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Representations of Knot Groups and Twisted Alexander Polynomials

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Abstract We present a twisted version of the Alexander polynomial associated with a matrix representation of the knot group. Examples of two knots with the same Alexander module but different twisted Alexander polynomials are given.

Keywords Twisted Alexander polynomial, Representation space, Knot group
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We present here a twisted version of Alexander polynomials of knots in S^3. Associated with each representation of the knot group, we can have a twisted Alexander polynomial for the corresponding knot. The classical Alexander polynomial is the one associated with 1-dimensional representations. We have examples of knots with the same Alexander polynomial but different twisted Alexander polynomials associated with representations into $SU(2,\mathbb{C})$.

We will use regular Seifert surfaces to define twisted Alexander polynomials for knots. A Seifert surface of a knot is regular if it has a spine (call it regular) which, being thought of as a bouquet of circles embedded in S^3, is isotopic to the standard embedding. Certainly, a regular Seifert surface is a free (unknotted) Seifert surface. But a free (unknotted) Seifert surface is not necessarily regular.

Similarly to the fact that two Seifert surfaces of a knot are thus called S-equivalent, we will show (Theorem 1.7) that two regular Seifert surfaces of a knot are regularly S-equivalent. Stronger than S-equivalence via a sequence of regular Seifert surfaces, regular S-equivalence also requires that one only uses unknotted handles to perform handle-addition or handle subtraction. This fact that regular Seifert surfaces are regularly S-equivalent is used to prove that our definition of twisted Alexander polynomials does not depend on the choices of regular Seifert surfaces and therefore gives us knot invariants.

Our approach here leaves a lot of questions open. First of all, we are not yet able to overcome the technical difficulties involved in generalizing the definition of twisted Alexander polynomials

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from knots to links. Without such a generalization, it seems to be very difficult to find any relation between twisted Alexander polynomials and the HOMFLY polynomials [1]. Another important question is how to generalize the definition of twisted Alexander polynomials to knots in homology 3-spheres. A knot in a homology 3-sphere bounds an unknotted Seifert surface. With an appropriate definition of regular Seifert surfaces, we expect that Theorem 1.7 is still true in this setting. Noticing that an unknotted Seifert surface gives us a Heegard decomposition of the ambient homology 3-sphere, it is reasonable to expect that a generalization of twisted Alexander polynomials to homology 3-spheres will have something to do with Casson's invariant of homology 3-spheres (even in the generalized version of Cappell-Lee-Miller [2]).

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Note This article appeared first in 1990 as a Columbia University preprint. Since then further works on the topic discussed in this article have been published by several authors [3–7]. As we have been receiving frequent requests for this unpublished article, we think it might be worthwhile to have it published.

1 Regular Seifert Surfaces of a Knot

A *knot* in S^3 is a tame embedding of S^1 into S^3. Let K be an oriented knot in S^3. A *Seifert surface* of K is a compact, connected and oriented surface S embedded in S^3 such that $\partial S = K$. There is a classical algorithm of constructing a Seifert surface for a knot via a regular plane projection of that knot due to Seifert. Let us describe this algorithm briefly as follows.

Consider a regular plane projection P of a knot K. In a neighborhood of each crossing of P, we change the diagram as shown in Figure 1.1 so that the resulting diagram consists of finitely many disjoint oriented circles in the plane. These oriented circles are usually called *Seifert circles*.

Each Seifert circle bounds an oriented disk in the plane. Let us imagine these disks are stacked in different levels so that if one disk D_1 is contained in the other disk D_2 in the plane, then D_1 is stacked above D_2. We then connect these disks by half-twisted bands in such a way that it reverses the previous operation of changing the knot diagram to Seifert circles (see Figure 1.2). The resulting compact, connected and oriented surface is a Seifert surface of the knot K. We denote this Seifert surface by S_P since it is completely determined by the regular plane projection P.

The above construction has an important feature which we are going to explore. Suppose N is a closed tubular neighborhood of S_P. Then N is a handlebody with $2g$ handles, where $g = \text{genus}(S_P)$. It is easy to see that $\overline{S^3 \setminus N}$ is also a handlebody. In other words, S_P is an unknotted Seifert surface.

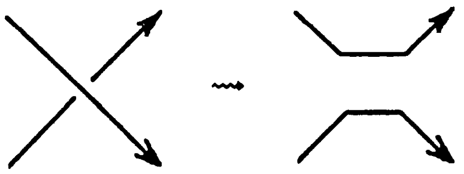

Figure 1.1

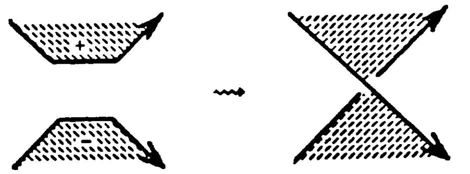

Figure 1.2

Let W_n be a bouquet of n circles, i.e., the space obtained from a disjoint union of n circles with the base points of each circle being identified with a single base point $*$. A *spine* of a compact surface S is a bouquet of circles $W \subset S$ such that it is a deformation retract of S. It is easy to see that for the Seifert surface S_P corresponding to the regular plane projection P, there is a spine W whose embedding in S^3 induced by $S_P \subset S^3$ is isotopic to the standard embedding. These motivate the following definitions:

Definition 1.1 *A Seifert surface S of a knot K is called free if $(n(S), \overline{S^3 \setminus n(S)})$ is a Heegard splitting of S^3. Here $n(S)$ is a closed tubular neighborhood of S in S^3.*

Definition 1.2 *A Seifert surface S of a knot K is called regular if it has a spine W whose embedding in S^3 induced by $S \subset S^3$ is isotopic to the standard embedding.*

So, for any regular plane projection P of the knot K, S_P is a regular Seifert surface of K.

Of course, regular Seifert surfaces are also free Seifert surfaces. But a free Seifert surface is not necessarily regular.

Let S be a Seifert surface of K. We can perform surgery on S to get new Seifert surfaces for K. Let d be an oriented arc in S^3 such that $d \cap S = \partial d$ and the intersection consists of the orientations of S and d. Add a tube to S along d. This operation gives us a new Seifert surface S'. We say S' is obtained from S by a *handle-addition*. On the other hand, S is obtained from S' by a *handle-subtraction*.

Two Seifert surfaces S and S' of a knot K are *S-equivalent* if there is a sequence of Seifert surfaces

$$S = S_1,\ S_2,\ \ldots,\ S_{m-1},\ S_m = S'$$

of K such that S_{i+1} is obtained from S_i by either a handle-subtraction or a handle-addition. We will think of isotopic Seifert surfaces as being the same. Then, a well-known fact (along with a somehow straightforward proof) is that *any* two Seifert surfaces of a knot K are S-equivalent (see [8]). We will have a more delicate version of this well-known fact.

Let S be a regular Seifert surface with a regular spine W. Let d be the arc in the definition of a handle-addition. We may assume that the two endpoints of d are identified with the base point of W.

Definition 1.3 *A handle-addition on S is called regular if $W \cup d \subset S^3$ is isotopic to the standard embedding. The inverse operation of a regular handle-addition is a regular handle-subtraction.*

Lemma 1.4 *Suppose that the regular spine W lies on a 2-sphere S^2. Assume that the arc d is a proper unknotted arc in $S^3\backslash S^2$ with two of its end points sitting on W. Then a handle-addition along d is regular.*

Proof On the surface, move the end points of d monotonically to the base point of W along different arcs of W. It is not hard to see that $W \cup d$ obtained in this way is isotopic to the standard embedding of a bouquet of circles.

Lemma 1.5 *If S' is obtained from a regular Seifert surface by a regular handle-addition, then S' is also a regular.*

Proof We can have a small disk D on S such that D only intersects W at its base point. Then $W' = W \cup d \cup \partial D$ is a spine of S'. It is easy to see that W' is isotopic to the standard embedding.

Definition 1.6 *Let S and S' be two regular Seifert surfaces of a knot K. They are regularly S-equivalent if there is a sequence of regular Seifert surfaces*

$$S = S_1,\ S_2,\ \ldots,\ S_{m-1},\ S_m = S'$$

of K such that S_{i+1} is obtained from S_i by either a regular handle-subtraction or a regular handle-addition.

