A THEOREM IN HOMOLOGICAL ALGEBRA AND STABLE HOMOTOPY OF PROJECTIVE SPACES

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Introduction. The paper exhibits a general change of rings theorem in homological algebra and shows how it enables to systematize the computation of the stable homotopy of projective spaces.

Chapter I considers the following situation: R and S are rings with unit, $h: R \to S$ is a ring homomorphism, M is a left S-module. If an S-free resolution of M and an R-free resolution of S are given, Theorem I.1. shows how to construct an R-free resolution of M.

Chapter II is devoted to computing the initial stable homotopy groups of projective spaces. Here the results of Chapter I are applied to the homomorphism $\alpha: A \to A$ of the Steenrod algebra over Z_2 (see I.3). The main tool in computing stable homotopy is the Adams spectral sequence [1]. Let RP^∞, CP^∞, HP^∞ be the real, complex, and quaternionic infinite-dimensional projective spaces, respectively. If X is a space, let $\Pi_m^S(X)$ denote the mth stable homotopy group of X [1]. Part of the results of Chapter II can be presented as follows:

m:	1	2	3	4	5	6	7	8
RP^∞:	Z_2	Z_2	Z_8	Z_2	0	Z_2	$Z_{16}\oplus Z_2$	$Z_2\oplus Z_2\oplus Z_2$
CP^∞:	0	Z	0	Z	Z_2	Z	Z_2	$Z\oplus Z_2$
HP^∞:	0	0	0	Z	Z_2	Z_2	0	Z

Chapter I. Homological algebra

1. **A change of rings theorem.** Let R and S be rings with unit, h: $R \to S$ a homomorphism of rings; under h, any left S-module can be considered as a left R-module.

Let M be a left S-module. Let Y be an S-free resolution of M: $Y = \sum_{q\geqq 0} Y_q$, with differential d' and augmentation ε'. Let X_q be an R-free resolution of Y_q: differential d'' and augmentation ε_q onto Y_q.

Let $C = \sum_{q\geqq 0} X_q$, $C_k = \sum_{q+r=k} X_{q\,r}$, and augmentation $\varepsilon = \varepsilon'(\sum_q \varepsilon_q)$. If

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$f: C \to C$ is a homomorphism which lowers total degree, then $f = \sum_{k=0}^{\infty} f_k$, where $f_k\colon X_q \to X_{q-k}$.

THEOREM I.1. *There exists a differential $d\colon C \to C$ such that $\{C, d, \varepsilon\}$ is an R-free resolution of M. The differential d can be chosen to have the properties*:

(1) *d_0 is induced from d''*,

(2) $d'\varepsilon_{q+1} = \varepsilon_q d_1$,

(3) $\sum_{i=0}^{k} d_i d_{k-i} = 0$;

conversely, any map with properties (1), (2), (3) *is a differential which makes C acyclic.*

REMARK. Let G be a finite group, H a normal subgroup of G, K a ring; let $R = K[G]$, $S = K[G/H]$, $M = K$. Theorem I.1 was proved by Wall [14] in this special case. The proof presented here is a straightforward translation to the general case.

Proof of Theorem I.1. Let us show that any d with properties (1), (2), (3) makes C acyclic. Filter C by $F^pC = \sum_{q \leqq p} X_q$. The differential d preserves filtration, and the associated spectral sequence converges to $H(C)$. The differential in E^0 is precisely d_0, hence $E^1 = Y$, with d^1 corresponding to d' because of (2). Since Y is a resolution of M, $E^2 = E^\infty = M$, hence C is acyclic.

To prove that d with properties (1), (2), (3) exists is easy. Since the X_k are R-free resolutions of Y_k, we can construct an R-map $d_1\colon X_{q,r} \to X_{q-1,r}$ such that $\varepsilon_{q-1} d_1 = d'\varepsilon_q$. To construct the maps d_k, $k \geqq 2$, we use induction on the total degree $q + r$ of $X_{q,r}$. We set $d_k = 0$ if it lands in $X_{q',r'}$ with $q' < 0$. Suppose d has been defined on $X_{q',r'}$ with $q' + r' < q + r$, and $d_0, \cdots, d_k$ have been defined on $X_{q,r}$. Let $f = -\sum_{i=1}^{k} d_i d_{k+1-i}$. We claim there exists a map d_{k+1} such that $d_0 d_{k+1} = f$. To prove this it suffices to prove that $d_0 f = 0$ and $\varepsilon_{q-k-1} f = 0$, but this is easy:

$$d_0 f = -\sum_{i=1}^{k+1} d_0 d_i d_{k+1-i} = \sum_{i=1}^{k+1} \sum_{j=1}^{i} d_j d_{i-j} d_{k+1-i}$$

$$= \sum_{j=1}^{k+1} d_j \sum_{i=j}^{k+1-j} d_{i-j} d_{k+1-i} = 0,$$

which completes the proof of Theorem I.1.

