ON ODD-DIMENSIONAL FIBRED KNOTS OBTAINED BY PLUMBING AND TWISTING

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Introduction

Plumbing a Hopf band and doing a Stallings twist on the fibre-surface of a given classical fibred link are two powerful ways to produce other such links [6, 15]. In fact Harer has shown that every classical fibred link can be constructed from the unknot using these two operations and their inverses. Melvin and Morton have proved using geometric arguments that there exist genus-two fibred classical knots that cannot be obtained by plumbing only (cf. [10]).

We consider in this paper the analogue of these two operations for high odd-dimensional knots. For instance Durfee [3] has shown that all knots that arise as the link of an isolated singularity of a complex hypersurface are obtained by plumbing. For this purpose it is necessary to widen the usual definition of a (2k-1)-dimensional knot to include (k-2)-connected (2k-1)-dimensional differentiable submanifolds N of S^{2k+1} . A knot is called spherical if N is homeomorphic to S^{2k-1} . The classical concepts of 'link' and 'knot' correspond therefore in high dimensions to those of 'knot' and 'spherical knot' respectively. This is the framework adopted in [3].

The use of the *h*-cobordism theorem is essential in many of the constructions so that we never consider 3-dimensional knots (that is, the case k = 2).

We obtain in particular the following results.

A simple high-dimensional fibred knot is obtained by plumbing if and only if it admits a unimodular triangular Seifert matrix (Proposition 2.4).

Let *l* be an integer greater than 1 and let *K* be a (4l-1)-dimensional spherical knot obtained by plumbing with positive definite intersection form *I*; then *I* is an orthogonal sum of copies of the form Γ_8 (Corollary 3.4).

For every $k \ge 3$ there exist (2k-1)-dimensional spherical fibred knots of arbitrarily high genus that cannot be obtained by plumbing and twisting (Theorem 5.5).

This result has the following classical counterpart.

Either there are unimodular Seifert forms that cannot be realised by fibred classical knots or there are fibred knots that cannot be obtained by plumbing and twisting only.

Finally we remark that the question whether all high-dimensional simple fibred knots are obtained by plumbing and deplumbing has an interesting formulation in terms of matrices, the answer to which we do not know.

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1. Definitions

In what follows, k will denote any positive integer different from 2. A knot is an oriented smooth (2k-1)-dimensional submanifold K of the oriented (2k+1)-dimensional sphere S^{2k+1} such that K is (k-2)-connected.

The knot K is called *spherical* if K is (k-1)-connected (K is then homeomorphic to S^{2k-1} using Poincaré duality and Smale's theorem). For k = 1, by convention, a knot is what is usually called a link and a spherical knot a 'true' (one component) knot.

Let T_K be a tubular neighbourhood of K in S^{2k+1} ; T_K is a trivial disc bundle over K. (To see this, note that the only obstruction to trivialise this bundle is its Euler class and that K may be thought to be embedded in \mathbb{R}^{2k+1} . Using excision it is easy to see that the Euler class of the normal bundle to an oriented submanifold of \mathbb{R}^n is zero.)

The knot K is simple if it bounds a (k-1)-connected oriented 2k-dimensional manifold F^{2k} embedded in S^{2k+1} , that is, if K possesses a (k-1)-connected Seifert 'surface'.

We say that K is fibred if there is a trivialisation $\phi: T_K \to K \times D^2$ such that

$$\operatorname{pr}_2 \circ \phi : \partial T_K \longrightarrow S^1$$

extends to a locally trivial fibration of $S^{2k+1} \setminus \hat{T}_K$ over S^1 . The inverse image of a point is (after collaring) a Seifert surface for K called the *fibre-surface*. Note that if K is simple, the infinite cyclic cover of K is (k-1)-connected (cf. [9, §2c]) and has the homotopy type of the fibre-surface, so that the fibre-surface is necessarily (k-1)-connected.

Let F be a Seifert surface for K; the normal bundle to F in S^{2k+1} is trivial and one can define two maps i_+ and $i_-: F \to S^{2k+1} \setminus F$ which send any point of F to a push-off of itself along the positive and negative direction of the bundle, respectively. (These directions are determined so that for all elements a and b of $H_k(F)$,

$$\mathscr{L}(a;i_+b) - \mathscr{L}(a;i_-b) = I_F(a;b),$$

where \mathscr{L} denotes linking number in S^{2k+1} and I_F is the intersection form of F.)

To any Seifert surface F for K there corresponds a Seifert form

$$A: H_k(F) \times H_k(F) \to \mathbb{Z}$$

defined by $A(x; y) = \mathcal{L}(x; i_+ y)$. Denote by A^T the transpose of A and recall that $A + (-1)^k A^T = I_F$. Hence if K is simple, K is spherical if and only if I_F is unimodular. Recall also that if K is any fibred knot with fibre-surface F, the monodromy of K induces an automorphism h of $H_k(F)$ which satisfies

$$A(x; y) = (-1)^{k+1} A^{T}(x; hy), \qquad I_{F}(x; y) = A((1-h)x; y).$$

The following lemma is well known for spherical knots, and its proof extends without change to the simple knots considered here (see [3, statement and proof of Theorem 3.1]).

LEMMA 1.1 (Levine, Durfee). Let k be an integer, $k \ge 3$.

(a) A simple (2k-1)-knot K is fibred if and only if K bounds a Seifert surface F such that the associated Seifert form is unimodular. Such a surface F is a fibre-surface for K.

- (b) There is a one-to-one correspondence between
 - (i) isotopy classes of (k−1)-connected (2k)-dimensional submanifolds F of S^{2k+1} such that
 K = ∂F is non-empty and (k−2)-connected,
 K is fibred and F is a fibre-surface for K;
 - (ii) isotopy classes of simple (k-2)-connected fibred knots in S^{2k+1} ;
 - (iii) congruence classes of integral unimodular bilinear forms defined on a free \mathbb{Z} -module of finite rank.

