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Surface embeddability of graphs via homology

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Abstract This paper presents a characterization of the embeddability on a surface of genus arbitrarily given for a graph. Its specific case for the surface of genus zero leads to the famous planarity theorems given independently by Whitney via duality, MacLane via cycle basis and Lefschetz via double covering at a time.

Keywords surface, graph, embedding, duality, homology

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1 Introduction

On a surface of genus 0, the embeddability of a graph is the planarity well-known in literature. Here, we only mention some classic theorems in topology related to the present topic in our own language.

What obtained by deleting an edge and/or identifying the two ends of an edge on a graph is called a *minor* of the graph.

Theorem A [3]. *A graph can be embedded into the plane if, and only if, neither K_5 nor $K_{3,3}$ is a minor of the graph.*

In this theorem, K_5 and $K_{3,3}$ are the complete graph of order 5 and the complete bipartite graph of order 6 with each part of 3 vertices, respectively.

Theorem B [9]. *A graph $G = (V, E)$ can be embedded into the plane if, and only if, there exists another graph $G^* = (V^*, E^*)$ with a bijection $\tau : E \to E^*$ such that for any circuit $C \subseteq E$, $\tau(C) = C^* \subseteq E^*$ is a cocircuit.*

The statement of this theorem is different from but equivalent to the original of Whitney's for the convenience of coming usage.

Theorem C [8]. *A graph G can be embedded into the plane if, and only if, there exists a set $\mathcal{C}$ of $\beta(G)$ circuits such that each edge of G occurs either once or twice in $\mathcal{C}$.*

In this theorem, $\beta(G)$ is the corank of G, i.e., the dimension of the cycle space of G.

Theorem D [4]. *A graph can be embedded into the plane if, and only if, there exists a set $\mathcal{C}$ of $\beta(G)$ circuits such that each edge of G occurs exactly twice in $\mathcal{C}$.*

The set $\mathcal{C}$ in this theorem is called a *double covering* of G.

Although the above four theorems were intended to employ as a basis for investigating the embeddability of a graph on some surface of genus not zero, only surface of genus 1 (nonorientable) yields a result which is rather complicated as seen in [2] on Theorem A and in [1] on Theorem B.

However, in monograph [5], a graph was considered as a set of polyhedra and then two mutual dual planarity characterizations were established via the homological and cohomological spaces introduced by the author himself. From these results, the above four theorems have been naturally deduced.

The purpose of this paper is to establish a characterization of embeddability on a surface of genus arbitrarily chosen for a graph and then to deduce the last three theorems (Theorems B, C and D) from the specific case of genus zero. Theorem A is kept for another paper to certain extent because of length limited.

2 Double coverings

Let $\mathcal{C}$ be a set of travels as a double covering of $G=(V,E)$. Given a vertex $v\in V$, for a pair of edges at v, if there exists a travel in $C\in\mathcal{C}$ such that they are successive on C, then the pair of edges are called *sharing* vertex v. This relation is, by the transitive law, extended as an equivalence on the set of edges at a vertex. If all edges at v are in the same class under this equivalence, then $\mathcal{C}$ is called *transitive* at v.

Lemma 1. *A double covering of a graph by travels determines an embedding of the graph if, and only if, it is transitive at all vertices.*

Proof. If double covering $\mathcal{C}$ forms an embedding of G, then its underlying graph is G. Because of edges at each vertex with a rotation, i.e., a cyclic permutation in the embedding, the double covering is transitive at all vertices. This is the necessity.

Conversely, assume that $\mathcal{C}$ is transitive at all vertices of G. Then all the edges at each vertex form a cyclic permutation. If this cyclic permutation is taken as a rotation, on the basis described in Subsection 2.1 of [6] (more clearly in Subsection 3.5 of [7]), $\mathcal{C}$ forms a polyhedron, and hence an embedding of G. This is the sufficiency. □

If a double covering $\mathcal{C}$ of G by travels is transitive at each vertex, then $\mathcal{C}$ is said to be *consistent*. $\mathcal{C}$ is called *orientable* if there exists an orientation for each travel such that the two occurrences of each edge in $\mathcal{C}$ are with different directions; *nonorientable*, otherwise.

In what follows, without loss of generality, all graphs are assumed to be nonseparable with neither loop nor isthmus.

Lemma 2. *Let $\mathcal{C}$ be an orientable double covering of G with $\beta(G)-2p+1$ travels, where p is a nonnegative integer. Then $\mathcal{C}$ is consistent if, and only if, the dimension of the space generated by travels in $\mathcal{C}$ is* $\dim\langle\mathcal{C}\rangle=\beta(G)-2p$.

Proof. Denote $k=\beta(G)-2p+1$, then $\dim\langle\mathcal{C}\rangle=k-1$.

