

# NON-CONNECTIVE $K$ - AND NIL-SPECTRA OF ADDITIVE CATEGORIES

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**ABSTRACT.** We present an elementary construction of the non-connective algebraic  $K$ -theory spectrum associated to an additive category following the contracted functor approach due to Bass. It comes with a universal property that easily allows us to identify it with other constructions, for instance with the one of Pedersen-Weibel in terms of  $\mathbb{Z}^i$ -graded objects and bounded homomorphisms.

## INTRODUCTION

In this paper we present a construction of the non-connective  $K$ -theory spectrum  $\mathbf{K}^\infty(\mathcal{A})$  associated to an additive category  $\mathcal{A}$ . Its  $i$ -th homotopy group is the  $i$ -th algebraic  $K$ -group of  $\mathcal{A}$  for each  $i \in \mathbb{Z}$ . The construction is a spectrum version of the construction of negative  $K$ -groups in terms of contracted functors due to Bass [11, §7 in Chapter XII].

The advantage of this construction is that it is elementary and comes with a universal property. Roughly speaking, the passage from the connective algebraic  $K$ -theory spectrum  $\mathbf{K}$  to the non-connective algebraic  $K$ -theory spectrum  $\mathbf{K}^\infty$  is up to weak homotopy equivalence uniquely determined by the properties that the Bass-Heller-Swan map for  $\mathbf{K}^\infty$  is a weak equivalence and the comparison map  $\mathbf{K} \rightarrow \mathbf{K}^\infty$  is bijective on homotopy groups of degree  $\geq 1$ . This universal property will easily allow us to identify our model of a non-connective algebraic  $K$ -theory spectrum with the construction due to Pedersen-Weibel [20] based on  $\mathbb{Z}^i$ -graded objects and bounded homomorphisms.

We will use this construction to explain how the twisted Bass-Heller-Swan decomposition for connective  $K$ -theory of additive categories can be extended to the non-connective version, compare [19]. We will also discuss that the compatibility of the connective  $K$ -theory spectrum with filtered colimits passes to the non-connective  $K$ -theory spectrum. Finally we deal with homotopy  $K$ -theory and applications to the  $K$ -theoretic Farrell-Jones Conjecture.

The paper is organized as follows:

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## 1. THE BASS-HELLER-SWAN MAP

Consider a covariant functor  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  from the category  $\text{Add-Cat}$  of additive categories to the category  $\text{Spectra}$  of (sequential) spectra. Our main example will be the functor  $\mathbf{K}$  which assigns to an additive category  $\mathcal{A}$  its (connective) algebraic  $K$ -theory spectrum  $\mathbf{K}(\mathcal{A})$ , with the property that  $K_i(\mathcal{A}) = \pi_i(\mathbf{K}(\mathcal{A}))$  for  $i \geq 0$  and  $\pi_i(\mathbf{K}(\mathcal{A})) = 0$  for  $i \leq -1$ . Let  $\mathcal{I}$  be the groupoid which has two objects 0 and 1 and for which the set of morphisms between any two objects consists of precisely one element. Equip  $\mathcal{A} \times \mathcal{I}$  with the obvious structure of an additive category. For an additive category  $\mathcal{A}$  let  $j_i: \mathcal{A} \rightarrow \mathcal{A} \times \mathcal{I}$  be the functor of additive categories which sends a morphism  $f: A \rightarrow B$  in  $\mathcal{A}$  to the morphism  $f \times \text{id}_i: A \times i \rightarrow B \times i$  for  $i = 0, 1$ .

**Condition 1.1.** *For every additive category  $\mathcal{A}$ , we require the existence of a map*

$$\mathbf{u}: \mathbf{E}(\mathcal{A}) \wedge [0, 1]_+ \rightarrow \mathbf{E}(\mathcal{A} \times \mathcal{I})$$

*such that  $\mathbf{u}$  is natural in  $\mathcal{A}$  and, for  $i = 0, 1$  and  $k_i: \mathbf{E}(\mathcal{A}) \rightarrow \mathbf{E}(\mathcal{A}) \times [0, 1]_+$  the obvious inclusion coming from the inclusion  $\{i\} \rightarrow [0, 1]$ , the composite  $\mathbf{u} \circ k_i$  agrees with  $\mathbf{E}(j_i)$ .*

The connective algebraic  $K$ -theory spectrum functor  $\mathbf{K}$  satisfies Condition 1.1, cf. [27, Proposition 1.3.1 on page 330].

**Lemma 1.2.** *Suppose that  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  satisfies Condition 1.1.*

- (i) *Let  $F_i: \mathcal{A} \rightarrow \mathcal{B}$ ,  $i = 0, 1$  be two functors of additive categories and let  $T: F_0 \xrightarrow{\cong} F_1$  be a natural isomorphism. Then we obtain a homotopy*

$$\mathbf{h}: \mathbf{E}(\mathcal{A}) \wedge [0, 1] \rightarrow \mathbf{E}(\mathcal{B})$$

*with  $\mathbf{h} \circ k_i = \mathbf{E}(F_i)$  for  $i = 0, 1$ . This construction is natural in  $F_0$ ,  $F_1$  and  $T$ ;*

- (ii) *An equivalence  $F: \mathcal{A} \rightarrow \mathcal{B}$  of additive categories induces a homotopy equivalence  $\mathbf{E}(F): \mathbf{E}(\mathcal{A}) \rightarrow \mathbf{E}(\mathcal{B})$ .*

*Proof.* (i) The triple  $(F_0, F_1, T)$  induces an additive functor  $H: \mathcal{A} \times \mathcal{I} \rightarrow \mathcal{B}$  with  $H \circ j_i = F_i$  for  $i = 0, 1$ . Define  $\mathbf{h}$  to be the composite  $\mathbf{E}(H) \circ \mathbf{u}$ .

(ii) Let  $F': \mathcal{B} \rightarrow \mathcal{A}$  be a functor such that  $F' \circ F$  is naturally isomorphic to  $\text{id}_{\mathcal{A}}$  and  $F \circ F'$  is naturally isomorphic to  $\text{id}_{\mathcal{B}}$ . Assertion (i) implies  $\mathbf{E}(F') \circ \mathbf{E}(F) \simeq \text{id}_{\mathbf{E}(\mathcal{A})}$  and  $\mathbf{E}(F) \circ \mathbf{E}(F') \simeq \text{id}_{\mathbf{E}(\mathcal{B})}$ .  $\square$

Let  $\mathcal{A}$  be an additive category. Define the *associated Laurent category*  $\mathcal{A}[t, t^{-1}]$  as follows. It has the same objects as  $\mathcal{A}$ . Given two objects  $A$  and  $B$ , a morphism  $f: A \rightarrow B$  in  $\mathcal{A}[t, t^{-1}]$  is a formal sum  $f = \sum_{k \in \mathbb{Z}} f_k \cdot t^k$ , where  $f_k: A \rightarrow B$  is a morphism in  $\mathcal{A}$  and only finitely many of the morphisms  $f_k$  are non-trivial. If

$f = \sum_{i \in \mathbb{Z}} f_i \cdot t^i: A \rightarrow B$  and  $g = \sum_{j \in \mathbb{Z}} g_j \cdot t^j: B \rightarrow C$  are morphisms in  $\mathcal{A}[t, t^{-1}]$ , we define the composite  $g \circ f: A \rightarrow C$  by

$$g \circ f := \sum_{k \in \mathbb{Z}} \left( \sum_{\substack{i, j \in \mathbb{Z}, \\ i+j=k}} g_j \circ f_i \right) \cdot t^k.$$

The direct sum and the structure of an abelian group on the set of morphism from  $A$  to  $B$  in  $\mathcal{A}[t, t^{-1}]$  is defined in the obvious way using the corresponding structures of  $\mathcal{A}$ .

Let  $\mathcal{A}[t]$  and  $\mathcal{A}[t^{-1}]$  respectively be the additive subcategory of  $\mathcal{A}[t, t^{-1}]$  whose set of objects is the set of objects in  $\mathcal{A}$  and whose morphism from  $A$  to  $B$  are given by finite Laurent series  $\sum_{k \in \mathbb{Z}} f_k \cdot t^k$  with  $f_k = 0$  for  $k < 0$  and  $k > 0$  respectively.

If  $\mathcal{A}$  is the additive category of finitely generated free  $R$ -modules, then  $\mathcal{A}[t]$  and  $\mathcal{A}[t^{-1}]$  respectively are equivalent to the additive category of finitely generated free modules over  $R[t]$  and  $R[t^{-1}]$  respectively.