The correspondence associates to each fibre-surface F the boundary of F and the Seifert form of F respectively.

Let K be a simple fibred knot with fibre-surface F; if K is spherical, $\operatorname{rk} H_k(F)$ is even. This is because if k is odd then $A + (-1)^k A^T$ is unimodular and skew-symmetric, and hence an orthogonal sum of hyperbolic planes, but if k is even, it is an even unimodular symmetric form (cf. [13, Chapter V]). For K spherical we define the genus of K to be $\frac{1}{2}\operatorname{rk} H_k(F)$.

2. Plumbing

Let K_1 and K_2 be two simple knots in S^{2k+1} bounding (k-1)-connected Seifert surfaces F_1 and F_2 . Divide S^{2k+1} into two hemispheres B_1 and B_2 intersecting in a (2k)-dimensional sphere S. Let $\Psi: D^k \times D^k \rightarrow S$ be an embedding and suppose that

(i) $F_i \subset B_i, i = 1, 2,$

(ii)
$$F_1 \cap S = F_2 \cap S = F_1 \cap F_2 = \Psi(D^k \times D^k)$$
,

- (iii) $\Psi(\partial D^k \times D^k) = \partial F_1 \cap \Psi(D^k \times D^k)$ and $\Psi(D^k \times \partial D^k) = \partial F_2 \cap \Psi(D^k \times D^k)$,
- (iv) the orientations on F_1 and F_2 match on $\Psi(D^k \times D^k)$.

Denote by $F_1 \square F_2$ the submanifold of S^{2k+1} obtained from $F_1 \cup F_2$ after smoothing the corners $(F_1 \square F_2$ depends of course on Ψ and the decomposition of S^{2k+1}).

DEFINITION. $F_1 \square F_2$ is said to be obtained by plumbing together the surfaces F_1 and F_2 .

PROPOSITION 2.1. If F_1 and F_2 are fibre-surfaces for K_1 and K_2 , then $K = \partial(F_1 \square F_2)$ is a simple fibred knot with fibre-surface $F_1 \square F_2$. Conversely, if K is fibred with fibre-surface $F_1 \square F_2$, then K_1 and K_2 are fibred with fibre-surfaces F_1 and F_2 .

If A_i denotes a Seifert matrix for F_i , i = 1, 2, there is a basis of $H_k(F_1 \Box F_2)$ for which the Seifert matrix of $F_1 \Box F_2$ looks like

$$\begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$$

for some integral matrix B.

Proof. For k = 1 (that is, classical links) see [4]. Let $k \ge 3$ and set

$$\Sigma = \Psi(D^k \times D^k) \text{ and } Y_i = \overline{\partial F_i \setminus \Sigma}, \quad i = 1, 2;$$

$$K = Y_1 \cup Y_2, \qquad Y_1 \cap Y_2 = \Psi(\partial D^k \times \partial D^k).$$

then

As $\partial F_1 = Y_1 \bigcup_{\Psi(\partial D^k \times \partial D^k)} \Psi(\partial D^k \times D^k)$ and $k \ge 3$, $\Pi_1(Y_1) \simeq \Pi_1(\partial F_1) = 1$; a similar result holds for $\Pi_1(Y_2)$. The exact sequence of the pair $(\partial F_i, Y_i)$ shows that $H_j(Y_i) = 0$ for $j \le k-2$. Using Mayer-Vietoris for $\partial(F_1 \square F_2) = Y_1 \cup Y_2$ it is then easy to see that $H_j(K) = 0$ for $j \le k-2$ and hence that K is (k-2)-connected.

To show that $F = F_1 \square F_2$ is the fibre-surface for K, note that F is clearly (k-1)-connected and that $H_k(F) \simeq H_k(F_1) \oplus H_k(F_2)$. The map $i_+: F \to S^{2k+1} \setminus F$ pushes all the points of Σ into one of the two hemispheres; suppose it is B_2 . For $x_i \in H_k(F_i)$, i = 1, 2, choose a cycle z_i with support in F_i ; z_1 and $i_+ z_2$ have their support contained in two distinct balls of S^{2k+1} separated by a small push-off of the sphere S, therefore $\mathscr{L}(x_1; i_+ x_2) = 0$.

The matrix A for the Seifert form of F relative to the bases of $H_k(F_1)$ and $H_k(F_2)$ is

$$A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

for some rk $H_k(F_2) \times \text{rk } H_k(F_2)$ matrix B.

If i_+ pushed Σ inside B_1 we would have

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}.$$

In both cases det $A = \det A_1 \det A_2$. Using Lemma 1.1, this shows that F is a fibre-surface if and only if both F_1 and F_2 are fibre-surfaces.

There are up to isotopy only two fibred classical links that possess a fibre-surface F with rk $H_1(F) = 1$; they are the right- and left-handed Hopf links respectively which each bound an annulus $S^1 \times D^2$ embedded in S^3 with ± 1 full twist. One can start from them and by repeated plumbing construct many new fibred knots or links. We shall see that the situation is entirely analogous in high dimensions for simple fibred knots.

Let $k \ge 3$ and K be a (not necessarily fibred) simple (2k-1)-dimensional knot and suppose that K bounds a (k-1)-connected surface F such that rk $H_k(F) = 1$; F is obtained by attaching a k-handle on a 2k-dimensional disc D^{2k} (cf. [14, Theorem 1.1] the fact that $k \ne 2$ is crucial here). The core of the handle together with a k-disc inside D^{2k} can be budged to an embedded differentiable sphere S^k and one can show that F is diffeomorphic to the total space of the normal disc bundle to S^k in F (cf. [17, proof of Theorem 1] for details).

Note that F is embedded in \mathbb{R}^{2k+1} , therefore the associated vector bundle over S^k is oriented and stably trivial. Recall that such bundles are classified according to the following lemma.

