. Then M^n embeds in P^{n+k} and*

$$[M^n \subset P^{n+k}] = \pi_{2^{\varphi(n+k-1)}-k}(V_{2^{\varphi(n+k-1)}+k-1,n+k}),$$

where $\varphi(n+k-1)$ is the number of m with $0 \leq m \leq n+k-1$ and $m \equiv 0,1,2,4$ mod 8, and $V_{q,\ell}$ is the Stiefel manifold of orthogonal ℓ-frames in $\mathbf{R}^q$.

Theorem 1.2. *Suppose that $3n \leq 4m-2$ and M^n is a $(2n-2m+1)$-connected compact manifold without boundary. Then M embeds in $L(p; q_1, \cdots, q_m)$, and*

$$[M^n \subset L(p; q_1, \cdots, q_m)] = \begin{cases} \bigoplus\limits_{1\leq j \leq \frac{1}{2}(p-1)} \pi^S_{2n-2m} & \text{if } p \text{ is odd,} \\ \pi_{2^{\varphi(2m)}-2m+n-1}(V_{2^{\varphi(2m)}+2m-n,2m+1}) \oplus \bigoplus\limits_{1\leq j<\frac{1}{2}p} \pi^S_{2n-2m} & \text{if } p \text{ is even,} \end{cases}$$

where π^S_k is the k-th stable homotopy group.

In particular, we have

Corollary 1.3. *If $n \geq 2$,*

$$[S^n \subset P^{2n+1}] = \begin{cases} \mathbf{Z} & \text{if } n \text{ is odd,} \\ \mathbf{Z}_2 & \text{if } n \text{ is even.} \end{cases}$$

Corollary 1.4. *If $n \geq 4$,*

$$[S^n \subset P^{2n}] = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 4, \\ \mathbf{Z}_4 & \text{if } n \equiv 1 \bmod 4, \\ \mathbf{Z}_2 & \text{if } n \equiv 2 \bmod 4, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

This result has been obtained by Larmore [6] using a different method based on Haefliger's embedding theory [7].

Corollary 1.5. *If $n \geq 8$,*

$$[S^n \subset P^{2n-2}] = \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } n \equiv 0 \bmod 4, \\ \mathbf{Z}_4 & \text{if } n \equiv 1 \bmod 4, \\ \mathbf{Z}_2 & \text{if } n \equiv 2 \bmod 4, \\ \mathbf{Z}_4 \oplus \mathbf{Z}_{16} & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

Corollary 1.6. *If* $n = m+1, m \geq 5$,

$$[S^n \subset L(p; q_1, \cdots, q_m)] = \begin{cases} \bigoplus_{\frac{1}{2}(p-1)} \mathbf{Z}_2, & \text{if } p \text{ is odd}, \\ (\bigoplus_{\frac{1}{2}p-1} \mathbf{Z}_2) \oplus \begin{cases} \mathbf{Z}_2 & \text{if } n \equiv 0 \bmod 4, \\ 0 & \text{if } n \equiv 1 \bmod 4, \\ \mathbf{Z}_8 & \text{if } n \equiv 2 \bmod 4, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } n \equiv 3 \bmod 4. \end{cases} & \text{if } p \text{ is even}. \end{cases}$$

Corollary 1.7. *If* $n = m+2, m \geq 8$,

$$[S^n \subset L(p; q_1, \cdots, q_m)] = 0 \quad \text{if } p \text{ is odd},$$

$$[S^n \subset L(p; q_1, \cdots, q_m)] = \begin{cases} \mathbf{Z}_8 & \text{if } n \equiv 0 \bmod 8. \\ 0 & \text{if } n \equiv 1 \bmod 8, \\ 0 & \text{if } n \equiv 2 \bmod 8, \\ \mathbf{Z}_2 & \text{if } n \equiv 3 \bmod 8, \\ \mathbf{Z}_{16} & \text{if } n \equiv 4 \bmod 8, \\ \mathbf{Z}_2 & \text{if } n \equiv 5 \bmod 8, \\ 0 & \text{if } n \equiv 6 \bmod 8, \\ \mathbf{Z}_2 & \text{if } n \equiv 7 \bmod 8, \end{cases} \quad \text{if } p \text{ is even}.$$

Remark. The special case of our Theorem 1.2 where p is odd and M^n is a sphere has been recently proved by F. Al-Rubaee in the categary of PL-manifolds [10].

§2. Salomonsen's Program for Embedding Problems in Normal Bordism Framework

Let M^n and U^{n+k} be manifolds and let $f: M \to U$ be a map. We define a space $EP(M) = (M \times M \times S^\infty)/\mathbf{Z}_2$, where $S^\infty = \bigcup_{n=1}^\infty S^n$ is the infinite unit sphere and the involution on $M \times M \times S^\infty$ is given by $(x_1, x_2, s) \longmapsto (x_2, x_1, -s)$. The space $EP(U)$ is defined in the same way and a map $EP(f): EP(M) \to EP(U)$ may be induced by $f \times f \times id$. The diagonal map $U \to U \times U$ induces an inclusion $d_U : U \times P^\infty \to EP(U)$. Then W_f may be defined as the following pull-back space in homotopy category:

$$\begin{array}{ccc} W_f & \xrightarrow{p_2} & U \times P^\infty \\ \downarrow p_1 & & \downarrow d_U \\ EP(M) & \xrightarrow{EP(f)} & EP(U) \end{array} \tag{2.1}$$

i.e. W_f consists of triples (u, v, ρ) with $u \in EP(M)$, $v \in U \times P^\infty$, and ρ a path in $EP(U)$ from $EP(f)(u)$ to $d_U(v)$. Note that $M \times P^\infty$ may be identified with a subspace of W_f which is the image of the injective map determined by a stationary homotopy of $EP(f) \circ d_M = d_U \circ (f \times id)$.

Suppose that M is compact without boundary and that $n \leq 2k-1$. Then the set of generic map forms an open, dense subspace of $C^\infty(M,U)$ ([8] and [1]). Therefore the problem of the existence of embeddings can be reduced to finding an embedding in the homotopy class of some generic map.