Define functors

$$z[t, t^{-1}], z[t], z[t^{-1}]: \text{Add-Cat} \rightarrow \text{Add-Cat}$$

by sending an object  $\mathcal{A}$  to the object  $\mathcal{A}[t, t^{-1}]$ ,  $\mathcal{A}[t]$  and  $\mathcal{A}[t^{-1}]$  respectively. Their definition on morphisms in  $\text{Add-Cat}$  is obvious. Next we define natural transformations of functors  $\text{Add-Cat} \rightarrow \text{Add-Cat}$

$$\begin{array}{ccccc} & & z[t] & & \\ & \text{ev}_0^+ \nearrow & & \searrow j_+ & \\ & i_+ & & & \\ \text{id}_{\text{Add-Cat}} & \xrightarrow{i_0} & z[t, t^{-1}] & & \\ & i_- \searrow & & \nearrow j_- & \\ & \text{ev}_0^+ \searrow & z[t^{-1}] & & \end{array}$$

We have to specify for every object  $\mathcal{A}$  in  $\text{Add-Cat}$  their values on  $\mathcal{A}$ . The functors  $i_0(\mathcal{A})$ ,  $i_+(\mathcal{A})$  and  $i_-(\mathcal{A})$  send a morphism  $f: A \rightarrow B$  in  $\mathcal{A}$  to the morphism  $f \cdot t^0: A \rightarrow B$ . The functors  $j_{\pm}(\mathcal{A})$  are the obvious inclusions. The functor  $\text{ev}_0^{\pm}(\mathcal{A}): \mathcal{A}_{\phi}[t^{\pm}] \rightarrow \mathcal{A}$  sends a morphism  $\sum_{k \geq 0} f_k \cdot t^k$  in  $\mathcal{A}[t]$  or  $\sum_{k \leq 0} f_k \cdot t^k$  in  $\mathcal{A}[t^{-1}]$  respectively to  $f_0$ . Notice that  $\text{ev}_0^+ \circ i_+ = \text{ev}_0^- \circ i_- = \text{id}_{\mathcal{A}}$  and  $i_0 = j_+ \circ i_+ = j_- \circ i_-$  holds.

**Notation 1.3** (Functors associated to  $\mathbf{E}$ ). Given a covariant functor  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  satisfying Condition 1.1, define covariant functors

$$\mathbf{BE}, \mathbf{B}_r \mathbf{E}, \mathbf{LE}, \mathbf{N}_{\pm} \mathbf{E}, \mathbf{Z}_{\pm} \mathbf{E}, \mathbf{ZE}: \text{Add-Cat} \rightarrow \text{Spectra}$$

and natural transformations

$$\begin{aligned} \mathbf{i}_0: \mathbf{E} &\rightarrow \mathbf{ZE}; \\ \mathbf{i}_{\pm}: \mathbf{E} &\rightarrow \mathbf{Z}_{\pm} \mathbf{E}; \\ \mathbf{j}_{\pm}: \mathbf{Z}_{\pm} \mathbf{E} &\rightarrow \mathbf{ZE}; \\ \mathbf{ev}_0^{\pm}: \mathbf{Z}_{\pm} \mathbf{E} &\rightarrow \mathbf{E}; \\ \mathbf{a}: \mathbf{E} \wedge (S^1)_+ &\rightarrow \mathbf{ZE}; \\ \mathbf{b}_{\pm}: \mathbf{N}_{\pm} \mathbf{E} &\rightarrow \mathbf{ZE}; \\ \mathbf{BHS}: \mathbf{BE} &\rightarrow \mathbf{ZE}; \\ \mathbf{BHS}_r: \mathbf{B}_r \mathbf{E} &\rightarrow \mathbf{ZE}; \\ \mathbf{s}: \mathbf{E} &\rightarrow \Omega \mathbf{LE}, \end{aligned}$$

as follows. Put

$$\begin{aligned}\mathbf{Z}_\pm \mathbf{E} &:= \mathbf{E} \circ z[t^{\pm 1}]; \\ \mathbf{Z}\mathbf{E} &:= \mathbf{E} \circ z[t, t^{-1}].\end{aligned}$$

The natural transformations  $\mathbf{i}_{\pm 1}$ ,  $\mathbf{i}_0$ ,  $\mathbf{j}_{\pm 1}$  and  $\mathbf{ev}_0^{\pm 1}$  are induced by applying  $\mathbf{E}$  to natural transformations  $i_\pm$ ,  $i_0$ ,  $j_\pm$  and  $\mathbf{ev}_0^{\pm 1}$ .

Next we define  $\mathbf{a}$ . Let  $T: i_0 \rightarrow i_0$  be the natural transformation of functors  $\mathcal{A} \rightarrow \mathcal{A}[t, t^{-1}]$  of additive categories whose value at an object  $A$  is given by the isomorphism  $\text{id}_A \cdot t: A \rightarrow A$  in  $\mathcal{A}[t, t^{-1}]$ . Because of Lemma 1.2 (i) it induces a homotopy  $\mathbf{h}: \mathbf{E}(\mathcal{A}) \wedge [0, 1]_+ \rightarrow \mathbf{E}(\mathcal{A}[t, t^{-1}])$  such that  $\mathbf{h}_0 = \mathbf{h}_1 = \mathbf{E}(i_0)$  holds, where  $\mathbf{h}_i := \mathbf{h} \circ k_i$  for the obvious inclusion  $k_i: \mathbf{E}(\mathcal{A}) \rightarrow \mathbf{E}(\mathcal{A}) \times [0, 1]_+$  for  $i = 0, 1$ . Since we have the obvious pushout

$$\begin{array}{ccc} \mathbf{E}(\mathcal{A}) \wedge \{0, 1\}_+ & \longrightarrow & \mathbf{E}(\mathcal{A}) \wedge [0, 1]_+ \\ \downarrow & & \downarrow \\ \mathbf{E} & \longrightarrow & \mathbf{E}(\mathcal{A}) \wedge (S^1)_+ \end{array}$$

we obtain a map  $\mathbf{a}(\mathcal{A}): \mathbf{E}(\mathcal{A}) \wedge (S^1)_+ \rightarrow \mathbf{E}(\mathcal{A}[t, t^{-1}])$ . This defines a transformation

$$\mathbf{a}: \mathbf{E} \wedge (S^1)_+ \rightarrow \mathbf{Z}\mathbf{E}.$$

In order to guarantee the existence of  $\mathbf{a}$ , we have imposed the Condition 1.1 which is stronger than just demanding that  $\mathbf{E}$  sends equivalences of additive categories to (weak) homotopy equivalences of spectra.

Define  $\mathbf{N}_\pm \mathbf{E}$  to be the homotopy fiber of the map of spectra  $\mathbf{ev}_0^\pm: \mathbf{Z}^\pm \mathbf{E} \rightarrow \mathbf{E}$ . Let

$$\mathbf{b}_\pm: \mathbf{N}_\pm \mathbf{E} \rightarrow \mathbf{Z}_\pm \mathbf{E} \xrightarrow{\mathbf{j}_\pm} \mathbf{Z}\mathbf{E}$$

be the composite map, where the first map is the canonical one.

Define

$$\begin{aligned}\mathbf{B}\mathbf{E} &= (\mathbf{E} \wedge (S^1)_+) \vee \mathbf{N}_+ \mathbf{E} \vee \mathbf{N}_- \mathbf{E}; \\ \mathbf{B}_r \mathbf{E} &= \mathbf{E} \vee \mathbf{N}_+ \mathbf{E} \vee \mathbf{N}_- \mathbf{E}.\end{aligned}$$

Put

$$\begin{aligned}\mathbf{B}\mathbf{H}\mathbf{S} &:= \mathbf{a} \vee \mathbf{b}_+ \vee \mathbf{b}_-; \\ \mathbf{B}\mathbf{H}\mathbf{S}_r &:= \mathbf{i}_0 \vee \mathbf{b}_+ \vee \mathbf{b}_-.\end{aligned}$$

We sometimes call  $\mathbf{B}\mathbf{H}\mathbf{S}$  the *Bass-Heller-Swan map* and  $\mathbf{B}\mathbf{H}\mathbf{S}_r$  the *restricted Bass-Heller-Swan map*.

We have the following commutative diagram

$$(1.4) \quad \begin{array}{ccc} \mathbf{E} & \xrightarrow{\mathbf{l}} & \mathbf{E} \wedge (S^1)_+ \\ \downarrow & & \downarrow \mathbf{a} \\ \mathbf{B}_r \mathbf{E} & \xrightarrow{\mathbf{B}\mathbf{H}\mathbf{S}_r} & \mathbf{Z}\mathbf{E} \end{array}$$

where the left vertical arrow is the canonical inclusion, and  $\mathbf{l}$  is the obvious inclusion coming from the identification  $\mathbf{E} = \mathbf{E} \wedge \text{pt}_+$  and the inclusion  $\text{pt}_+ \rightarrow (S^1)_+$ . It induces a map  $\mathbf{s}'': \text{hocolim}(\mathbf{l}) \rightarrow \text{hocolim}(\mathbf{B}\mathbf{H}\mathbf{S}_r)$ . Let  $\mathbf{pr}: \text{hocolim}(\mathbf{l}) \rightarrow \Sigma \mathbf{E} := \mathbf{E} \wedge S^1$  be the obvious projection which is a homotopy equivalence. Define  $\mathbf{L}\mathbf{E}$  to

be the homotopy pushout

$$(1.5) \quad \begin{array}{ccc} \mathrm{hocofib}(1) & \xrightarrow[\simeq]{\mathrm{pr}} & \Sigma \mathbf{E} \\ \downarrow s'' & & \downarrow s' \\ \mathrm{hocofib}(\mathbf{BHS}_r) & \xrightarrow[\mathrm{pr}]{\simeq} & \mathbf{LE} \end{array}$$

By construction we obtain a homotopy cofiber sequence

$$\mathbf{BHS}_r \mathbf{E} \xrightarrow{\mathbf{BHS}_r} \mathbf{ZE} \rightarrow \mathbf{LE}.$$

Denote by

$$s: \mathbf{E} \rightarrow \Omega \mathbf{LE}$$

the adjoint of  $s'$ .