LEMMA 2.2. Let ξ be a k-dimensional oriented stably trivial vector bundle over S^k . Let $D(\xi)$ denote the total space of the associated disc bundle and Σ its zero section. Let ε^1 be the trivial bundle of rank 1 over S^k . Then

(i) $\xi \oplus \varepsilon^1$ is trivial;

(ii) if k is even, ξ is classified by its Euler number $e(\xi)$ that can take any even value.

Let F be any embedding of $D(\xi)$ in S^{2k+1} and let a be the homology class in $H_k(F)$ carried by the image of Σ ; then $e(\xi) = 2\mathcal{L}(a; i_+a)$.

(iii) If k = 1, 3, 7, then ξ is trivial.

(iv) If k is odd, $k \neq 1, 3, 7$, then ξ is either the trivial or the tangent bundle over S^k (and these are distinct). Let F and a be as in (ii); it is the trivial bundle if $\mathcal{L}(a; i_+a)$ is even, the tangent bundle if $\mathcal{L}(a; i_+a)$ is odd.

Proof. (i) See [8, Lemma 3.5].

(ii) Let $\phi_n: SO(n) \hookrightarrow SO(n+1) \to S^n$ be the usual fibration of S^n . There is a one to one correspondence between oriented rank l bundles ξ over S^k and elements $\chi(\xi) \in \prod_{k-1} SO(l)$ [16, Theorem 18.5]. By (i), oriented stably trivial rank k bundles over S^k are classified by elements of Ker $i_*: \prod_{k-1} SO(k) \to \prod_{k-1} SO(k+1)$. Now ϕ_k (respectively ϕ_{k-1}) induces the following horizontal (respectively vertical) exact sequence.

Coherent orientations give identifications of $\prod_k S^k$ and $\prod_{k-1} S^{k-1}$ with \mathbb{Z} such that $\partial(1) = \chi(\tau)$, where τ is the tangent bundle over S^k . For any rank k bundle ξ over S^k , $j_{k-1}\chi(\xi) = e(\xi)$ is the Euler number of ξ so that $j_{k-1} \circ \partial$ is multiplication by $1 + (-1)^k$ [2, IV.1.9]. If k is even, this shows that $\operatorname{Im} j_k = 0$, $j_{k-1} | \operatorname{Ker} i_*$ is injective and the Euler number can take any even value.

To show the relation between the Euler number and $\mathcal{L}(a; i_+ a)$, recall that $\mathcal{L}(a; i_+ a) - \mathcal{L}(a; i_- a) = I(a; a)$, where I denotes the intersection form of F; for k even, $\mathcal{L}(a; i_+ a) = -\mathcal{L}(i_+ a; a) = -\mathcal{L}(a; i_- a)$ so that $2\mathcal{L}(a; i_+ a) = I(a; a) = e(\xi)$.

- (iii) This is similar to [2, Theorem IV 1.10].
- (iv) As $\xi \oplus \varepsilon^1$ is isomorphic to $S^k \times \mathbb{R}^{k+1}$, there exists a nowhere zero cross-section

$$S^k \longrightarrow S^k \times S^k \subset S^k \times \mathbb{R}^{k+1}, \qquad x \longrightarrow (x; \alpha(x))$$

such that $\alpha(x)$ is orthogonal to all elements in the fibre of ξ over x. Conversely, given such a map $\alpha: S^k \to S^k$, the orthogonal hyperplanes to $(x; \alpha(x))$ in $\{x\} \times \mathbb{R}^{k+1}$ form the total space of a stably trivial rank k bundle over S^k . The map $\partial: \Pi_k(S^k) \to \Pi_{k-1}$ SO(k) sends the homotopy class of α to the classifying element $\chi(\xi)$, so that ξ depends only on the degree of α .

Decompose S^{2k+1} as $(D^{k+1} \times S^k) \bigcup_{S^k \times S^k} (S^k \times D^{k+1})$. Since for $k \ge 2$ all embeddings of S^k in S^{2k+1} are isotopic (cf. [5, Théorème d'existence]), we may suppose that the embedding of F in S^{2k+1} sends the zero section to $S^k \times \{0\}$; $a \in H_k(F)$ is represented by $S^k \times \{0\}$ and $i_+ a$ by

 $S^k \longrightarrow S^k \times S^k, \qquad x \longrightarrow (x; \alpha(x)).$

It follows easily that $\mathscr{L}(a; i_+a) = \deg \alpha$.

Given $\rho \in \prod_k SO(k+1)$, the automorphism of the trivial bundle

$$S^k \times \mathbb{R}^{k+1} \longrightarrow S^k \times \mathbb{R}^{k+1}, \qquad (x; y) \longrightarrow (x; \rho(x) \cdot y)$$

describes an isomorphism between ξ and a bundle $\rho\xi$ associated to the map

 $\rho \alpha \colon S^k \longrightarrow S^k, \qquad x \longrightarrow \rho(x) \cdot \alpha(x)$

whose degree is $j_k(\rho) + \deg \alpha$. (To see this, decompose $\rho \alpha$ into the diagonal embedding of S^k in $S^k \times S^k$ followed by

$$S^k \times S^k \longrightarrow S^k$$
, $(x; y) \longrightarrow \rho(x) \cdot \alpha(y)$

whose bidegree is clearly $(j_k(\rho); \deg \alpha)$.) Since $\operatorname{Im} j_k$ has index 2 in $\Pi_k(S^k)$ for k odd, $k \neq 1, 3, 7$ [2, Corollary IV 1.11], one can change the degree of α to 0 or 1 without modifying the isomorphism type of the bundle ξ . Therefore it suffices to check that the trivial (respectively tangent) bundle of S^k is given by a map α of degree 0 (respectively 1). The constant map corresponds clearly to the trivial bundle while the identity map corresponds to the normal bundle to the diagonal of $S^k \times S^k$; this is precisely the tangent bundle of S^k .