Theorem 1.7 *Any two regular Seifert surfaces of a knot are regularly S-equivalent.*

We prove this theorem in two steps. First, we show that any regular Seifert surface S of a knot K is regularly S-equivalent to a regular Seifert surface of the form S_P for a certain regular plane projection P of the knot K. Then, we show that a Reidemeister move from a regular plane projection P to another regular plane projection P' changes S_P to $S_{P'}$ by either an isotopy or a handle-addition or a handle-subtraction. This shows that if P and P' are two regular plane projections of K, then S_P and $S_{P'}$ are regularly S-equivalent since P' can be obtained from P by a sequence of Reidemeister moves (see Figure 1.7).

Let S be a regular Seifert surface of a knot K. We need a certain normal form for S.

We consider *regular plane projections* of an embedding $W_n \subset S^3$. Here W_n is a bouquet of n circles with the base point $*$. Similarly to regular plane projection of knots, these plane projections have only double points as singular points. In addition, we assume that the base point $*$ of W_n is not a singular point.

For a regular plane projection of an embedding $W_n \subset S^3$, we can associate it with a compact, connected and orientable surface in the following way. We draw a disk D in the projection plane as a neighborhood of the base point $*$ of W_n such that the projection has no singular points in D. Then $D \cap W_n$ is a bouquet of $2n$ arcs and there are n arcs outside of D in the projection plane. Replace each of these n outside arcs with a narrow band, possibly having some full twists. At a singular point of the projection, these bands are over-crossing or under-crossing according to whether the corresponding singular point is an over-crossing point or an under-crossing point. The union of D and these bands is a surface *associated* with the given regular plane projection of $W_n \subset S^3$. See Figure 1.3.

For a regular Seifert surface S, let W_n be a spine of S such that it is isotopic to the standard embedding. Here $n = 2g$ with $g = \text{genus}(S)$. Let $a_1, b_1, \ldots, a_g, b_g$ be the oriented circles in W_n such that they form a symplectic basis of S. Then, we can assume that in a neighborhood of the base point $*$ of W_n in S, we see a bouquet of $4g$ arcs ordered and oriented as shown in Figure 1.4.

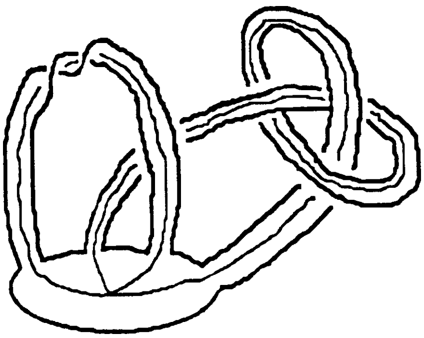

Figure 1.3

Lemma 1.8 *The induced embedding $W_n \subset S^3$ has a regular plane projection shown in Figure* 1.5*, where σ is a braid of index $2n$, such that the regular Seifert surface S is isotopic to an associated surface of this projection.*

Proof Begin with an arbitrary regular plane projection of the induced embedding $W_n \subset S^3$ such that in a disk neighborhood D of the base point $*$ of W_n in the projection plane, the arcs in $W_n \cap D$ are ordered and oriented in the same way as in a disk neighborhood of $*$ in S. Then S is isotopic to an associated surface of this projection. Since the induced embedding $W_n \subset S^3$ is isotopic to the standard one, after switching the arcs in $W_n \cap D$ appropriately, the arcs outside of D are isotopic to the standard embedding. Record the switches by a braid σ of index $2n$ and we have completed the proof.

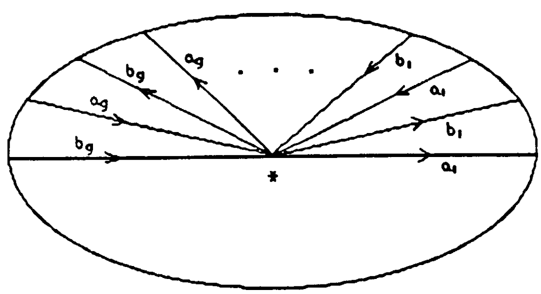

Figure 1.4

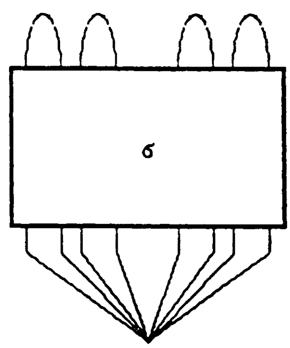

Figure 1.5

Proof of Theorem 1.7 Let us identify the knot K with the boundary of an associated surface S of the regular plane projection of $W_n \subset S^3$ described in Lemma 1.8. The regular plane projection of W_n induces a regular plane projection of $\partial S = K$. Denote this projection by P. We certainly have $S \neq S_P$ since S has a lot of band crossings. See Figure 1.6 where the $\pm$ signs indicate the orientation of the surface S. There are essentially two types of band crossings: in a type I band crossing, two bands have different signs; and in a type II band crossing, two bands have the same sign. But one can always transform a type II band crossing to a type I one by flipping (say) the underneath band over locally (see Figure 1.7).

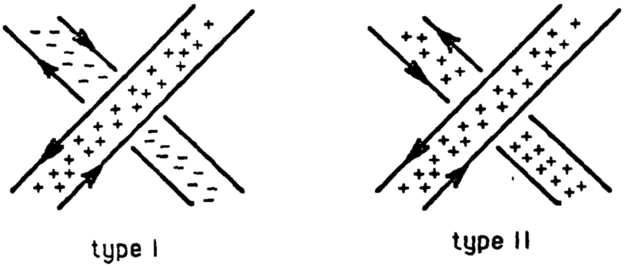

Figure 1.6

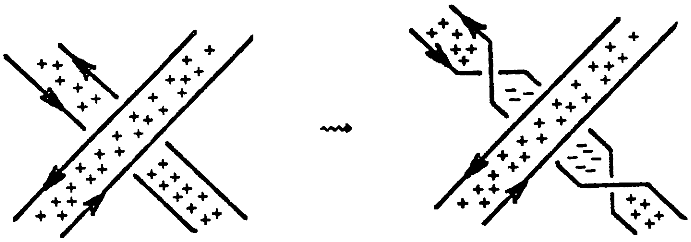

Figure 1.7

Let us change all type II band crossings of S to type I ones in the way shown in Figure 1.7. For a type I band crossing, we change S by a handle-addition as shown in Figure 1.8. Also in Figure 1.8, we see that locally, the resulting surface S' can be obtained by means of Seifert circles. We need to verify that S' is obtained from S by a regular handle-addition. In Figure 1.9, we draw the spine of S and the arc d in a neighborhood of the crossing where the handle-addition is performed. Back to the regular plane projection of W_n described in Lemma 1.8, we can connect the two endpoints of the arc d on S by sliding them down along the braid σ until they join the base point of W. This certainly gives us an unknotted circle. So, S' is obtained from S by a regular handle-addition.

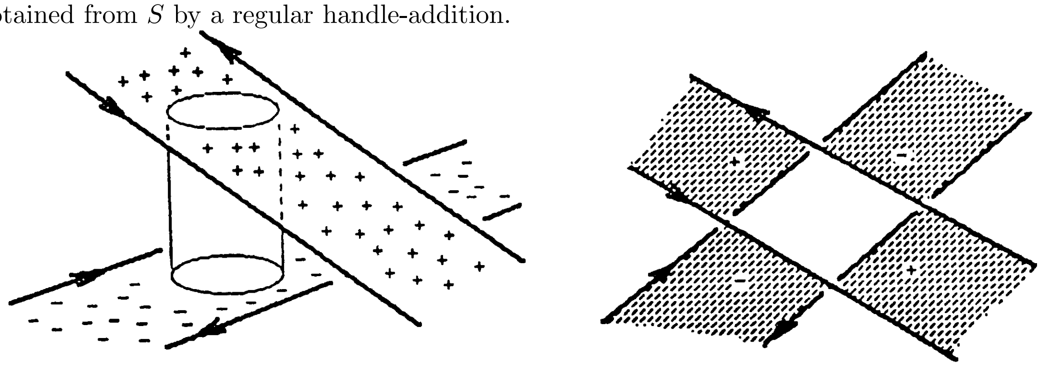

Figure 1.8

Now we add a handle at each band crossing of S (they are all of type I) as in Figure 1.8. The resulting surface is exactly S_P where P is the induced regular plane projection of $\partial S = K$.