For a generic map $f: M \longrightarrow U$, the union $\Delta' \coprod \Sigma$ of the double point set Δ' and the singular point set Σ forms an $(n-k)$-dimensional submanifold of M. The free involution T naturally defined on Δ' may be extended to $\Delta = \Delta' \coprod \Sigma$ with Σ as the fixed point set. Then $\bar{\Delta} = f(\Delta)$ is an $(n-k)$-dimensional submanifold of U with boundary $\partial\bar{\Delta} = f(\Sigma)$. We have the following commutative diagram:

$$\begin{array}{ccccc} & & \Delta' & \xrightarrow{\varphi'} & S^\infty \\ & & \downarrow f & & \downarrow \\ \bar{\Delta} & \xrightarrow{\gamma} & f(\Delta') & \xrightarrow{\bar{\varphi}} & P^\infty \end{array}, \tag{2.2}$$

where γ is a homotopy equivalence, $\bar{\varphi}$ a classifying map of the double covering $f: \Delta' \to f(\Delta')$ and φ' a $\mathbf{Z}_2$-equivariant map over $\bar{\varphi}$. Then the $\mathbf{Z}_2$-equivariant map $\Delta' \to M \times M \times S^\infty$ by $x \mapsto (x, T(x), \varphi'(x))$ induces a map $f(\Delta') \to EP(M)$ which clearly extends to some map $a: \bar{\Delta} \to EP(M)$. Another map $b: \bar{\Delta} \to U \times P^\infty$ may be defined by $z \mapsto (z, \bar{\varphi} \circ \gamma(z))$ so that $EP(f) \circ a = d_U \circ b$. Therefore, by the universal property of the pull-back diagram (2.1), the maps a and b together with the stationary homotopy of $EP(f) \circ a = d_U \circ b$ uniquely determine a map $\delta f: (\bar{\Delta}, \partial\bar{\Delta}) \to (W_f, M \times P^\infty)$.

Consider the total space of the tangent bundle τM as a manifold and let $\pi: \tau M \to M$ be the bundle projection. Then $EP(\pi): EP(\tau M) \to EP(M)$ is a vector bundle over $EP(M)$ of dimension $2n$. To describe an obstruction to the existence of embeddings in a normal bordism group, we define vector bundles $\psi_+ = p_1^* EP(\tau M)$ and $\psi_- = p_2^*(\tau U \tilde{\otimes} \lambda) \oplus \varepsilon^{n-k}$ over W_f, where λ is the canonical line bundle over P^∞ and p_1 and p_2 are maps defined in Diagram (2.1), and consider the normal bordism group of the space pair $(W_f, M \times P^\infty)$ with "coefficients" in the virtual bundle $\psi_+ - \psi_-$ in $\widetilde{KO}(W_f)$. It is proved in Lemma 4.1 of [1] that there is a split exact sequence of vector bundles

$$0 \longrightarrow \tau\bar{\Delta} \longrightarrow (\delta f)^*\psi_+ \longrightarrow (\delta f)^*(p_2^*(\tau U \tilde{\otimes} \lambda)) \longrightarrow 0.$$

The splitting of this sequence determines a stable bundle isomorphism

$$\overline{\delta f}: \tau\bar{\Delta} \oplus (\delta f)^*\psi_- \longrightarrow \varepsilon^{n-k} \oplus (\delta f)^*\psi_+$$

in the sense of [5].

Now the triple $(\bar{\Delta}, \delta f, \overline{\delta f})$ defines a ψ-manifold over $(W_f, M \times P^\infty)$ in the sense of [5], where $\psi = \psi_+ - \psi_-$. The normal bordism class of this triple is denoted by $\varepsilon(f) \in \Omega_{n-k}(W_f, M \times P^\infty; \psi)$. Note that, if $f: M \to U$ is itself an embedding, then $\varepsilon(f) = 0$ since the corresponding $\bar{\Delta}$ is an empty manifold.

Theorem 2.1(Corollary 5.2 of [1]). *Let $f: M^n \to U^{n+k}$ be a generic map with M compact without boundary. Suppose that $n \leq 2k-3$. Then f is homotopic to an embedding if and only if $\varepsilon(f) = 0$ in $\Omega_{n-k}(W_f, M \times P^\infty; \psi)$.*

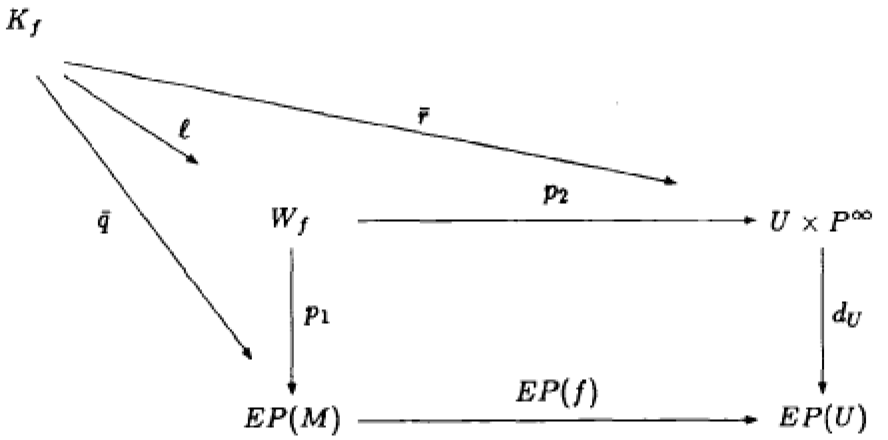
In [1] the obstruction $\varepsilon(f)$ is defined for every map $f : M \to U$ using isomorphism of the bordism group induced by a homotopy of f to a generic map. Therefore the above theorem holds for any map $f : M \to U$.

