ALGEBRAIC L-THEORY, II: LAURENT EXTENSIONS

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Introduction

In Part I ([4]), we defined the L-groups $U_n(A)$, $V_n(A)$, $W_n(A)$ of a ring with involution A, for n (mod 4).

The main result of Part II is that there exist natural direct sum decompositions

$$W_n(A_z) = W_n(A) \oplus V_{n-1}(A),$$

$$V_n(A_z) = V_n(A) \oplus U_{n-1}(A),$$

where $A_z = A[z, z^{-1}]$ is the Laurent extension ring of A, with involution $z \mapsto z^{-1}$. (Cf. Part III, [5], for the generalization to twisted Laurent extensions.)

Similar splittings arise in [3]—indeed, our method of proof follows that of [3], except that Novikov neglects 2-torsion in the L-groups, and assumes that 2 is invertible in A. In the geometrically realizable case $A = \mathbf{Z}[\pi]$, (π a finitely presented group), it is possible to obtain the decompositions by topological methods ([2], [6], and [8]).

Defining L-theories $L_*^{(m)}(A)$ for $m \leqslant 2$, n (mod 4) by

$$L_n^{(2)}(A) = W_n(A),$$

$$L_{n+1}^{(m+1)}(A_z) = L_{n+1}^{(m+1)}(A) \oplus L_n^{(m)}(A) \quad (m \leqslant 1),$$

it follows that $L_*^{(1)}(A) = V_*(A)$, $L_*^{(0)}(A) = U_*(A)$, and that

$$L_n^{(m)}(A_{z_1, z_2, \ldots, z_p}) = \sum_{r=0}^{p} \binom{p}{r} L_{(n-r)}^{(m-r)}(A),$$

where $A_{z_1, z_2, \ldots, z_p} = A[z_1, z_1{}^{-1}, z_2, z_2{}^{-1}, \ldots, z_p, z_p{}^{-1}]$ is the Laurent extension ring of A in p variables. It will be shown that we are dealing with natural isomorphisms

$$L_*^{(*)}(A_{z_1, z_2, \ldots, z_p}) \cong L_*^{(*)}(A) \otimes_{\mathbf{Z}} \Lambda_*(z_1, z_2, \ldots, z_p),$$

where $\Lambda_*(z_1, z_2, \ldots, z_p)$ is the graded exterior $\mathbf{Z}$-algebra on p generators $z_1, z_2, \ldots, z_p$ in degree 1. The appearance of exterior algebra in L-theory is explained in [3] in terms of the corresponding surgery operations.

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1. Laurent extensions

We refer to [4] as I. Notation and definitions are as in I. In particular, we are working over A, an associative ring with 1 and involution, and such that f.g. free A-modules have well-defined dimension.

Let z be an invertible indeterminate over A, which commutes with every element of A. The *Laurent extension* of A by z, A_z, is the ring of polynomials $\sum_{j=-\infty}^{\infty} a_j z^j$ in z, z^{-1} with only a finite number of the coefficients $a_j \in A$ non-zero. Then A_z is an associative ring with 1, under the usual addition and multiplication of polynomials. The function

$$\bar{}: A_z \to A_z; \quad a = \sum_{j=-\infty}^{\infty} a_j z^j \to \bar{a} = \sum_{j=-\infty}^{\infty} \bar{a}_j z^{-j}$$

is an involution of A_z. The projection

$$\varepsilon: A_z \to A; \quad \sum_{j=-\infty}^{\infty} a_j z^j \mapsto \sum_{j=-\infty}^{\infty} a_j$$

is a ring morphism which preserves unities and the involutions. Every f.g. free A_z-module Q has a well-defined dimension, namely that of the f.g. free A-module εQ.

Thus A_z satisfies all the conditions imposed above on A.

For example, if $A = \mathbf{Z}[\pi]$ (as in Example 0·1 of I), with $\pi = \pi_1(M)$ for some compact manifold M, then $A_z = \mathbf{Z}[\pi \times \mathbf{Z}]$, with $\pi \times \mathbf{Z} = \pi_1(M \times S^1)$.

The injection

$$\bar{\varepsilon}: A \to A_z; \quad a \mapsto a$$

splits ε, that is $\varepsilon\bar{\varepsilon} = 1_A$, and $\bar{\varepsilon}A$ is identified with A. Every A_z-module Q can be regarded as an A-module by restricting the action of A_z to one of A.

A *modular A-base* of an A_z-module Q is an A-submodule Q_0 of Q such that every $x \in Q$ has a unique expression as

$$x = \sum_{j=-\infty}^{\infty} z^j x_j \in Q \quad (x_j \in Q_0)$$

with $\{x_j \in Q_0 \mid x_j \neq 0\}$ finite, corresponding to an infinite direct sum

$$Q = \sum_{j=-\infty}^{\infty} z^j Q_0$$

of A-modules isomorphic to Q_0. Hence there is an A-module isomorphism

$$Q_0 \cong Q/(z-1)Q \quad (= \varepsilon Q)$$

and modular A-bases of isomorphic A_z-modules are isomorphic.

Given an A-module Q, define the A_z-module *freely generated* by Q, Q_z, to be the direct sum

$$Q_z = \sum_{j=-\infty}^{\infty} z^j Q$$

of a countable infinity of copies of Q with the action of A_z indicated—that is, $Q_z = \bar{\varepsilon}Q$. Then Q is a modular A-base of Q_z.

It is convenient to list here several properties of modular A-bases.

(i) Every modular A-base Q_0 of an A_z-module Q determines a dual modular A-base Q_0^* of Q^*, with

$$(z^k g)(z^j x) = g(x).z^{j-k} \in A_z \quad (g \in Q_0^*,\ x \in Q_0,\ j,k \in \mathbf{Z}).$$

(ii) For any A-modules P,Q, give $\mathrm{Hom}_A(P,Q)$ a left A-module structure by

$$A \times \mathrm{Hom}_A(P,Q) \to \mathrm{Hom}_A(P,Q);\quad (a,f) \mapsto (x \mapsto a.f(x))$$

and similary for A_z-modules.

Every $f \in \mathrm{Hom}_{A_z}(P_z,Q_z)$ defines $\sum_{j=-\infty}^{\infty} z^j f_j \in (\mathrm{Hom}_A(P,Q))_z$ by

$$f(x) = \sum_{j=-\infty}^{\infty} z^j f_j(x) \in A_z \quad (x \in P,\ f_j(x) \in Q),$$

and conversely, so that we may identify

$$\mathrm{Hom}_{A_z}(P_z,Q_z) = (\mathrm{Hom}_A(P,Q))_z.$$

Given $f \in \mathrm{Hom}_A(P,Q)$, let f also denote the element of $\mathrm{Hom}_{A_z}(P_z,Q_z)$ defined by

$$f\colon P_z \to Q_z;\quad \sum_{j=-\infty}^{\infty} z^j x_j \mapsto \sum_{j=-\infty}^{\infty} z^j f(x_j) \quad (x_j \in P).$$

(iii) The A_z-module Q_z is

$$\begin{cases}\text{f.g. projective}\\ \text{f.g. free}\end{cases} \text{if and only if } Q \text{ is a } \begin{cases}\text{f.g. projective}\\ \text{f.g. free}\end{cases} A\text{-module}.$$

A based A-module Q generates a based A_z-module Q_z in the obvious way. Conversely, a based A_z-module Q determines a based modular A-base Q.

(iv) Given an A-module Q define A-submodules

$$Q^+ = \sum_{j=0}^{\infty} z^j Q,\quad Q^- = \sum_{j=-\infty}^{-1} z^j Q$$

of Q_z. Then

$$\nu\colon Q_z = Q^+ \oplus Q^- \xrightarrow{(1\ 0)} Q^+$$

is the *positive projection* on Q.

(v) Let F, G be two modular A-bases of a f.g. free A_z-module Q. Then F, G are f.g. free A-modules and

$$z^N F^+ \subseteq G^+$$

for large enough integers $N \geqslant 0$. For such N define the A-module

$$B_N^+(F, G) = z^N F^- \cap G^+,$$

a direct summand of Q (regarded as an A-module), with

$$G^+ = z^N F^+ \oplus B_N^+(F, G).$$

If H is another modular A-base of Q, and if $M \geqslant 0$ is so large that $z^M G^+ \subseteq H^+$, then

$$B_{M+N}^+(F, H) = z^M B_N^+(F, G) \oplus B_M^+(G, H).$$

In particular, for $N_1 \geqslant 0$ so large that $z^{N_1} G^+ \subseteq F^+$,

$$z^N B_{N_1}^+(G, F) \oplus B_N^+(F, G) = B_{N+N_1}^+(G, G) = \sum_{j=0}^{N+N_1-1} z^j G$$

so that, as G is f.g. free, $B_N^+(F, G)$ is a f.g. projective A-module.

Moreover, as

$$B_{N+1}^+(F, G) = B_N^+(F, G) \oplus z^N F,$$

and F is f.g. free, the projective class $[B_N^+(F, G)] \in \tilde{K}_0(A)$ does not depend on N.

The A-module isomorphism

$$B_N^+(F^*, G^*) \to B_N^+(F, G)^*; \quad g \mapsto (x \mapsto [g(x)]_0)$$

is used as an identification, where $[a]_0 = a_0 \in A$ if $a = \sum_{j=-\infty}^{\infty} a_j z^j \in A_z$.

We now quote a principal result of algebraic K-theory ([**1**], Chapter XII; [**7**], p. 226).†

THEOREM. *There exists a natural direct sum decomposition*

$$\tilde{K}_1(A_z) = \tilde{K}_1(A) \oplus \tilde{K}_0(A) \oplus \mathrm{Nil}^+(A) \oplus \mathrm{Nil}^-(A),$$

where $\mathrm{Nil}^\pm(A)$ *is the subgroup of* $\tilde{K}_1(A_z)$ *generated by*

$$\{\tau((1 + \nu(z^{\pm 1} - 1)) : P_z \to P_z) \in \tilde{K}_1(A_z) \mid \nu \in \mathrm{Hom}_A(P, P) \text{ nilpotent}\}.$$

The splitting is by injections

$$\bar{\varepsilon}: \tilde{K}_1(A) \to \tilde{K}_1(A_z); \quad \tau(\alpha: F \to F) \mapsto \tau(\alpha: F_z \to F_z),$$

$$\bar{B}: \tilde{K}_0(A) \to \tilde{K}_1(A_z); \quad [P] \mapsto \tau\left(\zeta = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} : (P \oplus -P)_z \to (P \oplus -P)_z\right),$$

and by projections

$$\varepsilon: \tilde{K}_1(A_z) \to \tilde{K}_1(A); \quad \tau\left(\sum_{j=-\infty}^{\infty} z^j \alpha_j : F_z \to F_z\right) \mapsto \tau\left(\sum_{j=-\infty}^{\infty} \alpha_j : F \to F\right),$$

$$B: \tilde{K}_1(A_z) \to \tilde{K}_0(A); \quad \tau(\alpha: F_z \to F_z) \mapsto [B_N^+(F, \alpha(F))].$$

† See the Corrigendum on p. 156.

COROLLARY. *The diagram*

$$\begin{array}{ccc}
\tilde{K}_1(A_z) & \xrightarrow{*} & \tilde{K}_1(A_z) \\
\varepsilon \big\downarrow \big\uparrow \bar{\varepsilon} & & \varepsilon \big\downarrow \big\uparrow \bar{\varepsilon} \\
\tilde{K}_1(A) & \xrightarrow{*} & \tilde{K}_1(A)
\end{array}$$

commutes (in the sense that $*\varepsilon = \varepsilon *$, $*\bar{\varepsilon} = \bar{\varepsilon} *$*), and*

$$\begin{array}{ccc}
\tilde{K}_1(A_z) & \xrightarrow{*} & \tilde{K}_1(A_z) \\
B \big\downarrow \big\uparrow \bar{B} & & B \big\downarrow \big\uparrow \bar{B} \\
\tilde{K}_0(A) & \xrightarrow{*} & \tilde{K}_0(A)
\end{array}$$

skew-commutes ($*B = -B*$, $*\bar{B} = -\bar{B}*$), *where*

$$*: \tilde{K}_1(A) \to \tilde{K}_1(A); \quad \tau(\alpha: F \to F) \mapsto \tau(\alpha^*: F^* \to F^*)$$
$$*: \tilde{K}_0(A) \to \tilde{K}_0(A); \quad [P] \mapsto [P^*]$$

are the duality involutions.

Moreover,

$$*: \tilde{K}_1(A_z) \to \tilde{K}_1(A_z)$$

sends $\mathrm{Nil}^{\pm}(A)$ *onto* $\mathrm{Nil}^{\mp}(A)$.

