Kirby and the promised land of topological manifolds: memories and memorable arguments

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"Kirby and the promised land of
topological manifolds:
memories and memorable arguments"

- Try to explain:
  Kirby's handle-by-handle approach to
  Hauptvermutung.

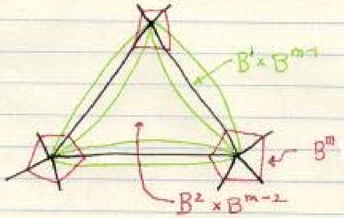
Earlier similar theories (i.e., handle-by-handle induction):
- Hirsch's smoothing theory:
  convert PL homeo to diffeo.
- Sullivan's thesis:
  convert homotopy equivalence to PL homeo.

PL topology: 1920's: Alexander $\longrightarrow$ Zeeman, Stallings, ...

Situation:
$$M^m_{PL} \xrightarrow[\approx]{h} M'^m_{PL}, \quad h \text{ 1-1 \& bicontinuous.}$$

Hauptvermutung:
Isotopically deform h to a PL homeo. h'

PL structure gives handle-decomposition:



$\exists$ open nbhd of any open simplex which is in product form: $\mathring{B}^k \times \mathbb{R}^{m-k}$

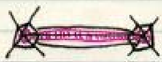
B^m around vertices

General handle: $B^k \times B^{m-k}$
$\subset B^k \times \mathbb{R}^{m-k}$

Initial boundary: $\partial_- = \partial B^k \times B^{m-k}$

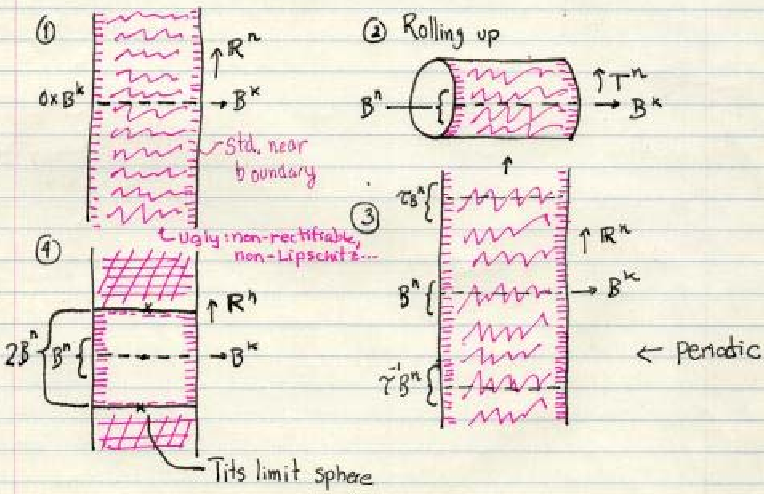
$\partial_+ = B^k \times \partial B^{m-k}$

Kirby's Annulus conjecture allowed to
Isotop h with compact support on
0-handle interior until it is PL
near the vertices (core of 0-handles)



1-handles:
want PL on nbhd of 1-skeleton,
etc.
By induction PL in open nbhd of $k-1$ skeleton

Solving k-handle problems: (see formulae below)



Handle problem:

(1) Handle: $(B^k \times \mathbb{R}^n)_\Sigma$ ← PL structure pulled back by h

⇒ Get a structure Σ' by Kirby or Novikov furling trick (needs at least Stallings engulfing), on:

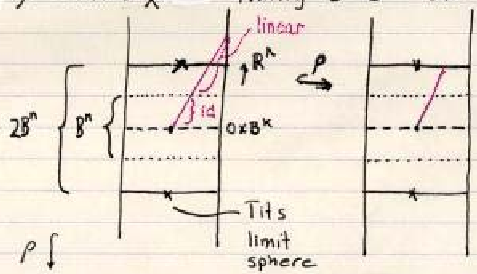
(2) $(B^k \times T^n)_{\Sigma'}$ $\quad T^n = \mathbb{R}^n/\lambda\mathbb{Z}$

Fact: can arrange that Σ' equals Σ on a $B^k \times B^n$ (near $B^k \times 0$) and on ∂_- as well (see Edwards' lecture)

Passing to universal cover:

(3) $(B^k \times \mathbb{R}^n)_{\tilde{\Sigma}'} \xhookrightarrow{\rho} B^k \times \mathbb{R}^n$

ρ radial re-embedding fixing $B^k \times B^n$ onto $(B^k \times 2B^n - 0 \times 2S^{n-1})$



$\rho \downarrow$

(4) $(B^k \times \mathbb{R}^n)$ $\quad \square$

Now, Surgery (& more...) shows:

If $k \neq 3$ then $\exists$ PL homeo g rel ∂_- : $B^k \times T^n \xrightarrow[\cong]{g} B^k \times T^n$

see below:
by surgery if $k \neq 3$ .....