2. **Hopf algebras.** Let E, F be graded, connected, associative Hopf algebras over field a K [12]. Suppose that F is a Hopf subalgebra of E. Then, according to Theorem 2.5 of [12], E is free as a right (or left) F-module. Therefore we have

PROPOSITION I.2. *$E \otimes_F$ is an exact functor of left F-modules into left E-modules.*

We shall say that F is normal in E if $\bar{F}E = E\bar{F}$, where $\bar{F}$ denotes the augmentation ideal of F. Let $B = E//F = E/E\bar{F}$.

PROPOSITION I.3. *If W is an F-free resolution of K, then $E \otimes_F W$ is an E-free resolution of B.*

Proof. Proposition I.2 and $E \otimes_F K = B$.

REMARK. Let $R = E$, $S = B$, and h: $R \to S$ the projection map. Let M be a B-module, $Y = B \otimes \bar{Y}$ a B-free resolution of M, $U = F \otimes \bar{U}$ an F-free resolution of K. Then, according to the proposition above, we can take for X_q in Theorem I.1. the complex $E \otimes \bar{Y}_q \otimes \bar{U}$ with the differential induced from U (see [10]).

3. **The Steenrod algebra.** Let A be the Steenrod algebra [11] over Z_2. The graded dual A^* is a polynomial algebra and the squaring map in A^* is a Hopf algebra map α^*. Let α: $A \to A$ be the dual of α^*; α is defined by $\alpha(Sq^{2^{r+1}}) = Sq^{2^r}$.

If I is a finitely nonzero sequence of non-negative integers, then we let Sq^I denote the Milnor basis element corresponding to I. Let Δ_i be the sequence consisting of 1 in the ith place and zeros elsewhere. Define the elements

$$Q_i = Sq^{\Delta_i}, \quad R_i = Sq^{2\Delta_i}.$$

Let C be the subalgebra of A generated by 1 and Q_k, $k = 0, 1, \cdots$; B the subalgebra of A generated by 1, Q_0, and R_k, $k = 0, 1, \cdots$.

PROPOSITION I.4. *B and C are normal Hopf subalgebras of A, and*

$$\text{Kernel } \alpha = A\bar{C},$$

$$\text{Kernel } \alpha \circ \alpha = A\bar{B}.$$

Proof. Immediate consequence of Lemma 2.4.2 of [2].

REMARKS. 1. The preceding proposition states that we may consider α and $\alpha \circ \alpha$ as the projection maps $A \to A//C$, $A \to A//B$, respectively.

2. The map α halves the grading. Let $\tilde{A}$ denote A with the grading of every element multiplied by two. Then $\alpha: A \to \tilde{A}$ preserves grading. The reader is asked to make such adjustments in the following pages.

It will be necessary to know the groups $\text{Ext}_C^{s,t}(Z_2, Z_2)$, $\text{Ext}_B^{s,t}(Z_2, Z_2)$. The first is easily determined, for C is a Grassman algebra:

$$\text{Ext}_C^{*,*}(Z_2, Z_2) = Z_2[q_0, \cdots, q_k, \cdots],$$

where the polynomial generator $q_k \in \text{Ext}^{1, 2^{k+1}-1}$.

We compute $\text{Ext}_B^{s,t}(Z_2, Z_2)$ using Theorem I.1. We shall use the standard minimal resolution of Z_2 over C. Generators will be in one-to- one correspondence with finitely nonzero sequences of integers I (the free C-generater corresponding to I will be denoted by $[I]$). Let $I = (i_0, i_1, \cdots, i_k, \cdots)$, then degree $[I]$

$=\Sigma_k i_k$, grade $[I]=\Sigma_k i_k(2^{k+1}-1)$. The differential in the minimal resolution is defined by

$$\tilde{d}[I]=\sum_{r=0}^{\infty} Q_r[I-\Delta_r],$$

where we set $[I-\Delta_r]=0$ if $i_r=0$.

According to Proposition I.4, $\operatorname{Ker}\alpha\,|\,B=B\bar{C}$, and C is normal in B. For the module $X_{i,j}$ in Theorem I.1 we take the free B-module on generators $[I]\otimes[J]$, where degree $[I]=i$, degree $[J]=j$, and grade $([I]\otimes[J])=2$ grade $[I]+$ grade $[J]$. The augmentation ε_i is defined by $\varepsilon_i([I]\otimes[J])=0$ if degree $[J]>0$, $\varepsilon_i([I]\otimes[J])=[I]$ if degree $[J]=0$. Both d_0 and d' are defined by the formula for $\tilde{d}$ above. An easy induction on the degree of $[J]$ shows that we can define the maps d_k for $k\geqq 1$ as follows:

$$\begin{aligned}
d_1[I]\otimes[J] &=\sum_k R_k[I-\Delta_k]\otimes[J]+\sum_k (j_{k+1}+1)[I-\Delta_k]\otimes[J-\Delta_0+\Delta_{k+1}],\\
d_2[I]\otimes[J] &=\sum_i (j_{i+1}+1)Q_0[I-\Delta_0-\Delta_i]\otimes[J+\Delta_{i+1}],\\
d_3[I]\otimes[J] &=\sum_{k<t}(j_{k+1}+1)(j_{t+1}+1)[I-\Delta_0-\Delta_k-\Delta_t]\otimes[J+\Delta_{k+1}+\Delta_{t+1}]\\
&\quad+\sum_t \binom{j_{t+1}+2}{2}[I-\Delta_0-2\Delta_t]\otimes[J+2\Delta_{t+1}],
\end{aligned}$$

$d_n=0$ for $n\geqq 4$.

Since we will only use the groups $\operatorname{Ext}_B^{s,t}(Z_2,Z_2)$ for $t-s<13$, it is sufficient to consider the generators $[I]\otimes[J]$ in the resolution for which $i_k=0$ for $k\geqq 2$, $j_r=0$ for $r\geqq 3$. Thus for $t-s<13$ $\operatorname{Ext}_B^{s,t}(Z_2,Z_2)$ is additively the homology of the bi-graded algebra $Z_2[x_0,x_1,y_0,y_1,y_2]$, where grade $(x_i)=2^{i+2}-2$, grade$(y_j)=2^{j+1}-1$, degree $(x_i)=$ degree$(y_j)=1$, under the differential $\delta_1+\delta_2$, where δ_1 is a derivation and

$$\delta_1(x_i)=0,\ \delta_1(y_0)=0,\ \delta_1(y_j)=y_0x_{j-1};$$

δ_2 is a map of $Z_2[x_0,x_1,y_0]$-modules with

$$\begin{aligned}
\delta_2(x_i) &= 0, \quad \delta_2(y_0)=0,\\
\delta_2(y_1^{m_1}y_2^{m_2}) &= m_1m_2x_0^2x_1y_1^{m_1-1}y_2^{m_2-1}+\binom{m_1}{2}x_0^3y_1^{m_1-2}y_2^{m_2}\\
&\quad+\binom{m_2}{2}x_0x_1^2y_1^{m_1}y_2^{m_2-2},
\end{aligned}$$

and $\delta_1\delta_2+\delta_2\delta_1=0$. We list some obvious cycles under $\delta_1+\delta_2$ in the following table, and give classes in Ext_{B_1} which they determine. (B_1 is the subalgebra of B generated by Q_0, R_0, R_1, and $\operatorname{Ext}_{B_1}^{s,t}(Z_2,Z_2)\cong\operatorname{Ext}_B^{s,t}(Z_2,Z_2)$ for $t-s<13$)$^{\cdot}$

TABLE

Cycle	Degree	Grade	Class
y_0	1	1	g_0
x_0	1	2	k_0
x_1	1	6	k_1
$x_0y_2 + x_1y_1$	2	9	γ
$y_0y_1^2 + x_0^2y_1$	3	7	τ_0
$y_0y_2^2 + x_0x_1y_2$	3	15	τ_1
y_1^4	4	12	ω_1
y_2^4	4	28	ω_2
$y_0y_1^2y_2^2 + x_0^2y_1y_2^2 + x_0x_1y_1^2y_2$	5	21	τ_{01}

PROPOSITION I.5. $\mathrm{Ext}_{B_1}^{s,t}(Z_2,Z_2)$ *is generated as an algebra by the classes*

$$g_0, k_0, k_1, \gamma, \tau_0, \tau_1, \tau_{01}, \omega_1, \omega_2.$$

Furthermore, it is a free $Z_2[\omega_1,\omega_2]$*-module with the following monomials as generators*:

$g_0^n,\ g_0^n\tau_0,\ g_0^n\tau_1,\ g_0^n\tau_{01},\ n \geqq 0,$

$k_0^ik_1^j,\ 0 \leqq i \leqq 2,\ 0 \leqq j$ *(if* $i > 0$, *then* $j \leqq 1$),

$k_0^ik_1^j\gamma,\ k_1^j\gamma^2,\ k_1^j\gamma^3.$

Proof. Find the homology under δ_1, decompose the homology into a tensor product of standard complexes under δ_2, and use the Künneth theorem over the ring $Z_2[x_0]$.

REMARK. Once $\mathrm{Ext}_{B_1}(Z_2,Z_2)$ is known, it is very easy to construct a minimal resolution for Z_2 over B_1. The task is left to the reader.