§3. A Simplified Construction

Let $f : M^n \to U^{n+k}$ be a map. Define Ω_f as $\{(x, y, \sigma) | f(x) = \sigma(-1), f(y) = \sigma(1)\}$, a subspace of the product $M \times M \times U^J$, where $J = [-1, 1]$. The natural projection $q : \Omega_f \to M \times M$ is a fibration with fibre $\Omega(U)$, the space of based loops in U. Furthermore, there is an involution $(x, y, \sigma) \mapsto (y, x, \sigma^{-1})$, where $\sigma^{-1}(t) = \sigma(-t)$. The manifold M may be identified with the fixed point subspace $\{(x, x, \sigma_{f(x)}) | x \in M\}$, where $\sigma_{f(x)}$ is the constant path at $f(x)$, of this $\mathbf{Z}_2$-action. We define an induced free involution T' on the product space $\Omega_f \times S^\infty$ by $(x, y, \sigma; s) \mapsto (y, x, \sigma^{-1}; -s)$, where s and $-s$ are a pair of antipodal points in S^∞. Denote by K_f the quotient space of $\Omega_f \times S^\infty$ by this free $\mathbf{Z}_2$-action. It is clear that $M \times P^\infty$ may be identified with the subspace of K_f consisting of points $\langle x, x, \sigma_{f(x)}; s \rangle$.

We shall show that the space pair $(W_f, M \times P^\infty)$ in Salomonsen's program may be replaced by $(K_f, M \times P^\infty)$ defined above.

The $\mathbf{Z}_2$-equivariant map $q \times id : \Omega_f \times S^\infty \to M \times M \times S^\infty$ induces a map $\bar{q} : K_f \to EP(M)$, and the $\mathbf{Z}_2$-equivariant map $r \times$ id: $\Omega_f \times S^\infty \to U \times S^\infty$ induces a map $\bar{r} : K_f \to U \times P^\infty$, where $r : \Omega_f \to U$ is defined by $r(x, y, \sigma) = \sigma(0)$. A homotopy $H : K_f \times I \to EP(U)$ from $EP(f) \circ \bar{q}$ to $d_U \circ \bar{r}$ may be defined by $(\langle x, y, \sigma; s \rangle, t) \mapsto [\sigma(t-1), \sigma(1-t), s]$, where $EP(f)$ and d_U are as defined in Diagram (2.1) of §2. A map $\ell : K_f \to W_f$ is thus determined uniquely by the universal property of the pull-back diagram with $p_1 \circ \ell = \bar{q}$ and $p_2 \circ \ell = \bar{r}$:



(3.1)

The restriction of ℓ on $M \times P^\infty$ is the identity map via the respective identifications of $M \times P^\infty$ in K_f and W_f.

Proposition 3.1. *The map ℓ embeds the space pair $(K_f, M \times P^\infty)$ into $(W_f, M \times P^\infty)$ as a deformation retract.*

Proof. Define a $\mathbf{Z}_2$-action on the path space $(S^\infty)^I$ via an involution $\overline{T}(\alpha)(t) = -\alpha(t)$. We consider K_f as a subspace of $\overline{K}_f = (\Omega_f \times (S^\infty)^I)/\mathbf{Z}_2$ by identifying each point in S^∞ with a constant path in $(S^\infty)^I$, so that S^∞ is a $\mathbf{Z}_2$-equivariant deformation retract of $(S^\infty)^I$. It follows that K_f is a deformation retract of $\overline{K}_f$.

We shall show that ℓ can be extended to a homeomorphism $L : \bar{K}_f \to W_f$ and thus complete the proof.

For $\langle x, y, \sigma; \alpha\rangle \in \overline{K}_f$, define $L : \overline{K}_f \to W_f$ by

$$L(\langle x, y, \sigma; \alpha\rangle) = ([x, y, \alpha(0)], (\sigma(0), \overline{\alpha(1)}), \rho),$$

where $\overline{\alpha(1)} \in P^\infty$ is the image of $\alpha(1) \in S^\infty$, and $\rho : I \to EP(U)$ is a path given by $\rho(t) = [\sigma(t-1), \sigma(1-t), \alpha(t)]$. It is straightforward to verify that L is a well-defined map and its restriction on K_f is the map ℓ.

To show that L is a homeomorphism, we define its inverse as follows.

For a point $([x, y, s], (z, \overline{s'}), \rho) \in W_f$, take a lifting $\widetilde{\rho} : I \to U \times U \times S^\infty$ of the path $\rho : I \to EP(U)$ so that $\widetilde{\rho}(0) = (f(x), f(y), s)$. Denote by pr_i the projection of $U \times U \times P^\infty$ on its i-th factor, $i = 1, 2, 3$. Since $pr_1 \circ \widetilde{\rho}\,(1) = z = pr_2 \circ \widetilde{\rho}(1)$ by the definition of the path ρ, we may define a path $\sigma : J \to U$ by

$$\sigma(t) = \begin{cases} pr_1 \circ \widetilde{\rho}(1+t), & -1 \le t \le 0, \\ pr_2 \circ \widetilde{\rho}(1-t), & 0 \le t \le 1. \end{cases}$$

The correspondence $L' : ([x, y, s], (z, \overline{s'}), \rho) \mapsto \langle z, y, \sigma; pr_3 \circ \widetilde{\rho}\rangle$ is a well-defined map $W_f \to \overline{K}_f$ and, clearly, is the inverse of L.

Recall the "coefficient" bundle $\psi = \psi_+ - \psi_-$ in §2, and define a virtual bundle $\phi = \phi_+ - \phi_-$ over K_f by $\phi_+ = \bar{q}^* EP(\tau M)$ and $\phi_- = \bar{r}^*(\tau U \widetilde{\otimes} \lambda) \oplus \varepsilon^{n-k}$. It is clear that $\ell^* \psi = \phi$. This, together with Proposition 3.1, implies

Proposition 3.2. *The homotopy equivalence $\ell : (K_f, M \times P^\infty) \to (W_f, M \times P^\infty)$ induces an isomorphism $\ell_* : \Omega_n(K_f, M \times P^\infty; \phi) \to \Omega_n(W_f, M \times P^\infty; \psi)$ for each $n \ge 0$.*

For a generic map $f : M^n \to U^{n+k}$, $\ell_*^{-1}(\varepsilon(f))$ is clearly the embedding obstruction in $\Omega_{n-k}(K_f, M \times P^\infty; \phi)$. More precisely, the bordism class $\ell_*^{-1}(\varepsilon(f))$ may be described as follows.