Recalling the definitions of the groups

$$\Omega_{\pm}(A) = \{\tau \in \tilde{K}_1(A) \mid \tau^* = \pm\tau \in \tilde{K}_1(A)\}/\{\omega \pm \omega^* \mid \omega \in \tilde{K}_1(A)\}$$
$$\Sigma_{\pm}(A) = \{[P] \in \tilde{K}_0(A) \mid [P^*] = \pm[P] \in \tilde{K}_0(A)\}/\{[Q] \pm [Q^*] \mid [Q] \in \tilde{K}_0(A)\}$$

from I, it follows that there are defined morphisms

$$\Omega_{\pm}(A) \underset{\varepsilon}{\overset{\bar{\varepsilon}}{\rightleftarrows}} \Omega_{\pm}(A_z) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} \Sigma_{\mp}(A)$$

and hence a splitting

$$\Omega_{\pm}(A_z) = \Omega_{\pm}(A) \oplus \Sigma_{\mp}(A).$$

We wish to establish an analogous result for algebraic L-theory.†

THEOREM 1.1. *There exists a diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \Omega_{(-)^{n+1}}(A) & \longrightarrow & W_n(A) & \longrightarrow & V_n(A) & \longrightarrow & \Omega_{(-)^n}(A) & \longrightarrow \cdots \\
& \bar{\varepsilon} \big\downarrow \big\uparrow \varepsilon & & \bar{\varepsilon} \big\downarrow \big\uparrow \varepsilon & & \bar{\varepsilon} \big\downarrow \big\uparrow \varepsilon & & \bar{\varepsilon} \big\downarrow \big\uparrow \varepsilon & \\
\cdots \longrightarrow & \Omega_{(-)^{n+1}}(A_z) & \longrightarrow & W_n(A_z) & \longrightarrow & V_n(A_z) & \longrightarrow & \Omega_{(-)^n}(A_z) & \longrightarrow \cdots \\
& B \big\downarrow \big\uparrow \bar{B} & & B \big\downarrow \big\uparrow \bar{B} & & B \big\downarrow \big\uparrow \bar{B} & & B \big\downarrow \big\uparrow \bar{B} & \\
\cdots \longrightarrow & \Sigma_{(-)^n}(A) & \longrightarrow & V_{n-1}(A) & \longrightarrow & U_{n-1}(A) & \longrightarrow & \Sigma_{(-)^{n-1}}(A) & \longrightarrow \cdots
\end{array}$$

† See the Corrigendum on p. 156.

of abelian groups and morphisms, defined for n (mod 4), in which squares of shape $\downarrow\rightrightarrows\downarrow$, $\uparrow\rightrightarrows\uparrow$ *commute. The rows are the exact sequences of Theorems* 4.3, 5.7 *in* I. *The columns are split short exact, with* $\varepsilon\bar{\varepsilon} = 1$, $B\bar{B} = 1$ *whenever defined, corresponding to direct sum decompositions*

$$W_n(A_z) = W_n(A) \oplus V_{n-1}(A),$$
$$V_n(A_z) = V_n(A) \oplus U_{n-1}(A).$$

The diagram is natural in A.

2. Proof of Theorem 1.1 (n odd)

Given A_z-modules P, Q and $\theta \in \mathrm{Hom}_{A_z}(P, Q^*)$, define

$$[\theta]_0 \in \mathrm{Hom}_A(P, \mathrm{Hom}_A(Q, A))$$

by

$$[\theta]_0(x)(y) = [\theta(x)(y)]_0 \in A \quad (x \in P,\ y \in Q),$$

where $[a]_0 = a_0 \in A$ if $a = \sum_{j=-\infty}^{\infty} a_j z^j \in A_z$.

Given A-modules P, Q and $\theta = \sum_{j=-\infty}^{\infty} z^j \theta_j \in \mathrm{Hom}_{A_z}(P_z, Q_z^*)$ (with $\theta_j \in \mathrm{Hom}_A(P, Q^*)$), $[\theta]_0 \in \mathrm{Hom}_A(P_z, Q_z^*)$ is given by

$$[\theta]_0(z^j x)(z^k y) = \theta_{k-j}(x)(y) \in A \quad (x \in P,\ y \in Q,\ j, k \in \mathbf{Z})$$

and

$$\theta(x)(y) = \sum_{j=-\infty}^{\infty} z^j([\theta]_0(x)(z^j y)) \in A_z \quad (x \in P,\ y \in Q).$$

LEMMA 2.1. *Let* (Q, φ) *be a non-singular* $\pm$ *form over* A_z, *and let* C, D *be complementary* A*-submodules of* Q *such that* C *is finitely generated and*

$$[\langle C, D\rangle_\varphi]_0 = \{0\} \subseteq A.$$

Then $(C, \iota^*[\varphi]_0\iota)$ *is a non-singular* $\pm$ *form over* A, *where* $\iota: C \to Q$ *is the inclusion.*

In general, $(C, \iota^*[\varphi]_0\iota)$ will be denoted by $(C, [\varphi]_0)$.

Define

$$B: V_{2i+1}(A_z) \to U_{2i}(A); \quad (Q, \varphi; F, G) \mapsto (B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*), [\varphi]_0),$$

where F and G are free, with modular A-bases F_0, G_0 respectively and $N \geqslant 0$ so large that

$$z^N(F_0 \oplus F_0^*)^+ \subseteq (G_0 \oplus G_0^*)^+$$

for some choice of hamiltonian complements F^*, G^* to F, G in (Q, φ) with dual modular A-bases F_0^*, G_0^*. Now

$$[\langle B_N^+(F_0 \oplus F_0^*,\ G_0 \oplus G_0^*), z^N(F_0 \oplus F_0^*)^+ \oplus (G_0 \oplus G_0^*)^-\rangle_\varphi]_0 = \{0\} \subseteq A$$

so that the hypotheses of Lemma 2.1 are satisfied, and

$$(B_N^+(F_0 \oplus F_0^*,\ G_0 \oplus G_0^*), [\varphi]_0)$$

is a non-singular $\pm$ form over A, and does represent an element of $U_{2i}(A)$. It does not depend on N because increasing N by 1 adds on $H_{\pm}(z^N F_0)$, which vanishes in $U_{2i}(A)$. Nor does the choice of F^* matter: for $N \geqslant 0$ so large that

$$z^N F_0^+ \subseteq (G_0 \oplus G_0^*)^+,$$

define the A-module

$$E_N^+(F_0, G_0 \oplus G_0^*) = \{x \in (G_0 \oplus G_0^*)^+ \mid [\langle z^N F_0^+, x\rangle_\varphi]_0 = \{0\} \subseteq A\}.$$

Observe that the $\pm$ form defined over A by

$$(E_N^+(F_0, G_0 \oplus G_0^*)/z^N F_0^+, [\varphi]_0)$$

coincides with $(B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*), [\varphi]_0)$ when N is so large that $z^N(F_0 \oplus F_0^*)^+ \subseteq (G_0 \oplus G_0^*)^+$, as then

$$E_N^+(F_0, G_0 \oplus G_0^*) = (F \oplus z^N F_0^{*-}) \cap (G_0 \oplus G_0^*)^+ = z^N F_0^+ \oplus B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*).$$

The choice of F^* did not enter in this new definition. The choice of G^* may be dealt with similarly.

Next, suppose $(Q, \varphi; F, G) = 0 \in V_{2i+1}(A_z)$, and consider the generic cases.

(i) F and G are hamiltonian complements in (Q, φ). Put $F_0^* = G_0$, $G_0^* = F_0$, $N = 0$ to obtain $B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*) = 0$, and so

$$B(Q, \varphi; F, G) = 0 \in U_{2i}(A).$$

(ii) F and G share a hamiltonian complement in (Q, φ). Put $F_0^* = G_0^*$ to obtain

$$\begin{aligned} B(Q, \varphi; F, G) &= B(Q, \varphi; F^*, G^*) \quad \text{(by symmetry of definition)} \\ &= 0 \in U_{2i}(A) \quad \text{(taking } N = 0\text{)}. \end{aligned}$$

It follows that $B(Q, \varphi; F, G) = 0 \in U_{2i}(A)$ whenever

$$(Q, \varphi; F, G) = 0 \in V_{2i+1}(A_z).$$

It now remains only to verify that the choice of modular A-bases F_0, G_0 for F, G is immaterial to $B(Q, \varphi; F, G) \in U_{2i}(A)$.

Let $\hat{F}_0$ be another modular A-base of F, with dual modular A-base $\hat{F}_0^*$ of F^*, and let $\hat{N} \geqslant 0$ be so large that

$$z^{\hat{N}}(\hat{F}_0 \oplus \hat{F}_0^*)^+ \subseteq (F_0 \oplus F_0^*)^+.$$

Then

$$\begin{aligned} &(B_{N+\hat{N}}^+(\hat{F} \oplus \hat{F}^*, G \oplus G^*), [\varphi]_0) \\ &\qquad = (z^N B_{\hat{N}}^+(\hat{F} \oplus \hat{F}^*, F \oplus F^*), [\varphi]_0) \oplus (B_N^+(F \oplus F^*, G \oplus G^*), [\varphi]_0) \\ &\qquad = H_{\pm}(z^N B_{\hat{N}}^+(\hat{F}, F)) \oplus (B_N^+(F \oplus F^*, G \oplus G^*), [\varphi]_0) \\ &\qquad = (B_N^+(F \oplus F^*, G \oplus G^*), [\varphi]_0) \in U_{2i}(A), \end{aligned}$$

so that $\hat{F}$ will do as well as F. Similarly, the choice of G is immaterial. Hence

$$B\colon V_{2i+1}(A_z) \to U_{2i}(A)$$

is well-defined.

The composition

$$V_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} V_{2i+1}(A_z) \xrightarrow{B} U_{2i}(A)$$

is 0 because

$$B\bar{\varepsilon}(Q,\varphi;\, F, G) = B(Q_z,\varphi;\, F_z, G_z) = (B_0^+(F\oplus F^*, G\oplus G^*),\varphi) = 0 \in U_{2i}(A).$$

The diagram

$$\begin{array}{ccc} V_{2i+1}(A_z) & \longrightarrow & \Omega_-(A_z) \\ \Big\downarrow B & & \Big\downarrow B \\ U_{2i}(A) & \longrightarrow & \Sigma_+(A) \end{array}$$

commutes, because given $(Q,\varphi;\, F, G) \in V_{2i+1}(A_z)$ and

$$\pi^{-1}(Q,\varphi;\, F,G) = ((\alpha,\chi)\colon (Q,\varphi) \to (Q,\varphi)) \in \mathscr{U}_\pm(A_z)/\mathscr{H}_\pm(A_z)$$

with $\alpha(F) = G$ (in the notation of Theorem 4.2 of I), then

$$\begin{aligned} B(\tau(\alpha)) &= [B_N^+(F_0\oplus F_0^*, \alpha(F_0\oplus F_0^*))] \\ &= [B_N^+(F_0\oplus F_0^*, G_0\oplus G_0^*)] \in \Sigma_+(A), \end{aligned}$$

for any modular A-base F_0 of F, with $G_0 = \alpha(F_0)$.

Define

$\bar{B}\colon U_{2i}(A) \to V_{2i+1}(A_z)$;

$$(Q,\varphi) \to ((Q_z\oplus Q_z, \varphi\oplus -\varphi)\oplus H_\pm(-Q_z);\, \Delta_{(Q_z,\varphi)}\oplus -Q_z, \zeta\Delta_{(Q_z,\varphi)}\oplus -Q_z),$$

where $-Q$ is any f.g. projective A-module such that $Q\oplus -Q$ is free and

$$\zeta = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \colon Q_z\oplus Q_z \to Q_z\oplus Q_z.$$

This is well defined because

$$\begin{aligned} &\{((x,0),(0,g),(0,y)) \in (P\oplus P^*)_z \oplus (P\oplus P^*)_z \\ &\qquad \oplus (-(P\oplus P^*)_z \oplus (-(P\oplus P^*)_z)^*) \mid x\in P_z,\ g\in P_z^*,\ y\in(-(P\oplus P^*)_z)^*\} \end{aligned}$$

is a hamiltonian complement to both $\Delta_{H_\pm(P_z)}\oplus -(P\oplus P^*)_z$ and $\zeta(\Delta_{H_\pm(P_z)})\oplus -(P\oplus P^*)_z$ in $H_\pm(P_z)\oplus -H_\pm(P_z)\oplus H_\pm(-(P\oplus P^*)_z)$, so that

$$\bar{B}(H_\pm(P)) = 0 \in V_{2i+1}(A_z)$$

for any f.g. projective A-module P.