**Definition 1.6** (Compatible transformations). *Let  $E, F: \mathrm{Add-Cat} \rightarrow \mathrm{Spectra}$  be two functors satisfying Condition 1.1. A natural transformation  $\phi: \mathbf{E} \rightarrow \mathbf{F}$  is called compatible if the obvious diagram*

$$\begin{array}{ccc} \mathbf{E}(\mathcal{A}) \wedge [0, 1]_+ & \xrightarrow{u} & \mathbf{E}(\mathcal{A} \times \mathcal{I}) \\ \downarrow \phi & & \downarrow \phi \\ \mathbf{F}(\mathcal{A}) \wedge [0, 1]_+ & \xrightarrow{u} & \mathbf{F}(\mathcal{A} \times \mathcal{I}) \end{array}$$

is commutative. The category

$$\mathrm{func}_c(\mathrm{Add-Cat}, \mathrm{Spectra})$$

is the category of functors satisfying Condition 1.1 whose morphisms are compatible natural transformations.

We leave the proof of the following lemma to the reader:

- Lemma 1.7.** (i) *If  $\mathbf{E}: \mathrm{Add-Cat} \rightarrow \mathrm{Spectra}$  satisfies Condition 1.1 then so do  $\mathbf{E} \wedge X$  and  $\mathrm{map}(X, \mathbf{E})$  for any space  $X$ .*  
(ii) *If  $\mathbf{E}$  and  $\mathbf{E}'$  satisfy Condition 1.1 then so does  $\mathbf{E} \vee \mathbf{E}'$ .*  
(iii) *If*

$$\mathbf{E}_1 \rightarrow \mathbf{E}_0 \leftarrow \mathbf{E}_2$$

*is a diagram in  $\mathrm{func}_c(\mathrm{Add-Cat}, \mathrm{Spectra})$ , then its homotopy pullback satisfies Condition 1.1.*

- (iv) *If  $\mathcal{J}$  is a small category and  $F: \mathcal{J} \rightarrow \mathrm{func}_c(\mathrm{Add-Cat}, \mathrm{Spectra})$  is a functor, then  $\mathrm{hocolim} F$  satisfies Condition 1.1.*

**Remark 1.8.** The category  $\mathrm{Spectra}$  is the “naive” one with strict morphisms of spectra as described for instance in [12]. Our model for  $\Omega \mathbf{E}$  is the spectrum  $\mathrm{map}(S^1, \mathbf{E})$  defined levelwise, and analogously for the homotopy pushout, homotopy pullback, homotopy fiber, and more general for homotopy colimits and homotopy limits over arbitrary index categories. For more details see for instance [12, 16].

As an application of Lemma 1.7, we deduce:

**Lemma 1.9.** *If  $\mathbf{E}: \mathrm{Add-Cat} \rightarrow \mathrm{Spectra}$  satisfies Condition 1.1, then so do  $\mathbf{BE}$ ,  $\mathbf{B}_r \mathbf{E}$ ,  $\mathbf{LE}$ ,  $\mathbf{N}_\pm \mathbf{E}$ ,  $\mathbf{Z}_\pm \mathbf{E}$ , and  $\mathbf{ZE}$ .*

We will apply this as well as the following result without further remarks.

**Lemma 1.10.** *Let  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$  be a transformation of functors  $\mathrm{Add-Cat} \rightarrow \mathrm{Spectra}$ . Suppose that it is a weak equivalence, i.e.,  $\mathbf{f}(\mathcal{A})$  is a weak equivalence for any object  $\mathcal{A}$  in  $\mathrm{Add-Cat}$ . Then the same is true for the transformations  $\mathbf{Bf}$ ,  $\mathbf{B}_r \mathbf{f}$ ,  $\mathbf{Lf}$ ,  $\mathbf{N}_\pm \mathbf{f}$ ,  $\mathbf{Z}_\pm \mathbf{f}$ , and  $\mathbf{Zf}$ .*

## 2. CONTRACTED FUNCTORS

Let  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  be a covariant functor satisfying Condition 1.1.

**Definition 2.1** (*c*-contracted). *For  $c \in \mathbb{Z}$ , we call  $\mathbf{E}$   $c$ -contracted if it satisfies the following two conditions:*

- (i) *For every  $i \in \mathbb{Z}$  the natural transformation  $\pi_i(\mathbf{BHS}_r): \pi_i(\mathbf{B}_r\mathbf{E}) \rightarrow \pi_i(\mathbf{Z}\mathbf{E})$  is split injective, i.e., there exists a natural transformation of functors from  $\text{Add-Cat}$  to the category of abelian groups*

$$\rho_i: \pi_i(\mathbf{Z}\mathbf{E}) \rightarrow \pi_i(\mathbf{B}_r\mathbf{E})$$

*such that the composite  $\pi_i(\mathbf{B}_r\mathbf{E}) \xrightarrow{\pi_i(\mathbf{BHS}_r)} \pi_i(\mathbf{Z}\mathbf{E}) \xrightarrow{\rho_i} \pi_i(\mathbf{B}_r\mathbf{E})$  is the identity;*

- (ii) *For  $i \in \mathbb{Z}, i \geq -c + 1$  the transformation*

$$\pi_i(\mathbf{BHS}): \pi_i(\mathbf{B}\mathbf{E}) \rightarrow \pi_i(\mathbf{Z}\mathbf{E})$$

*is an isomorphism, i.e., its evaluation at any additive category  $\mathcal{A}$  is bijective.*

*We call  $\mathbf{E}$   $\infty$ -contracted if  $\mathbf{BHS}: \mathbf{B}\mathbf{E} \rightarrow \mathbf{Z}\mathbf{E}$  is a weak homotopy equivalence.*

**Lemma 2.2.** *Let  $\mathbf{E}, \mathbf{E}': \text{Add-Cat} \rightarrow \text{Spectra}$  be covariant functors satisfying Condition 1.1.*

- (i) *Suppose that  $\mathbf{E}$  and  $\mathbf{E}'$  satisfy Condition 1.1. Consider  $i \in \mathbb{Z}$ . Then both  $\pi_i(\mathbf{BHS}(\mathbf{E}))$  and  $\pi_i(\mathbf{BHS}(\mathbf{E}'))$  are isomorphisms if and only if  $\pi_i(\mathbf{BHS}(\mathbf{E} \vee \mathbf{E}'))$  is an isomorphism;*
- (ii) *Suppose that  $\mathbf{E}$  and  $\mathbf{E}'$  satisfy Condition 1.1. Consider  $c \in \mathbb{Z}$ . Then  $\mathbf{E} \vee \mathbf{E}'$  is  $c$ -contracted if and only if both  $\mathbf{E}$  and  $\mathbf{E}'$  are  $c$ -contracted.*

*Proof.* The transformation

$$\mathbf{b}_\pm \vee \mathbf{i}_0: \mathbf{N}_\pm \mathbf{E} \vee \mathbf{E} \rightarrow \mathbf{Z}_\pm \mathbf{E}$$

is a weak equivalence, i.e., its evaluation at any additive category  $\mathcal{A}$  is a weak equivalence of spectra, since  $\mathbf{ev}_0^\pm \circ \mathbf{i}_0 = \text{id}_{\mathbf{E}}$  holds. Note that

$$\mathbf{Z}_\pm \mathbf{E} \vee \mathbf{Z}_\pm \mathbf{E}' = \mathbf{Z}_\pm (\mathbf{E} \vee \mathbf{E}')$$

so

$$\mathbf{N}_\pm \mathbf{E} \vee \mathbf{N}_\pm \mathbf{E}' = \mathbf{N}_\pm (\mathbf{E} \vee \mathbf{E}').$$

The obvious map

$$(\mathbf{E} \wedge (S^1)_+) \vee (\mathbf{E}' \wedge (S^1)_+) \rightarrow (\mathbf{E} \vee \mathbf{E}') \wedge (S^1)_+$$

is an isomorphism. Hence the following two obvious transformations are weak equivalences

$$\begin{aligned} \mathbf{B}_r \mathbf{E} \vee \mathbf{B}_r \mathbf{E}' &\rightarrow \mathbf{B}_r (\mathbf{E} \vee \mathbf{E}'); \\ \mathbf{B} \mathbf{E} \vee \mathbf{B} \mathbf{E}' &\rightarrow \mathbf{B} (\mathbf{E} \vee \mathbf{E}'). \end{aligned}$$