Observe that the correspondence $x \mapsto (x, T(x), \sigma_{f(x)}; \varphi'(x))$ defines a $\mathbf{Z}_2$-equivariant map $\Delta' \to \Omega_f \times S^\infty$, where $\varphi' : \Delta' \to S^\infty$ is as defined in Diagram (2.2) in §2. There is an induced map $f(\Delta') \to K_f$ which clearly extends to $\delta' f : \overline{\Delta} \to K_f$. It is straightforward to verify that, by definitions, $\delta f = \ell \circ \delta' f$ and, hence, there corresponds a commutative diagram of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau\bar{\Delta} & \longrightarrow & (\delta' f)^* \phi_+ & \longrightarrow & (\delta' f)^*(p_2^*(\tau U \widetilde{\otimes} \lambda)) & \longrightarrow & 0 \\ & & \| & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \tau\bar{\Delta} & \longrightarrow & (\delta f)^* \psi_+ & \longrightarrow & (\delta f)^*(p_2^*(\tau U \widetilde{\otimes} \lambda)) & \longrightarrow & 0 \end{array},$$

which defines a stable bundle isomorphism in the sense of [5]

$$\overline{\delta' f} : \tau\bar{\Delta} \oplus (\delta' f)^* \phi_- \longrightarrow \varepsilon^{n+k} \oplus (\delta' f)^* \phi_+.$$

The bordism class $\varepsilon'(f)$ of the triple $(\bar{\Delta}, \delta' f, \overline{\delta' f})$ satisfies $\ell_*(\varepsilon'(f)) = \varepsilon(f)$. We have proved

Theorem 3.3. *Let $f : M^n \to U^{n+k}$ be a generic map with M compact without boundary. Suppose that $n \leq 2k-3$. Then f is homotopic to an embedding if and only if $\varepsilon'(f) = 0$.*

Let U^M be the space of maps $M \to U$ and let E denote the subspace consisting of all embeddings. Let $[M \subset U]_f$ denote the set of isotopy classes of embeddings with a specific homotopy to f, i.e. $\pi_1(U^M, E, f)$. If f is homotopic to an embedding, the set of isotopy classes of embeddings homotopic to f is the orbit set of $[M \subset U]_f$ under the obviously-defined action of the group $\pi_1(U^M, f)$(see [1] and [6]). Combining Proposition 3.2 and the results in §6 of [1], we have

Theorem 3.4. *Let $f : M^n \to U^{n+k}$ be a map with M compact without boundary. Suppose that $n \leq 2k-4$. If f is homotopic to an embedding, then there is a bijection*

$$\theta : \Omega_{n-k+1}(K_f, M \times P^\infty; \phi) \longrightarrow [M \subset U]_f$$

which induces a bijection

$$\bar{\theta} : \Omega_{n-k+1}(K_f, M \times P^\infty; \phi)/\pi \longrightarrow [M \subset U]_f/\pi,$$

where $\pi = \pi_1(U^M, f)$.

§4. Comparison with Dax's Approach

Let (X, A) be a pair of spaces and let ξ^N be an N-dimensional vector bundle over X with $N \geq i+2$. Dax's version of a normal bordism group is an Abelian group $\overline{\Omega}_i(X, A; \xi^N)$ consisting of bordism classes of triples (Δ, b, B), where Δ^i is a compact submanifold of the disc D^{N+i} with $\partial\Delta = \Delta \cap S^{N+i-1}$, $b : (\Delta, \partial\Delta) \to (X, A)$ is a map and $B : \nu^N(\Delta) \to \xi^N$ is a vector bundle isomorphism over the map b of the normal bundle of $(\Delta, \partial\Delta)$ in (D^{N+i}, S^{N+i-1}). We refer to Chap. I of [2] for the detailed definition and properties of $\overline{\Omega}_i(X, A; \xi^N)$. Here we use the notation $\overline{\Omega}_i$ for Dax's version to distinguish it from Salomonsen's $\Omega_i(X, A; \phi_+ - \phi_-)$. The following proposition may be obtained by comparing the two definitions.

Proposition 4.1. *Suppose that there is a k'-dimensional vector bundle ϕ'_+ over X with $k' \geq i+1$ such that $\phi'_+ \oplus \phi_+ = \varepsilon^{k+k'}$. Then $\overline{\Omega}_i(X, A; \phi_- \oplus \phi'_+)$ is isomorphic to $\Omega_i(X, A; \phi_+ - \phi_-)$.*

In Chap. IV of [2], the embedding problem is converted into the study of the groups $\overline{\Omega}_i(\mathcal{C}_f, \partial W; \theta_f)$ for $i \geq 0$. We describe the construction of the space pair $(\mathcal{C}_f, \partial W)$ briefly as follows.

Let $f : M^n \to U^{n+k}$ be a map with M compact without boundary. Denote by T_ε, for $\varepsilon \geq 0$, the quotient space of $S(\tau M) \times [-\varepsilon, \varepsilon]$ by the equivalence relation $(s, u) \sim (-s, -u)$. The manifold $\widetilde{W}^{2n}$ is defined as the adjunction space $T_\varepsilon \bigcup_e (M \times M - \Delta_M)$ for a sufficiently small $\varepsilon > 0$, where $e : T_\varepsilon - T_0 \to M \times M - \Delta_M$ is an embedding given by $[s, u] \mapsto (\exp(-us), \exp(us))$. There is an obvious involution $\widetilde{T}$ on $\widetilde{W}^{2n}$ such that $\widetilde{T}(x, y) = (y, x)$ when restricted on $M \times M \times -\Delta_M$. The induced quotient manifold W^{2n} is compact with ∂W identified with the fixed point subspace of $\widetilde{T}$. Note that $\widetilde{T}$ induces a $\mathbf{Z}_2$-action $\widetilde{T}_1$ on $W \times \mathbf{R}$ with $\widetilde{T}_1(x, u) = (\widetilde{T}(x), -u)$ so that there is a unique line bundle ω over W and a vector bundle isomorphism $r_1 : \varepsilon^1 \to \omega$ over the quotient map $r : \widetilde{W} \to W$. Since $x \mapsto r_1(x,1)$ defines an embedding of $\widetilde{W}$ into $S(\omega)$, there is a map $S(\omega) \to \widetilde{W} \to M \times M$. Denote by $\Omega(M_f; M)$ the space of maps $c : J \to M_f$ of $J = [-1,1]$ to the mapping cylinder with $c(\{-1,1\}) \subset M$. The natural projection $\Omega(M_f;M) \to M \times M$ given by $c \mapsto (c(-1), c(1))$ is a fibration. Finally, we consider the fibre product $S(\omega) \times_{M\times M} \Omega(M_f; M)$ over $M \times M$ on which there is a free involution $(x, c) \mapsto (-x, c^{-1})$, where c^{-1} denotes the inverse path of c. The space pair $(\mathcal{C}_f, \partial W)$ is then defined as the quotient space obtained from the pair $(S(\omega) \times_{M\times M} \Omega(M_f; M), S(\omega|\partial W) \times_{M\times M} M)$, in which M is identified with the subspace of constant paths.