If $k \ge 3$, Lemma 1.1 shows that for $\varepsilon = \pm 1$ there is, up to isotopy, a unique surface F in S^{2k+1} such that $K = \partial F$ is (k-2)-connected, rk $H_k(F) = 1$ and the associated Seifert 'matrix' is (ε); moreover F is a fibre-surface for the fibred knot K. By Lemma 2.2, for $k \ne 2$, F is the total space of the *tangent* disc bundle of S^k and K is the total space of the corresponding sphere bundle.

DEFINITIONS. We shall call such a surface F a Hopf band and its boundary K a Hopf knot.

A knot K is said to be *obtained by plumbing* if there is a sequence of (2k)-manifolds F_0, F_1, \ldots, F_s embedded in S^{2k+1} such that $F_0 = D^{2k}, \partial F_s = K$ and F_{i+1} is obtained by plumbing together F_i and a Hopf band.

REMARKS. (i) By Proposition 2.1, K is a simple fibred knot.

(ii) For $k \neq 2$, any (k-1)-connected (2k)-manifold with boundary is obtained from D^{2k} by attaching handles of index k (cf. [14, Theorem 1.2]), so that any simple (2k-1)-knot bounds a Seifert surface obtained by 'abstract' (that is non-embedded) plumbing.

PROPOSITION 2.4. Let K be a simple (2k-1)-knot in S^{2k+1} . If $k \ge 3$, K is obtained by plumbing if and only if it admits a unimodular lower triangular Seifert matrix. For k = 1, the above condition is necessary.

Proof. Let K be obtained by plumbing, let F be the fibre-surface and let A be the corresponding Seifert matrix. We shall prove the proposition by induction on $r = \operatorname{rk} H_k(F)$. If r = 1, it is clear; if $r \ge 2$, let F' be the fibre-surface obtained by plumbing the first r-1 Hopf bands and let A' be the corresponding Seifert matrix. By induction A' is, with respect to some basis, a lower triangular matrix with ± 1 on the diagonal; A looks like

$$\begin{pmatrix} & & 0\\ & 0\\ A' & 0\\ \vdots\\ & \\ \hline \alpha_1 \alpha_2 \dots \alpha_{r-1} & \varepsilon \end{pmatrix} \text{ or } \begin{pmatrix} & \alpha_1\\ & \alpha_2\\ \vdots\\ & \\ \hline \alpha_{r-1}\\ \hline 0 & 0 & 0 \dots & 0 \\ \varepsilon \end{pmatrix}$$

where $\varepsilon = \pm 1$ (cf. Proposition 2.1). In the first instance A is lower triangular and unimodular. In the second, we can perform row and column operations on A to

eliminate the α_i in the last column at the expense of changing the last row to $(-\alpha_1 \dots - \alpha_{r-1}, \epsilon)$.

Conversely, suppose that $k \ge 3$ and K has a lower triangular Seifert matrix A with ± 1 on the diagonal. Let F be the corresponding Seifert surface. As det $A = \pm 1$, F is a fibre-surface (Lemma 1.1). We shall show by induction on $\rho = \operatorname{rk} A$ that A is the Seifert matrix of a knot obtained by plumbing. If $\rho = 1$, it is evident; if $\rho \ge 2$, by induction, we can represent the $(\rho - 1) \times (\rho - 1)$ matrix A' on the upper left-hand corner of A as the Seifert matrix of a fibre-surface F' obtained by plumbing Hopf bands. The matrix A looks like

$$\begin{pmatrix} & 0 \\ 0 \\ \vdots \\ 0 \\ \hline \alpha_1 \dots \alpha_{p-1} | \varepsilon \end{pmatrix}$$

with respect to some basis $\{e_i\}_{i=1}^{p-1}$ of $H_k(F')$. Look at the unimodular intersection pairing $J: H_k(F'; \partial F') \times H_k(F') \to \mathbb{Z}$ and let $c \in H_k(F'; \partial F')$ be the unique relative homology class such that $J(c; e_i) = \alpha_i$, i = 1, ..., p-1. Represent c by a proper embedding of a k-disc D^k in F' (that is, ∂D^k is embedded in $\partial F'$). This is possible since the connectedness hypotheses on F' and $\partial F'$ imply that $\prod_k(F'; \partial F') \simeq H_k(F'; \partial F')$; therefore any element of $H_k(F'; \partial F')$ can be represented by a proper map $f: D^k \to F'$ which we can homotop, using Whitney's theorems, to a proper embedding of D^k in F'. Look at the trivialisation $D^k \times D^k$ of the normal disc bundle to this embedding in F' and fatten it to a 2k+1 ball B on the positive side of F'. Use this $D^k \times D^k$ and the ball B to plumb an ε -handed Hopf band and call \overline{F} the resulting surface.

CLAIM. The matrix A is the Seifert matrix of \overline{F} .

As a basis for $H_k(\overline{F})$, take the previous basis of $H_k(F')$ together with the homology class e_ρ of the core of the new handle. Denote by \overline{A} the Seifert matrix of \overline{F} relative to this basis. Proposition 1.2 shows that

$$\overline{A} = \begin{pmatrix} & 0 \\ 0 \\ \vdots \\ 0 \\ \hline x_1 \dots x_{\rho-1} | \varepsilon \end{pmatrix},$$

where $x_i \in \mathbb{Z}$.

The integers x_i are determined by the intersection form $I_{\vec{F}}$ of \vec{F} . On one hand we have

$$I_{\overline{F}}(e_{\rho}; e_i) = J(c; e_i) = \alpha_i \text{ for } i = 1, \dots, \rho - 1;$$

on the other hand,

$$I_{\overline{F}}(e_{\rho};e_i) = \overline{A}(e_{\rho};e_i) + (-1)^k \,\overline{A}(e_i;e_{\rho}) = x_i.$$

This shows that $A = \overline{A}$ and proves the claim.