Now the claim follows from the following two commutative diagrams

$$\begin{array}{ccc} \pi_i(\mathbf{B}\mathbf{E}) \oplus \pi_i(\mathbf{B}\mathbf{E}') & \xrightarrow{\cong} & \pi_i(\mathbf{B}(\mathbf{E} \vee \mathbf{E}')) \\ \pi_i(\mathbf{BHS}(\mathbf{E})) \oplus \pi_i(\mathbf{BHS}(\mathbf{E}')) \downarrow & & \downarrow \pi_i(\mathbf{BHS}(\mathbf{E} \vee \mathbf{E}')) \\ \pi_i(\mathbf{Z}\mathbf{E}) \oplus \pi_i(\mathbf{Z}\mathbf{E}') & \xrightarrow{\cong} & \pi_i(\mathbf{Z}(\mathbf{E} \vee \mathbf{E}')) \end{array}$$

and

$$\begin{array}{ccc}
\pi_i(\mathbf{B}_r \mathbf{E}) \oplus \pi_i(\mathbf{B}_r \mathbf{E}') & \xrightarrow{\cong} & \pi_i(\mathbf{B}_r(\mathbf{E} \vee \mathbf{E}')) \\
\downarrow \pi_i(\mathbf{BHS}_r(\mathbf{E})) \oplus \pi_i(\mathbf{BHS}_r(\mathbf{E}')) & & \downarrow \pi_i(\mathbf{BHS}_r(\mathbf{E} \vee \mathbf{E}')) \\
\pi_i(\mathbf{Z}\mathbf{E}) \oplus \pi_i(\mathbf{Z}\mathbf{E}') & \xrightarrow{\cong} & \pi_i(\mathbf{Z}(\mathbf{E} \vee \mathbf{E}'))
\end{array}$$

□

Define

$$(2.3) \quad \mathbf{K}: \text{Add-Cat} \rightarrow \text{Spectra}$$

to be the connective  $K$ -theory spectrum functor in the sense of Quillen [21, page 95] by regarding  $\mathcal{A}$  as an exact category or in the sense of Waldhausen [27, page 330] by regarding  $\mathcal{A}$  as a Waldhausen category. (These approaches are equivalent, see [27, Section 1.9]).

**Theorem 2.4** (Bass-Heller-Swan Theorem for  $\mathbf{K}$ ). *The functor  $\mathbf{K}$  is 1-contracted in the sense of Definition 2.1.*

*Proof.* The proof that the Bass-Heller-Swan map induces bijections on  $\pi_i$  for  $i \geq 1$  can be found in [19, Theorem 0.4 (i)] provided that  $\mathcal{A}$  is idempotent complete. Denote by  $\eta: \mathcal{A} \rightarrow \text{Idem}(\mathcal{A})$  the inclusion of  $\mathcal{A}$  into its idempotent completion. By cofinality [26, Theorem A.9.1] the maps  $\mathbf{Z}\mathbf{K}(\eta)$  and  $\mathbf{B}_r \mathbf{K}(\eta)$  induce isomorphisms on  $\pi_1$  for  $i \geq 1$ ; the map  $\mathbf{BK}(\eta)$  induces isomorphisms at least for  $i \geq 2$ . The commutativity of the diagram

$$\begin{array}{ccc}
\mathbf{BK}(\mathcal{A}) & \xrightarrow{\mathbf{BHS}} & \mathbf{ZK}(\mathcal{A}) \\
\downarrow \mathbf{BK}(\eta) & & \downarrow \mathbf{ZK}(\eta) \\
\mathbf{BK}(\text{Idem}(\mathcal{A})) & \xrightarrow{\mathbf{BHS}} & \mathbf{ZK}(\text{Idem}(\mathcal{A}))
\end{array}$$

shows that the Bass-Heller-Swan map for  $\mathcal{A}$  induces isomorphisms of  $\pi_i$  for  $i \geq 2$  and that the restricted Bass-Heller-Swan map for  $\mathcal{A}$  is split injective on  $\pi_i$  for  $i \geq 1$ .

Since all spectra are connective, it remains to show that the restricted Bass-Heller-Swan map for  $\mathcal{A}$  is split injective on  $\pi_0$ . Notice that

$$\pi_0(\mathbf{K}(\mathcal{A})) \rightarrow \pi_0(\mathbf{K}(\mathcal{A}[t]))$$

is surjective as both categories  $\mathcal{A}$  and  $\mathcal{A}[t]$  have the same objects. Hence  $\pi_0(\mathbf{NK}(\mathcal{A})) = 0$  and we need to show that the map induced by the inclusion

$$\pi_0(\mathbf{K}(\mathcal{A})) \rightarrow \pi_0(\mathbf{K}(\mathcal{A}[t, t^{-1}]))$$

is split mono. Such a split is given by evaluation at  $t = 1$ . □

Denote by  $\text{Idem}: \text{Add-Cat} \rightarrow \text{Add-Cat}$  the idempotent completion functor, and let

$$\mathbf{K}_{\text{Idem}} := \mathbf{K} \circ \text{Idem}: \text{Add-Cat} \rightarrow \text{Spectra}.$$

**Example 2.5** (Algebraic  $K$ -theory of a ring  $R$ ). Given a ring  $R$ , then the idempotent completion  $\text{Idem}(\mathcal{R})$  of the additive category  $\mathcal{R}$  of finitely generated free  $R$ -modules is equivalent to the additive category of finitely generated projective  $R$ -modules. Moreover, the map  $\mathbb{Z} \rightarrow \pi_0 \mathbf{K}(\mathcal{R})$  sending  $n$  to  $[R^n]$  is surjective (even bijective if  $R^n \cong R^m$  implies  $m = n$ ), whereas  $\pi_0 \mathbf{K}_{\text{Idem}}(\mathcal{R})$  is the projective class group of  $R$ .

For an additive category we define its algebraic  $K$ -group

$$(2.6) \quad K_i(\mathcal{A}) := \pi_i(\mathbf{K}_{\text{Idem}}(\mathcal{A})) \quad \text{for } i \geq 0.$$

We already showed that by cofinality, the map induced by the inclusion

$$\pi_i \mathbf{K}(\mathcal{A}) \rightarrow \pi_i \mathbf{K}_{\text{Idem}}(\mathcal{A}) = K_i(\mathcal{A})$$

is an isomorphism for  $i \geq 1$ .

**Theorem 2.7** (Bass-Heller-Swan Theorem for connective algebraic  $K$ -theory). *The functor  $\mathbf{K}_{\text{Idem}}$  is 0-contracted in the sense of Definition 2.1.*

*Proof.* In view of the proof of Theorem 2.4, the Bass-Heller-Swan map is bijective on  $\pi_i$  for  $i \geq 1$ . It remains to show split injectivity on  $\pi_0$ .

We will abbreviate  $\mathcal{B} = \mathcal{A}[s, s^{-1}]$ . Notice for the sequel that  $\mathcal{B}[t^{\pm 1}] = (\mathcal{A}[t^{\pm 1}])[s, s^{-1}]$  and  $\mathcal{B}[t, t^{-1}] = (\mathcal{A}[t, t^{-1}])[s, s^{-1}]$ . Put

$$\begin{aligned} NK_i(\mathcal{A}[t^{\pm 1}]) &= \pi_i(\mathbf{N}_{\pm 1} \mathbf{K}_{\text{Idem}}(\mathcal{A})) = \ker(K_i(\mathcal{A}[t^{\pm 1}] \rightarrow K_i(\mathcal{A})); \\ NK_i(\mathcal{B}[t^{\pm 1}]) &= \pi_i(\mathbf{N}_{\pm 1} \mathbf{K}_{\text{Idem}}(\mathcal{B})) = \ker(K_i(\mathcal{B}[t^{\pm 1}] \rightarrow K_i(\mathcal{B})). \end{aligned}$$

Because of Lemma 2.2 also the Bass-Heller-Swan map for  $\mathbf{N}_{\pm 1} \mathbf{K}_{\text{Idem}}$  induces isomorphisms on  $\pi_1$ .

In particular we get split injections

$$\begin{aligned} \alpha: K_0(\mathcal{A}) &\rightarrow K_1(\mathcal{B}); \\ \beta_{\pm}: NK_0(\mathcal{A}[t^{\pm 1}]) &\rightarrow NK_1(\mathcal{B}[t^{\pm 1}]); \\ j: K_0(\mathcal{A}[t, t^{-1}]) &\rightarrow K_1(\mathcal{B}[t, t^{-1}]). \end{aligned}$$

We obtain the following commutative diagram

$$\begin{array}{ccc} K_0(\mathcal{A}) \oplus NK_0(\mathcal{A}[t]) \oplus NK_0(\mathcal{A}[t^{-1}]) & \xrightarrow{\pi_0(\mathbf{BHS}_r(\mathcal{A}))} & K_0(\mathcal{A}[t, t^{-1}]) \\ \alpha \oplus \beta_+ \oplus \beta_- \downarrow & & \downarrow j \\ K_1(\mathcal{B}) \oplus NK_1(\mathcal{B}[t]) \oplus NK_1(\mathcal{B}[t^{-1}]) & \xrightarrow[\cong]{\pi_1(\mathbf{BHS}_r(\mathcal{B}))} & K_1(\mathcal{B}[t, t^{-1}]) \end{array}$$

which is compatible with the splitting. So  $\pi_0(\mathbf{BHS}_r(\mathcal{A}))$  is a split mono, being a retract of the split mono  $\pi_1(\mathbf{BHS}_r(\mathcal{B}))$ .  $\square$

**Lemma 2.8.** *If  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  is  $c$ -contracted, then  $\Omega \mathbf{LE}: \text{Add-Cat} \rightarrow \text{Spectra}$  is  $(c+1)$ -contracted and the map  $\pi_i(\mathbf{s}): \pi_i(\mathbf{E}) \rightarrow \pi_i(\Omega(\mathbf{LE}))$  is bijective for  $i \geq -c$ .*

*Proof.* Obviously it suffices to show that  $\mathbf{LE}$  is  $c$ -contracted and that the map  $\mathbf{s}': \Sigma \mathbf{E} \rightarrow \mathbf{LE}$ , which is the adjoint of  $\mathbf{s}$ , induces an isomorphism on  $\pi_i$  for  $i \geq -c+1$ .