The following proposition reveals that the different approaches to the embedding problems in the metastable range are essentially the same.

Proposition 4.2. *There exists an $(n-1)$-equivalence $\mu : (\mathcal{C}_f, \partial W) \to (K_f, M \times P^\infty)$ which induces an isomorphism $\mu_* : \overline{\Omega}_i(\mathcal{C}_f, \partial W; \theta_f) \to \Omega_i(K_f, M \times P^\infty; \phi_+ - \phi_-)$ for $i < n-1$.*

Proof. We first observe that $S(\omega)$ may be identified with the adjunction space $(S(\tau M) \times [0,\varepsilon]) \cup_d (M \times M - \Delta_M)$ for a sufficiently small $\varepsilon > 0$, where $d : S(\tau M) \times (0, \varepsilon] \to M \times M - \Delta_M$ is an embedding given by $(s,u) \to (\exp(-us), \exp(us))$. The free involution on $S(\omega)$ determines the double covering $S(\omega) \to W$ with classifying map $\bar{\varphi} : W \to P^\infty$. We also observe that the map $\alpha : \Omega(M_f; M) \to \Omega_f$ defined by $\alpha(c) = (c(-1), c(1), p_U \circ c)$ is a $\mathbf{Z}_2$-equivariant fibre homotopy equivalence over $M \times M$, where $p_U : M_f \to U$ is the projection. The diagrams of maps

$$\begin{array}{ccc} (S(\omega) \times_{M\times M} \Omega(M_f;M), S(\tau M)) & \xrightarrow{id\times\alpha} & (S(\omega)\times_{M\times M} \Omega_f, S(\tau M)) \\ \downarrow pr_1 & & \downarrow pr_1 \\ (S(\omega), S(\tau M)) & \xrightarrow{id} & (S(\omega), S(\tau M)) \end{array} \quad , \tag{4.1}$$

$$\begin{array}{ccc} (S(\omega) \times_{M\times M} \Omega_f, S(\tau M)) & \xrightarrow{\varphi\times id} & (S^\infty \times \Omega_f, S^\infty \times M) \\ \downarrow pr_1 & & \downarrow id \times q \\ (S(\omega), S(\tau M)) & \xrightarrow{\varphi\times\pi} & (S^\infty \times M \times M, S^\infty \times \Delta_M) \end{array} \tag{4.2}$$

and

$$\begin{array}{ccc} (S(\omega) \times_{M\times M} \Omega(M_f;M), S(\tau M)) & \xrightarrow{\varphi\times\alpha} & (S^\infty \times \Omega_f, S^\infty \times M) \\ \downarrow & & \downarrow \\ (\mathcal{C}_f, \partial M) & \xrightarrow{\mu} & (K_f, P^\infty \times M) \end{array} \tag{4.3}$$

are commutative, where $S(\tau M) = S(\omega|\partial W) \times_{M\times M} M$, and $\varphi : S(\omega) \to S^\infty$ is the $\mathbf{Z}_2$-equivariant map over the classifying map $\bar{\varphi} : W \to P^\infty$.

We shall show that $\varphi \times \alpha$ in (4.3) is an $(n-1)$-equivalence so that μ is an $(n-1)$-equivalence which we are searching for.

Diagram (4.2) is a pull-back diagram of fibration pair $id \times q$. Hence, exploiting the homotopy sequences and the Five Lemma, we know that $\varphi \times id$ is an $(n-1)$-equivalence if and only if $\varphi \times \pi$ is. Since $id \times \alpha$ is a homotopy equivalence, it remains to prove that $\varphi \times \pi$ is an $(n-1)$ -equivalence.

Observe that the composition $g : M \times M - \Delta_M \hookrightarrow S(\omega) \xrightarrow{\varphi\times\pi} S^\infty \times M \times M \xrightarrow{pr_2} M \times M$ is an inclusion. Since every map of a sphere of dimension $\leq n-1$ to a $2n$-manifold is homotopic to an embedding, it follows from the transversality theorem that g must be an $(n-1)$-equivalence. The inclusion $M \times M - \Delta_M \hookrightarrow S(\omega)$ is a homotopy equivalence because $S(\omega) - (M \times M - \Delta_M)$ is just the boundary of $S(\omega)$. This implies that $\varphi\times\pi : S(\omega) \to S^\infty \times M \times M$ is an $(n-1)$-equivalence. The restriction $\varphi\times\pi : S(\tau M) \to S^\infty \times M$ is also an $(n-1)$-equivalence, for $pr_2 \circ (\varphi\times\pi) : S(\tau M) \to M$ is a fibration with fibre S^{n-1}.

Take νM as the n-dimensional normal bundle of M. Then $EP(\tau M) \oplus EP(\nu M)$ is a trivial vector bundle. It follows from Proposition 4.1 that, to complete the proof, we need only to verify $\mu^*(\bar{r}^*(\tau U \tilde{\otimes} \lambda) \oplus \varepsilon^{n-k} \oplus \bar{q}^* EP(\nu M)) = \theta_f$.