The composite

$$U_{2i}(A) \xrightarrow{\bar{B}} V_{2i+1}(A_z) \xrightarrow{\varepsilon} V_{2i+1}(A)$$

is 0 because

$$\varepsilon\bar{B}(Q,\varphi) = ((Q\oplus Q,\varphi\oplus -\varphi)\oplus H_{\pm}(-Q);\Delta_{(Q,\varphi)}\oplus -Q,\Delta_{(Q,\varphi)}\oplus -Q)$$
$$= 0 \in V_{2i+1}(A).$$

The diagram

$$\begin{array}{ccc} U_{2i}(A) & \longrightarrow & \Sigma_{+}(A) \\ \bar{B}\downarrow & & \downarrow\bar{B} \\ V_{2i+1}(A_z) & \longrightarrow & \Omega_{-}(A_z) \end{array}$$

commutes because, given $(Q,\varphi)\in U_{2i}(A)$ (with π as in Theorem 4.2 of Part I),

$$\tau(\pi^{-1}\bar{B}(Q,\varphi))$$
$$= \tau(\zeta\oplus 1\colon (Q_z\oplus Q_z)\oplus(-Q_z\oplus -Q_z^*)\to(Q_z\oplus Q_z)\oplus(-Q_z\oplus -Q_z^*))$$
$$= \bar{B}([Q])\in\Omega_{-}(A_z).$$

The composite

$$U_{2i}(A)\xrightarrow{\bar{B}} V_{2i+1}(A)\xrightarrow{B} U_{2i}(A)$$

is the identity because, for each $(Q,\varphi)\in U_{2i}(A)$,

$$B\bar{B}(Q,\varphi) = B((Q_z\oplus Q_z,\varphi\oplus -\varphi)\oplus H_{\pm}(-Q_z);\Delta_{(Q_z,\varphi)}\oplus -Q_z,\zeta\Delta_{(Q_z,\varphi)}\oplus -Q_z)$$
$$= (B_1^{+}(\Delta_{(Q,\varphi)}\oplus\Delta^{*}_{(Q^*,\psi)},\zeta(\Delta_{(Q,\varphi)}\oplus\Delta^{*}{}_{(Q^*,\psi)})),\varphi\oplus -\varphi)\oplus H_{\pm}(-Q)$$
$$= (B_1^{+}(Q\oplus Q,Q\oplus zQ),\varphi\oplus -\varphi)\oplus H_{\pm}(-Q)$$
$$= (Q,\varphi)\in U_{2i}(A),$$

where $\Delta^{*}_{(Q^*,\psi)}$ is any hamiltonian complement to $\Delta_{(Q,\varphi)}$ in $(Q\oplus Q,\varphi\oplus -\varphi)$, in the terminology of Lemma 1.4 of I.

It now remains only to verify that the sequence

$$V_{2i+1}(A)\xrightarrow{\bar{\varepsilon}} V_{2i+1}(A_z)\xrightarrow{B} U_{2i}(A)$$

is exact. This will be done by first characterizing the $\pm$ formations over A_z equivalent to ones obtained from $\pm$ formations over A via $\bar{\varepsilon}\colon A\to A_z$ (in Lemma 2.2 below), and then using the hamiltonian transformation of Lemma 2.3 to show that every element of $\ker(B\colon V_{2i+1}(A_z)\to U_{2i}(A))$ has a representative satisfying that criterion.

LEMMA 2.2. *A $\pm$ formation $(Q,\varphi;F,G)$ over A_z is equivalent to $\bar{\varepsilon}(Q_0,\varphi_0;F_0,G_0)$ for some $\pm$ formation $(Q_0,\varphi_0;F_0,G_0)$ over A if and only if F has a modular A-base F_0 such that, for some hamiltonian complement F^* to F in (Q,φ), the positive projection on $F_0\oplus F_0^*$,*

$$\nu\colon Q = F\oplus F^*\to(F_0\oplus F_0^*)^{+},$$

preserves G, that is $\nu(G)\subseteq G$.

Proof. It is clear that $\bar{\varepsilon}(Q_0, \varphi_0; F_0, G_0)$ satisfies the condition, for any $\pm$ formation $(Q_0, \varphi_0; F_0, G_0)$ over A.

Conversely, assume that the condition holds for $(Q, \varphi; F, G)$, a $\pm$ formation over A_z.

The A-module morphism

$$\xi = z(1-\nu)z^{-1}\nu : Q \to Q$$

sends Q onto $F_0 \oplus F_0^*$, and has the property that

$$x = \sum_{j=-\infty}^{\infty} z^j \xi z^{-j} x \in (F_0 \oplus F_0^*)_z = Q$$

for every $x \in Q$.

Now $\nu(G) \subseteq G$, so that

$$\xi(G) = G \cap (F_0 \oplus F_0^*),$$

and $G_0 = \xi(G)$ is therefore a modular A-base of G contained in $F_0 \oplus F_0^*$. Thus, up to equivalence of $\pm$ formations over A_z,

$$(Q, \varphi; F, G) = (H_\pm(F); F, G) = \bar{\varepsilon}(H_\pm(F_0); F_0, G_0).$$

LEMMA 2.3. *Given a morphism of $\pm$ forms over A*

$$(f, \chi): (P, \theta) \to (Q, \varphi),$$

define the self-equivalence

$$H(f) = \left(\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ f^*(\varphi \pm \varphi^*) & -\theta & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\varphi f & 0 \\ 0 & \chi & 0 \\ 0 & 0 & 0 \end{pmatrix} \right):$$

$$(Q, \varphi) \oplus H_\pm(P) \to (Q, \varphi) \oplus H_\pm(P).$$

If (Q, φ) is non-singular, the self-equivalence $h' = H(f) \oplus 1$ of

$$(Q', \varphi') = ((Q, \varphi) \oplus H_\pm(P)) \oplus ((Q, -\varphi) \oplus H_\pm(-P) \oplus H_\pm(-Q))$$

is a hamiltonian transformation, that is

$$(Q', \varphi'; L', h'(L')) = 0 \in V_{2i+1}(A)$$

for any free lagrangian L' of (Q', φ').

Proof. The self-equivalence $h': (Q', \varphi') \to (Q', \varphi')$ preserves the free lagrangian

$$L = \{(x, y, x) \in Q \oplus (P \oplus P^*) \oplus Q \mid x \in Q,\ y \in P^*\} \oplus -P^* \oplus -Q$$

so that it is necessarily a product

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \beta \mp \beta^* \\ 0 & 1 \end{pmatrix} : L \oplus L^* \to L \oplus L^*$$

of elementary hamiltonian transformations, for any hamiltonian complement L^* (cf. Theorem 4.2 of I).

We now prove the exactness of

$$V_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} V_{2i+1}(A_z) \xrightarrow{B} U_{2i}(A).$$

Given $(Q,\varphi;F,G) \in \ker(B\colon V_{2i+1}(A_z) \to U_{2i}(A))$, there exists $N > 0$ so large that $(B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*), [\varphi]_0)$ is trivial, for some choice of modular A-bases F_0, G_0 and hamiltonian complements F^*, G^* for F, G respectively. Denoting the A-module $B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*)$ by P_0, let $P = (P_0)_z$, the f.g. projective A_z-module freely generated by P_0. Define an A_z-module morphism

$$f\colon P \to Q$$

by sending elements of the modular A-base P_0 to themselves in Q, and extending A_z-linearly. Then $f^*\varphi f \in \mathrm{Hom}_{A_z}(P,P^*)$ can be expressed as

$$f^*\varphi f = [\varphi]_+ + [\varphi]_0 + [\varphi]_- \in \mathrm{Hom}_{A_z}(P,P^*)$$

with

$$[\varphi]_+(P_0) \subseteq \sum_{j=1}^{\infty} z^j P_0^*, \quad [\varphi]_-(P_0) \subseteq \sum_{j=-\infty}^{-1} z^j P_0^*, \quad [\varphi]_0(P_0) \subseteq P_0^*.$$

Choose hamiltonian complements L_0, L_0^* in $(P_0, [\varphi]_0)$, and let $L = (L_0)_z$. Denote $H_\pm(L)$ by (P,ψ), so that

$$[\varphi]_0 - \psi = \chi \mp \chi^*\colon P \to P^*$$

for some $\mp$ form (P,χ) over A_z (of the type $\bar{\varepsilon}(P_0,\chi_0)$, for some $\mp$ form (P_0,χ_0) over A).

Consider now the self-equivalence

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -f\zeta & 0 \\ 0 & 1 & 0 \\ \zeta^* f^*(\varphi \pm \varphi^*) & -\zeta^*\theta\zeta & 1 \end{pmatrix}\colon$$
$$(Q,\varphi) \oplus H_\pm(P) \to (Q,\varphi) \oplus H_\pm(P),$$

where

$$\eta = \begin{pmatrix} 0 & \mp z \\ z^{-1} & 0 \end{pmatrix}\colon P^* = L^* \oplus L \to L \oplus L^* = P,$$

$$\zeta = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}\colon P = L \oplus L^* \to L \oplus L^* = P,$$

and

$$\theta = [\varphi]_+ \pm [\varphi]_-^* + \psi \in \mathrm{Hom}_{A_z}(P,P^*).$$

Defining the positive projection

$$\nu\colon Q \oplus (P \oplus P^*) \to ((G_0 \oplus G_0^*) \oplus (P_0 \oplus P_0^*))^+$$

and also the A-module projection

$$\beta: Q = P_0 \oplus (z^N(F_0 \oplus F_0^*)^+ \oplus (G_0 \oplus G_0^*)^-) \xrightarrow{(1\,0)} P_0,$$

note that

$$\nu h(x,y) = \begin{cases} h(x,y) & (x \in z^N F_0^+,\ y \in P_0^+), \\ h(0, \zeta^{-1}\beta(f\zeta(y)-x)) & (x \in z^N F_0^-,\ y \in P_0^-), \end{cases}$$

whence

$$\nu h(F \oplus P) \subseteq h(F \oplus P).$$

The product decomposition used to define h shows that the self-equivalence $h' = h \oplus 1$ of $(Q',\varphi') = ((Q,\varphi) \oplus H_\pm(P)) \oplus H_\pm(-P)$ is a hamiltonian transformation over A_z. The matrix involving the even $\mp$ product $\eta \in \mathrm{Hom}_{A_z}(P, P^*)$ is an elementary hamiltonian transformation, while the other is the hamiltonian transformation generated (in the sense of Lemma 2.3) by the morphism of $\pm$ forms over A_z

$$(f\zeta, \zeta^*([\varphi]_- + \chi)\zeta): (P, \zeta^*\theta\zeta) \to (Q, \varphi).$$

The lagrangians $F' = F \oplus P \oplus -P$, $G' = G \oplus P \oplus -P$ of (Q',φ') are such that

$$(Q,\varphi;\, F, G) = (Q',\varphi';\, F', G') = (Q',\varphi';\, h'(F'), G') \in V_{2i+1}(A_z),$$

using the V-theory sum formula of Lemma 3.3 of I. The last representative $\pm$ formation satisfies the hypothesis of Lemma 2.2 with the roles played by F and G reversed—this is clearly all right for non-singular $\pm$ formations. Thus

$$(Q,\varphi;\, F, G) \in \mathrm{im}(\bar{\varepsilon}: V_{2i+1}(A) \to V_{2i+1}(A_z)),$$

completing the proof of the part of Theorem 1.1 relating to $V_n(A_z)$ with n odd.

We now give the analogous constructions for W-theory.

Define

$$B: W_{2i+1}(A_z) \to V_{2i}(A);\quad (Q,\varphi;\, F, G) \mapsto (B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*), [\varphi]_0),$$

where F_0 is the modular A-base generated by the given A_z-base of F, and similarly for G_0, G. Then

$$[B_N^+(F_0 \oplus F_0^*, G_0 \oplus G_0^*)] = 0 \in \tilde{K}_0(A)$$

because it is the image under $B: \tilde{K}_1(A_z) \to \tilde{K}_0(A)$ of an automorphism of Q taking a hamiltonian base extending F to one extending G, which is simple by construction (cf. § 5 of I), so that $B: W_{2i+1}(A_z) \to V_{2i}(A)$ is well defined.

The composite

$$W_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} W_{2i+1}(A_z) \xrightarrow{B} V_{2i}(A)$$

is 0, as for V-theory.

The square

$$\begin{array}{ccc} \Omega_+(A_z) & \longrightarrow & W_{2i+1}(A_z) \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle B} \\ \Sigma_-(A) & \longrightarrow & V_{2i}(A) \end{array}$$

commutes, sending $\tau(\alpha: F_z \to F_z) \in \Omega_+(A_z)$ to $H_\pm(B_N^+(F, \alpha(F))) \in V_{2i}(A)$ both ways.

Define

$$\bar{B}: V_{2i}(A) \to W_{2i+1}(A_z); \quad (Q, \varphi) \mapsto ((Q \oplus Q)_z, \varphi \oplus -\varphi; \Delta_{(Q_z,\varphi)}, \zeta\Delta_{(Q_z,\varphi)}),$$

where $\zeta = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}: Q_z \oplus Q_z \to Q_z \oplus Q_z$, Q is free with any base, and $(Q \oplus Q, \varphi \oplus -\varphi)$ is any hamiltonian base extending $\Delta_{(Q,\varphi)}$. Then $\bar{B}(Q, \varphi)$ is just

$$\pi'((\zeta, 0): (Q_z \oplus Q_z, \varphi \oplus -\varphi) \to (Q_z \oplus Q_z, \varphi \oplus -\varphi)),$$

in the terminology of Theorem 5.6 of I, as

$$\tau(\zeta) = \bar{B}(-[Q]) = 0 \in \tilde{K}_1(A_z),$$

so that we are dealing with an element of the special unitary group $\mathscr{SU}_\pm(A_z)$.

The composites

$$V_{2i}(A) \xrightarrow{\bar{B}} W_{2i+1}(A_z) \xrightarrow{\varepsilon} W_{2i+1}(A),$$

$$V_{2i}(A) \xrightarrow{\bar{B}} W_{2i+1}(A_z) \xrightarrow{B} V_{2i}(A)$$

are 0, 1 as for V-theory.