Since  $\mathbf{E}$  is  $c$ -contracted,  $\mathbf{ZE}$ ,  $\mathbf{Z}_+ \mathbf{E}$  and  $\mathbf{Z}_- \mathbf{E}$  are  $c$ -contracted. We have the obvious cofibration sequence  $\mathbf{E} \rightarrow \mathbf{E} \wedge (S^1)_+ \rightarrow \mathbf{E} \wedge S^1$  and the retraction  $\mathbf{E} \wedge (S^1)_+ \rightarrow \mathbf{E}$ . There is a weak equivalence  $\mathbf{N}_{\pm} \mathbf{E} \vee \mathbf{E} \rightarrow \mathbf{Z}_{\pm} \mathbf{E}$ . We conclude from Lemma 2.2 that  $\mathbf{N}_{\pm} \mathbf{E}$ ,  $\mathbf{B}_r \mathbf{E}$  and  $\mathbf{BE}$  are  $c$ -contracted.

By construction we have the homotopy cofibration sequence  $\mathbf{B}_r \mathbf{E} \rightarrow \mathbf{ZE} \rightarrow \mathbf{LE}$ . It induces a long exact sequence of homotopy groups. The existence of the retractions  $\rho_i$  imply that it breaks up into short split exact sequences of transformations of functors from  $\text{Add-Cat}$  to the category of abelian groups

$$0 \rightarrow \pi_i(\mathbf{B}_r \mathbf{E}) \rightarrow \pi_i(\mathbf{ZE}) \rightarrow \pi_i(\mathbf{LE}) \rightarrow 0.$$

We obtain a commutative diagram with short split exact rows as vertical arrows, where the retractions from the middle term to the left term are also compatible



with the vertical maps.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \pi_i(\mathbf{Z}_{\pm} \mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z}_{\pm} \mathbf{Z} \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z}_{\pm} \mathbf{L} \mathbf{E}) \longrightarrow 0 \\
& & \downarrow \pi_i(\mathbf{ev}_0^{\pm}(\mathbf{B}_r \mathbf{E})) & & \downarrow \pi_i(\mathbf{ev}_0^{\pm}(\mathbf{Z} \mathbf{E})) & & \downarrow \pi_i(\mathbf{ev}_0^{\pm}(\mathbf{L} \mathbf{E})) \\
0 & \longrightarrow & \pi_i(\mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{L} \mathbf{E}) \longrightarrow 0
\end{array}$$

Since we have the isomorphism

$$\pi_i(\mathbf{b}_{\pm}) \oplus \pi_i(\mathbf{i}_{+}) : \pi_i(\mathbf{N}_{\pm} \mathbf{E}) \oplus \pi_i(\mathbf{E}) \xrightarrow{\cong} \pi_i(\mathbf{Z}_{\pm} \mathbf{E}),$$

and  $\pi_i(\mathbf{ev}_0^{\pm}) \circ \pi_i(\mathbf{i}_{+}) = \text{id}$ , we obtain the short split exact sequence

$$0 \rightarrow \pi_i(\mathbf{N}_{\pm} \mathbf{B}_r \mathbf{E}) \rightarrow \pi_i(\mathbf{N}_{\pm} \mathbf{Z} \mathbf{E}) \rightarrow \pi_i(\mathbf{N}_{\pm} \mathbf{L} \mathbf{E}) \rightarrow 0.$$

We have the obvious short split exact sequences

$$0 \rightarrow \pi_i(\mathbf{B}_r \mathbf{E} \wedge (S^1)_{+}) \rightarrow \pi_i(\mathbf{Z} \mathbf{E} \wedge (S^1)_{+}) \rightarrow \pi_i(\mathbf{L} \mathbf{E} \wedge (S^1)_{+}) \rightarrow 0.$$

Taking direct sums shows that we obtain short split exact sequences

$$0 \rightarrow \pi_i(\mathbf{B} \mathbf{B}_r \mathbf{E}) \rightarrow \pi_i(\mathbf{B} \mathbf{Z} \mathbf{E}) \rightarrow \pi_i(\mathbf{B} \mathbf{L} \mathbf{E}) \rightarrow 0,$$

and

$$0 \rightarrow \pi_i(\mathbf{B}_r \mathbf{B}_r \mathbf{E}) \rightarrow \pi_i(\mathbf{B}_r \mathbf{Z} \mathbf{E}) \rightarrow \pi_i(\mathbf{B}_r \mathbf{L} \mathbf{E}) \rightarrow 0.$$

Thus we obtain for all  $i \in \mathbb{Z}$  a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \pi_i(\mathbf{B} \mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{B} \mathbf{Z} \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{B} \mathbf{L} \mathbf{E}) \longrightarrow 0 \\
& & \downarrow \pi_i(\mathbf{BHS}(\mathbf{B}_r \mathbf{E})) & & \downarrow \pi_i(\mathbf{BHS}(\mathbf{Z} \mathbf{E})) & & \downarrow \pi_i(\mathbf{BHS}(\mathbf{L} \mathbf{E})) \\
0 & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{Z} \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{L} \mathbf{E}) \longrightarrow 0
\end{array}$$

Since  $\pi_i(\mathbf{BHS}(\mathbf{B}_r \mathbf{E}))$  and  $\pi_i(\mathbf{BHS}(\mathbf{Z} \mathbf{E}))$  are isomorphisms for  $i \geq -c + 1$ , the same is true for  $\pi_i(\mathbf{BHS}(\mathbf{L} \mathbf{E}))$  by the Five-Lemma.

The following diagram commutes and has exact rows

$$\begin{array}{ccccccc}
(2.9) \quad 0 & \longrightarrow & \pi_i(\mathbf{B}_r \mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{B}_r \mathbf{Z} \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{B}_r \mathbf{L} \mathbf{E}) \longrightarrow 0 \\
& & \downarrow \pi_i(\mathbf{BHS}_r(\mathbf{B}_r \mathbf{E})) & & \downarrow \pi_i(\mathbf{BHS}_r(\mathbf{Z} \mathbf{E})) & & \downarrow \pi_i(\mathbf{BHS}_r(\mathbf{L} \mathbf{E})) \\
0 & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{Z} \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{L} \mathbf{E}) \longrightarrow 0
\end{array}$$

The first two vertical arrows are split injective. We claim that the retractions fit into the following commutative square

$$\begin{array}{ccc}
(2.10) \quad \pi_i(\mathbf{Z} \mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{Z} \mathbf{Z} \mathbf{E}) \\
\rho_i(\mathbf{B}_r \mathbf{E}) \downarrow & & \downarrow \rho_i(\mathbf{Z} \mathbf{E}) \\
\pi_i(\mathbf{B}_r \mathbf{B}_r \mathbf{E}) & \longrightarrow & \pi_i(\mathbf{B}_r \mathbf{Z} \mathbf{E})
\end{array}$$

This follows from the fact that we have the commutative diagram with isomorphisms as horizontal arrows

$$\begin{array}{ccc}
\pi_i(\mathbf{Z} \mathbf{E}) \oplus \pi_i(\mathbf{Z} \mathbf{N}_{+} \mathbf{E}) \oplus \pi_i(\mathbf{Z} \mathbf{N}_{-} \mathbf{E}) & \xrightarrow[\cong]{\pi_i(\mathbf{Z} \mathbf{i}_0) \oplus \pi_i(\mathbf{Z} \mathbf{b}_{+}) \oplus \pi_i(\mathbf{Z} \mathbf{b}_{-})} & \pi_1(\mathbf{Z} \mathbf{B}_r \mathbf{E}) \\
\downarrow \rho_i(\mathbf{E}) \oplus \rho_i(\mathbf{N}_{+} \mathbf{E}) \oplus \rho_i(\mathbf{N}_{-} \mathbf{E}) & & \downarrow \rho_i(\mathbf{B}_r \mathbf{E}) \\
\pi_i(\mathbf{B}_r \mathbf{E}) \oplus \pi_i(\mathbf{B}_r \mathbf{N}_{+} \mathbf{E}) \oplus \pi_i(\mathbf{B}_r \mathbf{N}_{-} \mathbf{E}) & \xrightarrow[\cong]{\pi_i(\mathbf{B}_r \mathbf{i}_0) \oplus \pi_i(\mathbf{B}_r \mathbf{b}_{+}) \oplus \pi_i(\mathbf{B}_r \mathbf{b}_{-})} & \pi_i(\mathbf{B}_r \mathbf{B}_r \mathbf{E})
\end{array}$$