Recall from p.339 of [2] that $\theta_f = P^*(\omega \tilde{\otimes} \tau U \oplus (\nu(W)^{N-2n} \times U))$, where $P = P_1 \times P_2 : C_f \to W \times U$ and N may be taken as $4n$ which is large enough for our purpose. (Here P_1 is the map p defined in [2], which is the quotient of the $\mathbf{Z}_2$-equivariant map $pr_2 : S(\omega) \times_{M\times M} \Omega(M_f; M) \to S(\omega)$, and P_2 is the map q defined in [2] which may be identified with $pr_1 \circ \bar{r} \circ \mu$, where $pr_1 : U \times P^\infty \to U$ and $\bar{r} : K_f \to U \times P^\infty$ is as in (3.1) of §3). It follows that $P^*(\omega \tilde{\otimes} \tau U) \cong P_1^*\omega \otimes P_2^* \tau U = P_1^* \bar{\varphi}^* \lambda \otimes \mu^* \circ \bar{r}^* \circ pr_1^* \tau U = \mu^* \circ \bar{r}^* \circ pr_2^* \lambda \otimes \mu^* \circ \bar{r}^* \circ pr_1^* \tau U = \mu^*(\bar{r}^*(\tau U \tilde{\otimes} \lambda))$, where $\bar{\varphi}$ is the classifying map of the line bundle ω. Denote by W' the submanifold of W which is the quotient of $S(\omega) - S(\tau M) \times [0,\varepsilon)$. The restriction $\nu W|_{W'}$ may be identified with $(\nu M \times \nu M)/\mathbf{Z}_2|_{W'}$ since $S(\omega) - S(\tau M) \times [0,\varepsilon) \subset M \times M - \Delta_M$ so that $W' \subset (M \times M - \Delta_M)/\mathbf{Z}_2$. There is a deformation retraction $r' : W \to W'$ such that $r'^*((\nu M \times \nu M)/\mathbf{Z}_2|_{W'}) \cong \nu W|_{W'}$. We observe that in

$$\begin{array}{ccccc} C_f & \xrightarrow{P_1} & W & \xrightarrow{r'} & W \\ \downarrow \bar{q}\circ\mu & & \downarrow \bar{h} & & \downarrow id \\ EP(M) & \xrightarrow{\overline{pr}_2} & (M\times M)/\mathbf{Z}_2 & \xhookleftarrow{j} & W \end{array} \tag{4.4}$$

the square is commutative while the triangle on the right is homotopy commutative. Note that $\overline{pr}_2^*(\nu M \times \nu M)/\mathbf{Z}_2) = EP(\nu M)$, where $pr_2 : S^\infty \times M \times M \to M \times M$ is the projection. Therefore, $P^*((\nu W) \times U) = P_1^* W \cong P_1^* \circ r'^*(\nu W|_{W'}) \cong P_1^* \circ r'^*((\nu M \times \nu M)/\mathbf{Z}_2|_{W'}) \cong P_1^* \circ r'^* \circ j^*((\nu M \times \nu M)/\mathbf{Z}_2) \cong \mu^* \circ \bar{q}^* \circ \overline{pr}_2^*((\nu M \times \nu M)/\mathbf{Z}_2) \cong \mu^* \circ \bar{q}^* EP(\nu M)$.

The proof is completed.

§5. Computations of $[M^n \subset P^{n+k}]$ and $[M^n \subset L(p; q_1, \cdots, q_m)]$

To demonstrate by examples the facilitation to computation resulting from the simpler construction, we first discuss the relation between the fundamental groups of the manifolds M and U and the set of path components of the space K_f.

The group $\pi_1(M) \times \pi_1(M)$ operates on $\pi_1(U)$ as a set in the way $c \mapsto f_*(a^{-1}) \cdot c \cdot f_*(b)$, where $(a,b) \in \pi_1(M) \times \pi_1(M)$ and $c \in \pi_1(U)$. We denote by A the set of orbits of this operation and define an equivalence relation on A by $\langle c \rangle \sim \langle c^{-1} \rangle$.

Proposition 5.1. *There is a one-one correspondence between the set of equivalence classes on A and the set of path components of the space K_f.*

Proof. Take a base point x_0 in M. We observe that each path component of Ω_f contains a point of the form (x_0, x_0, σ), where σ is a loop in U at $f(x_0)$. Suppose that the points (x_0, x_0, σ) and (x_0, x_0, σ') lie in the same component. There exist loops τ_1 and τ_{-1} in M at x_0 and a homotopy $F : J \times I \to U$ from σ to σ' such that $F(-1,t) = f \circ \tau_{-1}(t)$ and $F(1,t) = f \circ \tau_1(t)$. This imples that $f_*([\tau_{-1}]^{-1}) \cdot [\sigma] \cdot f_*([\tau_1]) = [\sigma']$.

Conversely, if $f_*(a^{-1}) \cdot [\sigma] \cdot f_*(b) = [\sigma']$ for some $a, b \in \pi_1(M)$, then (x_0, x_0, σ) and (x_0, x_0, σ') lie in the same component. Therefore, the set A corresponds one-one to the set of path components of Ω_f.

Consider the double covering $\Omega_f \times S^\infty \to K_f$. The points $((x_0, x_0, \sigma); s)$ and $(x_0, x_0, \sigma^{-1}); -s)$ lie in the same fibre. This implies that $\langle[\sigma]\rangle$ and $\langle[\sigma^{-1}]\rangle$ corresponds to the same path component of K_f.

Example 5.2 (Embedding in real projective space). Let $f : M^n \to P^{n+k}$ be a null homotopic map. Then

$$K_f \simeq (M \times M \times \Omega_0(P^{n+k}) \times S^\infty)/\mathbf{Z}_2 \coprod (M \times M \times \Omega_1(P^{n+k}) \times S^\infty)/\mathbf{Z}_2,$$

where $\Omega_0(P^{n+k})$ and $\Omega_1(P^{n+k})$ are the two path components of the loop space $\Omega(P^{n+k})$.