We have shown so far that there exists a surface \overline{F} obtained by plumbing Hopf bands which has the same Seifert form as F. As both surfaces are (k-1)-connected, they are isotopic (cf. Lemma 1.1) and K is obtained by plumbing.

3. Intersection forms of spherical knots obtained by plumbing

Let K be a simple spherical fibred (2k-1)-dimensional knot with fibre-surface F and Seifert form A. Let I_F denote the intersection form on $H_k(F)$. The forms A and I_F are unimodular and satisfy $I_F = A + (-1)^k A^T$, so that I_F is $(-1)^k$ -symmetric. If K is further assumed to be obtained by plumbing, then A is triangulable.

There are therefore two natural questions we can ask. What $(-1)^k$ -symmetric unimodular bilinear forms S can be decomposed as $S = A + (-1)^k A^T$

- (a) with A unimodular?
- (b) with A unimodular and triangulable?

Question (a) is the subject of [7] and we investigate question (b) below relying on the ideas used there. The investigation breaks into three parts: (I) k odd, (IIa) k even and S indefinite, (IIb) k even and S definite. We shall denote by $T_1 \oplus T_2$ the orthogonal sum of the forms T_1 and T_2 and by nT the orthogonal sum of n copies of the form T.

(I), where k is odd. Here the form S is unimodular and skew symmetric and is therefore an orthogonal sum of hyperbolic planes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

all such forms satisfy (a) and (b).

(II), where k is even. Let S be a unimodular bilinear form defined on a free \mathbb{Z} -module L. We have the following obvious condition.

Condition 3.1. There exists a unimodular triangular form A such that $S = A + A^T$ if and only if S is symmetric and there exists a basis $\{e_i\}$ of L such that $S(e_i, e_i) = \pm 2$. In particular S is even (that is, S assumes only even values).

(IIa), where k is even and S is indefinite. If S is even symmetric unimodular and indefinite, then $\pm S$ is isometric to an orthogonal sum of m copies of the hyperbolic plane $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and n copies of the form

$$\Gamma_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

(cf. [13, chapter V, Theorem 5]). As can be easily checked, H does not decompose as $H = A + A^T$ with A unimodular. On the other hand we show the following.

PROPOSITION 3.2. If S is an even symmetric unimodular indefinite form of rank greater than 2, there exists a unimodular triangular form A such that $S = A + A^T$.

Proof. By the above classification, it suffices to prove the proposition for 2H, 3H, Γ_8 and $H \boxdot \Gamma_8$. Let us denote by $E_{\alpha}(i;j)$ the elementary operation which consists of adding α times the *i*-th row and column to the *j*-th row and column.

(a) The sequence $E_1(2; 1)$, $E_1(4; 3)$, $E_1(1; 4)$, $E_1(3; 2)$ changes 2H to

$$J = \begin{pmatrix} 2 & 1 & 0 & 2 \\ 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix}$$

which satisfies condition 3.1.

(b) The above sequence followed by $E_1(6; 5)$, $E_1(1; 6)$ changes 3H to

$$\begin{pmatrix} & & & 0 & 2 \\ & & & 0 & 1 \\ J & & 0 & 0 \\ & & & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 & 1 & 2 \end{pmatrix}$$

which satisfies 3.1.

(c) Γ_8 obviously satisfies 3.1.

(d) The sequence $E_1(2; 1)$, $E_1(3; 2)$ also changes $H \boxplus \Gamma_8$ to a matrix that satisfies condition 3.1.

(IIb), where k is even and S is definite. After a change of signs in S, we may suppose that S is positive definite. Let S be a (non-necessarily unimodular) even and positive definite symmetric form defined on a free Z-module L and let R(S) denote the 'root system' of S, that is, $R(S) = \{x \in L \mid S(x;x) = 2\}$. Positive definite forms S such that R(S) generates L are completely classified: they are orthogonal sums of the (irreducible) forms corresponding to the root systems: A_n $(n \ge 1)$, D_n $(n \ge 4)$, E_6 , E_7 , E_8 (cf. [12] for details and the following facts about these root systems). These forms have discriminants n+1, 4, 3, 2 and 1 respectively and the form associated to E_8 is precisely Γ_8 . These considerations lead to the following.

PROPOSITION 3.3. Let S be a positive definite unimodular form. There exists a unimodular triangular bilinear form A such that $S = A + A^T$ if and only if S is an orthogonal sum of copies of Γ_8 .

Proof. As Γ_8 decomposes as $A + A^T$ with A unimodular and triangular, the same remains true for any orthogonal sum of Γ_8 . Conversely, if $S = A + A^T$ with A unimodular and triangular, condition 3.1 shows that the root system of S forms a basis for L. The above classification shows that S is an orthogonal sum of forms corresponding to the root systems in the list. If det S = 1 then the only form that can appear as an orthogonal summand is Γ_8 .

COROLLARY 3.4. Let k be an even integer greater than 3 and let K be a (2k-1)-dimensional spherical knot obtained by plumbing with positive definite intersection form I; then I is an orthogonal sum of copies of the form Γ_8 .

We conclude this section by a summary of the results contained in [7, 1] that we shall use in the sequel.

LEMMA 3.5. Let S be a unimodular symmetric form defined on a free \mathbb{Z} -module L. There exists a unimodular bilinear form A such that $S = A + A^T$ if and only if there exists an isometry $t: L \to L$ of S such that 1 - t is an isomorphism.

Kervaire calls such an isometry a *perfect* isometry. Note that an isometry t is perfect if and only if its characteristic polynomial χ satisfies $\chi(1) = \pm 1$.

Even symmetric positive definite unimodular forms of rank at most 24 are characterised by their root systems [12, 13]. There are 1 form in dimension 8 corresponding to E_8 , 2 forms in dimension 16 corresponding to $2E_8$ and D_{16} , 24 forms in dimension 24, one of which is associated with the 'Leech lattice' Λ in \mathbb{R}^{24} and has an empty root system.