Kirby Tower:

$$\begin{tikzcd}
B^k \times T^n \arrow[r, "g \text{ rel } \partial"', "\cong"] & (B^k \times \mathbb{R}^n)_\Sigma \arrow[d, dash] \\
B^k \times \mathbb{R}^n \arrow[u, "T^n"] \arrow[r, "\tilde{g}"] \arrow[d, "p"] \arrow[dr, phantom, "\bigcirc"] & (B^k \times T^n)_{\Sigma'} \arrow[d] \\
B^k \times \mathbb{R}^n \arrow[r, "H"'] & (B^k \times \mathbb{R}^n)_\Sigma
\end{tikzcd}$$

} furling

$g \simeq id$
$\Downarrow$
$\tilde{g}$ bdd. So:

H is a homeo.

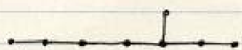
By definition H is Id. outside of $B^k \times 2B^n$
So H a PL homeo near pre-image of $B^k \times B^n$
$\because$ H id. outside $B^k \times 2B^n$.
Apply Alexander isotopy
H isotopic to identity, fixing nbhd of ∂_-
and complement of $B^k \times 2B^n$
Isotopy $H_t : 0 \le t \le 1$
$H_0 = id$ and $H_1 = H$
Carry structure Σ along from id to H
& can go on to $k+1$

Claim: If not caught in $k=3$ exception we have the hauptvermutung.

id: $B^k \times T^n \longrightarrow (B^k \times T^n)_{\Sigma'}$ homotopy PL structure
In Casson's lecture (almost!)
$\mathcal{S}(B^k \times T^n, \text{ rel } \partial B^k \times T^n) = H^3(B^k \times T^n, \partial; \mathbb{Z}_2) \cong H^{3-k}(T^n; \mathbb{Z}_2)$
$\therefore$ $k > 3$ is 0
$k = 3$ bad, $\mathbb{Z}_2$ (and covering invariant)
$k = 0, 1, 2$ $\mathbb{Z}_2$'s but die under 2^n fold std. covering (in fact, <u>is</u> 0 for $k = 0,1,2$ by S-cobord. trick)
but suff. to use the fact it so dies

Q: Why the problem with core dimension 3?
What does it mean?
How can it be solved?

Obstruction comes from Rohlin's Theorem
For M^4 closed PL + spin
$$\sigma(M^4) \equiv 0 \ (16)$$
Find a sort of 'homotopy violation' of it:

P^4 from E_8 

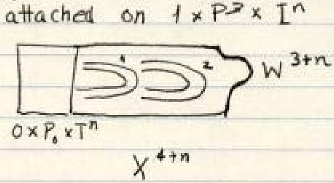
8 plumbed unit disk bundles of S^2  ← plumbing

$\partial P^4 = P^3$ Poincaré
$\pi_1 P^3 = \widetilde{A}_5 \subset$ Quat.
$H_*(P^3) = H_*(S^3)$
$\sigma(P^4) = 8 \underset{\text{Rohlin}}{\Rightarrow}$ Too small σ for a closed (smooth or PL) manifold.

Bad Guy:
$g: B^3 \times T^n \overset{g^{-1}}{\rightleftarrows} W^{3+n}$, $g \not\simeq$ PL homeo rel ∂

Build quickly using suggestion of Casson from punctured P^3, called P_0^3

W end of surgery: $I \times P_0^3 \times T^n \cup$ (2-handle) $\cup$ (3-handle -handes attached on $1 \times P^3 \times I^n$



g not $\simeq$ rel ∂ to PL homeo, by Rohlin, even after finite cover

BUT

$g \simeq$ rel ∂ to homeo, since after finite cover it
mapping cylinder of <u>small</u> PL autom $\gamma : B^2 \times T^n \circlearrowleft$
topologically isotopic to identity rel ∂
(Černavski Kirby Edwards)

Bad guy: $B^3 \times T^n \xrightarrow[\simeq]{g} W^{n+3}$
PL iso on ∂
rep. nonzero elt. of $H^3(B^3 \times T^n, \partial; \mathbb{Z}_2) = \mathbb{Z}_2$

<u>Claim</u>: g is $\simeq$ to a homeomorphism

<u>Proof</u> By s-cobordism theorem (identifying
$B^3 = [0,1] \times B^2$ have

$$f : [0,1] \times B^2 \xrightarrow{\cong} W^{n+3}$$

that equals g on $0 \times B^2 \cup [0,1] \times \partial B^2$
consider $f^{-1} g : [0,1] \times B^2 \times T^n \to [0,1] \times B^2 \times T^n$
is id on $0 \times B^2 \cup [0,1] \times \partial B^2 \times T^n$
but <u>not</u> PL isot to id on $B^2 \times 1 \times T^n$ rel ∂
(else g good guy!)