4. **Operations of** Ext **and the Adams spectral sequence.** Let A be the Steenrod algebra over Z_p, L a left A-module. There is a natural map

$$\mu: \mathrm{Ext}_A^{q,u}(L,Z_p) \otimes \mathrm{Ext}_A^{r,v}(Z_p,Z_p) \to \mathrm{Ext}_A^{q+r,u+v}(L,Z_p)$$

which makes $\mathrm{Ext}_A(L,Z_p)$ into a right $\mathrm{Ext}_A(Z_p,Z_p)$-module. For the definition of μ see, for example, [2]. We write $\alpha * \beta$ for $\mu(\alpha \otimes \beta)$.

For the Adams spectral sequence see [1].

THEOREM I.6 (ADAMS). *The spectral sequence for the sphere S^0 operates on the spectral sequence for any arbitrary space X. In particular, if*

$$h \in \operatorname{Ext}_A^{s,t}(Z_p, Z_p), \qquad a \in \operatorname{Ext}_A^{u,v}(H^*(X), Z_p),$$

and $d_j(h) = 0,\ j = 2, \cdots, r,\ d_k(a) = 0,\ k = 2, \cdots, r-1,$ *then*

$$d_r(\{a * h\}) = \{d_r a\} * h.$$

Proof. The proof of Theorem 2.2 of [1]; see also Théorème IIB, Exposé 19 of [6].

CHAPTER II. STABLE HOMOTOPY OF PROJECTIVE SPACES

1. **The prime** $p = 2$. Let RP^∞, CP^∞, HP^∞ be the real, complex, and quaternionic infinite-dimensional projective spaces, respectively. It is well known that

$$H^*(RP^\infty; Z_2) = Z_2[x], \tag{1}$$

$$H^*(CP^\infty; Z_2) = Z_2[y], \tag{2}$$

$$H^*(HP^\infty; Z_2) = Z_2[u], \tag{3}$$

where x, y, u are the nonzero 1, 2, 4-dimensional classes, respectively. Let L, M, N be the elements of positive degree in (1), (2), (3), in the order given. Let α: $A \to A$ be the dual of the squaring map (see Proposition I.3).

PROPOSITION II.1. *There are Z_2-isomorphisms f: $M \to L$, g: $N \to M$ such that the following diagram is commutative*:

$$\begin{array}{ccc} A \otimes N & \longrightarrow & N \\ {\scriptstyle \alpha \otimes g} \downarrow & & \downarrow {\scriptstyle g} \\ A \otimes M & \longrightarrow & M \\ {\scriptstyle \alpha \otimes f} \downarrow & & \downarrow {\scriptstyle f} \\ A \otimes L & \longrightarrow & L, \end{array} \tag{4}$$

where the horizontal arrows indicate the action of A.

Proof. According to [11], if $\theta \in A$, then

$$\theta x = \sum_{n=0}^{\infty} \langle \xi_n, \theta \rangle x^{2^n}. \tag{5}$$

Let $h: RP^\infty \to CP^\infty$ be a map such that $h^*(y) = x^2$; h^* is a monomorphism. Thus from (5) and h^*

$$\theta y = \sum_{n=0}^{\infty} \langle \xi_n^2, \theta \rangle y^{2^n}.$$

Let $f\colon M \to L$ be the algebra map given by $f(y) = x$. Then $f(\theta y) = \alpha(\theta) f(y)$, for $\langle \xi_n^2, \theta\rangle = \langle \alpha^*(\xi_n), \theta\rangle = \langle \xi_n, \alpha(\theta)\rangle$. With this choice for f, the bottom rectangle of (4) is commutative. The proof is completed by defining $g(u) = y$ and considering a map $k\colon CP^\infty \to HP^\infty$ such that $k^*(u) = y^2$.

REMARK. Proposition II.1 is used by S. P. Novikov in his investigation of Thom spectra (dissertation – unpublished).

According to the proposition M and N are isomorphic to L as A-modules through the homomorphisms $\alpha, \alpha \circ \alpha$, respectively. We are all set to apply the change of rings Theorem I.1. since we know the cohomology of the subalgebras C and B (at least in low dimensions, see Proposition I.3).

Before we introduce the results, let us define some elements in

$$\mathrm{Ext}_A(Z_2 Z_2)\colon\ g_0 \in \mathrm{Ext}^{1,1}, \quad h_i \in \mathrm{Ext}^{1,2^{i+1}}, \quad i = 0, 1, \cdots$$

(the element g_0 corresponds to the element h_0 of [2]; our h_i corresponds to h_{i+1} of [2]).