and the following commutative diagrams

$$\begin{array}{ccc}
\pi_i(\mathbf{ZN}_{\pm}\mathbf{E}) & \xrightarrow{\pi_i(\mathbf{Zb}_{\pm})} & \pi_i(\mathbf{ZZE}) \\
\rho_i(\mathbf{N}_{\pm}\mathbf{E}) \downarrow & & \downarrow \rho_i(\mathbf{ZE}) \\
\pi_i(\mathbf{B}_r\mathbf{N}_{\pm}\mathbf{E}) & \xrightarrow{\pi_i(\mathbf{B}_r\mathbf{b}_{\pm})} & \pi_i(\mathbf{B}_r\mathbf{ZE})
\end{array}$$

and

$$\begin{array}{ccc}
\pi_i(\mathbf{ZE}) & \xrightarrow{\pi_i(\mathbf{Zi}_0)} & \pi_i(\mathbf{ZZE}) \\
\rho_i(\mathbf{E}) \downarrow & & \downarrow \rho_i(\mathbf{ZE}) \\
\pi_i(\mathbf{B}_r\mathbf{E}) & \xrightarrow{\pi_i(\mathbf{B}_r\mathbf{i}_0)} & \pi_i(\mathbf{B}_r\mathbf{ZE})
\end{array}$$

The two diagrams (2.9) and (2.10) imply that  $\pi_i(\mathbf{BHS}_r(\mathbf{LE})): \pi_i(\mathbf{B}_r\mathbf{LE}) \rightarrow \pi_i(\mathbf{ZLE})$  is split injective for all  $i \in \mathbb{Z}$ . This finishes the proof that  $\mathbf{LE}$  is  $c$ -contracted.

We have the following diagram which has homotopy cofibration sequences as vertical arrows and which commutes up to homotopy.

$$\begin{array}{ccccc}
\mathbf{B}_r\mathbf{E} & \longrightarrow & \mathbf{BE} & \longrightarrow & \Sigma\mathbf{E} \\
\downarrow \text{id} & & \downarrow \mathbf{BHS} & & \downarrow s' \\
\mathbf{B}_r\mathbf{E} & \xrightarrow{\mathbf{BHS}_r} & \mathbf{ZE} & \longrightarrow & \mathbf{LE}
\end{array}$$

The long exact homotopy sequences associated to the rows and the fact that  $\pi_i(\mathbf{BHS}_r): \pi_i(\mathbf{B}_r\mathbf{E}) \rightarrow \pi_i(\mathbf{ZE})$  is split injective for  $i \in \mathbb{Z}$  imply that we obtain for all  $i \in \mathbb{Z}$  a commutative diagram with exact rows.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \pi_i(\mathbf{B}_r\mathbf{E}) & \longrightarrow & \pi_i(\mathbf{BE}) & \longrightarrow & \pi_i(\Sigma\mathbf{E}) \longrightarrow 0 \\
& & \downarrow \text{id} & & \downarrow \pi_i(\mathbf{BHS}) & & \downarrow s' \\
0 & \longrightarrow & \pi_i(\mathbf{B}_r\mathbf{E}) & \longrightarrow & \pi_i(\mathbf{ZE}) & \longrightarrow & \pi_i(\mathbf{LE}) \longrightarrow 0
\end{array}$$

Since  $\pi_i(\mathbf{BHS})$  is bijective for  $i \geq -c+1$  by assumption, the same is true for  $\pi_i(s')$ . This finishes the proof of Lemma 2.8.  $\square$

### 3. THE DELOOPING CONSTRUCTION

Let  $\mathbf{E}: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$  be a covariant functor satisfying Condition 1.1. Next we define inductively a sequence of spectra

$$(3.1) \quad (\mathbf{E}[n])_{n \geq 0}$$

together with maps of spectra

$$(3.2) \quad \mathbf{s}[n]: \mathbf{E}[n] \rightarrow \mathbf{E}[n+1] \quad \text{for } n \geq 0.$$

We define  $\mathbf{E}[0]$  to be  $\mathbf{E}$ . In the induction step we have to explain how we construct  $\mathbf{E}[n+1]$  and  $\mathbf{s}[n]$  provided that we have defined  $\mathbf{E}[n]$ . Define  $\mathbf{E}[n+1] = \Omega\mathbf{LE}[n]$  and let  $\mathbf{s}[n]$  be the map  $\mathbf{s}: \mathbf{E}[n] \rightarrow \Omega\mathbf{LE}[n]$  associated to  $\mathbf{E}[n]$ .

**Definition 3.3** (Delooping  $\mathbf{E}[\infty]$ ). *Define the delooping  $\mathbf{E}[\infty]$  of  $\mathbf{E}$  to be the homotopy colimit of the sequence*

$$\mathbf{E} = \mathbf{E}[0] \xrightarrow{\mathbf{s}[0]} \mathbf{E}[1] \xrightarrow{\mathbf{s}[1]} \mathbf{E}[2] \xrightarrow{\mathbf{s}[2]} \dots$$

Define the map of spectra

$$\mathbf{d}: \mathbf{E} \rightarrow \mathbf{E}[\infty]$$

to be the zero-th structure map of the homotopy colimit.

**Theorem 3.4** (Main property of the delooping construction). *Fix an integer  $c$ . Suppose that  $\mathbf{E}$  is  $c$ -contracted. Then*

- (i) *The map  $\pi_i(\mathbf{d}): \pi_i(\mathbf{E}) \rightarrow \pi_i(\mathbf{E}[\infty])$  is bijective for  $i \geq -c$ ;*
- (ii)  *$\mathbf{E}[\infty]$  is  $\infty$ -contracted;*
- (iii)  *$\mathbf{E}$  is  $\infty$ -contracted if and only if  $\mathbf{d}: \mathbf{E} \rightarrow \mathbf{E}[\infty]$  is a weak equivalence.*

*Proof.* (i) This follows from the conclusion of Lemma 2.8 that  $\pi_i(\mathbf{s}[n]): \pi_i(\mathbf{E}[n]) \rightarrow \pi_i(\mathbf{E}[n+1])$  is bijective for  $i \geq c$  and from  $\text{colim}_{n \rightarrow \infty} \pi_i(\mathbf{E}[n]) = \pi_i(\mathbf{E}[\infty])$ .

(ii) over  $n$  we conclude from Lemma 2.8 that  $\mathbf{E}[n]$  is  $(n+c)$ -contracted for  $n \geq 0$ . Obviously  $\text{hocolim}$  and  $\mathbf{Z}^+$  commute as well as  $\text{hocolim}$  and  $\mathbf{Z}$ . Hence  $\text{hocolim}$  and  $\mathbf{N}_\pm$  commute up to weak equivalence, since  $\text{hocolim}$  is compatible with  $\vee$  up to weak equivalence and we have a natural equivalence  $\mathbf{E} \vee \mathbf{N}_\pm \mathbf{E} \rightarrow \mathbf{Z}_\pm \mathbf{E}$ . This implies that  $\text{hocolim}$  and  $\mathbf{B}$  commute up to weak equivalence. Obviously  $\text{hocolim}$  commutes with  $-\wedge (S^1)_+$ . Hence we obtain for each  $i \in \mathbb{Z}$  the following commutative diagram with isomorphisms as horizontal maps

$$\begin{array}{ccc} \text{colim}_{n \rightarrow \infty} \pi_i(\mathbf{BE}[n]) & \xrightarrow{\cong} & \pi_i(\mathbf{BE}[\infty]) \\ \downarrow \text{colim}_{n \rightarrow \infty} \pi_i(\mathbf{BHS}(\mathbf{E}[n])) & & \downarrow \pi_i(\mathbf{BHS}(\mathbf{E}[\infty])) \\ \text{colim}_{n \rightarrow \infty} \pi_i(\mathbf{ZE}[n]) & \xrightarrow{\cong} & \pi_i(\mathbf{ZE}[\infty]) \end{array}$$

Since  $\mathbf{E}[n]$  is  $(n+c)$ -contracted, the left arrow and hence the right arrow are isomorphisms for all  $i \in \mathbb{Z}$ .