According to [7], the root systems of those that admit a perfect isometry are exactly the following ones:

1 in dimension 8, namely E_8 ,

1 in dimension 16, namely $2E_8$,

10 in dimension 24, namely A_{24} , $2A_{12}$, $3A_8$, $4A_6$, $6A_4$, $12A_2$, $6D_4$, $4E_6$, $3E_8$, \emptyset .

Following Eva Bayer, we call an integer *m* mixed if *m* is not of the form p^r or $2p^r$ with *p* prime. Recall that $\Phi_m(1) \cdot \Phi_m(-1) = \pm 1$ if and only if *m* is mixed, where Φ_m denotes the *m*-th cyclotomic polynomial of degree $\phi(m)$. We quote the following theorem from [1].

THEOREM [1, Theorem 1.1 and Corollary 2.3]. If m is a mixed integer such that $\phi(m)$ is divisible by 8 there is an even unimodular positive definite symmetric form S of rank $\phi(m)$ together with an isometry t of S whose characteristic polynomial is Φ_m . If, moreover, m is square-free and $\phi(m) > 8$, then S does not represent 2.

4. Examples of spherical simple fibred knots not obtained by plumbing

The preceding section shows that the intersection form can give an obstruction for a spherical knot to be obtained by plumbing only if k is even and the form is definite. The above results can none the less be applied for k odd if one assumes that the symmetrised Seifert form is also unimodular.

Case I, in which k is even and greater than 3. Let S be one of the forms in dimension 24 that possess a perfect isometry (except $3\Gamma_8$) or one of the forms not representing 2 whose existence is given by Eva Bayer's theorem. The associated perfect isometry yields a unimodular form A such that $S = A + A^T$ and therefore, as k is even, k > 3, a simple spherical (2k-1)-dimensional knot (Lemma 1.1) which cannot be obtained by plumbing (Corollary 3.4).

Case II, in which k is odd and greater than 1. The same ideas can in principle be used for (2k-1)-spherical knots with k odd. If such a knot is fibred it admits a

unimodular Seifert matrix A such that $\det(A-A^T) = \pm 1$. However, $S = A+A^T$ is no longer necessarily unimodular. Also, there is an isometry t of S (given in matrix form by $P = (A^T)^{-1}A$) which satisfies $S = A^T(1+P)$ so that $\det S = \pm \det(1+P)$. As $A^T(1-P) = A^T - A$, $\det(1-P) = \pm 1$, and t is again perfect. The converse is unfortunately not true; if t is a perfect isometry of a symmetric form S such that $\det(1+t) = \det S$, we can still define a unimodular form A by the equation $S = A^T(1+t)$ which will satisfy $S = A + A^T$ and $\det(A - A^T) = \pm 1$, but there is in general no way to be sure that A is integral. This will, however, be the case if we assume S to be unimodular and 1+t to be an isomorphism. We set accordingly the following definition.

DEFINITION. Let S be a symmetric unimodular form defined on a free \mathbb{Z} -module L. An isometry t of S is doubly perfect if 1+t and 1-t are isomorphisms of L.

Note that an isometry of S is doubly perfect if and only if its characteristic polynomial χ satisfies $\chi(1) = \pm 1$ and $\chi(-1) = \pm 1$.

Here is an easy way to construct doubly perfect isometries.

LEMMA 4.1. If S admits a perfect isometry, $S \oplus S$ admits a doubly perfect isometry.

Proof. Let t be a perfect isometry of S and define $u: L \oplus L \to L \oplus L$ by u(x; y) = (y; tx); u is clearly an isometry of $S \oplus S$ and $1 - u^2 = (1 - t) \oplus (1 - t)$ so that 1 - u and 1 + u are isomorphisms.

The geometric realisation of these forms gives the following.

THEOREM 4.2. There exist for any $k \ge 3$ simple spherical fibred (2k-1)-knots of arbitrarily high genus that cannot be obtained by plumbing.

Proof. Let S be one of the 9 forms of dimension 24 that possess a perfect isometry (with $3\Gamma_8$ excluded). By Lemma 4.1, S \boxplus S admits a doubly perfect isometry, so there exists a unimodular form A such that $A + A^T = S \boxplus S$ and det $A - A^T = \pm 1$. As $k \ge 3$, Lemma 1.1 associates to A a simple fibred knot K which is spherical for k both even and odd. If the connected sum of a number of copies of K were obtained by plumbing, there would exist an integer n such that $n(S \boxplus S)$ is isomorphic to $6n\Gamma_8$ (Proposition 3.3), but this would contradict the uniqueness of decomposition of positive definite forms (cf. [11, Theorem 6.4]).

REMARK. Applying Eva Bayer's theorem to the values m = 35, 51, 55, one obtains forms of rank 24, 32 and 40 that admit (since m is mixed) a doubly perfect isometry. Realising these forms and taking connected sums one can produce for any $k \ge 3$ and $l \ge 3$ simple spherical fibred (2k-1)-knots of genus 4l that cannot be obtained by plumbing.

5. Twisting

Let * be a base point of S^k and let $\rho: S^k \to SO(k+1)$ be a differentiable map such that ρ is the identity near *. The composition $\overline{\rho}: D^k \to D^k/\partial D^k \xrightarrow{\rho} SO(k+1)$ gives a diffeomorphism

$$T(\rho): S^k \times D^k \longrightarrow S^k \times D^k, \qquad (x; y) \longrightarrow (\bar{\rho}(y) \cdot x; y)$$

which is the identity near $S^k \times \partial D^k$. Let p be a point on ∂D^k and c (respectively d) denote the k-chain $S^k \times \{p\}$ (respectively $\{*\} \times D^k\}$ of $S^k \times D^k$; $(T(\rho) - 1)d$ is a k-cycle of $S^k \times D^k$ and is therefore homologous to βc for some $\beta \in \mathbb{Z}$.