PROPOSITION II.2. *As an* $\mathrm{Ext}_A(Z_2, Z_2) \in$ *module,* $\mathrm{Ext}_A^{s,t}(L, Z_2)$ *has the following elements as generators for* $t - s \leqq 10$ (*if* $s \leqq 2$) *and* $t - s \leqq 9$ (*if* $s > 2$):

$$e_{0,1}, e_{0,3}, e_{0\ 7}, e_{2,10}, e_{4,13}$$

where $e_{s,t}$ *denotes a nontrivial class in* $\mathrm{Ext}_A^{s,t}(L, Z_2)$. *A* Z_2*-basis in these dimensions is given by the following set of classes*:

$$\begin{array}{lllll} e_{0,1}, & e_{0,1} * h_0, & e_{0\ 1} * h_1, & e_{0,1} * h_2, & e_{0\ 1} * h_0 h_2, \\ e_{0,1} * h_1^2, & e_{0,3}, & e_{0,3} * g_0, & e_{0,3} * g_0^2, & e_{0,3} * h_1, \\ e_{0.3} * h_2, & e_{0\ 3} * g_0 h_2, & e_{0,7} * g_0^k, & & k = 0, 1, 2, 3, \\ e_{0.7} * h_0, & e_{0,7} * h_0^2, & e_{2,10}, & e_{2,10} * h_0, & e_{4,13}. \end{array}$$

Proof. Explicit minimal resolution, using the methods of [8].

REMARKS. Compare Proposition II.2 with the results of Adams vanishing Theorem [4]. Also $e_{4,13} = P e_{0\ 1}$ (see Theorem 5 of [4]).

PROPOSITION II.3. $\mathit{Ext}_A^{s,t}(M, Z_2)$ *has the following* Z_2*-basis for* $t - s \leqq 11$:

$$\begin{array}{llllll} e_{0\ 2} * g_0^n, & e_{0,6} * g_0^n, & e_{1\ 5} * g_0^n, & e_{2\ 12} * g_0^n, & e_{3,11} * g_0^n, & n = 0, 1, \cdots, \\ e_{0,2} * h_1, & e_{0,2} * g_0 h_1, & e_{0,2} * h_2 g_0^k, & k = 0,1,2,3, & & \\ e_{0,6} * h_0, & e_{0\ 6} * h_0^2, & e_{2.13} * g_0^k, & k = 0, 1, 2, 3, & e_{3,14}. & \end{array}$$

PROPOSITION II.4. $\mathrm{Ext}_A^{s,t}(N, Z_2)$ *for* $t - s \leqq 13$ *has the following* Z_2*-basis*:

$$e_{0,4} * g_0^n, \quad e_{0,12} * g_0^n, \quad e_{3,11} * g_0^n, \qquad n = 0, 1, 2, \cdots,$$

$$e_{0,4} * h_0, \quad e_{0,4} * h_0^2, \quad e_{1,10},$$

$$e_{1,10} * h_0, \quad e_{1,10} * h_0^2, \quad e_{1,12} * g_0^k, \qquad k = 0, 1, 2, 3,$$

$$e_{1,12} * h_0, \quad e_{2\ 13}, \quad e_{2,13} * h_0, \quad e_{2,13} * h_0^2, \quad e_{5,18}, \quad e_{0,12} * h_0.$$

PROPOSITION II.3 and II.4 are proved by using the constructions of Theorem I.1. In the proof of Proposition II.3 we take an A-minimal resolution Y of L and take the tensor product of Y with a minimal resolution of Z_2 over C. In the proof of Proposition II.4 the tensor product of Y with a minimal resolution of Z_2 over B is examined. In both cases, for the range of s and t given, only the map d_1 need be examined.

We give a sample computation. The minimal resolution of L over A for $t - s \leqq 5$ can be taken as follows:

$$0 \leftarrow L \overset{\varepsilon}{\leftarrow} C_0 \overset{d}{\leftarrow} C_1 \overset{d}{\leftarrow} C_2 \overset{d}{\leftarrow} 0 \leftarrow 0 \cdots,$$

where C_0 is free on $c_{0,1}, c_{0,3}$, C_1 is free on $c_{1,3}, c_{1,4}, c_{1,5}$, C_2 is free on $c_{2,5}$; the maps ε, d are defined to be

$$\begin{aligned} \varepsilon(c_{0,1}) &= x, \varepsilon(c_{0,3}) = x^3, \\ d(c_{1,3}) &= a_1 c_{0,1}, \\ d(c_{1,4}) &= Q_0 c_{0\ 3} + Q_1 c_{0,1}, \\ d(c_{1,5}) &= a_2 c_{0,1}, \\ d(c_{2,5}) &= Q_0 c_{1,4} + a_1 c_{1.3}, \end{aligned}$$

where $a_i = Sq^{2^i}$, $Q_{i+1} = [a_{i+1}, Q_i]$.