Example 5.3 (Embedding in lens space). Let $S^{2m+1} = \{(z_0, z_1, \cdots, z_m) \in \mathbf{C}^{m+1} \mid \sum |z_i|^2 = 1\}$ and let $q_1, \cdots, q_m$ be integers relatively prime to p. The homeomorphism $h : S^{2m+1} \to S^{2m+1}$ with period p by $h(z_0, z_1, \cdots, z_m) = (e^{2\pi i/p} z_0, e^{2\pi i q_1/p} z_1, \cdots, e^{2\pi i q_m/p} z_m)$ determines a free $\mathbf{Z}_p$-action on S^{2m+1}; the orbit space in the lens space $L(p; q_1, \cdots, q_m)$. Let $f : M^n \to L(p; q_1, \cdots, q_m)$ be a null homotopic map. Denote by $\Omega_j L$ the path component of $\Omega L(p; q_1, \cdots, q_m)$ consisting of the loops which represent $j \in \mathbf{Z}_p$. Then,

$$K_f \simeq \begin{cases} Q \coprod \coprod_{1 \le j \le \frac{1}{2}(p-1)} (M \times M \times \Omega_j L) & \text{if } p \text{ is odd}, \\ Q \coprod (M \times M \times \Omega_{p/2} L \times S^\infty)/\mathbf{Z}_2 \coprod \coprod_{1 \le j < \frac{p}{2}} (M \times M \times \Omega_j L) & \text{if } p \text{ is even}, \end{cases}$$

where $Q = (M \times M \times \Omega_0 L \times S^\infty)/\mathbf{Z}_2$.

Proof of Theorem 1.1 *and Theorem* 1.2. First of all we observe that there exists an embedding of M^n in $\mathbf{R}^{n+k}$ by Theorem 4.1 of [8] and, hence, M^n embeds in P^{n+k}. Similarly, we know that M^n embeds in $L(p; q_1, \cdots, q_m)$.

Since M is simply connected, any map $f : M^n \to P^{n+k}$ (or $f : M^n \to L(p; q_1, \cdots, q_m)$ is null homotopic. We need only to compute the $(n-k+1)$-th normal bordism group (resp. $(2n-2m)$-th normal bordism group) of the space pair $(K_f, M \times P^\infty)$ in Example 5.2 (resp. Example 5.3) for a constant map.

It is clear that the inclusions

$$M \times P^\infty \to (M \times M \times \Omega_0(P^{n+k}) \times S^\infty)/\mathbf{Z}_2$$

and

$$\Omega_1(P^{n+k})/\mathbf{Z}_2 \to (M \times M \times \Omega_1(P^{n+k}) \times S^\infty)/\mathbf{Z}_2$$

are $(n-k+2)$-equivalences. It follows that

$$\Omega_{n-k+1}((M \times M \times \Omega_0(P^{n+k}) \times S^\infty)/\mathbf{Z}_2, M \times P^\infty; \phi) = 0$$

and

$$i_{1*} : \Omega_{n-k+1}(\Omega_1(P^{n+k})/\mathbf{Z}_2; i_1^*\phi) \to \Omega_{n-k+1}((M \times M \times \Omega_1(P^{n+k}) \times S^\infty)/\mathbf{Z}_2; \phi)$$

is an isomorphism.

We observe that $\Omega_1(P^{n+k})$ may be identified with $P(S^{n+k}; s_0, -s_0)$, the space of paths in S^{n+k} starting at s_0 and ending at the antipodal point $-s_0$. Thus the equator S^{n-k+1} is embedded in $\Omega_1(P^{n+k})$ as a $\mathbf{Z}_2$-invariant subspace if one identifies each point s in S^{n+k-1} with the path naturally defined along the great circle $(\widehat{s_0, s, -s_0})$. In this way we obtain an $(n-k+2)$-equivalence $i_2 : P^{n+k-1} \to \Omega_1(P^{n+k})/\mathbf{Z}_2$.

The composition $\bar{q} \circ i_1 \circ i_2 : P^{n+k-1} \to EP(M)$ is the map induced from a $\mathbf{Z}_2$-equivariant map $S^{n+k-1} \to M \times M \times S^\infty$ by $s \mapsto (x_0, x_0, i(s))$. Therefore,

$$\begin{aligned}
&(\bar{q} \circ i_1 \circ i_2)^* EP(\tau M) \cong (S^{n+k-1} \times \mathbf{R}^n \times \mathbf{R}^n)/\mathbf{Z}_2 \\
&\cong (S^{n+k-1} \times \mathbf{R}^1 \times \mathbf{R}^1)/\mathbf{Z}_2 \oplus \cdots \oplus (S^{n+k-1} \times \mathbf{R}^1 \times \mathbf{R}^1)/\mathbf{Z}_2 \\
&\cong (\epsilon^1 \oplus \lambda_{n+k-1}) \oplus \cdots \oplus (\epsilon^1 \oplus \lambda_{n+k-1}) \cong \epsilon^n \oplus n\lambda_{n+k-1},
\end{aligned}$$

where the action of $\mathbf{Z}_2$ on $\mathbf{R}^1 \times \mathbf{R}^1$ is defined by

$$((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \mapsto ((y_1, \cdots, y_n)((x_1, \cdots, x_n))$$

or, equivalently, by

$$((x_1, y_1), (x_n, y_n)) \mapsto ((y_1, x_1), \cdots, (y_n, x_n)).$$

The composition $\bar{r} \circ i_1 \circ i_2 : P^{n+k-1} \to P^{n+k} \times P^\infty$ may be identified with the product map of the natural inclusions $P^{n+k-1} \hookrightarrow P^{n+k}$ and $P^{n+k-1} \hookrightarrow P^\infty$. Therefore,

$$(\bar{r} \circ i_1 \circ i_2)^*(\tau P^{n+k} \tilde{\otimes} \lambda) \cong (\tau P^{n+k-1} \oplus \lambda_{n+k-1}) \otimes \lambda_{n+k-1}.$$

It follows that $(i_1 \circ i_2)^*(\phi_+ - \phi_-) = (n+1)\lambda_{n+k-1} - \epsilon^{n+1}$ in $\widetilde{KO}\,(P^{n+k-1})$ and hence, by Theorem 3.4,

$$[M^n \subset P^{n+k}]_f = \Omega_{n-k+1}(P^{n+k-1}; (n+1)\lambda_{n+k-1} - \epsilon^{n+1}).$$

This bordism group may be identified with $\pi_{2^{\varphi(n+k-1)}-k}(V_{2^{\varphi(n+k-1)}+k-1, n+k})$ by Proposition 4.1 and a usual Thom-Pontrjagin procedure shown in Proposition I.7.3. of [2].