(iii) If  $\mathbf{d}$  is a weak equivalence, then  $\mathbf{BHS}: \mathbf{BE} \rightarrow \mathbf{ZE}$  is a weak equivalence by assertion (ii) and the fact that the following diagram commutes and has weak equivalences as horizontal arrows

$$\begin{array}{ccc} \mathbf{BE} & \xrightarrow[\cong]{\mathbf{Bd}} & \mathbf{BE}[\infty] \\ \downarrow \mathbf{BHS}(\mathbf{E}) & & \downarrow \mathbf{BHS}(\mathbf{E}[\infty]) \\ \mathbf{ZE} & \xrightarrow[\cong]{\mathbf{Zd}} & \mathbf{ZE}[\infty] \end{array}$$

Suppose that  $\mathbf{BHS}: \mathbf{BE} \rightarrow \mathbf{ZE}$  is a weak equivalence. Then  $\mathbf{E}$  is  $c$ -contracted for all  $c \in \mathbb{Z}$ . Because of Lemma 2.8  $\mathbf{E}[n]$  is  $c$ -contracted for all  $c \in \mathbb{Z}$  and  $\pi_i(\mathbf{s}[n]): \pi_i(\mathbf{ZE}[n]) \rightarrow \pi_i(\mathbf{ZE}[n+1])$  is bijective for all  $i \in \mathbb{Z}$  and  $n \geq 0$ . This implies that  $\pi_i(\mathbf{d})$  is bijective for all  $i \in \mathbb{Z}$ .  $\square$

**Remark 3.5** (Retraction needed in all degrees). One needs the retractions  $\rho_i$  appearing in Definition 2.1 in each degree  $i \in \mathbb{Z}$  and not only in degree  $-c$ . The point is that one has a  $c$ -contracted functor  $\mathbf{E}$  and wants to prove that  $\mathbf{E}[n]$  is  $(c+n)$ -contracted. For this purpose one needs the retraction to split up certain long exact sequence into pieces in dimensions  $i \geq -c$  to verify bijectivity on  $\pi_i$  for  $i \geq -c$ , but also in degree  $i = -c-1$ , to construct the retraction for  $\mathbf{E}[1]$  in degree  $-c-1$ . For this purpose one needs the retraction for  $\mathbf{E}$  also in degree  $-c-2$ . In order to be able to iterate this construction, namely, to pass from  $\mathbf{E}[1]$  to  $\mathbf{E}[2]$ , we have the same problem, the retraction for  $\mathbf{E}[1]$  must also be present in degree  $-c-3$ . Hence we need a priori the retractions for  $\mathbf{E}$  also in degree  $-c-3$ . This argument goes on and on and forces us to require the retractions in all degrees.

One needs retractions and not only injective maps in the definition of  $c$ -contracted. Injectivity would suffice to reduce the long exact sequences obtained after taking homotopy groups to short exact sequences and most of the arguments involve the Five-Lemma where only short exact but not split short exact is needed. However,

at one point we want to argue for a commutative diagram with exact rows of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 \longrightarrow 0 \end{array}$$

that the third vertical arrow admits a retraction if the first and the second arrow admit retractions compatible with the two first horizontal arrows. This is true. But the corresponding statement is wrong if we replace “admitting a retraction” by “injective”.

**Lemma 3.6.** *Suppose that the covariant functors  $\mathbf{E}, \mathbf{F}: \text{Add-Cat} \rightarrow \text{Spectra}$  satisfy Condition 1.1 and are  $\infty$ -contracted. Let  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$  be a compatible transformation. Suppose that there exists an integer  $N$  such that  $\pi_i(\mathbf{f}(\mathcal{A}))$  is bijective for all  $i \geq N$  and all objects  $\mathcal{A}$  in  $\text{Add-Cat}$ .*

*Then  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$  is a weak equivalence.*

*Proof.* We show by induction over  $i$  that  $\pi_i(\mathbf{f}(\mathcal{A}))$  is bijective for  $i = N, N-1, N-2$  and all objects  $\mathcal{A}$  in  $\text{Add-Cat}$ . The induction beginning  $i = N$  is trivial, the induction step from  $i$  to  $i-1$  done as follows.

We have the following commutative diagram whose horizontal arrows come from the Bass-Heller-Swan maps and hence are bijective by assumption

$$\begin{array}{ccc} \pi_{i-1}(\mathbf{E}(\mathcal{A})) \oplus \pi_i(\mathbf{E}(\mathcal{A})) \oplus \pi_i(\mathbf{N}_+\mathbf{E}(\mathcal{A})) \oplus \pi_i(\mathbf{N}_-\mathbf{E}(\mathcal{A})) & \xrightarrow{\cong} & \pi_i(\mathbf{Z}\mathbf{E}(\mathcal{A})) \\ \downarrow \pi_{i-1}(\mathbf{f}(\mathcal{A})) \oplus \pi_i(\mathbf{f}(\mathcal{A})) \oplus \pi_i(\mathbf{N}_+\mathbf{f}(\mathcal{A})) \oplus \pi_i(\mathbf{N}_-\mathbf{f}(\mathcal{A})) & & \downarrow \pi_i(\mathbf{Z}\mathbf{f}(\mathcal{A})) \\ \pi_{i-1}(\mathbf{F}(\mathcal{A})) \oplus \pi_i(\mathbf{F}(\mathcal{A})) \oplus \pi_i(\mathbf{N}_+\mathbf{F}(\mathcal{A})) \oplus \pi_i(\mathbf{N}_-\mathbf{F}(\mathcal{A})) & \xrightarrow{\cong} & \pi_i(\mathbf{Z}\mathbf{F}(\mathcal{A})) \end{array}$$

By the induction hypothesis  $\pi_i(\mathbf{Z}\mathbf{f}(\mathcal{A}))$  is bijective. Hence  $\pi_{i-1}(\mathbf{f}(\mathcal{A}))$  is bijective since it is a direct summand in  $\pi_i(\mathbf{Z}\mathbf{f}(\mathcal{A}))$ .  $\square$

Theorem 3.4 and Lemma 3.6 imply

**Corollary 3.7.** *Suppose that the covariant functors  $\mathbf{E}, \mathbf{F}: \text{Add-Cat} \rightarrow \text{Spectra}$  satisfy Condition 1.1. Suppose that  $\mathbf{E}$  and  $\mathbf{F}$  are  $c$ -contracted for some integer  $c$ . Let  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$  be a compatible transformation. Suppose that there exists an integer  $N$  such that  $\pi_i(\mathbf{f}): \pi_i(\mathbf{E}) \rightarrow \pi_i(\mathbf{F})$  is bijective for  $i \geq N$ .*

*Then the following diagram commutes*

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\mathbf{d}(\mathbf{E})} & \mathbf{E}[\infty] \\ \mathbf{f} \downarrow & & \downarrow \mathbf{f}[\infty] \\ \mathbf{F} & \xrightarrow{\mathbf{d}(\mathbf{F})} & \mathbf{F}[\infty] \end{array}$$

*and the right vertical arrow is a weak equivalence.*

**Remark 3.8** (Universal property of the delooping construction in the homotopy category). Suppose that the covariant functors  $\mathbf{E}, \mathbf{F}: \text{Add-Cat} \rightarrow \text{Spectra}$  satisfy Condition 1.1. Suppose that  $\mathbf{E}$  is  $c$ -contracted for some integer  $c$ , and let  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$  be a compatible transformation to an  $\infty$ -contracted functor.

Then, in the homotopy category (of functors  $\text{Add-Cat} \rightarrow \text{Spectra}$ ), the transformation  $\mathbf{f}$  factors uniquely through  $\mathbf{d}(\mathbf{E}): \mathbf{E} \rightarrow \mathbf{E}[\infty]$ :

$$\mathbf{f} = \mathbf{d}(\mathbf{F})^{-1} \circ \mathbf{f}[\infty] \circ \mathbf{d}(\mathbf{E}).$$

4. DELOOPING ALGEBRAIC  $K$ -THEORY OF ADDITIVE CATEGORIES

Now we treat our main example for  $\mathbf{E}$ , the functor  $\mathbf{K}: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$  that assigns to an additive category  $\mathcal{A}$  the connective  $K$ -theory spectrum  $\mathbf{K}(\mathcal{A})$  of  $\mathcal{A}$ .

**Definition 4.1** (Non-connective algebraic  $K$ -theory spectrum  $\mathbf{K}^\infty$ ). *We call the functor*

$$\mathbf{K}^\infty := \mathbf{K}[\infty]: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$$

*associated to  $\mathbf{K}: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$  in Definition 3.3 the non-connective algebraic  $K$ -theory functor.*

*If  $\mathcal{A}$  is an additive category, then  $K_i(\mathcal{A}) := \pi_i(\mathbf{K}^\infty(\mathcal{A}))$  is the  $i$ -th algebraic  $K$ -group of  $\mathcal{A}$  for  $i \in \mathbb{Z}$ .*

Notice that by Lemma 3.6 we could have as well defined  $\mathbf{K}^\infty$  to be  $\mathbf{K}_{\text{Idem}}[\infty]$ . In particular, by Theorem 2.7 and Theorem 3.4 (i), Definition 4.1 extends the previous definition  $K_i(\mathcal{A}) := \pi_i(\mathbf{K}_{\text{Idem}}(\mathcal{A}))$  for  $i \geq 0$  of (2.6).