This shows that a regular Seifert surface of a knot K is regularly S-equivalent to S_P for a certain regular plane projection P of K.

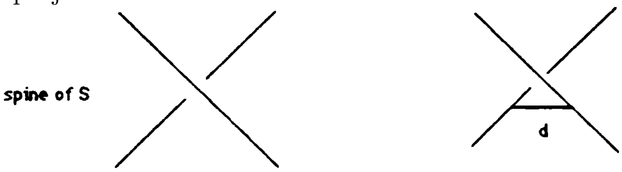

Figure 1.9

Let us now recall Reidemeister moves among regular plane projections of a knot. There are three types of moves (see Figure 1.10).

It is understood that these moves are to be performed locally on regular plane projections and no other strands are presented locally other than these depicted in the moves. A well-known fact is that two regular plane projections of a knot are related by a sequence of Reidemeister moves (see [9]).

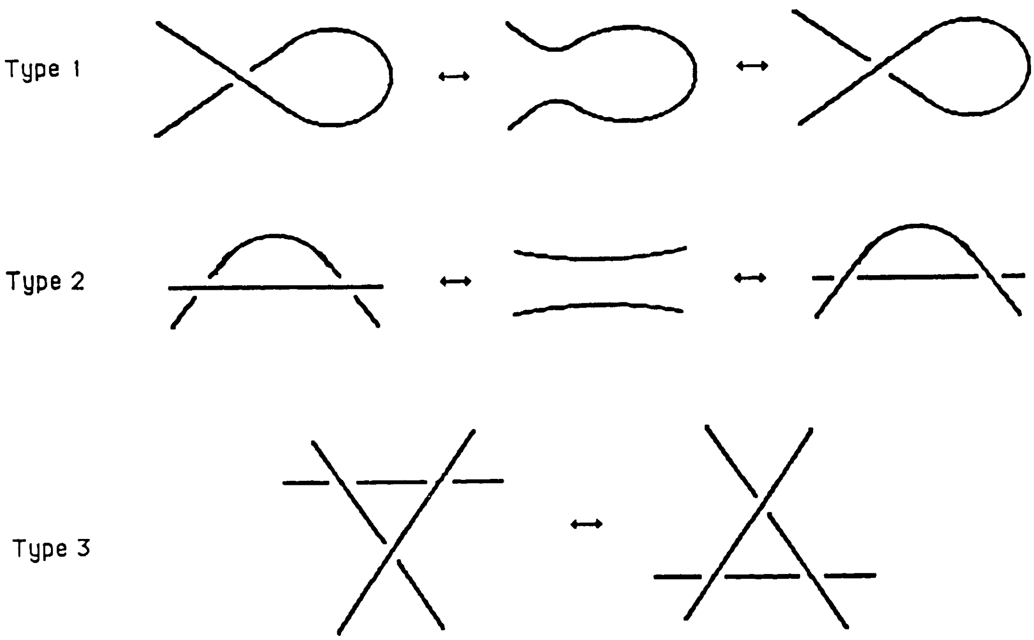

Figure 1.10

We want to see how S_P changes if the regular plane projection P is changed by a Reidemeister move.

Suppose P is changed by a Reidemeister move and the resulting new regular plane projection is P'.

Case 1 A type 1 move. It is quite clear in this case that $S_{P'}$ is isotopic to S_P.

Case 2 A type 2 move. In this case, although one needs to analyze several different situations, it is not hard to see by drawing pictures that $S_{P'}$ can be obtained from S_P by an isotopy or a regular handle-addition or a regular handle-subtraction.

Case 3 A type 3 move. It seems to be a surprise that in this case, $S_{P'}$ is isotopic to S_P.

In Figure 1.11 (b) (resp. (c)), c_1, c_2, c_3 are arcs on the Seifert circles of Figure 1.11 (a) (resp. (d)), and b_1, b_2, b_3 are half-twisted bands used to build S_P (resp. $S_{P'}$). By sliding b_3 up we can isotope (b) to (c). This shows that $S_{P'}$ is isotopic to S_P.

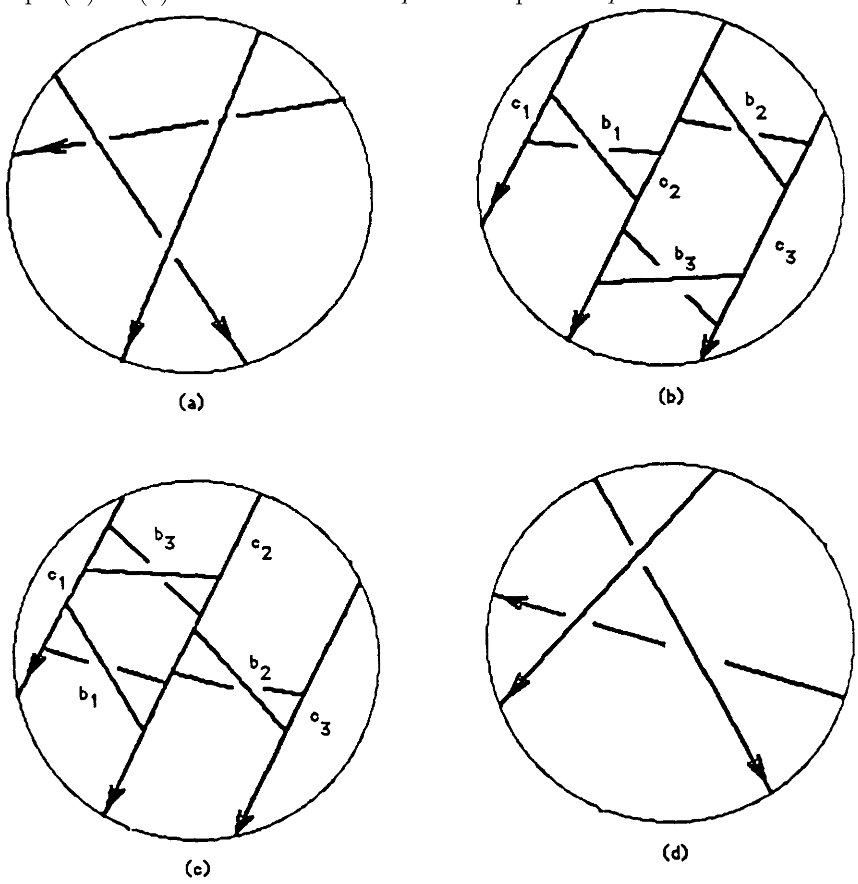

Figure 1.11

Thus, we see that if P and P' are two regular plane projections of a knot K, then S_P and $S_{P'}$ are regularly S-equivalent as regular Seifert surfaces of K.

Combining the above two steps, we have shown that any two regular Seifert surfaces of a knot are regularly S-equivalent. This finishes the proof of Theorem 1.7.

2 Presentations of Knot Groups via Regular Seifert Surfaces

Suppose S is a free Seifert surface of a knot K. Let $W = W_n$ be a spine of S, where $n = 2g$ with $g = \text{genus}(S)$. Denote the oriented circles in W by a_1, b_1, …, a_g, b_g. They form a symplectic basis (in that order) of S. Moreover, we assume that W is in the interior of S. The base point of W is denoted by $*$.

Let $S \times [-1, 1]$ be a bicollar of S given by the positive normal direction of S such that

$S = S \times 0$. Let $W^{\pm} = W \times \{\pm 1\}$, $a_i^{\pm} = a_i \times \{\pm 1\}$ and $b_i^{\pm} = b_i \times \{\pm 1\}$, $i = 1, \ldots, g$. On $\partial(S \times [-1,1])$, there is an arc connecting the base points of $W^{\pm}$ whose interior has no intersection with $W^{\pm}$. Up to an isotopy, such an arc is unique. Denote this arc by c.