Sufficiency. Since $\beta(G)=\epsilon(G)-\nu(G)+1$, we have $k=\epsilon(G)-\nu(G)-2p+2$. Assume that $\mathcal{C}$ is not consistent. Then $\mathcal{C}$ is a polyhedron Σ with at least $\nu(G)+1$ vertices. Because of $\epsilon(\Sigma)=\epsilon(G)$,

$$\begin{aligned}\dim(\Sigma)&=\beta(\Sigma)-2p=\epsilon(\Sigma)-\nu(\Sigma)+1-2p\\&\leqslant\epsilon(G)-(\nu(G)+1)-2p+1=\dim\langle\mathcal{C}\rangle-1.\end{aligned}$$

This is a contradiction to $\dim(\Sigma)=\dim\langle\mathcal{C}\rangle$. Therefore, $\mathcal{C}$ is consistent.

Necessity. Because of $\mathcal{C}$ transitive at each vertex of G, $\mathcal{C}$ as a polyhedron is an embedding of G and hence $\dim\langle\mathcal{C}\rangle=k-1=\beta(G)-2p$. □

Lemma 3. *Let $\mathcal{C}$ be an embedding of G on an orientable surface of genus p, then* $\dim\langle\mathcal{C}\rangle=\beta(G)-2p$.

Proof. Since $\mathcal{C}$ is an embedding of G on an orientable surface of genus p, by the Eulerian characteristic, this embedding has $\phi=|\mathcal{C}|=\beta(G)-2p+1$ faces. Let ∂f_i be the boundary of face f_i, $1\leqslant i\leqslant\phi$. Because of

$$\sum_{1\leqslant i\leqslant\phi}\partial f_i=0,$$

we have

$$\dim\langle\mathcal{C}\rangle \leqslant \beta(G) - 2p = \phi - 1.$$

From the connectedness of G, any $\phi - 1$ vectors in $\mathcal{C}$ are independent. (Otherwise, the $\phi - 1$ travels in $\mathcal{C}$ form a polyhedron. However, the skeleton of this polyhedron is a proper subgraph of G. A contradiction to the connectedness of G.) Therefore, $\dim\langle\mathcal{C}\rangle = \beta(G) - 2p$. □

3 Main results

Now, the main theorems and corollaries of this paper are shown in this section.

Theorem 1. *A graph G can be embedded on an orientable surface of genus p if, and only if, an orientable double covering $\mathcal{C}$ with $\beta(G) - 2p + 1$ travels exists on G such that $\dim\langle\mathcal{C}\rangle = \beta(G) - 2p$.*

Proof. Necessity. If Σ is an embedding of G on an orientable surface of genus p, let $\mathcal{C}$ be the set of all face boundaries of polyhedron Σ. From the Eulerian characteristic, $|\mathcal{C}| = \beta(G) - 2p + 1$. Naturally, $\mathcal{C}$ is an orientable double covering of G by travels. From Lemma 3, $\dim\langle\mathcal{C}\rangle = \beta(G) - 2p$.

Sufficiency. Assume that $\mathcal{C}$ is a double covering of G with $\beta(G) - 2p + 1$ travels such that $\dim\langle\mathcal{C}\rangle = \beta(G) - 2p$. Then from Lemmas 1–2, $\mathcal{C}$ is an embedding of G on an orientable surface of genus p. □

Since $\dim\langle\mathcal{C}\rangle = \beta(G) - 2p$ leads to that $\mathcal{C}$ has $\beta(G) - 2p + 1$ travels, this enables us to employ the former instead of the latter.

Corollary 1 (Theorem D in Section 1)**.** *A nonseparable graph G is planar if, and only if, there exists a set $\mathcal{C}$ of $\beta(G) + 1$ circuits which is a double covering of G.*

Proof. This is the case of $p = 0$ in Theorem 1 when $\dim\langle\mathcal{C}\rangle = \beta(G)$ and $|\mathcal{C}| = \beta(G) + 1$. □

Corollary 2 (Theorem C in Section 1)**.** *A nonseparable graph $G = (V, E)$ is planar if, and only if, there exists a set $\mathcal{C}$ of $\beta(G) + 1$ circuits in G such that each $e \in E$ occurs in at most two circuits of $\mathcal{C}$.*

Proof. Because of $p = 0$, we have $\dim\langle\mathcal{C}\rangle = \beta(G)$ and $\mathcal{C} + C_0$ is a double covering of G, where

$$C_0 = \sum_{C\in\mathcal{C}} C.$$

From Theorem 1, the corollary holds. □

Corollary 3. *A graph $G = (V, E)$ can be embedded on an orientable surface of genus $p \geqslant 0$ if, and only if, there exists another graph $G^*_p = (V^*, E^*)$ with a bijection $\tau: E \longrightarrow E^*$ such that $\dim\langle\tau E_v | \forall\, v \in V\rangle = \beta(G^*) - 2p$.*

Proof. In fact, this is the dual form of Theorem 1. □

Corollary 4 (Theorem B in Section 1)**.** *A connected graph $G = (V, E)$ is planar if, and only if, there exists another graph $G^*_0 = (V^*_0, E^*_0)$ with a bijection $\tau: E \longrightarrow E^*_0$ such that $\dim\langle\tau E_v | \forall\, v \in V\rangle = \beta(G^*)$.*

Proof. This is the case of $p = 0$ in Corollary 3. □

In this corollary, G^*_0 is equivalent to the algebraic dual of Whitney.