The square

$$\begin{array}{ccc} \Sigma_-(A) & \longrightarrow & V_{2i}(A) \\ {\scriptstyle \bar{B}}\downarrow & & \downarrow{\scriptstyle \bar{B}} \\ \Omega_+(A_z) & \longrightarrow & W_{2i+1}(A_z) \end{array}$$

commutes, sending $[P] \in \Sigma_-(A)$ to

$$((Q \oplus Q)_z, \varphi \oplus -\varphi; (P \oplus P^*)_z, \zeta(P \oplus P^*)_z) \in W_{2i+1}(A_z)$$

both ways round, where $(Q, \varphi) = H_\pm(P)$ with any base for $P \oplus P^*$.

The (split) exactness of

$$0 \longrightarrow W_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} W_{2i+1}(A_z) \xrightarrow{B} V_{2i}(A) \longrightarrow 0$$

follows from a diagram chase round:

$$
\begin{array}{ccccccccc}
 & & \Omega_+(A) & \xrightarrow{\alpha} & W_{2i+1}(A) & \xrightarrow{\beta} & V_{2i+1}(A) & \xrightarrow{\gamma} & \Omega_-(A) \\
 & & {\scriptstyle\varepsilon}\uparrow\downarrow{\scriptstyle\bar{\varepsilon}} & & \downarrow{\scriptstyle\bar{\varepsilon}} & & \downarrow{\scriptstyle\bar{\varepsilon}} & & \downarrow{\scriptstyle\bar{\varepsilon}} \\
V_{2i+2}(A_z) & \xrightarrow{\delta} & \Omega_+(A_z) & \xrightarrow{\alpha} & W_{2i+1}(A_z) & \xrightarrow{\beta} & V_{2i+1}(A_z) & \xrightarrow{\gamma} & \Omega_-(A_z) \\
{\scriptstyle\bar{B}}\uparrow & & {\scriptstyle\bar{B}}\uparrow\downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B} & & \\
U_{2i+1}(A) & \xrightarrow{\lambda} & \Sigma_-(A) & \xrightarrow{\mu} & V_{2i}(A) & \xrightarrow{\nu} & U_{2i}(A) & &
\end{array}
$$

in which all the squares commute, and the rows are the exact sequences of Theorems 4.3 and 5.7 of I. The inside left and right columns are exact—we wish to verify that the centre column is exact as well:

let $x \in W_{2i+1}(A_z)$ be such that $B(x) = 0 \in V_{2i}(A)$; then

$$B\beta(x) = \nu B(x) = 0 \in U_{2i}(A) \quad \text{and} \quad \beta(x) \in \ker B = \operatorname{im}\bar{\varepsilon} \subseteq V_{2i+1}(A_z);$$

let $y \in V_{2i+1}(A)$ be such that $\beta(x) = \bar{\varepsilon}(y) \in V_{2i+1}(A_z)$; then

$$\bar{\varepsilon}\gamma(y) = \gamma\bar{\varepsilon}(y) = \gamma\beta(x) = 0 \in \Omega_-(A_z) \quad \text{and}$$
$$y \in \ker\gamma = \operatorname{im}\beta \subseteq V_{2i+1}(A);$$

let $s \in W_{2i+1}(A)$ be such that $\beta(s) = y \in V_{2i+1}(A)$; then

$$\beta(x-\bar{\varepsilon}(s)) = (y-\beta(s)) = 0 \in V_{2i+1}(A_z),$$

and

$$(x-\bar{\varepsilon}(s)) \in \ker\beta = \operatorname{im}\alpha \subseteq W_{2i+1}(A_z);$$

let $t \in \Omega_+(A_z)$ be such that $\alpha(t) = x-\bar{\varepsilon}(s) \in W_{2i+1}(A_z)$; now

$$t = \bar{B}B(t) + \bar{\varepsilon}\varepsilon(t),$$

so

$$(x-\bar{\varepsilon}(s+\alpha\varepsilon(t))) = \alpha\bar{B}B(t) \in W_{2i+1}(A_z);$$

also

$$\mu B(t) = B\alpha(t) = B(x) - B\bar{\varepsilon}(s) = 0 \in V_{2i}(A);$$

and

$$B(t) \in \ker\mu = \operatorname{im}\lambda \subseteq \Sigma_-(A);$$

let $u \in U_{2i+1}(A)$ be such that $\lambda(u) = B(t) \in \Sigma_-(A)$; then

$$\alpha\bar{B}B(t) = \alpha\bar{B}\lambda(u) = \alpha\delta\bar{B}(u) = 0 \in W_{2i+1}(A_z);$$

hence

$$x = \bar{\varepsilon}(s+\alpha\varepsilon(t)) \in \operatorname{im}(\bar{\varepsilon}\colon W_{2i+1}(A) \to W_{2i+1}(A_z)).$$

This completes the proof of Theorem 1.1 for n odd.

3. Proof of Theorem 1.1 (n even)

We define $B\colon V_{2i}(A_z) \to U_{2i+1}(A)$, using

LEMMA 3.1. *Given a non-singular $\pm$ form (Q,φ) over A_z, and a modular A-base Q_0 for Q, let*

$$\nu\colon Q\oplus Q^* \to (Q_0\oplus Q_0^*)^+$$

be the positive projection, and let $N \geqslant 0$ be so large that

$$(\varphi \pm \varphi^*)(Q_0) \subseteq \sum_{j=-N}^{N} z^j Q_0^*, \quad (\varphi \pm \varphi^*)^{-1}(Q_0^*) \subseteq \sum_{j=-N}^{N} z^j Q_0.$$

Then the A-submodule

$$B_N(Q_0, \varphi)$$
$$= \{(z^N(1-\nu)z^{-N}x, \nu(\varphi \pm \varphi^*)x) \in Q \oplus Q^* \mid x \in B_N^+((\varphi \pm \varphi^*)^{-1}Q_0^*, Q_0)\}$$

of $Q \oplus Q^*$ is a lagrangian of $H_\mp(\sum_{j=0}^{N-1} z^j Q_0)$ such that

$$\left(H_\mp\left(\sum_{j=0}^{N-1} z^j Q_0\right); \sum_{j=0}^{N-1} z^j Q_0, B_N(Q_0, \varphi)\right) \in U_{2i-1}(A)$$

does not depend on N and Q_0.

Proof. The hessian $\pm$ product on $B_N(Q_0, \varphi)$ in $H_\mp(\sum_{j=0}^{N-1} z^j Q_0)$ is given by

$$B_N(Q_0, \varphi) \to B_N(Q_0, \varphi)^*;$$
$$(z^N(1-\nu)z^{-N}x, \nu(\varphi \pm \varphi^*)x) \mapsto ((z^N(1-\nu)z^{-N}x', \nu(\varphi \pm \varphi^*)x') \mapsto \langle x, x' \rangle_{[\varphi]_0}),$$

which is clearly even, as required for a lagrangian.

A hamiltonian complement to $B_N(Q_0, \varphi)$ in $H_\mp(\sum_{j=0}^{N-1} z^j Q_0)$ is given by

$$B_N^*(Q_0, \varphi) = \{(-\nu y, \nu(\varphi \pm \varphi^*)(1-\nu)y) \in Q_0 \oplus Q_0^* \mid y \in B_N^+(Q_0, (\varphi \pm \varphi^*)^{-1}Q_0^*)\}.$$

Every $(s, t) \in (\sum_{j=0}^{N-1} z^j Q_0) \oplus (\sum_{j=0}^{N-1} z^j Q_0^*)$ can be expressed as

$$(s, t) = (z^N(1-\nu)z^{-N}x, \nu(\varphi \pm \varphi^*)x) + (-\nu y, \nu(\varphi \pm \varphi^*)(1-\nu)y)$$
$$\in B_N(Q_0, \varphi) \oplus B_N^*(Q_0, \varphi)$$

with

$$x = \nu(\varphi \pm \varphi^*)^{-1}((1-\nu)(\varphi \pm \varphi^*)s + t) \in B_N^+((\varphi \pm \varphi^*)^{-1}Q_0^*, Q_0),$$
$$y = (-(\varphi \pm \varphi^*)^{-1}\nu(\varphi \pm \varphi^*)s + z^N(1-\nu)z^{-N}(\varphi \pm \varphi^*)^{-1}t) \in B_N^+(Q_0, (\varphi \pm \varphi^*)^{-1}Q_0^*).$$

The associated $\mp$ product of $H_\mp(\sum_{j=0}^{N-1} z^j Q_0)$ restricts to an A-module isomorphism

$$B_N^*(Q_0, \varphi) \to B_N(Q_0, \varphi)^*;$$
$$(-\nu y, \nu(\varphi \pm \varphi^*)(1-\nu)y) \mapsto ((z^N(1-\nu)z^{-N}x, \nu(\varphi \pm \varphi^*)x) \mapsto \langle y, x \rangle_{[\varphi]_0})$$

so that we are dealing with hamiltonian complements.

Increasing N by 1, we have

$$B_{N+1}(Q_0, \varphi) = B_N(Q_0, \varphi) \oplus \{(z^{N+1}(1-\nu)z^{-(N+1)}x, (\varphi \pm \varphi^*)(x)) \mid$$
$$x \in (\varphi \pm \varphi^*)^{-1}(z^N Q_0^*)\}.$$

Now $B_N^*(Q_0,\varphi)\oplus z^N Q_0$ is a hamiltonian complement in $H_\pm(\sum_{j=0}^{N-1} z^j Q_0)$ to both $B_{N+1}(Q_0,\varphi)$ and $B_N(Q_0,\varphi)\oplus z^N Q_0^*$, so that

$$\left(H_\mp\left(\sum_{j=0}^{N-1} z^j Q_0\right);\ \sum_{j=0}^{N-1} z^j Q_0, B_N(Q_0,\varphi)\right)$$

$$= \left(H_\mp\left(\sum_{j=0}^{N} z^j Q_0\right);\ \sum_{j=0}^{N} z^j Q_0, B_N^*(Q_0,\varphi)\oplus z^N Q_0^*\right)$$

$$= \left(H_\mp\left(\sum_{j=0}^{N} z^j Q_0\right);\ \sum_{j=0}^{N} z^j Q_0, B_{N+1}(Q_0,\varphi)\right) \in U_{2i-1}(A).$$

Hence the choice of N is immaterial.

Let $\hat{Q}_0$ be another modular A-base of Q, with

$$\hat{\nu}: Q\oplus Q^* \to (\hat{Q}_0\oplus \hat{Q}_0^*)^+$$

the new positive projection. Let $M \geqslant 0$ be so large that

$$\hat{Q}_0 \subseteq \sum_{j=-M}^{M} z^j Q_0,\quad Q_0 \subseteq \sum_{j=-M}^{M} z^j \hat{Q}_0.$$

Then $\hat{N} = N+2M$ is large enough for $B_{\hat{N}}(\hat{Q}_0,\varphi)$ to be defined, and

$$B_{\hat{N}}^+((\varphi\pm\varphi^*)^{-1}\hat{Q}_0^*,\hat{Q}_0)$$
$$= (\varphi\pm\varphi^*)^{-1}(z^{M+N}B_M^+(\hat{Q}_0^*,Q_0^*))\oplus z^M B_N^+((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)\oplus B_M^+(Q_0,\hat{Q}_0)$$

so that

$$\begin{aligned}B_{\hat{N}}(\hat{Q}_0,\varphi) = \{(z^{\hat{N}}(1-\hat{\nu})z^{-\hat{N}}x, (\varphi\pm\varphi^*)x) \mid x\in(\varphi\pm\varphi^*)^{-1}(z^{M+N}B_M^+(\hat{Q}_0^*,Q_0^*))\}\\ \oplus\{(x,(\varphi\pm\varphi^*)x)\mid x\in z^M B_N^+((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)\}\\ \oplus\{(x,\hat{\nu}(\varphi\pm\varphi^*)x)\mid x\in B_M^+(Q_0,\hat{Q}_0)\}.\end{aligned}$$

Moreover,

$$\sum_{j=0}^{\hat{N}-1} z^j\hat{Q}_0 = z^{M+N}B_M^+(\hat{Q}_0,Q_0)\oplus z^M\left(\sum_{j=0}^{N-1} z^j Q_0\right)\oplus B_M^+(Q_0,\hat{Q}_0)$$

and

$$z^{M+N}B_M^+(\hat{Q}_0,Q_0)\oplus z^M B_N^*(Q_0,\varphi)\oplus B_M^+(Q_0^*,\hat{Q}_0^*)$$

is a hamiltonian complement in $H_\mp(\sum_{j=0}^{\hat{N}-1} z^j\hat{Q}_0)$ to both $B_{\hat{N}}(\hat{Q}_0,\varphi)$ and $z^{M+N}B_M^+(\hat{Q}_0^*,Q_0^*)\oplus z^M B_N(Q_0,\varphi)\oplus B_M^+(Q_0,\hat{Q}_0)$.

Thus

$$\left(H_\mp\left(\sum_{j=0}^{\hat{N}-1} z^j\hat{Q}_0\right);\ \sum_{j=0}^{\hat{N}-1} z^j\hat{Q}_0, B_{\hat{N}}(\hat{Q}_0,\varphi)\right)$$

$$= \left(H_\mp\left(\sum_{j=0}^{\hat{N}-1} z^j\hat{Q}_0\right);\ \sum_{j=0}^{\hat{N}-1} z^j\hat{Q}_0, z^{M+N}B_M^+(\hat{Q}_0^*,Q_0^*)\oplus z^M B_N(Q_0,\varphi)\oplus B_M(Q_0,\hat{Q}_0)\right)$$

$$= \left(H_\mp\left(\sum_{j=0}^{N-1} z^j Q_0\right);\ \sum_{j=0}^{N-1} z^j Q_0, B_N(Q_0,\varphi)\right) \in U_{2i-1}(A).$$

Hence there is independence of choice of Q_0.