Since $\overline{S^3 \setminus S \times [-1,1]}$ is a handlebody, its fundamental group is a free group. Think of c as the base point and choose a basis $x_1, \ldots, x_{2g}$ for $\pi_1(\overline{S^3 \setminus S \times [-1,1]})$, where $g = \text{genus}(S)$. Then, a_i^+ (resp. a_i^-) determines a word α_i (resp. β_i) in $x_1, \ldots, x_{2g}$ for each i. Call these words, $\{\alpha_i, \beta_i \ ; \ i = 1, \ldots, 2g\}$, a set of *induced words* of the spine W.

Lemma 2.1 *The fundamental group $\pi_1(S^3 \setminus K)$ has a presentation*

$$\langle \ x_1, \ldots, x_{2g}, z \ ; \ z\alpha_i z^{-1} = \beta_i, \ i = 1, \ldots, 2g \ \rangle,$$

where z is represented by a meridian of K.

Proof Let K' be a simple closed curve on S parallel to $\partial S = K$ and disjoint with W. We can identify the knots K' and K and think of $m = \{*\} \times [-1,1] \cup c$ as a meridian of K' or K. Let us identify the fundamental group of $S^3 \setminus S$ with the free group F generated by $x_1, \ldots, x_{2g}$ and let $z \in \pi_1(S^3 \setminus K')$ be represented by m. Then it is quite easy to see that $\pi_1(S^3 \setminus K)$ has the presentation described in the lemma.

We now discuss how a set of induced words depends on the choices of bases of $\pi_1(S^3 \setminus S)$ and spines of S for a fixed free Seifert surface S.

Suppose we choose another basis for $\pi_1(S^3 \setminus S)$, say $y_1, \ldots, y_{2g}$. The set of induced words of W in terms of this basis is $\{\alpha_i', \beta_i' \ ; \ i = 1, \ldots, 2g\}$. Let F' be the free group generated by $y_1, \ldots, y_{2g}$.

Lemma 2.2 *There is an isomorphism $\phi : F' \to F$ such that $\phi(\alpha_i') = \alpha_i$, $\phi(\beta_i') = \beta_i$ for each $i = 1, \ldots, 2g$.*

Proof The basis transformation from $\{x_1, \ldots, x_{2g}\}$ to $\{y_1, \ldots, y_{2g}\}$ gives us the desired isomorphism.

Suppose we choose another spine W' of S and the set of induced words of W' in terms of the basis $\{x_1, \ldots, x_{2g}\}$ is $\{\alpha_i', \beta_i' \ ; \ i = 1, \ldots, 2g\}$. The deformation retraction $S \to W$ gives us a map $W' \to W$ which induces an isomorphism between the corresponding fundamental groups. Let us assume that W' and W have the same base point $*$ and the map $W' \to W$ preserves the base point. The induced isomorphism, $\phi : \pi_1(W', *) \to \pi_1(W, *)$, corresponds to a basis transformation for the free group $\pi_1(W, *)$. We can think of ϕ as an isomorphism

$$\phi : \ F(\alpha_1', \ldots, \alpha_{2g}') \longrightarrow F(\alpha_1, \ldots, \alpha_{2g})$$

as well as an isomorphism

$$\phi : \ F(\beta_1', \ldots, \beta_{2g}') \longrightarrow F(\beta_1, \ldots, \beta_{2g}),$$

where $F(\cdots)$ is the free group generated by the corresponding letters.

Lemma 2.3 *The diagrams*

$$\begin{array}{ccc} F(\alpha'_1,\ldots,\alpha'_{2g}) & \longrightarrow & F(x_1,\ldots,x_{2g}) \\ {\scriptstyle\phi}\big\downarrow & & \big\| \\ F(\alpha_1,\ldots,\alpha_{2g}) & \longrightarrow & F(x_1,\ldots,x_{2g}) \end{array}$$

and

$$\begin{array}{ccc} F(\beta'_1,\ldots,\beta'_{2g}) & \longrightarrow & F(x_1,\ldots,x_{2g}) \\ {\scriptstyle\phi}\big\downarrow & & \big\| \\ F(\beta_1,\ldots,\beta_{2g}) & \longrightarrow & F(x_1,\ldots,x_{2g}) \end{array}$$

commute. Here the horizontal arrows are homomorphisms given by the induced words of (S, W) *and* (S, W') *respectively.*

Proof This is quite clear.

In the case where S is a regular Seifert surface, i.e. S has a regular spine W, the fundamental group of $S^3 \setminus S$ has a more or less natural basis and we can interpret a set of induced words of W geometrically in terms of this natural basis.

By definition, a regular spine $W \subset S^3$ is isotopic to the standard embedding. Thus, the oriented circles a_1, b_1, …, a_g, b_g in W bound the oriented disks D_1, D_2, …, D_{2g-1}, D_{2g}, respectively, such that $D_i \cap D_j = \{*\}$ for $i \neq j$. We assume that these disks have no intersections with the arc c. Denote this collection of disks spanned by W by $\mathcal{D}$.

Let $F = F(x_1, x_2, \ldots, x_{2g-1}, x_{2g})$. We can record the intersection of a_1^+ with the disks in $\mathcal{D}$ by a word $\alpha_1 \in F$. Assume a_1^+ intersects the disks in $\mathcal{D}$ transversally at finitely many points. We name each intersection point of a_1^+ with D_i by x_i or x_i^{-1}, according to whether the intersection number is 1 or -1. Write down all these intersection points on a_1^+ successively from left to right, beginning with the first one after the base point of a_1^+ in the positive direction. This gives us a word $\alpha_1 \in F$. Similarly, we get words α_2, …, α_{2g-1}, α_{2g} from b_1^+, …, a_g^+, b_g^+ respectively. Also, we get words β_1, β_2, …, β_{2g-1}, β_{2g} in F from a_1^-, b_1^-, …, a_g^-, b_g^- respectively. Call this set of words $\{\alpha_1, \ldots, \alpha_{2g}, \beta_1, \ldots, \beta_{2g}\}$ a set of *dual words* of the regular spine W. Of course, it is also a set of induced words of the spine W.

We need to understand how a set of dual words change when the regular Seifert surface S is changed by a regular handle-addition resulting a new regular Seifert surface $\overline{S}$.

Choose a regular spine W for S, and let $\{\alpha_i, \beta_i \ ; \ i = 1, \ldots, 2g\}$ be a set of dual words of (S, W). We draw the unknotted handle added to S as in Figure 2.1. Let $\overline{S}$ be the new regular Seifert surface. The union of W and the circles a_{g+1} and b_{g+1} depicted in Figure 2.1 gives us a regular spine $\overline{W}$ for $\bar{S}$ where $g = \text{genus}(S)$.

Let $\{\bar{\alpha}_i, \bar{\beta}_i \ ; \ i = 1, \ldots, 2g+2\}$ be a set of dual words of $(\overline{S}, \overline{W})$. Notice that by adding some full twists to the band associated with a_{g+1} if necessary, we can assume that a_{g+1}^+ and a_{g+1}^- have no intersections with the disk bounded by a_{g+1}.

Lemma 2.4 *For $i = 1, \ldots, 2g$, $\bar{\alpha}_i$ and $\bar{\beta}_i$ are words in $x_1, \ldots, x_{2g}, x_{2g+1}$ and they reduce to α_i and β_i respectively when we set $x_{2g+1} = 1$. Moreover, we have $\bar{\alpha}_{2g+1} = \gamma x_{2g+2}$, $\bar{\alpha}_{2g+2} = 1$, $\bar{\beta}_{2g+1} = \delta$ and $\bar{\beta}_{2g+2} = x_{2g+1}$ where γ and δ are words in $x_1, \ldots, x_{2g}$.*

Proof All these conclusions are quite easy to see from Figure 2.1 and the assumption that $a^{\pm}_{g+1}$ have no intersections with the disk bounded by a_{g+1}.