Lemma 4. *Let $\mathcal{C}$ be a nonorientable double covering of G with $\beta(G) - q + 1$ travels, where q is a positive integer. If the dimension of the space generated by $\mathcal{C}$ is $\dim\langle\mathcal{C}\rangle = \beta(G) - q$, then $\mathcal{C}$ is consistent.*

Proof. Denote $k = \beta(G) - q + 1$. Then $\dim\langle\mathcal{C}\rangle = k - 1$. Because of $\beta(G) = \epsilon(G) - \nu(G) + 1$, we have $k = \epsilon(G) - \nu(G) - q + 2$. Assume $\mathcal{C}$ is not consistent. Then the polyhedron Σ represented by $\mathcal{C}$ has at least $\nu(G) + 1$ vertices. Because of $\epsilon(\Sigma) = \epsilon(G)$, we have

$$\dim(\Sigma) = \epsilon(\Sigma) - \nu(\Sigma) + 1 - q \leqslant \epsilon(G) - (\nu(G) + 1) - q + 1 = \dim\langle\mathcal{C}\rangle - 1.$$

This is a contradiction to $\dim(\Sigma) = \dim\langle\mathcal{C}\rangle$. Therefore, $\mathcal{C}$ is consistent. □

Lemma 5. *Let $\mathcal{C}$ be an embedding of G on a nonorientable surface of genus q. Then*

$$\dim\langle\mathcal{C}\rangle = \beta(G) - q.$$

Proof. Since $\mathcal{C}$ is an embedding of G on a nonorientable surface of genus q, from the Eulerian characteristic, this embedding has $\phi = |\mathcal{C}| = \beta(G) - q + 1$ faces. Let ∂f_i be the boundary of face f_i, $1 \leqslant i \leqslant \phi$. Because of

$$\sum_{1\leqslant i\leqslant \phi} \partial f_i = 0,$$

we have

$$\dim\langle \mathcal{C}\rangle \leqslant \beta(G) - 2p = \phi - 1.$$

In virtue of the connectedness of G, any $\phi - 1$ elements of $\mathcal{C}$ are independent. (Otherwise, the $\phi - 1$ travels in $\mathcal{C}$ form a polyhedron themselves. However, the skeleton of this polyhedron is a proper subgraph of G. This contradicts to the connectedness of G.) Therefore, $\dim\langle \mathcal{C}\rangle = \beta(G) - q$. □

Theorem 2. *A graph G can be embedded on a nonorientable surface of genus q if, and only if, a nonorientable double covering by $\beta(G) - q + 1$ travels exists on G such that* $\dim\langle \mathcal{C}\rangle = \beta(G) - q$.

Proof. Necessity. If Σ is an embedding of G on a nonorientable surface of genus q, let $\mathcal{C}$ be the set of all face boundaries of polyhedron Σ. From the Eulerian characteristic, $|\mathcal{C}| = \beta(G) - q + 1$. In virtue of Lemma 5, $\dim\langle \mathcal{C}\rangle = \beta(G) - q$.

Sufficiency. Assume that $\mathcal{C}$ is a nonorientable double covering of G with $\beta(G) - q + 1$ travels and $\dim\langle \mathcal{C}\rangle = \beta(G) - q$. Then from Lemma 1 and Lemma 4, $\mathcal{C}$ is an embedding of G on a nonorientable surface of genus q. □

4 Research notes

(i) For the embeddability of a graph on torus, double torus etc. or in general orientable surface of small genus, more efficient characterizations are still necessary to be further investigated on the basis of Theorems 1 and 2.

(ii) For planarity of a graph, it suffices to discuss nonseparable graphs instead of general ones and hence Corollaries 1, 2 and 4 are, respectively, corresponding to Theorems D, C and B.

(iii) A number of quotient spaces seem useful for observing global structural properties of a graph. Homological and cohomological spaces are only shown as an example. More others are still necessary to be paid in certain attention to.

References

1 Abrams L, Slilaty D C. An algebraic characterization of projective-planar graphs. J Graph Theory, 2003, 42: 320–331
2 Archdeacon D. A Kuratowski theorem for the projective plane. J Comb Theory, 1981, 5: 243–246
3 Kuratowski K. Sur le problem des courbes Gauches en topologie. Fund Math, 1930, 15: 271–283
4 Lefschetz S. Planar graphs and related topics. Proc Nat Acad Sci, 1965, 54: 1763–1765
5 Liu Y P. Embeddability in Graphs. Dordrecht-Boston-London: Kluwer, 1995
6 Liu Y P. Theory of Polyhedra. Beijing: Science Press, 2008
7 Liu Y P. Topological Theory on Graphs. Hefei: Univ Sci Tech China Press, 2008
8 MacLane S. A combinatorial condition for planar graphs. Fund Math, 1937, 28: 22–32
9 Whitney H. Planar graphs. Fund Math, 1933, 21: 73–84