Define

$$B\colon V_{2i}(A_z) \to U_{2i-1}(A);\quad (Q,\varphi) \mapsto \left(H_{\mp}\left(\sum_{j=0}^{N-1} z^j Q_0\right);\ \sum_{j=0}^{N-1} z^j Q_0, B_N(Q_0,\varphi)\right)$$

for any modular A-base Q_0 of Q (which may be assumed to be free). As shown in Lemma 3.1 this does not depend on the choices made of N and Q_0.

Given a f.g. free A_z-module F, with modular A-base F_0, we have

$$B(H_{\pm}(F)) = \left(H_{\mp}(0);\ 0, B_0\left(F_0 \oplus F_0^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\right) = 0 \in U_{2i-1}(A).$$

Hence $B(Q,\varphi) = 0 \in U_{2i-1}(A)$ whenever $(Q,\varphi) = 0 \in V_{2i}(A_z)$, and

$$B\colon V_{2i}(A_z) \to U_{2i-1}(A)$$

is well defined.

The composite

$$V_{2i}(A) \xrightarrow{\bar{\varepsilon}} V_{2i}(A_z) \xrightarrow{B} U_{2i-1}(A)$$

is 0, because it sends $(Q,\varphi) \in V_{2i}(A)$ to

$$B\bar{\varepsilon}(Q,\varphi) = (H_{\mp}(0);\ 0, B_0(Q,\varphi)) = 0 \in U_{2i-1}(A).$$

The square

$$\begin{array}{ccc} V_{2i}(A_z) & \longrightarrow & \Omega_+(A_z) \\ B\big\downarrow & & \big\downarrow B \\ U_{2i-1}(A) & \longrightarrow & \Sigma_-(A) \end{array}$$

commutes, for given $(Q_z,\varphi) \in V_{2i}(A_z)$, with Q a f.g. free A-module

$$[B_N(Q,\varphi)] = [B_N^+(Q^*, (\varphi \pm \varphi^*)Q)] = B_T(Q_z,\varphi) \in \Sigma_-(A).$$

We define $\bar{B}\colon U_{2i-1}(A) \to V_{2i}(A_z)$, using

LEMMA 3.2. *Let (Q,φ) be a trivial $\mp$ form over A, with lagrangian L, and a hamiltonian complement L^*, so that*

$$\varphi = \begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix} : L \oplus L^* \to L^* \oplus L,$$

where $\gamma \mp \delta^* = 1\colon L^* \to L^*$.

Then the equivalence class of the $\pm$ form over A_z,

$$\left(Q_z = L_z \oplus L_z^*, \theta = \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix} : L_z \oplus L_z^* \to L_z^* \oplus L_z\right),$$

does not depend on the choice of L^*.

If $(Q,\varphi) = H_{\mp}(P)$, then (Q_z,θ) is a non-singular $\pm$ form over A_z such that

$$((Q_2,\theta) \oplus H_{\pm}(-L_z)) \in \ker(\varepsilon\colon V_{2i}(A_z) \to V_{2i}(A))$$

with torsion

$$\bar{B}([L]-[P^*]) \in \Omega_+(A_z).$$

Moreover,

$$((Q_z, \theta) \oplus H_\pm(-L_z)) = 0 \in V_{2i}(A_z)$$

if L is a hamiltonian complement in (Q, φ) to either P or P^*.

Proof. Change of hamiltonian complement L^* corresponds to an automorphism

$$\alpha = \begin{pmatrix} 1 & \kappa \pm \kappa^* \\ 0 & 1 \end{pmatrix} : L \oplus L^* \to L \oplus L^*,$$

for some $\pm$ form (L^*, κ). The $\pm$ form over A_z, (Q_z, θ'), determined by this new choice of hamiltonian complement to L is given by

$$\theta' = \begin{pmatrix} \lambda & -z\gamma' \\ \delta' & (1-z)(\lambda_1' \pm \lambda_1'^*) \end{pmatrix} : L_z \oplus L_z^* \to L_z^* \oplus L_z,$$

where $\gamma', \delta', \lambda_1'$ are defined by

$$\alpha^* \varphi \alpha = \begin{pmatrix} \lambda \pm \lambda^* & \gamma' \\ \delta' & \lambda_1' \pm \lambda_1'^* \end{pmatrix} : L \oplus L^* \to L^* \oplus L.$$

Now

$$\left(\begin{pmatrix} 1 & (1-z)(\kappa \pm \kappa^*) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -(\lambda \pm z\lambda^*)(\kappa \pm \kappa^*) \\ 0 & (1-z)(\kappa^* \pm \kappa)\lambda(\kappa \pm \kappa^*) \end{pmatrix} \right) :$$

$$(L_z \oplus L_z^*, \theta) \to (L_z \oplus L_z^*, \theta')$$

is an equivalence of $\pm$ forms over A_z. Hence the choice of L^* is immaterial.

Define $\omega \in \mathrm{Hom}_{A_z}(Q_z, Q_z)$ by

$$\omega = \begin{pmatrix} 1 & 0 \\ 0 & 1-z \end{pmatrix} : L_z \oplus L_z^* \to L_z \oplus L_z^*$$

and $\tilde{\omega} \in \mathrm{Hom}_{A_z}(Q_z^*, Q_z^*)$ by

$$\tilde{\omega} = \begin{pmatrix} 1-z^{-1} & 0 \\ 0 & 1 \end{pmatrix} : L_z^* \oplus L_z \to L_z^* \oplus L_z.$$

Note that there is an identity

$$\tilde{\omega}(\theta \pm \theta^*) = (\varphi \mp z^{-1}\varphi^*)\omega : Q_z \to Q_z^*.$$

Similarly, defining

$$\tilde{\varphi} = (\varphi \mp \varphi^*)^{-1}\varphi(\varphi \mp \varphi^*)^{-1} = \begin{pmatrix} \mp(\lambda_1 \pm \lambda_1^*) & \delta \\ \gamma & \mp\lambda(\pm\lambda^*) \end{pmatrix} : L^* \oplus L \to L \oplus L^*$$

and

$$\tilde{\theta} = \begin{pmatrix} \mp(1-z^{-1})(\lambda_1 \pm \lambda_1^*) & \delta \\ -z^{-1}\gamma & \mp\lambda \end{pmatrix} : L_z^* \oplus L_z \to L_z \oplus L_z^*,$$

there is an identity

$$\omega(\tilde{\theta} \pm \tilde{\theta}^*) = (\tilde{\varphi} \mp z\tilde{\varphi}^*)\tilde{\omega} : Q_z^* \to Q_z.$$

If $(Q,\varphi) = H_{\mp}(P)$, then

$$\varphi \mp z^{-1}\varphi^* = \begin{pmatrix} 0 & 1 \\ \mp z^{-1} & 0 \end{pmatrix} : P_z \oplus P_z^* \to P_z^* \oplus P_z$$

and combining the two identities above, we obtain

$$\begin{aligned} \omega(\tilde{\theta} \pm \tilde{\theta}^*)(\theta \pm \theta^*) &= (\tilde{\varphi} \mp z\tilde{\varphi}^*)\tilde{\omega}(\theta \pm \theta^*) \\ &= (\tilde{\varphi} \mp z\tilde{\varphi}^*)(\varphi \mp z^{-1}\varphi^*)\omega = \omega : Q_z \to Q_z \end{aligned}$$

and similarly

$$\tilde{\omega}(\theta \pm \theta^*)(\tilde{\theta} \pm \tilde{\theta}^*) = \tilde{\omega} : Q_z^* \to Q_z^*.$$

Both $\omega \in \mathrm{Hom}_{A_z}(Q_z, Q_z)$ and $\tilde{\omega} \in \mathrm{Hom}_{A_z}(Q_z^*, Q_z^*)$ are monomorphisms, so

$$\tilde{\theta} \pm \tilde{\theta}^* = (\theta \pm \theta^*)^{-1} : Q_z^* \to Q_z$$

and (Q_z, θ) is a non-singular $\pm$ form over A_z.

The projection $\varepsilon : V_{2i}(A_z) \to V_{2i}(A)$ sends $((Q_z,\theta) \oplus H_{\pm}(-L_z))$ to

$$\left(\left(L \oplus L^*, \begin{pmatrix} \lambda & -\gamma \\ \delta & 0 \end{pmatrix}\right) \oplus H_{\pm}(-L)\right) \in V_{2i}(A)$$

which vanishes in $V_{2i}(A)$ because $L^* \oplus -L^*$ is a free lagrangian. Thus the component of

$$\tau(((Q_z,\theta) \oplus H_{\pm}(-L_z))) \in \Omega_+(A_z) = \bar{\varepsilon}\Omega_+(A) \oplus \bar{B}\Sigma_-(A)$$

in $\bar{\varepsilon}\Omega_+(A)$ is 0, and

$$\begin{aligned} \tau((Q_z,\theta) \oplus H_{\pm}(-L_z)) &= \bar{B}B\tau((Q_z,\theta) \oplus H_{\pm}(-L_z)) \\ &= \bar{B}[B_1^+(Q^* \oplus (-L^* \oplus -L), \\ &\qquad (\theta \pm \theta^*)Q \oplus (-L^* \oplus -L))] \in \Omega_+(A_z). \end{aligned}$$

Computing directly,

$$\begin{aligned} &B_1^+((\theta \pm \theta^*)^{-1}Q^*, Q) \\ &= \left\{(x,y) \in L^+ \oplus L^{*+} \,\middle|\, \begin{array}{l} (\lambda \pm \lambda^*)x + (-z\gamma \pm \delta^*)y \in zL^{*-} \\ (\delta \mp z^{-1}\gamma^*)x + (1-z)(1-z^{-1})(\lambda_1 \pm \lambda_1^*)y \in zL^- \end{array}\right\} \\ &= L \oplus \{((1-z)x, y) \in L_z \oplus L_z^* \mid x \in L,\ y \in L^*,\ \varphi(x,y) = 0 \in Q^*\}. \end{aligned}$$

Now $\ker(\varphi : Q \to Q^*) = P$ and $Q = L \oplus L^* = P \oplus P^*$ so

$$\begin{aligned} \tau((Q_z,\theta) \oplus H_{\pm}(-L_z)) &= \bar{B}([L] + [P] + [-P \oplus -P^*]) \\ &= \bar{B}([L] - [P^*]) \in \Omega_+(A_z). \end{aligned}$$

Finally, suppose that L is a hamiltonian complement to either P or P^*, choosing L^* accordingly. Then $\lambda_1 = 0$ and the annihilator of L_z^* in

$$(Q_z,\theta) = \left(L_z \oplus L_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & 0 \end{pmatrix}\right)$$

is given by

$$\begin{aligned} L_z^{*\perp} &= \{(x,y) \in L_z \oplus L_z^* \mid (\delta \mp z^{-1}\gamma^*)(x) = 0\} \\ &= L_z^* \oplus \ker((\gamma^* \mp z\delta) : L_z \to L_z). \end{aligned}$$

Let $x \in \ker((\gamma^* \mp z\delta): L_z \to L_z)$. As $\gamma \mp \delta^* = 1: L^* \to L^*$,

$$x = (z-1)(\pm \delta x) \in (z-1)L_z$$

and $(\pm \delta x) \in \ker(\gamma^* \mp z\delta): L_z \to L_z)$ as well. By induction on N, $x \in (z-1)^N L_z$ for every $N \geqslant 1$. This is impossible unless $x = 0$. Thus $L_z^{*\perp} = L_z^*$ and $L_z^* \oplus -L_z^*$ is a free lagrangian of

$$\left(L_z \oplus L_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & 0 \end{pmatrix}\right) \oplus H_{\pm}(-L_z),$$

making it vanish in $V_{2i}(A_z)$.

Define

$\bar{B}: U_{2i-1}(A) \to V_{2i}(A_z)$;

$$(Q, \varphi; F, G) \mapsto \left(\left(G_z \oplus G_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix}\right) \oplus H_{\pm}(-G_z)\right)$$

by choosing hamiltonian complements F^*, G^* to F, G in (Q, φ), and expressing

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}: F \oplus F^* \to F^* \oplus F$$

as

$$\begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix}: G \oplus G^* \to G^* \oplus G.$$

We have already shown, in Lemma 3.2 above, that this does not depend on the choice of G^*, and that $\bar{B}(Q, \varphi; F, G) = 0 \in V_{2i}(A_z)$ if

$$(Q, \varphi; F, G) = 0 \in U_{2i-1}(A).$$

Hence the choice of hessian $\pm$ form (G, λ) in (Q, φ) is also immaterial: for

$$\bar{B}(Q \oplus Q, \varphi \oplus -\varphi; F \oplus F^*, G \oplus G^*) = 0 \in V_{2i}(A_z),$$

so that

$$\bar{B}(Q, \varphi; F, G) = -\bar{B}(Q, -\varphi; F^*, G^*) \in V_{2i}(A_z),$$

and $-\bar{B}(Q, -\varphi; F^*, G^*)$ can be defined without a choice of (G, λ).