The proof for lens spaces is similar. We need only to point out the following facts.

(1) $\Omega_{2n-2m}(M \times M \times \Omega_j L; \phi) \cong \pi^S_{2n-2m}$ by the corollary to Proposition 7.1 of [2] because $M \times M \times \Omega_j L$ is $(2n-2m+1)$-connected.

(2) If p is even, $q_1, \cdots, q_m$ are all odd numbers. The composition of homeomorphisms $h^{\frac{1}{2}p} : S^{2m+1} \to S^{2m+1}$ is just the antipodal map. Therefore, $\Omega_{\frac{1}{2}p} L$ may be identified with $P(S^{2m+1}; s_0, -s_0)$ and hence we have a $(2n-2m+1)$-equivalence

$i_2 : P^{2m} \to \Omega_{\frac{1}{2}p}L/\mathbf{Z}_2$. The composition $\bar{r} \circ i_1 \circ i_2 : P^{2m} \to L \times P^\infty$ is the product map of the standard immersion $g : P^{2m} \to L(p; q_1, \cdots, q_m)$ and the natural inclusion $P^{2m} \to P^\infty$, where g is defined by $[a_0, z_1, \cdots, z_m] \mapsto \langle a_0, z_1, \cdots, z_m\rangle$ for $(a_0, z_1, \cdots, z_m)$ in S^{2m} with a_0 the real part of z_0. The normal bundle ν^1 of the immersion g is isomorphic to λ_{2m} over P^{2m} because the lens space is an orientable manifold and hence $w_1(\nu^1) \neq 0$. Thus,

$$(\bar{r} \circ i_1 \circ i_2)^*(\tau L \tilde{\otimes} \lambda) \cong (\tau P^{2m} \oplus \nu^1) \otimes \lambda_{2m} \cong (\tau P^{2m} \oplus \lambda_{2m}) \otimes \lambda_{2m}.$$

The proof of the theorems will be completed after we prove the following lemmas.

Lemma 5.4. *Let M be a compact manifold without boundary.*

(i) *If $\dim(M) < 2m$, the action of $\pi_1(L^M, f)$ on $\pi_1(L^M, E, f) = [M \subset L]_f$ is trivial for any map $f : M \to L(p; q_1, \cdots, q_m)$.*

(ii) *If $\dim(M) < 2m - 1$, $\pi_1((P^{2m})^M, f)$ acts on $[M \subset P^{2m}]_f$ trivially for any null homotopic map $f : M \to P^{2m}$.*

Proof. It follows from Lemma 1 of [9] that $\pi_1(L^M, f) \cong \pi_1(L) = \mathbf{Z}_p$. There is a C^∞-flow Φ_t on $L(p; q_1, \cdots, q_m)$ such that, for any fixed x in L, the closed orbit $\Phi_t(x), t \in [0,1]$, represents the generator of $\pi_1(L^M, f)$. For $j = 0, 1, \cdots, p-1$, define $f_t^j : M \times [0,1] \to L$ by $f_t^j(x) = \Phi_{jt}(f(x))$. Then $f_0^j = f = f_t^j$, and $f_t^0, f_t^1, \cdots, f_t^{p-1}$ represent all the elements of $\pi_1(L^M, f)$.

The action of $\pi_1(L^M, f)$ may be considered as transformations along flow lines. More precisely, for a class $\alpha \in \pi_1(L^M, E, f)$ represented by a homotopy $F : M \times I \to L$ with $F_0 = f$ and F_1 an embedding, $j \cdot \alpha$, for any $j \in \mathbf{Z}_p$, has a representative $F^j : M \times I \to L$

$$F_t^j(x) = \begin{cases} f_{2t}^j(x), & 0 \le t \le \frac{1}{2}; \\ F_{2t-1}(x), & \frac{1}{2} \le t \le 1, \end{cases}$$

and there is a homotopy $F \underset{K}{\simeq} F^j$ by

$$K(x,t,u) = \begin{cases} \Phi_{2jut}(f(x)), & u \ge 2t, \\ \Phi_{ju}(F_{2t-u}(x)), & u \le 2t. \end{cases}$$

This implies $j \cdot \alpha = \alpha$ and hence $\pi_1(L^M, f)$ acts on $[M \subset L]_f$ trivially.

Suppose that $\dim(M) \le 2m - 1$ and $f : M \to P^{2m}$ is a constant map. The homotopy $F : M \times I \to P^{2m}$ representing a class in $[M \subset P^{2m}]_f$ is not onto because it is also null homotopic. Then there is $C^\infty-$ flow Φ_t on $P^{2m} \setminus \{*\}$, where $*$ is a point in P and $F(M \times I) \subset P^{2m} \setminus \{*\}$, such that the closed orbit $\Phi_t(f(x))$, $t \in [0,1]$, represents the generator of $(\pi_1(P^{2m})^M, f) = \mathbf{Z}_2$. The rest of the proof is similar to that for lens spaces.

Remark 5.5. If we replace P^{n+k} in Theorem 1.1 by $P^l \times \mathbf{R}^{n-l+k}$ for $l > n$, the same argument shows that

$$[M^n \subset P^l \times \mathbf{R}^{n-l+k}] = \pi_{2^{\varphi(l-1)}-k}(V_{2^{\varphi(l-1)}-n+l-1,l}).$$

For example,

$$[S^4 \subset P^5 \times \mathbf{R}^3] = 0 \quad \text{and} \quad [S^6 \subset P^7 \times \mathbf{R}^4] = \mathbf{Z}_8.$$

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