Take generators $[I]$ of a minimal resolution W of Z_2 over C in one-to-one correspondence with finitely nonzero sequence I of non-negative integers. We denote by Δ_i the sequence consisting of 1 in the ith place and zeroes elsewhere; we let $I - J$ be the sequence of term-by-term differences (we set $[I - J] = 0$ if at least one entry is negative). The differential d'' in W is defined by

$$d''[I] = \sum_{i=0}^{\infty} Q_i[I - \Delta_i].$$

Let us show as an example that we can define d_1 on $c_{1,5} \otimes [n\Delta_0]$ as

$$\begin{aligned} &a_3 c_{0,1} \otimes [n\Delta_0] + a_1 a_2 c_{0\ 1} \otimes [(n-1)\Delta_0 + \Delta_1] + a_1 c_{0\ 1} \otimes [(n-1)\Delta_0 + \Delta_2] \\ &\quad + a_2 c_{0\ 1} \otimes [(n-2)\Delta_0 + 2\Delta_1] + c_{0\ 1} \otimes [(n-2)\Delta_0 + \Delta_1 + \Delta_2] \qquad (6) \\ &\quad + a_1 c_{0\ 1} \otimes [(n-3)\Delta_0 + 3\Delta_1] + c_{0,1} \otimes [(n-4)\Delta_0 + 4\Delta_1]. \end{aligned}$$

We shall need relations in addition to those exhibited in Chapter I.3:

$$Q_0 a_3 = a_3 Q_0 + a_1 a_2 Q_1 + a_1 Q_2,$$

$$Q_0 a_2 = a_2 Q_0 + a_1 Q_1.$$

The proof that (6) is admissible by induction on n. Since $\alpha(a_3) = a_2$, (6) is fine for $n = 0$. Suppose (6) is acceptable for $n > 0$:

$$\begin{aligned} d_1 d_0(c_{1,5} \otimes [(n+1)\Delta_0]) &= d_1 Q_0 c_{1,5} \otimes [n\Delta_0] \\ &= Q_0 a_3 c_{0,1} \otimes [n\Delta_0] \\ &\quad + Q_0 a_1 a_2 c_{0,1} \otimes [(n-1)\Delta_0 + \Delta_1] \\ &\quad + Q_0 a_1 c_{0,1} \otimes [(n-1)\Delta_0 + \Delta_2] \\ &\quad + Q_0 a_2 c_{0,1} \otimes [(n-2)\Delta_0 + 2\Delta_1] \\ &\quad + Q_0 c_{0,1} \otimes [(n-2)\Delta_0 + \Delta_1 + \Delta_2] \\ &\quad + Q_0 a_1 c_{0,1} \otimes [(n-3)\Delta_0 + 3\Delta_1] \\ &\quad + Q_0 c_{0,1} \otimes [(n-4)\Delta_0 + 4\Delta_1] \\ &= (a_3 Q_0 + a_1 a_2 Q_1 + a_1 Q_2) c_{0,1} \otimes [n\Delta_0] \\ &\quad + (a_1 a_2 Q_0 + a_2 Q_1 + Q_2) c_{0,1} \otimes [(n-1)\Delta_0 + \Delta_1] \\ &\quad + (a_1 Q_0 + Q_1) c_{0,1} \otimes [(n-1)\Delta_0 + \Delta_2] \\ &\quad + (a_2 Q_0 + a_1 Q_1) c_{0,1} \otimes [(n-2)\Delta_0 + 2\Delta_1] \\ &\quad + Q_0 c_{0,1} \otimes [(n-2)\Delta_0 + \Delta_1 + \Delta_2] \\ &\quad + (a_1 Q_0 + Q_1) c_{0,1} \otimes [(n-3)\Delta_0 + 3\Delta_1] \\ &\quad + Q_0 c_{0,1} \otimes [(n-4)\Delta_0 + 4\Delta_1], \end{aligned}$$

which is precisely d_0 of (6) for $n+1$, which completes the inductive step.

Let $\Pi_m^S(X; p)$ be the mth stable homotopy group of X [1] modulo the subgroup of elements having finite order prime to p. $\Pi_m^S(X; p)$ may be computed up to extensions by the Adams spectral sequence for the prime p; the extension can often be determined if we remark that $*g_0$ corresponds to multiplication by p in Π_*^S.

PROPOSITION II.5. *In the Adams spectral sequence* ($p=2$) *for* RP^∞ *all differentials vanish in total degrees* $\leqq 10$.

Proof. Since in the Adams spectral sequence for the sphere $d_r(g_0) = d_r(h_0) = d_r(h_1) = d_r(h_2) = 0$ for all r [4] according to Theorem I.6. it suffices to prove that all differentials vanish on $e_{0\,1}$, $e_{0,3}$, $e_{0\,7}$, $e_{2\,10}$, $e_{4\,13}$, but this is easy for the differentials land on groups which are zero according to Proposition II.2.