It remains to verify the invariance of the definition under changes of F^*. In order to do this, it is convenient to have available a more intrinsic characterization of the $\pm$ product over A_z

$$\theta \pm \theta^*: Q_z \to Q_z^*$$

associated with the $\pm$ form (Q_z, θ) defined in Lemma 3.2, as follows: given a lagrangian L of a $\mp$ form (Q, φ) over A, let

$$\psi: L_z \oplus Q_z \to L_z^* \oplus Q_z^*$$

be the unique A_z-linear extension of the even $\pm$ product

$$(1-z)\varphi \pm (1-z^{-1})\varphi^*: Q_z \to Q_z^*$$

such that

$$\psi(R) = 0,$$

where $R = \{((z-1)e, e) \in L_z \oplus Q_z \mid e \in L\}$.

Then ψ induces an even $\pm$ product over A_z

$$\psi: (L_z \oplus Q_z)/R_z \to ((L_z \oplus Q_z)/R_z)^*: [e,x] \mapsto ([f,y] \mapsto \psi(e,x)(f,y)),$$

writing $[e,x]$ for the residue class $\bmod R_z$ of $(e,x) \in L_z \oplus Q_z$. A choice of hamiltonian complement L^* to L in (Q,φ) determines an A_z-module isomorphism

$$\eta: L_z \oplus L_z^* \to (L_z \oplus Q_z)/R_z; \quad (e,v) \mapsto [e,v]$$

such that

$$\eta^*\psi\eta = \theta \pm \theta^*: Q_z \to Q_z^*.$$

Now let $(Q,\varphi) = H_{\mp}(P)$, and let $\hat{P}^*$ be any hamiltonian complement to P in $H_{\mp}(P)$, so that $\hat{P}^* = \Gamma_{(P^*,\mu)}$ for some $\pm$ form (P^*,μ) (by Lemma 1.3 of I). Then the isomorphism

$$L_z \oplus Q_z \to L_z \oplus Q_z; \ (e,x) \mapsto (e, x+\mu(\beta(-e+(z-1)x)))$$

induces, via η, an equivalence of $\pm$ forms over A_z

$$(Q_z, \theta) \to (Q_z, \hat{\theta}),$$

where β is the projection on P^* along P, and $(Q_z,\hat{\theta})$ is defined as (Q_z,θ), but with $\hat{P}^*$ in place of P^*.

Thus $\bar{B}(Q,\varphi: F, G) \in V_{2i}(A_z)$ does not depend on the representative $\mp$ formation of $(Q,\varphi; F, G) \in U_{2i-1}(A)$. In other words

$$\bar{B}: U_{2i-1}(A) \to V_{2i}(A_z)$$

is well defined.

It should be noted that we can give a more symmetric definition.

$\bar{B}: U_{2i-1}(A) \to V_{2i}(A_z)$;

$$(Q,\varphi; F, G) \mapsto \left(\left(G_z \oplus G_z^*, \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix}\right) \oplus H_{\pm}(-G_z)\right)$$

$$\oplus - \left(\left(F_z \oplus F_z^*, \begin{pmatrix} \mu & -z\alpha \\ \beta & (1-z)(\mu_1 \pm \mu_1^*) \end{pmatrix}\right) \oplus H_{\pm}(-F_z)\right),$$

where $(Q,\varphi) = H_{\mp}(P)$ and

$$\varphi = \begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix}: G \oplus G^* \to G^* \oplus G,$$

$$\varphi = \begin{pmatrix} \mu \pm \mu^* & \alpha \\ \beta & \mu_1 \pm \mu_1^* \end{pmatrix}: F \oplus F^* \to F^* \oplus F$$

for hamiltonian complements F^*, G^* to F, G in (Q, φ). The two definitions agree because

$$\begin{aligned}(H_{\mp}(P);\, F, G) &= (H_{\mp}(P);\, P, G) \oplus (H_{\mp}(P);\, F, P) \\ &= (H_{\mp}(P);\, P, G) \oplus -(H_{\mp}(P);\, P, F) \in U_{2i-1}(A)\end{aligned}$$

by the sum formula for U-theory of Lemma 3.3 of I.

It is immediate from Lemma 3.2 that the composite

$$U_{2i-1}(A) \xrightarrow{\bar{B}} V_{2i}(A_z) \xrightarrow{\varepsilon} V_{2i}(A)$$

is 0, and that the diagram

$$\begin{array}{ccc} U_{2i-1}(A) & \longrightarrow & \Sigma_{-}(A) \\ \bar{B}\big\downarrow & & \big\downarrow \bar{B} \\ V_{2i}(A_z) & \longrightarrow & \Omega_{+}(A_z) \end{array}$$

commutes.

Lemma 3.3. *The composite*

$$U_{2i-1}(A) \xrightarrow{\bar{B}} V_{2i}(A_z) \xrightarrow{B} U_{2i-1}(A)$$

is the identity.

Proof. Given $(Q, \varphi;\, F, G) \in U_{2i-1}(A)$ we may assume $(Q, \varphi) = H_{\mp}(F)$, so that

$$\bar{B}(Q, \varphi;\, F, G) = ((Q_z, \theta) \oplus H_{\pm}(-G_z)) \in V_{2i}(A_z),$$

where

$$\varphi = \begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix} : G \oplus G^* \to G^* \oplus G$$

and

$$\theta = \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix} : G_z \oplus G_z^* \to G_z^* \oplus G_z$$

for some hamiltonian complement G^* to G in (Q, φ). Thus

$$\begin{aligned} B\bar{B}&(Q, \varphi;\, F, G) \\ &= B((Q_z, \theta) \oplus H_{\pm}(-F_z)) \\ &= (H_{\mp}(Q);\, Q, B_1(Q, \theta)) \oplus (H_{\mp}(-F \oplus -F^*);\, -F \oplus -F^*, \Gamma_{H_{\pm}(-F)}) \\ &= (H_{\mp}(Q);\, Q, B_1(Q, \theta)) \in U_{2i-1}(A), \end{aligned}$$

where

$$B_1(Q, \theta) = \{(z(1-\nu)z^{-1}x, \nu(\theta \pm \theta^*)x) \in Q \oplus Q^* \mid x \in B_1^+((\theta \pm \theta^*)^{-1}Q^*, Q)\}$$

with $\nu\colon (Q\oplus Q^*)_z \to (Q\oplus Q^*)^+$ the positive projection. As in the proof of Lemma 3.2,

$$B_1^+((\theta\pm\theta^*)^{-1}Q^*,Q) = G\oplus\{(1-z)x+y\in Q_z \mid x\in G,\ y\in G^*,\ x+y\in F\},$$

so that

$$B_1(Q,\theta) = \{(x,\varphi x)\in Q\oplus Q^* \mid x\in G\}\oplus\{(y,\ \pm\varphi^*y)\in Q\oplus Q^* \mid y\in F\}.$$

The equivalence of $\mp$ forms over A,

$$\left(\begin{pmatrix} 1 & 1\\ \varphi & \pm\varphi^*\end{pmatrix},\ \begin{pmatrix} 0 & 0\\ \varphi & \varphi\end{pmatrix}\right)\colon (Q\oplus Q,\varphi\oplus-\varphi)\to H_{\mp}(Q),$$

sends $F\oplus F^*$ onto Q, and $G\oplus F$ onto $B_1(Q,\theta)$. So

$$\begin{aligned} B\bar{B}(Q,\varphi;\ F,G) &= (H_{\mp}(Q);\ Q,B_1(Q,\theta))\\ &= (Q\oplus Q,\varphi\oplus-\varphi;\ F\oplus F^*,G\oplus F)\\ &= (Q,\varphi;\ F,G)\in U_{2i-1}(A).\end{aligned}$$

We need just one more result to prove that the sequence

$$0 \longrightarrow V_{2i}(A) \xrightarrow{\ \bar{\varepsilon}\ } V_{2i}(A_z) \xrightarrow{\ B\ } U_{2i-1}(A) \longrightarrow 0$$

is split short exact.

Let z_1, z_2 be independent commuting indeterminates over A. The *double Laurent extension* of A by (z_1,z_2), A_{z_1,z_2}, is the ring of polynomials in z_1, z_1^{-1}, z_2, z_2^{-1} with involution by $z_1\mapsto z_1^{-1}$, $z_2\mapsto z_2^{-1}$. It is clear that A_{z_1,z_2} may be regarded as either $(A_{z_1})_{z_2}$ or $(A_{z_2})_{z_1}$ and satisfies all the conditions imposed above on the ground ring A.

LEMMA 3.4. *The diagram*

$$\begin{array}{ccc} V_{2i}(A_{z_1}) & \xrightarrow{B(z_1)} & U_{2i-1}(A)\\ \bar{B}(z_2)\big\downarrow & & \big\downarrow\bar{B}(z_2)\\ W_{2i+1}(A_{z_1,z_2}) & \xrightarrow{B(z_1)} & V_{2i}(A_{z_2}) \end{array}$$

skew-commutes.

Proof. Given $(Q,\varphi)\in V_{2i}(A_{z_1})$, we may assume that Q is free, as usual. Choose a modular A-base Q_0 of Q, so that

$$\Delta_0 = \{(x,x)\in Q\oplus Q \mid x\in Q_0\}$$

is a modular A-base of $\Delta_{(Q,\varphi)}$.

Let (Q^*,ψ) be a $\pm$ form over A_{z_1} such that there is an equivalence

$$((\varphi\pm\varphi^*),\chi)\colon (Q,\varphi)\to(Q^*,\ \pm\psi)$$

(cf. Lemma 1.4 of I). Then

$$\psi \pm \psi^* = (\varphi \pm \varphi^*)^{-1}: Q^* \to Q$$

and

$$\Delta_0^* = \{(\psi t, \mp \psi^* t) \in Q \oplus Q^* \mid t \in Q_0^*\}$$

is the modular A-base dual to Δ_0 of the hamiltonian complement $\Delta^*_{(Q^*,\psi)}$ to $\Delta_{(Q,\varphi)}$ in $(Q \oplus Q, \varphi \oplus -\varphi)$.

Let $N \geqslant 0$ be an integer so large that

$$(\varphi \pm \varphi^*)(Q) \subseteq \sum_{j=-N}^{N} z_1{}^j Q_0^*, \quad (\psi \pm \psi^*)(Q_0^*) \subseteq \sum_{j=-N}^{N} z_1{}^j Q_0.$$

Adding on some even $\mp$ product to ψ, if necessary, it may be assumed that

$$\psi(Q_0^*) \subseteq \sum_{j=0}^{N} z_1{}^j Q_0.$$

This ensures that

$$z_1^N(\Delta_0)_{z_2}^{+1} \subseteq \zeta_2(\Delta_0 \oplus \Delta_0^*)_{z_2}^{+1},$$

where

$$\zeta_2 = \begin{pmatrix} 1 & 0 \\ 0 & z_2 \end{pmatrix}: (Q_0)_{z_2} \oplus (Q_0)_{z_2} \to (Q_0)_{z_2} \oplus (Q_0)_{z_2},$$

because every $(s, s) \in z_1{}^N(\Delta_0)_{z_2}^{+1}$ can be expressed as

$$(s, s) = (x, z_2 x) + (\psi y, \mp z_2 \psi y^*) \in \zeta_2(\Delta_0)_{z_2}^{+1} \oplus \zeta_2(\Delta_0^*)_{z_2}^{+1}$$

with

$$y = (1 - z_2{}^{-1})(\varphi \pm \varphi^*)(s) \in (Q_0^*)_{z_2}^{+1}, \quad x = (s - \psi y) \in (Q_0)_{z_2}^{+1}.$$

For any A_{z_1}-base of Q

$$\bar{B}(z_2)(Q, \varphi) = ((Q \oplus Q)_{z_2}, \varphi \oplus -\varphi; \Delta_{(Q_{z_2},\varphi)}, \zeta_2 \Delta_{(Q_{z_2},\varphi)}) \in W_{2i+1}(A_{z_1,z_2}).$$

Thus

$$B(z_1)\bar{B}(z_2)(Q, \varphi) = (E_N^{+1}((\Delta_0)_{z_2}, \zeta_2(\Delta_0)_{z_2} \oplus \zeta_2(\Delta_0)_{z_2})/z_1{}^N(\Delta_0)_{z_2}^{+1}, [\varphi \oplus -\varphi]_{z_1=0}) \in V_{2i}(A_{z_2}),$$

where

$$\begin{aligned} &E_N^{+1}((\Delta_0)_{z_2}, \zeta_2(\Delta_0)_{z_2} \oplus \zeta_2(\Delta_0)_{z_2}^*) \\ &= \{w \in \zeta_2(\Delta_0)_{z_2}^{+1} \oplus \zeta_2(\Delta_0^*)_{z_2}^{+1} \mid \langle z_1{}^N(\Delta_0)_{z_2}^{+1}, w\rangle_{[\varphi\oplus-\varphi]_{z_1=0}} = \{0\} \subseteq A_{z_2}\} \\ &= \{(a, (\psi \pm \psi^*)(\nu + z_2(1-\nu))(\varphi \pm \varphi^*)a) \in (Q_0)_{z_2} \oplus (Q_0)_{z_2} \mid a \in (Q_0)_{z_2}^{+1}\} \\ &\qquad \oplus \left\{(0, (\psi \pm \psi^*)b) \in (Q_0)_{z_2} \oplus (Q_0)_{z_2} \mid b \in \sum_{j=0}^{N-1} z_1{}^j (Q_0)_{z_2}^{+1}\right\} \end{aligned}$$

(using the alternative definition of $B: W_{2i+1}(A_z) \to V_{2i}(A)$ given for V-theory in § 2) with

$$\nu: Q \oplus Q^* \to (Q_0 \oplus Q_0^*)^{+1}$$

the positive projection.