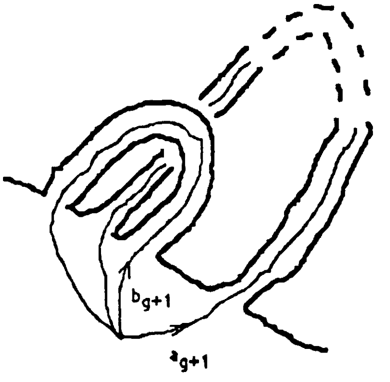

Figure 2.1

3 Twisted Alexander Modules

Let us first recall the definition of the *Alexander module* of a knot K (see [10]). Denote by M the knot complement $S^3 \setminus K$. Let $\widetilde{M} \to M$ be the infinite cyclic covering. Consider the homology group $H_1(\widetilde{M}) = H_1(\widetilde{M}; \mathbb{Z})$. Let

$$t: \ H_1(\widetilde{M}) \longrightarrow H_1(\widetilde{M})$$

be the isomorphism induced by a generator of the deck transformations. Then we can think of $H_1(\widetilde{M})$ as a $\mathbb{Z}[t, t^{-1}]$-module. This is the Alexander module of K which is denoted by $A(K)$.

Let us choose a regular Seifert surface S with a regular spine W for the knot K. Let $\{\alpha_i, \beta_i \ ; \ i = 1, \ldots, 2g\}$ be a set of dual words of (S, W), where $g = \text{genus}(S)$. Denote by $v_{i,j}$ the sum of indices of x_j in α_i and $u_{i,j}$ the sum of indices of x_j in β_i. Then we have two $2g \times 2g$ matrices $V = (v_{i,j})$ and $U = (u_{i,j})$ with integer entries. The following facts are standard:

(1) V is the so-called *Seifert matrix* of the Seifert surface S;

(2) $U = V^T$, where T stands for transpose;

(3) $V - V^T = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \ldots, \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$; and,

(4) $tV - V^T$ is a presentation matrix of the Alexander module of K.

A generator of the order ideal of the Alexander module of K is called an *Alexander polynomial* of K. We can take

$$\triangle_K(t) = \det\ (tV - V^T)$$

as an Alexander polynomial of K.

We need Fox's free differential calculus to get a twisted version of Alexander modules and Alexander polynomials (see [6]).

Let G be an arbitrary group, $\mathbb{Z}G$ the integer group ring of G. A *derivation* in $\mathbb{Z}G$ is an additive homomorphism $D : \mathbb{Z}G \to \mathbb{Z}G$ such that

$$D(g_1 g_2) = D(g_1) + g_1 D(g_2)$$

for any $g_1, g_2 \in G$. The set of all derivations in $\mathbb{Z}G$ can be thought of as a (left) $\mathbb{Z}G$-module in a natural manner. As for free groups, the structure of this module is quite clear.

Proposition 3.1 (Fox) *Let F be the free group generated by x_1, …, x_n. Then all derivations in $\mathbb{Z}F$ form a free $\mathbb{Z}F$-module generated by n derivations $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$ which are uniquely determined by the property*

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

Here δ_{ij} is the Kronecker symbol.

By Lemma 2.1, we have a presentation

$$\pi_1(S^3 \setminus K) = \langle\, x_1, \ldots, x_{2g}, z \;;\; z\alpha_i z^{-1} = \beta_i,\; i = 1, \ldots, 2g \,\rangle,$$

where

$$\alpha_i = \alpha_i(x_1, \ldots, x_{2g}), \quad \beta_i = \beta_i(x_1, \ldots, x_{2g}), \quad i = 1, \ldots, 2g$$

are dual words of a regular Seifert surface of the knot K. We will consider the Jacobian matrices

$$\left(\frac{\partial \alpha_i}{\partial x_j}\right)_{2g\times 2g} \qquad \text{and} \qquad \left(\frac{\partial \beta_i}{\partial x_j}\right)_{2g\times 2g}$$

of the dual words.

Suppose $\rho : \pi_1(S^3 \setminus K) \to GL(n, \mathbb{C})$ is a representation. Such a representation is determined by the matrices $\rho(x_i)$ for $i = 1, \ldots, 2g$ and $\rho(z)$ and they should be subject to the relations in $\pi_1(S^3 \setminus K)$. We denote by

$$\left(\frac{\partial \alpha_i}{\partial x_j}\right)^{\rho} \qquad \text{and} \qquad \left(\frac{\partial \beta_i}{\partial x_j}\right)^{\rho}$$

the $2ng \times 2ng$ matrices obtained from the corresponding Jacobians by replacing each x_i with the $n \times n$ matrix $\rho(x_i)$. We construct a $\mathbb{C}[t, t^{-1}]$-module as follows:

Definition 3.2 *The twisted Alexander module associated with a representation ρ of the knot group $\pi_1(S^3 \setminus K)$ is a $\mathbb{C}[t, t^{-1}]$-module given by the presentation matrix*

$$t \cdot \operatorname{diag}\left(\rho(z), \ldots, \rho(z)\right) \cdot \left(\frac{\partial \alpha_i}{\partial x_j}\right)^{\rho} - \left(\frac{\partial \beta_i}{\partial x_j}\right)^{\rho}.$$

We denote this module by $A(K; \rho)$.

Theorem 3.3 *The isomorphism class of the module $A(K; \rho)$ is independent of the various choices we made prior to Definition 3.2, i.e., $A(K; \rho)$ is an invariant of the knot K.*

Proof Let us first extend the definition of a twisted Alexander module to any free Seifert surfaces by using their induced words. Let S be a free Seifert surface and W be a spine of S. Choose a basis $\{x_1, \ldots, x_{2g}\}$ for $\pi_1(S^3 \setminus S)$ and denote by $\{\alpha_i, \beta_i \;;\; i = 1, \ldots, 2g\}$ the set of induced words. Then we can also consider the presentation matrix

$$t \cdot \operatorname{diag}\left(\rho(z), \ldots, \rho(z)\right) \cdot \left(\frac{\partial \alpha_i}{\partial x_j}\right)^{\rho} - \left(\frac{\partial \beta_i}{\partial x_j}\right)^{\rho}.$$

Let us first consider a basis transformation in $F(x_1, \ldots, x_{2g})$. Suppose the new basis is $\{y_1, \ldots, y_{2g}\}$. Then, we have

$$\left(\frac{\partial \alpha_i}{\partial y_j}\right)^{\rho} = \left(\frac{\partial \alpha_i}{\partial x_j}\right)^{\rho} \cdot \left(\frac{\partial x_i}{\partial y_j}\right)^{\rho}$$

and

$$\left(\frac{\partial \beta_i}{\partial y_j}\right)^{\rho} = \left(\frac{\partial \beta_i}{\partial x_j}\right)^{\rho} \cdot \left(\frac{\partial x_i}{\partial y_j}\right)^{\rho},$$

where $\left(\frac{\partial x_i}{\partial y_j}\right)^{\rho}$ is an invertible matrix. So this will not change the isomorphism class of $A(K;\rho)$.

Next, we consider the case when the spine W is changed to another spine W'. Using the notations in Lemma 2.3, we have

$$\left(\frac{\partial \alpha'_i}{\partial x_j}\right)^{\rho} = \left(\frac{\partial \alpha'_i}{\partial \alpha_j}\right)^{\rho} \cdot \left(\frac{\partial \alpha_i}{\partial x_j}\right)^{\rho}$$

and

$$\left(\frac{\partial \beta'_i}{\partial x_j}\right)^{\rho} = \left(\frac{\partial \beta'_i}{\partial \beta_j}\right)^{\rho} \cdot \left(\frac{\partial \beta_i}{\partial x_j}\right)^{\rho}.$$

Since the transformations from $\{\alpha_1, \ldots, \alpha_{2g}\}$ to $\{\alpha'_1, \ldots, \alpha'_{2g}\}$ and from $\{\beta_1, \ldots, \beta_{2g}\}$ to $\{\beta'_1, \ldots, \beta'_{2g}\}$ are the same and

$$\rho(z) \cdot \rho(\alpha_i) \cdot \rho(z^{-1}) = \rho(\beta_i), \quad i = 1, \ldots, 2g,$$

we have

$$\mathrm{diag}\,(\rho(z), \ldots, \rho(z)) \cdot \left(\frac{\partial \alpha'_i}{\partial \alpha_j}\right)^{\rho} \cdot \mathrm{diag}\,\left(\rho(z^{-1}), \ldots, \rho(z^{-1})\right) = \left(\frac{\partial \beta'_i}{\partial \beta_j}\right)^{\rho}.$$

Thus, the isomorphism class of $A(K;\rho)$ is also unchanged in this case.