Since $RP^\infty = K(Z_2, 1)$ we do not have to consider the spectral sequences for p odd: they are all zero. Since $*g_0$ corresponds to multiplication by 2 we have:

THEOREM II.6. *The stable homotopy groups $\Pi_k^S(RP^\infty)$ are as follows for $k \leqq 9$:*

k:	Π_k^S:
0	0
1	Z_2
2	Z_2
3	Z_8
4	Z_2
5	0
6	Z_2
7	$Z_{16} \oplus Z_2$
8	$Z_2 \oplus Z_2 \oplus Z_2$
9	$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$.

We precede the next theorem by a proposition about stable secondary cohomology operations.

PROPOSITION II.7 (ADAMS). *There exists a stable secondary cohomology operation Ψ of degree 4 such that if $y \in H^2(CP^\infty; Z_2)$ then $\Psi(y)$ is defined and*

$$\Psi(y) = y^3 \text{ modulo zero.}$$

Proof. This is Theorem 4.4.1 of [2].

THEOREM II.8. *In the Adams spectral sequence for CP^∞ $(p=2)$ the only nontrivial differential in total degrees $\leqq 9$ is*

$$d_2(e_{0\,6}) = e_{0\,2} * g_0 h_1.$$

Furthermore, the groups $\Pi_m^S(CP^\infty; Z_2)$ are as follows for $m \leqq 8$:

m:	0	1	2	3	4	5	6	7	8
Π_m^S:	0	0	Z	0	Z	Z_2	Z	Z_2	$Z \oplus Z_2$.

Proof. Suppose $a * g_0^j = 0$ for some j. Then if $d_r(a) = b$, $b * g_0^j = 0$ in E_r, according to Theorem I.6. This settles all differentials in total degrees $\leqq 9$, except $d_2(e_{0,6})$. According to Proposition II.7, $e_{0,6}$ cannot be a d_r-cycle for all r, since it is not in the image of the mod 2 Hurewicz homomorphism. This implies that $r=2$, for d_r, $r>2$ is automatically zero on $e_{0,6}$.

THEOREM II.9. *In the Adams spectral sequence for HP^∞ $(p=2)$ all differentials vanish in total degrees $\leqq 11$. Furthermore, the groups $\Pi_m^S(HP^\infty;2)$ for $m \leqq 10$ are as follows*:

m:	0	1	2	3	4	5	6	7	8	9	10
Π_m^S:	0	0	0	0	Z	Z_2	Z_2	0	Z	Z_2	Z_2.

Proof. Proposition II.4 and argument as for Theorem II.8.

2. **The primes** $p>2$. In order to complete our study of the initial stable homotopy of projective spaces, we must examine the Adams spectral sequences for CP^∞ HP^∞, for primes $p>2$.

The following two propositions are proved by constructing minimal resolutions for low total degrees. The task is straightforward and is left to the reader.

Let $M=\tilde{H}^*(CP^\infty;Z_p)$ the augmented cohomology of CP^∞, p an odd prime, A the Steenrod algebra over Z_p.

PROPOSITION II.10. *A Z_p-basis $(p>2)$ for $\mathrm{Ext}_A^{s,t}(M,Z_p)$ for $t-s \leqq 6p-4$ is furnished by classes*

$$e_{0,2j} * g_0^n, \quad e_{1,2k+2p-1} * g_0^n, \quad e_{2,2r+4p-2} * g_0^n,$$
$$e_{1\ 4p-2}, \quad e_{1,4p-2} * g_0,$$

where $j=1,\cdots,p-1,2p-1$, $k=1,\cdots,p-1$, $r=2,\cdots,p-1$ $(p>3$ for $r)$, $n=0,1,\cdots$; if $p=3$, we have in addition

$$e_{0,2} * h_1, \quad e_{1,4p-2} * h_0, \quad e_{0,2} * h_1 g_0, \quad e_{0,2} * h_1 g_0^2.$$

Let $N=\tilde{H}^*(HP^\infty;Z_p)$.

PROPOSITION II.11. *Let $p>2$. Then $\mathrm{Ext}_A^{s,t}(N,Z_p)$ for $t-s \leqq 6p-2$ has the following elements as a Z_p-basis*:

$$e_{0,4k} * g_0^n, \quad e_{1,4j+2p-1} * g_0^n, \quad e_{2,4j+4p-2} * g_0^n,$$
$$e_{0,4} * h_0, \quad e_{0\ 4} * h_0 g_0, \quad e_{0\ 4} * h_0 g_0^2,$$

where $n=0,1,\cdots$, $k=1,\cdots,\frac{1}{2}(p-1)$, $\frac{1}{2}(3p-1)$, $j=1,\cdots,\frac{1}{2}(p-1)$.

We are now ready to examine the Adams spectral sequence for CP^∞, HP^∞ for an odd prime p.