Next, let $P = \sum_{j=0}^{N-1} z_1{}^j Q_0$ (an A-module), and define an A_{z_2}-module isomorphism

$$f\colon P_{z_2} \oplus P^*_{z_2} \to E_N^{+1}((\Delta_0)_{z_2}, \zeta_2(\Delta_0 \oplus \Delta_0^*)_{z_2})/z_1{}^N(\Delta_0)_{z_2}^{+1};$$

$$(a,b) \mapsto (a, (\psi \pm \psi^*)((\nu + z_2(1-\nu))(\varphi \pm \varphi^*)a + b)),$$

so that

$$B(z_1)\bar{B}(z_2)(Q,\varphi) = (P_{z_2} \oplus P^*_{z_2}, f^*[\varphi \oplus -\varphi]_{z_1=0} f) \in V_{2i}(A_{z_2}).$$

Define a $\pm$ form over A_{z_2}, $(P_{z_2} \oplus P^*_{z_2}, \theta)$, by

$$\begin{aligned}\theta(a,b)(a',b') &= [((\varphi \pm \varphi^*)(a)(a') - (\nu(\varphi \pm \varphi^*)a)((\psi \pm \psi^*)\nu(\varphi \pm \varphi^*)a'))(1 - z_2{}^{-1}) \\ &\quad - b((\psi \pm \psi^*)\nu(\varphi \pm \varphi^*)a') - ((1-\nu)(\varphi \pm \varphi^*)a)((\psi \pm \psi^*)b')z_2{}^{-1} \\ &\quad - b(\psi b')]_{z_1=0} \in A_{z_2}\ (a, a' \in P_{z_2},\ b, b' \in P^*_{z_2}).\end{aligned}$$

It is not difficult to verify that θ differs from $f^*[\varphi \oplus -\varphi]_{z_1=0} f$ by an even $\mp$ product (over A_{z_2}), and also that

$$\theta = \begin{pmatrix} (1-z_2)(\lambda_1 \pm \lambda_1^*) & -\delta \\ z_2\gamma & \lambda \end{pmatrix} : P_{z_2} \oplus P^*_{z_2} \to P^*_{z_2} \oplus P_{z_2},$$

where

$$\begin{pmatrix} \lambda_1 \pm \lambda_1^* & \delta \\ \gamma & \lambda \pm \lambda^* \end{pmatrix} : P \oplus P^* \to P^* \oplus P$$

is an expression for

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : B_N(Q_0,\varphi) \oplus B_N^*(Q_0,\varphi) \to B_N(Q_0,\varphi)^* \oplus B_N^*(Q_0,\varphi)^*$$

with $B_N(Q_0,\varphi)$, $B_N^*(Q_0,\varphi)$ the hamiltonian complements in $H_\mp(P)$ of Lemma 3.1.

Defining the A-module isomorphism

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : P \oplus P^* \to P \oplus P^*,$$

note that

$$\eta^*\theta\eta = \begin{pmatrix} (1-z_2)(\lambda_1 \pm \lambda_1^*) & \delta \\ -z_2\gamma & \lambda \end{pmatrix} : P_{z_2} \oplus P^*_{z_2} \to P^*_{z_2} \oplus P_{z_2}.$$

Finally,

$$\begin{aligned}B(z_1)\bar{B}(z_2)(Q,\varphi) &= \bar{B}(z_2)(H_\mp(P); B_N^*(Q_0,\varphi), P^*) \\ &= \bar{B}(z_2)(-(H_\mp(P); P, B_N(Q_0,\varphi))) \\ &= -\bar{B}(z_2)B(z_1)(Q,\varphi) \in V_{2i}(A_{z_2}),\end{aligned}$$

using the U-theory sum formula of Lemma 3.3 of I.

Applying $B(z_2)$ to the decomposition

$$W_{2i+1}(A_{z_1,z_2}) = \bar{\varepsilon}(z_1)W_{2i+1}(A_{z_2}) \oplus \bar{B}(z_1)V_{2i}(A_{z_2})$$

obtained in § 2, it is now immediate that

$$V_{2i}(A_{z_1}) = \bar{\varepsilon}(z_1)V_{2i}(A) \oplus \bar{B}(z_1)U_{2i-1}(A).$$

This proves the part of Theorem 1.1 relating to $V_n(A_z)$ for n even.

To complete the proof, we give analogous constructions for W-theory. Define

$$B\colon W_{2i}(A_z) \to V_{2i-1}(A);$$

$$(Q,\varphi) \mapsto \left(H_{\mp}\left(\sum_{j=0}^{N-1} z^j Q_0\right);\ \sum_{j=0}^{N-1} z^j Q_0, B_N(Q_0,\varphi)\right)$$

with Q_0 the modular A-base of Q generated by the given A_z-base, and $B_N(Q_0,\varphi)$ as in Lemma 3.1. Then

$$[B_N(Q_0,\varphi)] = B\tau(Q,\varphi) = 0 \in \tilde{K}_0(A),$$

as required for V-theory, since

$$\tau(Q,\varphi) = 0 \in \tilde{K}_1(A_z)$$

by construction of $W_{2i}(A_z)$ (cf. § 5 of I).

The composite

$$W_{2i}(A) \xrightarrow{\bar{\varepsilon}} W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A)$$

is 0, as for V-theory.

The square

$$\begin{array}{ccc} \Omega_-(A_z) & \longrightarrow & W_{2i}(A_z) \\ B\big\downarrow & & \big\downarrow B \\ \Sigma_+(A) & \longrightarrow & V_{2i-1}(A) \end{array}$$

commutes: for

$$\Omega_-(A_z) = \bar{\varepsilon}\Omega_-(A) \oplus \bar{B}\Sigma_+(A)$$

and elements of $\bar{\varepsilon}\Omega_-(A)$ are sent to 0 both ways round the square, while the composition

$$\Sigma_+(A) \xrightarrow{\bar{B}} \Omega_-(A_z) \longrightarrow W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A)$$

sends $[P] \in \Sigma_+(A)$ to

$$B\left((P\oplus -P)_z \oplus (P\oplus -P)_z^*, \left(\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{matrix} 0 \\ \\ \end{matrix} \begin{matrix} 0 \\ 0 \end{matrix}\right)\right)$$

$$\begin{aligned} &= (H_{\mp}(P^*\oplus P\oplus -P\oplus -P^*);\ P^*\oplus P\oplus -P\oplus -P^*, \\ &\qquad\qquad \Gamma_{H_\pm(P^*)}\oplus(-P)^*\oplus -P^*) \\ &= (H_{\mp}(P^*\oplus P\oplus -P\oplus -P^*);\ P^*\oplus P\oplus -P\oplus -P^*, \\ &\qquad\qquad (P^*)^*\oplus (P)^*\oplus -P\oplus(-P^*)^*) \\ &= (H_{\mp}(P\oplus -P);\ P\oplus -P, P^*\oplus -P) \\ &= (H_{\mp}(P\oplus -P);\ P\oplus -P, P\oplus -P^*) \in V_{2i-1}(A), \end{aligned}$$

agreeing with the map $\Sigma_+(A) \to V_{2i-1}(A)$ defined in Theorem 4.3 of I.

Next, define

$$\bar{B}\colon V_{2i-1}(A) \to W_{2i}(A_z);$$
$$(Q,\varphi\colon F, G) \mapsto \left((Q_z,\theta)\oplus\left(R\oplus R^*, \begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix}\right)\right),$$

where $(Q,\varphi) = H_{\mp}(F)$ for any base of F (assumed free) and (Q_z,θ) is the $\pm$ form over A_z defined in Lemma 3.2 (with F, G replacing P, L respectively), so that

$$\tau(Q_z,\theta) = \bar{B}([G]-[F^*]) = 0 \in \Omega_+(A_z),$$

and

$$\psi\colon R \to R$$

is an automorphism of a based A_z-module R such that

$$\tau(Q_z,\theta)+\tau(\psi)+\tau(\psi^*) = 0 \in \tilde{K}_1(A_z), \quad \tau(\psi) \in \mathrm{Nil}^+(A).$$

The composites

$$V_{2i-1}(A) \xrightarrow{\bar{B}} W_{2i}(A_z) \xrightarrow{\varepsilon} W_{2i}(A),$$

$$V_{2i-1}(A) \xrightarrow{\bar{B}} W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A)$$

are 0, 1 as for V-theory.

The exactness of

$$0 \longrightarrow W_{2i}(A) \xrightarrow{\bar{\varepsilon}} W_{2i}(A_z) \xrightarrow{B} V_{2i-1}(A) \longrightarrow 0$$

follows from that of

$$0 \longrightarrow V_{2i}(A) \xrightarrow{\bar{\varepsilon}} V_{2i}(A_z) \xrightarrow{B} U_{2i-1}(A) \longrightarrow 0$$

by diagram chasing as at the end of § 2. The square

$$\begin{array}{ccc} \Sigma_+(A) & \longrightarrow & V_{2i-1}(A) \\ \bar{B}\downarrow & & \downarrow\bar{B} \\ \Omega_-(A_z) & \longrightarrow & W_{2i}(A_z) \end{array}$$

commutes because its commutator lies in

$$\ker(\varepsilon\colon W_{2i}(A_z) \to W_{2i}(A)) \cap \ker(B\colon W_{2i}(A_z) \to V_{2i-1}(A)) = \{0\}.$$

This completes the proof of Theorem 1.1.

4. Multiple Laurent extensions

Let $T(p)$ be the free abelian group of rank p, for $p \geqslant 0$, written multiplicatively. The group ring $A[T(p)]$, with involution

$$^{-}\colon A[T(p)] \to A[T(p)]; \quad \sum_{g\in T(p)} a_g g \mapsto \sum_{g\in T(p)} \bar{a}_g g^{-1} \quad (a_g \in A)$$

is the *p-fold Laurent extension* of A. We may identify

$$A[T(0)] = A, \quad A[T(1)] = A_z, \quad A[T(2)] = A_{z_1,z_2}$$

and also

$$(A[T(p)])[T(q)] = A[T(p+q)] \quad (p,q \geqslant 0),$$

so that each $A[T(p)]$ satisfies the conditions imposed on the ground ring A. Denoting some set of generators of $T(p)$ by $z_1, z_2, \ldots, z_p$ (for $p \geqslant 1$), we can also write

$$A[T(p)] = A_{z_1,z_2,\ldots,z_p},$$

extending the previous notation.

In order to give a full description of the L-theory of $A_{z_1,z_2,\ldots,z_p}$ we recall first the 'lower K-theory' of Chapter XII of **[1]**, involving K-groups $\tilde{K}_m(A)$ for $m < 0$, and subgroups $N_m^+(A)$, $N_m^-(A)$ of $\tilde{K}_{m+1}(A_z)$. There are defined morphisms

$$\tilde{K}_{m+1}(A_z) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} \tilde{K}_m(A) \quad (m < 0)$$

such that

$$B\bar{B} = 1\colon \tilde{K}_m(A) \to \tilde{K}_m(A),$$

giving natural direct sum decompositions

$$\tilde{K}_{m+1}(A_z) = \tilde{K}_{m+1}(A) \oplus \tilde{K}_m(A) \oplus N_m^+(A) \oplus N_m^-(A) \quad (m < 0).$$

Duality involutions

$$*\colon \tilde{K}_m(A) \to \tilde{K}_m(A)$$

are defined for all $m < 0$, with

$$\begin{array}{ccc} \tilde{K}_m(A_z) & \xrightarrow{*} & \tilde{K}_m(A_z) \\ \varepsilon \downarrow \uparrow \bar{\varepsilon} & & \varepsilon \downarrow \uparrow \bar{\varepsilon} \\ \tilde{K}_m(A) & \xrightarrow{*} & \tilde{K}_m(A) \end{array}$$

commuting, and

$$\begin{array}{ccc} \tilde{K}_{m+1}(A_z) & \xrightarrow{*} & \tilde{K}_{m+1}(A_z) \\ B \downarrow \uparrow \bar{B} & & B \downarrow \uparrow \bar{B} \\ \tilde{K}_m(A) & \xrightarrow{*} & \tilde{K}_m(A) \end{array}$$

skew-commuting. Moreover, the duality involution on $\tilde{K}_{m+1}(A_z)$ sends $N^{\pm}_{m}(A)$ onto $N^{\mp}_{m}(A)$ for all $m < 0$. In short, $\tilde{K}_{m+1}(A_z)$ is related to $\tilde{K}_m(A)$ in exactly the same way for $m < 0$ as for $m = 0$.