CLAIM. $\beta = j_k(\rho)$.

(For a definition of j_k , see Lemma 2.2.) Now $T(\rho)$ induces a diffeomorphism of $S^k \times D^k / \partial D^k$ whose matrix in homology is $\begin{pmatrix} 1 & j_k(\rho) \\ 0 & 1 \end{pmatrix}$ with respect to the basis $S^k \times \{p\}$, $\{*\} \times D^k / \partial D^k$, so that $\begin{pmatrix} 0 & j_k(\rho) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and therefore $j_k(\rho) = \beta$.

Let K be a simple fibred (2k-1)-dimensional knot in S^{2k+1} bounding a fibre-surface F^{2k} and let C be the image of an embedding of S^k in F. Set $\alpha = \mathcal{L}(C; i_+ C)$ and suppose that the normal bundle to C in F is trivial. Recall (Lemma 2.2) that this is always the case for k = 1, 3, 7 and is equivalent to $\alpha = 0$ for k even, $\alpha \equiv 0(2)$ for k odd, $k \neq 1, 3, 7$. Using a trivialisation of the tubular neighbourhood $\tau(C)$ of C in F and the diffeomorphism $T(\rho)$, we get a diffeomorphism $D(C; \rho)$ of F which is the identity outside $\tau(C)$. We call such a diffeomorphism a Dehn twist along C.

LEMMA 5.1. $D(C; \rho)$ induces the transvection

$$H_k(F) \longrightarrow H_k(F), \qquad x \longmapsto x + j_k(\rho) I(x; C) C.$$

Proof. By transversality x can be represented by a cycle z which meets $\tau(C)$ in meridinal discs $\{pt\} \times D^k$ with total algebraic intersection number equal to I(x; C); $(D_*(C; \rho) - 1)z$ is then a cycle in $\tau(C)$ and the above claim shows that it is homologous to $I(x; C)j_k(\rho)C$. This proves the lemma.

Let $h: F \to F$ be the monodromy of K; $h' = h \circ D(C; \rho)$ is the monodromy of a new fibred knot K' bounding a simple fibre-surface F' in a (2k + 1)-dimensional manifold M. We shall show that M is obtained by surgery in the following sense. Let γ be the image of an embedding of S^k in S^{2k+1} and let N be a tubular neighbourhood of γ . Now N is diffeomorphic to the torus $S^k \times D^{k+1}$ and there are two well-defined classes m and l of $H_k(\partial N)$ called the meridian and preferred longitude respectively, such that $\mathscr{L}(m;\gamma) = +1$ and m is represented by a k-cycle in ∂N which bounds in N, and I(m;l) = +1 and l is represented by a k-cycle which bounds in $S^{2k+1} \setminus N$. Let $\phi: \partial D^{k+1} \times S^k \to \partial N$ be a diffeomorphism and consider the manifold

$$M = (S^{2k+1} \backslash \mathring{N}) U_{\phi}(D^{k+1} \times S^k).$$

Call μ (respectively λ) the homology classes carried by $\partial D^{k+1} \times \{pt\}$ (respectively $\{pt\} \times S^k$) in $H_k(\partial D^{k+1} \times S^k)$; these are the meridian respectively longitude of the 'new' torus $D^{k+1} \times S^k$. Now ϕ induces an isomorphism $H_k(\partial D^{k+1} \times S^k) \to H_k(\partial N)$ such that $\phi_{\star}(\mu) = am + bl$, $\phi_{\star}(\lambda) = cm + dl$, ad - bc = 1. Call $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ the surgery matrix associated to ϕ .

associated to φ .

PROPOSITION 5.2. M is a manifold obtained by surgery on the k-sphere $i_+(C)$ with surgery matrix

$$\begin{pmatrix} 1+j_k(\rho) \,\alpha & \alpha \\ j_k(\rho) & 1 \end{pmatrix}.$$

Proof. $M \setminus \mathring{T}_{K'}$ is diffeomorphic to $F \times [0, 1]/h \circ D(C; \rho)$ and therefore to $(F \times [0, \frac{1}{2}] U_{\psi} F \times [\frac{1}{2}, 1])/h$, where

$$\psi: F \times \{\frac{1}{2}\} \longrightarrow F \times \{\frac{1}{2}\}, \qquad (x; \frac{1}{2}) \longrightarrow (D(C; \rho) x; \frac{1}{2}).$$

We may identify $i_+(C)$ with $\{C\} \times \{\frac{1}{4}\}$ and N with $\tau(C) \times [0, \frac{1}{2}]$. As $D(C; \rho)$ is the identity outside $\tau(C)$, it follows that $M \setminus \mathring{T}_{K'}$ is diffeomorphic to $S^{2k+1} \setminus \mathring{T}_{K}$ surgered along N. To determine the surgery matrix, take a transverse k-disc δ to C in $\tau(C)$; the meridian of the new torus is $\mu = \delta \times \{0\} \cup \operatorname{bd} \delta \times [0, \frac{1}{2}] \cup \delta \times \{\frac{1}{2}\}$ and the attaching map ψ sends μ to $\delta \times \{0\} \cup \operatorname{bd} \delta \times [0, \frac{1}{2}] \cup D(C; \rho) \delta \times \{\frac{1}{2}\}$. This chain is homologous in ∂N to $m + j_k(\rho) C^+$, where C^+ is a push-off of C at level $\frac{1}{2}$. As $\mathscr{L}(i_+C; C^+) = \mathscr{L}(C; i_+C) = \alpha$, it follows that $\phi_*(\mu) = m + j_K(\rho) (l + \alpha m)$. In a similar way $\phi_*(\lambda) = C = l + \alpha m$.

COROLLARY 5.3. *M* is again S^{2k+1} provided that $1+j_k(\rho)\alpha = \pm 1$ for $k \ge 3$, $1+j_k(\rho)\alpha = \pm 1$ and *C* is unknotted for k = 1.