At this moment, we can say that the isomorphism class of $A(K;\rho)$ might only depend on the choices of free Seifert surfaces. To prove that the isomorphism class of $A(K;\rho)$ is invariant under a regular handle-addition, we must use regular spines and their dual words.

Suppose the regular Seifert surface S is altered by a regular handle-addition yielding a new regular Seifert surface $\overline{S}$. We use the notations in Lemma 2.4.

Since $\bar{\alpha}_{2g+2} = 1$ and $\bar{\beta}_{2g+2} = x_{2g+1}$, $\rho(x_{2g+1})$ must be the identity matrix E_n. We then have

$$\left(\frac{\partial \bar{\alpha}_i}{\partial x_j}\right)^{\rho} = \begin{pmatrix} \multicolumn{3}{c}{\multirow{3}{*}{$\left(\frac{\partial \bar{\alpha}_i}{\partial x_j}\right)^{\rho}_{2g\times 2g}$}} & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & 0 & \rho(\gamma) \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \multicolumn{3}{c}{\multirow{3}{*}{$\left(\frac{\partial \alpha_i}{\partial x_j}\right)^{\rho}_{2g\times 2g}$}} & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & 0 & \rho(\gamma) \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and

$$\left(\frac{\partial \bar{\beta}_i}{\partial x_j}\right)^{\rho} = \begin{pmatrix} \multicolumn{3}{c}{\multirow{3}{*}{$\left(\frac{\partial \bar{\beta}_i}{\partial x_j}\right)^{\rho}_{2g\times 2g}$}} & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & E_n & 0 \end{pmatrix} = \begin{pmatrix} \multicolumn{3}{c}{\multirow{3}{*}{$\left(\frac{\partial \beta_i}{\partial x_j}\right)^{\rho}_{2g\times 2g}$}} & * & 0 \\ & & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & E_n & 0 \end{pmatrix}.$$

From these equalities, it is quite easy to see that the isomorphism class of $A(K;\rho)$ is invariant under a regular handle-addition. Thus, by Theorem 1.7, $A(K;\rho)$ is a knot invariant.

Notice that if two representations ρ and ρ' of the knot group are conjugate, then $A(K;\rho)$ and $A(K;\rho')$ are isomorphic. Also, if ρ reduces to 1-dimensional representations, then $A(K;\rho)$ is a direct sum of n copies of the (complexified) Alexander module $A(K)\otimes\mathbb{C}$.

Now we can have a twisted version of Alexander polynomials of knots.

Definition 3.4 *A twisted Alexander polynomial of a knot K associated with a representation $\rho : \pi_1(S^3 \setminus K) \to GL(n)$ is a generator of the order ideal of the twisted Alexander module $A(K;\rho)$.*

We can take

$$\Delta_K(t;\rho) = \det\left(t\cdot \mathrm{diag}(\rho(z),\dots,\rho(z))\cdot\left(\frac{\partial\alpha_i}{\partial x_j}\right)^{\rho} - \left(\frac{\partial\beta_i}{\partial x_j}\right)^{\rho}\right)$$

as a twisted Alexander polynomial of K associated with the representation ρ.

We can also have a twisted version of the potential function of a knot K.

Definition 3.5 *Suppose $\rho : \pi(S^3 \setminus K) \to SL(n,\mathbb{C})$ is a representation. Then the twisted potential function of K associated with ρ is defined to be*

$$\Omega_K(s;\rho) = \det\left(s\cdot \mathrm{diag}\left(\rho(z),\dots,\rho(z)\right)\cdot\left(\frac{\partial\alpha_i}{\partial x_j}\right)^{\rho} - s^{-1}\cdot\left(\frac{\partial\beta_i}{\partial x_j}\right)^{\rho}\right).$$

Theorem 3.6 *The twisted potential function $\Omega_K(s;\rho)$ is a well-defined knot invariant.*

Proof The proof of this theorem is essentially the same as the proof of Theorem 3.3. So we will only give an outline. Notice that the notations in the proof of Theorem 3.3 will be used in the follow discussion.

First, let us consider a basis transformation from $\{x_1,\dots,x_{2g}\}$ to $\{y_1,\dots,y_{2g}\}$. Notice that ρ is a representation into $SL(n,\mathbb{C})$. So we have

$$\det\left(\left(\frac{\partial x_i}{\partial y_j}\right)^{\rho}\right) = 1.$$

Thus $\Omega_K(s;\rho)$ is invariant in this case.

Next, suppose a spine is changed to another one. We need to consider only a basis transformation from $\{\beta_1,\dots,\beta_{2g}\}$ to $\{\beta'_1,\dots,\beta'_{2g}\}$. Then, we have

$$\det\left(\left(\frac{\partial \beta'_i}{\partial \beta_j}\right)^{\rho}\right) = 1.$$

Again, $\Omega_K(s;\rho)$ is invariant.

Finally, assume that we have changed a regular Seifert surface by a regular handle-addition resulting a new regular Seifert surface. Since

$$\det\begin{pmatrix} 0 & s\rho(z\gamma) \\ s^{-1}E_n & 0 \end{pmatrix} = 1,$$

we see that $\Omega_K(s;\rho)$ remains the same under a regular handle-addition.

Thus, $\Omega_K(s;\rho)$ is a well-defined knot invariant.

4 Metabelian Representations and Examples

The presentation of the knot group $\pi_1(S^3 \setminus K)$ described in Lemma 2.1 has an interesting consequence. Let us first have a notion about group representations.

Definition 4.1 *Let ρ be a representation of a group G. We call ρ a metabelian representation if $\rho([G,G])$ is abelian.*

The following result is essentially due to Fox (see [11]):

Proposition 4.2 *The number of conjugacy classes of irreducible metabelian representations of $\pi_1(S^3 \setminus K)$ into $SU(2,\mathbb{C})$ is*

$$\frac{1}{2}\left(|\Delta_K(-1)| - 1\right).$$

Proof Consider the presentation of $\pi_1(S^3 \setminus K)$ given by Lemma 2.1:

$$\langle\ x_1, \ldots, x_{2g}, z\ ;\ z\alpha_i z^{-1} = \beta_i,\ i = 1, \ldots, 2g\ \rangle,$$

where α_i's and β_i's are dual words of a regular Seifert surface of K. It is not hard to see that $x_1, \ldots, x_{2g}$ are all commutators of $\pi_1(S^3 \setminus K)$. Let $\rho : \pi_1(S^3 \setminus K) \to SU(2,\mathbb{C})$ be an irreducible metabelian representation. Up to a conjugation, we can assume that

$$\rho(x_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \bar{\lambda}_i \end{pmatrix}, \qquad i = 1, \ldots, 2g.$$

Since $\rho(zx_iz^{-1})$ should also be a diagonal matrix, we must have

$$\rho(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\rho(zx_iz^{-1}) = \begin{pmatrix} \bar{\lambda}_i & 0 \\ 0 & \lambda_i \end{pmatrix}, \qquad i = 1, \ldots, 2g.$$

Putting these matrices into the defining relations of $\pi_1(S^3 \setminus K)$, we see that $\lambda_1, \ldots, \lambda_{2g}$ must satisfy the following equations:

$$\lambda_1^{w_{i,1}} \lambda_2^{w_{i,2}} \cdots \lambda_{2g}^{w_{i,2g}} = 1, \qquad i = 1, \ldots, 2g, \tag{$*$}$$

where

$$(w_{i,j})_{2g \times 2g} = V + V^T.$$

On the other hand, a non-trivial solution $\{\lambda_1, \ldots, \lambda_{2g}\}$ of the equations $(*)$ will certainly produce an irreducible metabelian representation of $\pi_1(S^3 \setminus K)$ into $SU(2,\mathbb{C})$.