Regarding $\tilde{K}_m(A)$ as a $\mathbf{Z}_2$-module via $*$, there are defined Tate cohomology groups

$$\begin{aligned} H^{(m)}_n(A) &\equiv H^n(\mathbf{Z}_2; \tilde{K}_m(A)) \\ &= \{x \in \tilde{K}_m(A) \mid *x = (-)^n x\}/\{y + (-)^n * y \mid y \in \tilde{K}_m(A)\} \end{aligned}$$

depending only on $n \pmod 2$, which are abelian of exponent 2. This generalizes to $m < 0$ the definitions of

$$\Omega_{(-)^n}(A) = H^{(1)}_n(A), \quad \Sigma_{(-)^n}(A) = H^{(0)}_n(A).$$

The induced maps

$$H^{(m)}_n(A) \underset{\varepsilon}{\overset{\bar{\varepsilon}}{\rightleftarrows}} H^{(m)}_n(A_z) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} H^{(m-1)}_{n-1}(A)$$

give natural splittings

$$H^{(m)}_n(A_z) = H^{(m)}_n(A) \oplus H^{(m-1)}_{n-1}(A) \quad (m \leqslant 0,\ n \pmod 2)$$

as for $m = 1$.

We now define the 'lower L-groups'

$$L^{(m)}_n(A) = \ker(\varepsilon\colon L^{(m+1)}_{n+1}(A_z) \to L^{(m+1)}_{n+1}(A))$$

for $m \leqslant 1$, $n \pmod 4$ with $L^{(2)}_*(A) = W_*(A)$. It is clear from Theorem 1.1 that $L^{(1)}_*(A) = V_*(A)$, $L^{(0)}_*(A) = U_*(A)$ and that there is a natural exact sequence

$$\ldots \to H^{(m)}_{n+1}(A) \to L^{(m+1)}_n(A) \to L^{(m)}_n(A) \to H^{(m)}_n(A) \to \ldots$$

of abelian groups and morphisms for $m = 0, 1$. Hence all the L-theories differ in 2-torsion only. More precisely:

THEOREM 4.1. *There is a natural exact sequence of abelian groups*

$$\dots \to H_{n+1}^{(m)}(A) \to L_n^{(m+1)}(A) \to L_n^{(m)}(A) \to H_n^{(m)}(A) \to \dots$$

defined for all $m \leqslant 1$, $n \pmod 4$.

Proof. Use induction on m, downwards.

THEOREM 4.2. *There is defined an isomorphism of graded abelian groups*

$$L_*^{(*)}(A[T(p)]) \cong L_*^{(*)}(A) \otimes_{\mathbf{Z}} \Lambda_*(p),$$

where $\Lambda_*(p)$ *is the graded exterior* $\mathbf{Z}$*-algebra on* p *generators* $z_1, z_2, \dots, z_p$ *of degree* 1. *The isomorphism has components*

$$L_n^{(m)}(A[T(p)]) \cong \sum_{r=0}^{p} \sum_{1 \leqslant i_1 < \dots < i_r \leqslant p} L_{n-r}^{(m-r)}(A) \otimes (z_{i_1} \wedge \dots \wedge z_{i_r}) \quad (m \leqslant 2,\ n (\mathrm{mod}\ 4))$$

(*interpreting* $z_{i_1} \wedge \dots \wedge z_{i_r}$ *as* 1 *if* $r = 0$) *and is natural in both* A *and* $T(p)$.

Proof. It is sufficient to consider the case $W_*(A_{z_1, z_2})$, the others following by induction on p.

We need first the odd-dimensional counterpart to the result of Lemma 3.4, that the diagram

$$\begin{array}{ccc} V_{2i+1}(A_{z_1}) & \xrightarrow{B(z_1)} & U_{2i}(A) \\ {\scriptstyle \bar{B}(z_2)}\downarrow & & \downarrow{\scriptstyle \bar{B}(z_2)} \\ W_{2i+2}(A_{z_1, z_2}) & \xrightarrow{B(z_1)} & V_{2i+1}(A_{z_2}) \end{array}$$

skew-commutes. The proof of this is left to the reader. [It is known that

$$V_{2i+1}(A_{z_1}) = \bar{\varepsilon}(z_1) V_{2i+1}(A) \oplus \bar{B}(z_1) U_{2i}(A).$$

The elements of $\bar{\varepsilon}(z_1) V_{2i+1}(A)$ are sent to 0 in $V_{2i+1}(A_{z_2})$ both ways round the square, so it is sufficient to verify that the composite

$$U_{2i}(A) \xrightarrow{\bar{B}(z_1)} V_{2i+1}(A_{z_1}) \xrightarrow{\bar{B}(z_2)} W_{2i+2}(A_{z_1, z_2}) \xrightarrow{B(z_1)} V_{2i+1}(A_{z_2})$$

coincides with

$$-\bar{B}(z_2) \colon U_{2i}(A) \to V_{2i+1}(A_{z_2}).]$$

Thus

$$B(z_1) \bar{B}(z_2) = -\bar{B}(z_2) B(z_1) \colon V_n(A_{z_1}) \to V_n(A_{z_2})$$

for all $n \pmod 4$, and as

$$\bar{B}B + \bar{\varepsilon}\varepsilon = 1 \colon W_n(A_z) \to W_n(A_z)$$

it follows that

$$\begin{aligned} \bar{B}(z_1) \bar{B}(z_2) &= (\bar{B}(z_2) B(z_2) + \bar{\varepsilon}(z_2) \varepsilon(z_2)) \bar{B}(z_1) \bar{B}(z_2) \\ &= \bar{B}(z_2)(-\bar{B}(z_1) B(z_2)) \bar{B}(z_2) + (\bar{\varepsilon}(z_2) \bar{B}(z_1))(\varepsilon(z_2) \bar{B}(z_2)) \\ &= -\bar{B}(z_2) \bar{B}(z_1) \colon U_{n-2}(A) \to W_n(A_{z_1, z_2}), \end{aligned}$$

and similarly that

$$B(z_1)B(z_2) = -B(z_2)B(z_1)\colon W_n(A_{z_1,z_2}) \to U_{n-2}(A).$$

Accordingly, we have an isomorphism of abelian groups

$$L_n^{(2)}(A_{z_1,z_2}) \cong \sum_{j=0}^{2} L_{n-j}^{(2-j)}(A) \otimes_{\mathbf{Z}} \Lambda_j(2) \quad (n \ (\mathrm{mod}\ 4))$$

sending

$$\begin{aligned} &\bar{\varepsilon}(z_1)\bar{\varepsilon}(z_2)L_n^{(2)}(A) \text{ to } L_n^{(2)}(A) \otimes 1,\\ &\bar{B}(z_1)\bar{\varepsilon}(z_2)L_{n-1}^{(1)}(A) \text{ to } L_{n-1}^{(1)}(A) \otimes z_1,\\ &\bar{\varepsilon}(z_1)\bar{B}(z_2)L_{n-1}^{(1)}(A) \text{ to } L_{n-1}^{(1)}(A) \otimes z_2,\\ &\bar{B}(z_1)\bar{B}(z_2)L_{n-2}^{(0)}(A) \text{ to } L_{n-2}^{(0)}(A) \otimes (z_1 \wedge z_2).\end{aligned}$$

Naturality with respect to $T(2)$, and more generally $T(p)$, follows on noting that a morphism

$$f\colon T(p) \to T(q)$$

is determined by the $p \times q$ integer matrix $(f_{jk})_{1 \leqslant j \leqslant p,\, 1 \leqslant k \leqslant q}$ such that

$$f(z_j) = \prod_{k=1}^{q} z_k^{f_{jk}} \quad (1 \leqslant j \leqslant p,\ f_{jk} \in \mathbf{Z}),$$

the composition of such morphisms corresponding to multiplication of the matrices. Every such matrix can be expressed as the product of elementary matrices, such as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ N\ (\in \mathbf{Z}), \ldots$$

or their enlargements

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ldots.$$

It is easy to show directly that, for $p, q \leqslant 2$, the elementary $p \times q$ matrices induce the corresponding morphisms

$$1 \otimes f\colon L_*^{(*)}(A) \otimes \Lambda_*(p) \to L_*^{(*)}(A) \otimes \Lambda_*(q)$$

in the exterior algebra, where

$$f\colon \Lambda_*(p) \to \Lambda_*(q);\ z_{j_1} \wedge z_{j_2} \wedge \ldots \wedge z_{j_r} \mapsto \bigwedge_{m=1}^{r} \left(\sum_{k=1}^{q} f_{j_m k} z_k \right) \quad (1 \leqslant r \leqslant p).$$

Naturality with respect to A is obvious.

Corrigendum (added in proof 24 March 1973). I am grateful to M. K. Siu for pointing out the following error in the statement (on p. 129) of the theorem on the K-theory of Laurent extensions. The original theorem ([**1**], Chapter XII, 7.4) states that

$$K_1(A_z) = K_1(A) \oplus K_0(A) \oplus \mathrm{Nil}^+(A) \oplus \mathrm{Nil}^-(A).$$

In passing to the reduced groups $\tilde{K}_i(A) = \mathrm{coker}(K_i(\mathbf{Z}) \to K_i(A))$ $(i = 0, 1)$, there is obtained a direct sum decomposition

$$\bar{K}_1(A_z) = \tilde{K}_1(A) \oplus \tilde{K}_0(A) \oplus \mathrm{Nil}^+(A) \oplus \mathrm{Nil}^-(A)$$

of

$$\bar{K}_1(A_z) = \mathrm{coker}(K_1(\mathbf{Z}_z) \to K_1(A_z)),$$

where

$$\mathbf{Z}_z \to A_z;\ \sum_{j=-\infty}^{\infty} n_j z^j \mapsto \sum_{j=-\infty}^{\infty} (n_j.1) z^j,$$

and not of

$$\tilde{K}_1(A_z) = \mathrm{coker}(K_1(\mathbf{Z}) \to K_1(A_z))$$

as stated. The corresponding decomposition of the Tate cohomology groups is

$$\bar{\Omega}_{\pm}(A_z) = \Omega_{\pm}(A) \oplus \Sigma_{\mp}(A),$$

where

$$\bar{\Omega}_{(-)^n}(A_z) = H^n(\mathbf{Z}_2; \bar{K}_1(A_z))\ (n \,(\mathrm{mod}\, 2)).$$

For n (mod 4) let $\bar{W}_n(A_z)$ be the abelian groups defined as $W_n(A_z)$ except that torsions are to vanish in $\bar{K}_1(A_z)$ rather than $\tilde{K}_1(A_z)$. Theorem 3.3 of III, [**5**], shows that there is defined an exact sequence

$$\ldots \to \bar{\Omega}_{(-)^{n+1}}(A_z) \to \bar{W}_n(A_z) \to V_n(A_z) \to \bar{\Omega}_{(-)^n}(A_z) \to \ldots$$

(by analogy with the sequence of Theorem 5.7 of I).

The statement and proof of Theorem 1.1 become valid on the application of the following

Correction. *For $\tilde{K}_1(A_z)$, $\Omega_{\pm}(A_z)$, $W_n(A_z)$ read $\bar{K}_1(A_z)$, $\bar{\Omega}_{\pm}(A_z)$, $\bar{W}_n(A_z)$ throughout.*

As $K_m(\mathbf{Z}) = 0$ for $m < 0$, there is no need to correct the decomposition

$$\tilde{K}_{m+1}(A_z) = \tilde{K}_{m+1}(A) \oplus \tilde{K}_m(A) \oplus N_m^+(A) \oplus N_m^-(A).$$

(In fact, $\tilde{K}_m(A) = K_m(A)$ for $m < 0$. The reduced notation is used for uniformity with $\tilde{K}_0, \tilde{K}_1$.) The correction for $\tilde{K}_1(A_z)$ does not affect the lower L-theories $L_*^{(*)}(A)$, except in the case $m = 2$ of Theorem 4.2, where $L_n^{(2)}(A[T(p)])$ is to be interpreted as the group defined in the same way as $W_n(A[T(p)])$ but with torsions vanishing in

$$\mathrm{coker}(K_1(\mathbf{Z}[T(p)]) \to K_1(A[T(p)])$$

rather than $\tilde{K}_1(A[T(p)])$. Theorem 3.3 of III, [5], gives an exact sequence

$$\ldots \to H^{n+1}(\mathbf{Z}_2;\ \tilde{K}_1(\mathbf{Z}[T(p)])) \to W_n(A[T(p)]) \to L_n^{(2)}(A[T(p)]) \to H^n(\mathbf{Z}_2;\ \tilde{K}_1(\mathbf{Z}[T(p)])) \to \ldots.$$

Now

$$\tilde{K}_1(\mathbf{Z}[T(p)]) = \sum_{j=1}^{p} \bar{B}(z_j)K_0(\mathbf{Z}), \quad K_0(\mathbf{Z}) = \mathbf{Z},$$

so that

$$H^n(\mathbf{Z}_2;\ \tilde{K}_1(\mathbf{Z}[T(p)])) = \begin{cases} 0 & n \equiv 0 \pmod 2 \\ p\mathbf{Z}_2 & n \equiv 1 \pmod 2). \end{cases}$$

In particular, for $p = 1$ there are exact sequences

$$0 \to W_{2i+1}(A_z) \to \overline{W}_{2i+1}(A_z) \to \mathbf{Z}_2 \to W_{2i}(A_z) \to \overline{W}_{2i}(A_z) \to 0$$

for $i \pmod 2$.

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