Proof. The condition above implies that, for $k \ge 3$, M is a homology sphere that is simply connected and for k = 1 a lens space such that $H_1(M) = 0$; M is therefore at least homeomorphic to S^{2k+1} . A more precise analysis of the attaching map shows that M is in fact diffeomorphic to S^{2k+1} .

DEFINITION. A Dehn twist along C satisfying the condition of Corollary 5.3 is called a *Stallings twist* along C. (Stallings was the first to consider these twists for classical links in [15].)

Observe that if $j_k(\rho) = 0$ or C = 0 in $H_k(F)$ the intersection forms and algebraic monodromies of F and F' are the same; this implies that if K and K' are spherical, the Seifert forms are the same and, as both F and F' are simple, K and K' are isotopic for $k \ge 3$ by Lemma 1.1.

Recall (proof of Lemma 2.2) that $\text{Im } j_k = 0$ if k is even, so that Stallings twists do not yield new fibred knots in these dimensions.

If $j_k(\rho) \neq 0$ the allowable values of α are at most 0, ± 1 , ± 2 . This shows that to perform a non-trivial Stallings twist on F there must exist a non-zero homology class $C \in H_k(F)$ such that A(C; C) = 0, ± 1 , ± 2 , where A denotes the Seifert form of F. We shall show that there exist Seifert forms for which this condition is never satisfied. I thank Eva Bayer for suggesting the following approach.

DEFINITION. Let S be a positive definite symmetric form on a free \mathbb{Z} -module L, define m(S) to be the integer $\min_{x \in L \setminus \{0\}} S(x; x)$.

LEMMA 5.4. If S is an even symmetric unimodular positive definite form, so is $S \otimes \Gamma_8$: $(L \otimes \mathbb{Z}^8) \times (L \otimes \mathbb{Z}^8) \to \mathbb{Z}$; if t is a doubly perfect isometry of S, then $T = t \otimes id$ is a doubly perfect isometry of $S \otimes \Gamma_8$; furthermore, the following equality holds: $m(S \otimes \Gamma_8) = 2m(S)$.

Proof. The tensor product of two unimodular forms defined on a free Z-module is again unimodular [11, Lemma 5.3]; T is clearly an isometry of $S \otimes \Gamma_8$ and the equation $1 \pm T = (1 \pm t) \otimes 1$ shows that T is doubly perfect. The equality $m(S \otimes \Gamma_8) = 2m(S)$ is due to Steinberg (cf. [11, Theorem 9.6]).

DANIEL LINES

In [6], John Harer proves that every classical fibred link can be obtained by plumbing, doing Stallings twists and deplumbing. He also asks $[6, \S 5]$ whether deplumbing is necessary. We shall show that there exist high-dimensional fibred knots that cannot be obtained by plumbing and twisting only. As the method involved is algebraic, we get only a partial result for classical knots.

THEOREM 5.5. For every $k \ge 3$ there exist spherical fibred knots of arbitrarily high genus that cannot be obtained by plumbing and twisting. For k = 1, either there are unimodular Seifert forms that cannot be realised by fibred classical knots or deplumbing is a necessary operation.

Proof. Take one of the forms S admitting a doubly perfect isometry and such that $m(S) \ge 4$ (cf. §4). By lemma 5.4 $m(S \otimes \Gamma_8) \ge 8$ and there exists a unimodular form A such that $A - A^T$ is unimodular and $A + A^T = S \otimes \Gamma_8$. For $k \ge 3$ we realise this form by a fibre-surface $F \subset S^{2k+1}$ (Lemma 1.1); $K = \partial F$ is a spherical fibred knot. If it were obtained by plumbing and twisting, there would exist a sequence $F_0 = D^{2k}, F_1, \ldots, F_{s-1}, F_s = F$ where F_{i+1} would be obtained from F_i using one of the two operations. Suppose that F were obtained from F_{s-1} by doing a Stallings twist, F_{s-1} would clearly be also be obtained from F in a similar way, so there would exist $C \in H_k(F)$ such that $C \neq 0$ and $A(C; C) = 0, \pm 1, \pm 2$, hence $S(C; C) = 0, \pm 2, \pm 4$, which would contradict $m(S \otimes \Gamma_8) \ge 8$. If F were obtained from F_{s-1} by plumbing, there would exist $C \in H_k(F)$ such that $A(C; C) = \pm 1$, so that $S(C; C) = \pm 2$, again a contradiction. Taking connected sums of K gives the result. Note that the least genus that the above method can achieve is $\frac{1}{2}(24\cdot8) = 96$.

For k = 1, nothing more can be said, since we do not know whether every unimodular Seifert form is the form of some classical fibred knot.

REMARK. It is interesting to note that Theorem 5.5 can be proved using only the form Γ_8 . Take the obvious triangular unimodular form A such that $\Gamma_8 = A + A^T$; then A corresponds to a perfect isometry (Lemma 3.5). Use Lemma 4.1 to obtain a doubly perfect isometry of $2\Gamma_8$ and Lemma 5.4 twice. This provides the form $S = (2\Gamma_8) \otimes \Gamma_8 \otimes \Gamma_8$ (which has minimum 8) with a doubly perfect isometry; then take orthogonal sums of S.

FINAL REMARK. It is not known in the classical case whether every fibred link is stably obtained by plumbing, that is, whether by plumbing enough Hopf bands to a fibre-surface one gets a surface which can be obtained by plumbing (cf. [6, \S 5]). The analogous statement in high dimensions is equivalent, using Proposition 2.4, to the following algebraic question whose answer I do not know.

If A is any $m \times m$ integral unimodular matrix, does there exist a lower triangular unimodular $n \times n$ matrix T and an integral $n \times m$ matrix B such that $\begin{pmatrix} A & 0 \\ B & T \end{pmatrix}$ is congruent to a triangular matrix?

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