PROPOSITION II.12. *There exists a stable secondary cohomology operation Λ of* *degree* $4p-4$, *defined on cohomology classes* x *such that* $Q_0 x = 0$, $Q_1 x = 0$, $P^2 x = 0$, *such that*

$$\Lambda(y) = by^{2p-1} \text{ modulo zero,}$$

where $b \neq 0$ *and* $y \in H^2(CP^\infty; Z_p)$.

PROPOSITION II.13. *There exists a stable secondary cohomology operation* Γ *of degree* $6p-6$ *such that*

(i) Γ *is defined on* $y \in H^2(CP^\infty; Z_p)$ $u \in H^4(HP^\infty; Z_p)$

(ii) $\Gamma(y) = cy^{3p-2}$, *modulo zero, where* $c \neq 0$ *in* Z_p,

(iii) $\Gamma(u) = 2cu^{(3p-1)/2}$, *modulo zero.*

PROPOSITIONS II.11 and II.12 are proved as in [9] using [2].

PROPOSITION II.14. (i) *The only nontrivial differential in the Adams spectral sequence for* CP^∞ *and* $p \geqq 5$ *for total degree* $\leqq 6p-4$ *is given by*

$$d_2(e_{0,4p-2}) = be_{1,4p-2} * g_0,$$

where $b \neq 0$ *in* Z_p.

(ii) *Statement* (i) *is valid for* $p = 3$ *in total degrees* $\leqq 13$.

Proof. Consider the case $p \geqq 5$. According to Proposition II.10 all nonzero elements of $\mathrm{Ext}_w(M, Z_p)$ have even total degree—except $e_{1,4p-2}$ and $e_{1,4p-2} * g_0$. The only elements in total degree $4p-4$ are the basis elements $e_{1,2p-2+2p-1} * g_0^n$. Theorem I.6. shows that all differentials vanish on $e_{1,4p-2}$ for $e_{1,4p-2} * g_0^2 = 0$. In order to prove (i) it remains to show that the stable mod p Hurewicz homomorphism is zero in dimension $4p-2$. This is taken care of by Proposition II.12.

THEOREM II.15. (i) *If* $p \geqq 5$ *the groups* $\Pi_k^S(CP^\infty; p)$ *for* $k \leqq 6p-4$ *are as follows*:

$$\Pi_k^S(CP^\infty; p) = Z \qquad \text{if } k = 2_i,\ 1 \leqq i \leqq 3p-2,$$

$$\Pi_k^S(CP^\infty; p) = 0 \qquad \text{if } k = 2i+1,\ i \neq 2p-2$$

$$\Pi_k^S(CP^\infty; p) = Z_p \qquad \text{if } k = 4p-3;$$

(ii) *the groups* $\Pi_k^S(CP^\infty; 3)$ *for* $k \leqq 12$ *are as follows*:

k:	2	3	4	5	6	7	8	9	10	11	12
Π_k^S:	Z	0	Z	0	Z	0	Z	Z_3	Z	0	$Z \oplus Z_3$.

Proof. Propositions II.10, II.14.

PROPOSITION II.16. *In the Adams spectral sequence for* HP^∞ *and* $p \geqq 3$ *the only nontrivial differential for total degrees* $\leqq 6p-2$ *is*

$$d_2(e_{0,6p-2}) = be_{0,4} * h_0 g_0,$$

where $b \neq 0$ *in* Z_p.

Proof. According to Proposition II.11 the only elements of odd total degree $\leqq 6p-2$ are the classes $e_{0,4} * h_0 g_0^r$, $r = 0, 1, 2$. All differentials on $e_{0,4}$ vanish, thus we only need to evaluate d_2 and d_3 on $e_{0,6p-2}$. Proposition II.13 implies that one of these two differentials is nonzero on $e_{0,6p-2}$. We use a folk theorem, which can be proved using the approach of [8] to the Adams spectral sequence: suppose a stable secondary cohomology operation corresponding to an element $u \in E_2^{2,*}$, has a minimal A-generator as image; suppose this generator determines the class $v \in E_2^{0,*}$ then $d_2(v) = u$. The proof is completed by remarking that the operation Γ of Proposition II.13 corresponds to $e_{0\,4} * h_0 g_0$.

THEOREM II.17. *If $p \geqq 3$, the groups $\Pi_m^S(HP^\infty; p)$ for $m \leqq 6p-2$ are as follows*

$$\Pi_{4k}^S(HP^\infty; p) = Z \qquad 0 < 4k \leqq 6p-2,$$

$$\Pi_{2j-1}^S(HP^\infty; p) = 0 \qquad 2j-1 \leqq 6p-2, \quad j \neq 3p-1,$$

$$\Pi_{6p-3}^S(HP^\infty; p) = Z_p.$$

Proof. Proposition II. 16.

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