With some linear algebra, one can see that the number of non-trivial solutions of $(*)$ is

$$\left|\det\,(V+V^T)\right|-1=|\Delta_K(-1)|-1.$$

But if $\{\lambda_1,\ldots,\lambda_{2g}\}$ is a non-trivial solution of $(*)$, then $\{\bar{\lambda}_1,\ldots,\bar{\lambda}_{2g}\}$ is another non-trivial solution of $(*)$. The irreducible metabelian representations given by these two solutions respectively are conjugate. Moreover, that is the only situation when the irreducible metabelian representations produced by the non-trivial solutions of $(*)$ are conjugate. So there are totally

$$\frac{1}{2}\left(|\Delta_K(-1)|-1\right)$$

conjugacy classes of irreducible metabelian representations.

Let $\rho:\pi_1(S^3\setminus K)\to SU(2,\mathbb{C})$ be a metabelian representation. We can always assume that $\rho(z)=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Then $\bar{\rho}$ is the conjugation of ρ by $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$.

Proposition 4.3 *If ρ is a metabelian representation of $\pi_1(S^3\setminus K)$ into $SU(2,\mathbb{C})$, then the coefficients of $\Delta_K(t;\rho)$ and $\Omega_K(s;\rho)$ are all real.*

Proof Conjugating the presentation matrix of $A(K;\rho)$ by

$$\mathrm{diag}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ldots,\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)$$

amounts to changing ρ to $\bar{\rho}$. But

$$\overline{\Delta_K(t;\rho)}=\Delta_K(t;\bar{\rho})=\Delta_K(t;\rho).$$

Therefore, the coefficients of $\Delta_K(t;\rho)$ are all real. Similarly, we can show that the coefficients of $\Omega_K(s;\rho)$ are all real.

Let us now calculate twisted Alexander polynomials associated with representations into $SU(2,\mathbb{C})$ for the trefoil knot.

Consider a class of knots $K(p,q)$ with $p,q\in\mathbb{Z}$ and $p,q\neq 0$ constructed as follows:

Begin with a standard punctured torus in S^3 and specify a spine of it consisting of two oriented circles a and b (see Figure 4.1 (1)). Add p full twists to the band associated with a and q full twists to the band associated with b (see Figure 4.1 (2)). Denote the resulting surface by $S(p,q)$. Then

$$K(p,q)=\partial S(p,q).$$

Here the signs of p and q are determined so that $K(1,1)$ is the right trefoil knot. Thus, $K(-1,-1)$ is the left trefoil knot. In general, $K(-p,-q)$ is the mirror image of $K(p,q)$.

The Seifert form of $K(p,q)$ is

$$V=\begin{pmatrix}-p&1\\0&-q\end{pmatrix}.$$

So, one can easily get $pq(1-t)^2+t$ as an Alexander polynomial of $K(p,q)$.

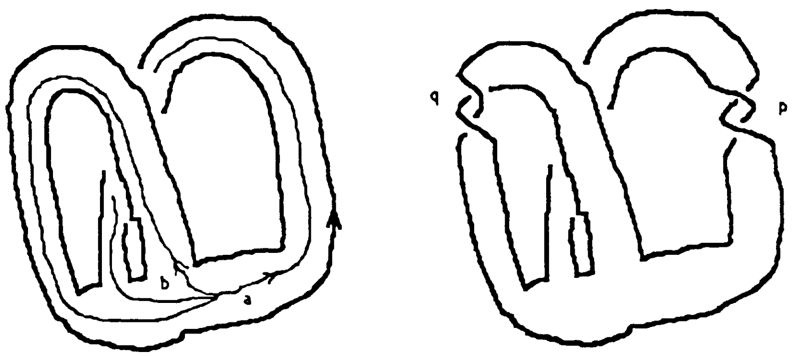

Figure 4.1

Notice that $S(p,q)$ is a regular Seifert surface of the knot $K(p,q)$ and $W = a \cup b$ is a regular spine of $S(p,q)$. We have disks D_a and D_b such that $D_a \cap S(0,0) = \partial D_a = a$ and $D_b \cap S(0,0) = \partial D_b = b$. We may choose $\{D_a, D_b\}$ as the collection $\mathcal{D}$ of disks spanned by W. The induced words of the triple $(S(p,q), W, \mathcal{D})$ are

$$\alpha_1 = x_1^{-p}x_2, \quad \alpha_2 = x_2^{-q}, \quad \beta_1 = x_1^{-p}, \quad \beta_2 = x_1x_2^{-q}.$$

We consider the right trefoil knot $K(1,1)$. The corresponding Jacobians are given by

$$\left(\frac{\partial \alpha_i}{\partial x_j}\right) = \begin{pmatrix} -x_1^{-1} & x_1^{-1} \\ 0 & -x_2^{-1} \end{pmatrix}, \qquad \left(\frac{\partial \beta_i}{\partial x_j}\right) = \begin{pmatrix} -x_1^{-1} & 0 \\ 1 & -x_1x_2^{-1} \end{pmatrix}.$$

Let ρ be a representation of the knot group of $K(1,1)$. Then, we have

$$\begin{aligned}
&\det\left(t\cdot \operatorname{diag}(\rho(z),\rho(z))\cdot\left(\frac{\partial \alpha_i}{\partial x_j}\right)^{\rho} - \left(\frac{\partial \beta_i}{\partial x_j}\right)^{\rho}\right) \\
&= \det\left(t\begin{pmatrix} \rho(z) & 0 \\ 0 & \rho(z)\end{pmatrix}\begin{pmatrix} -\rho(x_1^{-1}) & \rho(x_1^{-1}) \\ 0 & -\rho(x_2^{-1})\end{pmatrix} - \begin{pmatrix} \rho(x_1^{-1}) & 0 \\ E_2 & -\rho(x_1x_2^{-1})\end{pmatrix}\right) \\
&= \det\begin{pmatrix} -t\rho(zx_1^{-1}) + \rho(x_1^{-1}) & t\rho(zx_1^{-1}) \\ -E_2 & -t\rho(zx_2^{-1}) + \rho(x_1x_2^{-1})\end{pmatrix} \\
&= \det\left(\left(-t\rho(zx_1^{-1}) + \rho(x_1^{-1})\right)\left(-t\rho(zx_2^{-1}) + \rho(x_1x_2^{-1})\right) + t\rho(zx_1^{-1})\right) \\
&= \det\left(t^2\rho(zx_1^{-1}z) - t\rho(z) + E_2\right).
\end{aligned}$$

Here we used the relation $zx_1^{-1}x_2z^{-1} = x_1^{-1}$.

Let us assume that ρ is a representation into $SU(2,\mathbb{C})$. Then, up to a conjugation, we can assume

$$Z = \rho(z) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}.$$

Also, we can write

$$X_1 = \rho(x_1), \qquad X_2 = \rho(x_2).$$

With these assumptions, one can easily calculate the determinant and get (with $K = K(1,1)$)

$$\Delta_K(t;\rho) = t^4 - \mathrm{tr}(Z)t^3 + (1 + \mathrm{tr}(Z^{-2}X_1))t^2 - \mathrm{tr}(Z)t + 1.$$

Here $\mathrm{tr}(\cdot)$ stands for the trace of the corresponding matrix.

Lemma 4.4 *We have* $\mathrm{tr}(Z^{-2}X_1) = 1$ *if* ρ *is an irreducible representation.*

Proof We will use the trace identity

$$\mathrm{tr}(XY) + \mathrm{tr}(XY^{-1}) = \mathrm{tr}(X)\mathrm{tr}(Y)$$

for any $X, Y \in SU(2,\mathbb{C})$. For example, we have

$$\mathrm{tr}(Z^{-2}X_1) = \mathrm{tr}(Z)^2 - \mathrm{tr}(X_1).$$

The defining relations of the knot group of $K(1,1)$ is

$$zx_1^{-1}x_2z^{-1} = x_1^{-1}, \quad zx_2^{-1}z^{-1} = x_1x_2^{-1}.$$

Applying the trace function to these relations and using the trace identity, we find $\mathrm{tr}(X_1)$ is the solution of the equation

$$(x-2)(x - \mathrm{tr}(Z)^2 + 1) = 0.$$

The solution $x = 2$ corresponds to the reducible representation with both X_1 and X_2 being the identity matrix. So, for ρ to be irreducible, we must have

$$\mathrm{tr}(X_1) = \mathrm{tr}(Z)^2 - 1.$$

This finishes the proof.

Thus, we have

$$\Delta_K(t;\rho) = t^4 - \mathrm{tr(Z)}t^3 + 2t^2 - \mathrm{tr(Z)}t + 1$$

with K the (right) trefoil knot and $Z = \rho(z)$.

If ρ is an irreducible metabelian representation, we have $\mathrm{tr}(Z) = 0$. So the associated twisted Alexander polynomial becomes $t^4 + 2t^2 + 1$. On the other hand, if ρ is a reducible representation with $Z = \rho(z) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \bar\lambda \end{smallmatrix}\right)$, the associated "twisted" Alexander polynomial is

$$(\lambda^2t^2 - \lambda t + 1)(\bar\lambda^2 - \bar\lambda t + 1) = t^4 - \mathrm{tr}(Z)t^3 + (\mathrm{tr}(Z)^2 - 1)t^2 - \mathrm{tr}(Z)t + 1.$$

Finally, we notice that the identity

$$\mathrm{tr}(X_1) = \mathrm{tr}(Z)^2 - 1$$

in the proof of Lemma 4.4 shows that for ρ to be an irreducible representation, we must have

$$|\mathrm{tr}(Z)| < \sqrt{3}.$$

Moreover, the space of conjugacy classes of irreducible representations $\pi_1(S^3 \setminus K) \to SU(2,\mathbb{C})$ with $K = K(1,1)$ the trefoil knot can be parameterized by $\mathrm{tr}(Z)$ with $-\sqrt{3} < \mathrm{tr}(Z) < \sqrt{3}$.

To see this, notice that the conjugacy classes of (X_1, X_2) with X_1, X_2 not simultaneously diagonalizable are determined by $\mathrm{tr}(X_1)$, $\mathrm{tr}(X_2)$ and $\mathrm{tr}(X_1X_2)$, whereas all of them can be expressed as functions of $\mathrm{tr}(Z)$. See Figure 4.2 where the vertical line segment corresponds to the conjugacy classes of reducible representations and the semicircle to that of irreducible ones.

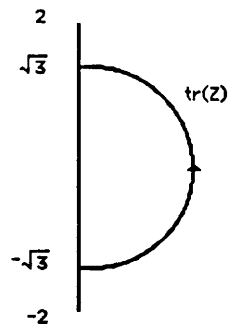

Figure 4.2

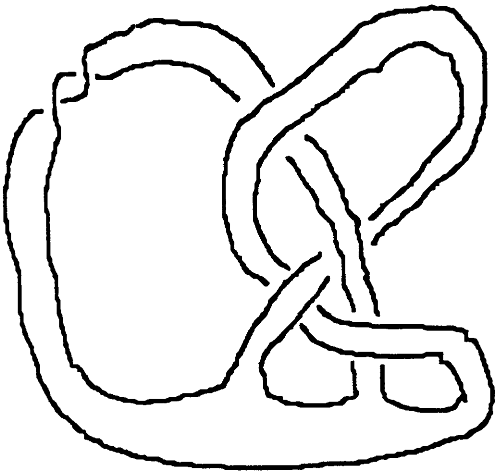

Figure 4.3

Let us now point out an example of a knot whose Alexander module is isomorphic to that of the trefoil knot but whose twisted Alexander polynomial (associated with the only conjugacy class of metabelian representations) is not the same as that of the trefoil knot.

The knot in our example is depicted in Figure 4.3 with an obvious regular Seifert surface. It is quite easy to see that this knot and the trefoil knot have the same Seifert form.

We can write down the induced words as follows:

$$\begin{aligned} \alpha_1 &= x_1x_2^{-1}x_1^{-1}x_2^2x_1^{-1}, & \beta_1 &= x_1x_2^{-1}x_1^{-1}x_2^2x_1^{-1}x_2^{-1}, \\ \alpha_2 &= x_1x_2^{-1}x_1^{-1}x_2x_1^{-1}x_2^{-1}x_1, & \beta_2 &= x_1x_2^{-1}. \end{aligned}$$

We have calculated that the twisted Alexander polynomial associated with the only metabelian representation is $4(1 + 2t^2 + t^4)$. Here one should notice that in this case, the twisted Alexander module associated with the metabelian representation can be thought of as a module over the ring $\mathbb{Z}[\omega][t, t^{-1}]$ with ω a primitive cubic root of the unit. Certainly, $4 \in \mathbb{Z}[\omega]$ is not invertible. So, this is an example of two knots with the same Alexander module but different twisted Alexander modules.

To finish this section, we show that twisted Alexander polynomials can be used to distinguish $K(p, q)$ and $K(p', q')$ with $pq = p'q'$. Notice that such knots can not be distinguished by their Alexander polynomials.

Similarly to the calculation in the case of $p, q = 1$, we have calculated that

$$\Delta_{K(p,q)}(t; \rho) = \xi_p\bar{\xi}_p\zeta_q\bar{\zeta}_q(1 + 2t^2 + t^4),$$

for ρ, an irreducible metabelian representation into $SU(2, \mathbb{C})$. Here

$$\begin{pmatrix} \xi_p & 0 \\ 0 & \bar{\xi}_p \end{pmatrix} = \left(\frac{\partial x_1^{-p}}{\partial x_1}\right)^{\rho}, \quad \begin{pmatrix} \zeta_q & 0 \\ 0 & \bar{\zeta}_q \end{pmatrix} = \left(\frac{\partial x_2^{-q}}{\partial x_2}\right)^{\rho}.$$

Denoting $\xi_p\bar{\xi}_p\zeta_q\bar{\zeta}_q$ by $c_{p,q} = c_{p,q}(\omega)$ with $\omega^{4pq-1} = 1$ and $\omega \neq 1$, we have

$$\begin{aligned} c_{p,q} = (1 + \omega + \cdots + \omega^{|p|-1})(1 + \bar{\omega} + \cdots + \bar{\omega}^{|p|-1}) \\ \cdot (1 + \omega^{2|p|} + \cdots + \omega^{2|p|(|q|-1)})(1 + \bar{\omega}^{2|p|} + \cdots + \bar{\omega}^{2|p|(|q|-1)}). \end{aligned}$$

For example, we have

$$\begin{aligned} c_{1,4} &= (1 + \omega^2 + \omega^4 + \omega^6)(1 + \bar{\omega}^2 + \bar{\omega}^4 + \bar{\omega}^6) \\ &= 4 + 3(\omega^2 + \bar{\omega}^2) + 2(\omega^4 + \bar{\omega}^4) + (\omega^6 + \bar{\omega}^6) \end{aligned}$$

and

$$\begin{aligned} c_{2,2} &= (1 + \omega)(1 + \bar{\omega})(1 + \omega^4)(1 + \bar{\omega}^4) \\ &= 4 + 2(\omega + \bar{\omega}) + (\omega^3 + \bar{\omega}^3) + 2(\omega^4 + \bar{\omega}^4) + (\omega^5 + \bar{\omega}^5) \end{aligned}$$

with $\omega^{15} = 1$ and $\omega \neq 1$. For $K(1,4)$ and $K(2,2)$ having the same twisted Alexander polynomial in the ring $\mathbb{Z}[\omega]$, the only possibility is that the two sets of real numbers $\{c_{1,4}\}$ and $\{c_{2,2}\}$ are the same. This is certainly not the case.

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