We conclude from Theorem 2.4 and Theorem 3.4 (ii)

**Theorem 4.2** (Bass-Heller-Swan-Theorem for non-connective algebraic  $K$ -theory). *The Bass-Heller-Swan transformation*

$$\mathbf{BHS}: \mathbf{K}^\infty \wedge (S^1)_+ \vee \mathbf{N}_+ \mathbf{K}^\infty \vee \mathbf{N}_- \mathbf{K}^\infty \xrightarrow{\cong} \mathbf{ZK}^\infty$$

*is a weak equivalence.*

*In particular we get for every  $i \in \mathbb{Z}$  and every additive category  $\mathcal{A}$  an in  $\mathcal{A}$ -natural isomorphism*

$$K_{i-1}(\mathcal{A}) \oplus K_i(\mathcal{A}) \oplus NK_i(\mathcal{A}[t]) \oplus NK_i(\mathcal{A}[t^{-1}]) \xrightarrow{\cong} K_i(\mathcal{A}[t, t^{-1}]),$$

*where  $NK_i(\mathcal{A}[t^\pm])$  is defined as the kernel of  $K_i(\text{ev}_0^\pm): K_i(\mathcal{A}[t^\pm]) \rightarrow K_i(\mathcal{A})$ .*

We will extend Theorem 4.2 later to the twisted case.

**Remark 4.3** (Fundamental sequence). Theorem 4.2 is equivalent to the statement that there exists for each  $i \in \mathbb{Z}$  a fundamental sequence of algebraic  $K$ -groups

$$\begin{aligned} 0 \rightarrow K_i(\mathcal{A}) &\xrightarrow{(K_i(i_+), -K_i(i_-))} K_i(\mathcal{A}[t]) \oplus K_i(\mathcal{A}[t^{-1}]) \\ &\xrightarrow{K_i(j_+) \oplus K_i(j_-)} K_i(\mathcal{A}[t, t^{-1}]) \xrightarrow{\partial_i} K_{i-1}(\mathcal{A}) \rightarrow 0 \end{aligned}$$

which comes with a splitting  $s_{i-1}: K_{i-1}(\mathcal{A}) \rightarrow K_i(\mathcal{A}[t, t^{-1}])$  of  $\partial_i$ , natural in  $\mathcal{A}$ .

**Remark 4.4** (Identification with the original negative  $K$ -groups). Bass [11, page 466 and page 462] (see also [22, Chapter 3, Section 3]) defines negative  $K$ -groups  $K_i(\mathcal{A})$  for  $i = -1, -2, \dots$  inductively by putting

$$K_{i-1}(\mathcal{A}) := \text{cok} (K_i(j_+) \oplus K_i(j_-) \oplus K_i(\mathcal{A}[t]) \oplus K_i(\mathcal{A}[t^{-1}]) \rightarrow K_i(\mathcal{A}[t, t^{-1}])).$$

We conclude from Remark 4.3 that the negative  $K$ -groups of Definition 4.1 are naturally isomorphic to the negative  $K$ -groups defined by Bass.

**Remark 4.5** (Identification with the construction of Pedersen-Weibel). Pedersen-Weibel [20] construct another transformation  $\mathbf{K}_{\text{PW}}: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$  which models negative algebraic  $K$ -theory. We conclude from Corollary 3.7 that there exists weak equivalences

$$\mathbf{K}^\infty \xrightarrow{\cong} \mathbf{K}_{\text{PW}}[\infty] \xleftarrow{\cong} \mathbf{K}_{\text{PW}}$$

since there is a natural map  $\mathbf{K} \rightarrow \mathbf{K}_{\text{PW}}$  inducing on  $\pi_i$  bijections for  $i \geq 1$  and the Bass-Heller-Swan map for  $\mathbf{K}_{\text{PW}}$  is a weak equivalence as  $\pi_i(\mathbf{K}_{\text{PW}}[\infty])$  agrees in a natural way with the  $i$ -th homotopy groups of the connective  $K$ -theory for  $i \geq 1$  and with the negative  $K$ -groups of Bass for  $i \leq 0$ , see [20, Theorem A].

## 5. COMPATIBILITY WITH COLIMITS

Let  $\mathcal{J}$  be a small category, not necessarily filtered or finite. Recall the notation

$$\text{func}_c(\text{Add-Cat}, \text{Spectra})$$

from Definition 1.6. Consider a  $\mathcal{J}$ -diagram in  $\text{func}_c(\text{Add-Cat}, \text{Spectra})$ , i.e., a covariant functor  $\mathbf{E}: \mathcal{J} \rightarrow \text{func}_c(\text{Add-Cat}, \text{Spectra})$ . There is the functor homotopy colimit

$$\text{hocolim}_{\mathcal{J}}: \text{func}_c(\mathcal{J}, \text{Spectra}) \rightarrow \text{Spectra}$$

which sends a  $\mathcal{J}$ -diagram of spectra, i.e., a covariant functor  $\mathcal{J} \rightarrow \text{Spectra}$ , to its homotopy colimit. As a consequence of Lemma 1.7, it induces a functor, denoted by the same symbol

$$\text{hocolim}_{\mathcal{J}}: \text{func}(\mathcal{J}, \text{func}_c(\text{Add-Cat}, \text{Spectra})) \rightarrow \text{func}_c(\text{Add-Cat}, \text{Spectra}),$$

that sends a  $\mathcal{J}$ -diagram  $(\mathbf{E}(j))_{j \in \mathcal{J}}$  to the functor  $\text{Add-Cat} \rightarrow \text{Spectra}$  which assigns to an additive category  $\mathcal{A}$  the spectrum  $\text{hocolim}_{\mathcal{J}} \mathbf{E}(j)(\mathcal{A})$ .

**Theorem 5.1** (Compatibility of the delooping construction with homotopy colimits). *Given a  $\mathcal{J}$ -diagram  $\mathbf{E}$  in  $\text{func}_c(\text{Add-Cat}, \text{Spectra})$ , there is a morphism in  $\text{func}_c(\text{Add-Cat}, \text{Spectra})$ , natural in  $\mathbf{E}$ ,*

$$\gamma(\mathbf{E}): \text{hocolim}_{\mathcal{J}}(\mathbf{E}(j)[\infty]) \xrightarrow{\cong} (\text{hocolim}_{\mathcal{J}} \mathbf{E}(j))[\infty]$$

*that is a weak equivalence, i.e., its evaluation at any object in  $\text{Add-Cat}$  is a weak homotopy equivalence of spectra.*

The proof uses some well-known properties of homotopy colimits of spectra, which we record here for the reader's convenience.

**Lemma 5.2.** *Let  $\mathbf{E}$  and  $\mathbf{F}$  be  $\mathcal{J}$ -diagrams of spectra and let  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$  be a morphism between them.*

(i) *The canonical map*

$$(\text{hocolim}_{\mathcal{J}} \mathbf{E}) \vee (\text{hocolim}_{\mathcal{J}} \mathbf{F}) \xrightarrow{\cong} \text{hocolim}_{\mathcal{J}}(\mathbf{E} \vee \mathbf{F})$$

*is an isomorphism;*

(ii) *If  $Y$  is a pointed space, then we obtain an isomorphism, natural in  $\mathbf{E}$ ,*

$$\text{hocolim}_{\mathcal{J}}(\mathbf{E} \wedge Y) \xrightarrow{\cong} (\text{hocolim}_{\mathcal{J}} \mathbf{E}) \wedge Y;$$

(iii) *There is a weak homotopy equivalence, natural in  $\mathbf{E}$ ,*

$$\text{hocolim}_{\mathcal{J}}(\Omega \mathbf{E}) \xrightarrow{\cong} \Omega(\text{hocolim}_{\mathcal{J}} \mathbf{E});$$

(iv) *If  $\mathcal{K}$  is another small category and we have a  $\mathbf{J} \times \mathbf{K}$  diagram  $\mathbf{E}$  of spectra. Then we have isomorphisms of spectra, natural in  $\mathbf{E}$ ,*

$$\text{hocolim}_{\mathcal{J}}(\text{hocolim}_{\mathcal{K}} \mathbf{E}) \xrightarrow{\cong} \text{hocolim}_{\mathcal{J} \times \mathcal{K}} \mathbf{E} \xleftarrow{\cong} \text{hocolim}_{\mathcal{K}}(\text{hocolim}_{\mathcal{J}} \mathbf{E});$$

(v) *Let  $\text{hofib}(\mathbf{f})$  and  $\text{hocolim}(\mathbf{f})$  respectively be the  $\mathcal{J}$ -diagram of spectra which assigns to an object  $j$  in  $\mathcal{J}$  the homotopy fiber and homotopy cofiber respectively of  $\mathbf{f}(j): \mathbf{E}(j) \rightarrow \mathbf{F}(j)$ .*

*Then there are weak homotopy equivalences, natural in  $\mathbf{f}$ ,*

$$\text{hocolim}_{\mathcal{J}} \text{hofib}(\mathbf{f}) \xrightarrow{\cong} \text{hofib}(\text{hocolim}_{\mathcal{J}} \mathbf{f});$$

$$\text{hocolim}_{\mathcal{J}} \text{hocolim}(\mathbf{f}) \xrightarrow{\cong} \text{hocolim}(\text{hocolim}_{\mathcal{J}} \mathbf{f});$$

- (vi) If  $\mathcal{J}$  is filtered, i.e., for any two objects  $j_0$  and  $j_1$  there exists a morphism  $u: j \rightarrow j'$  in  $\mathcal{J}$  such that there exists morphisms from both  $j_0$  and  $j_1$  to  $j$  and for any two morphisms  $u_0: j_0 \rightarrow j$  and  $u_1: j_1 \rightarrow j$  we have  $u \circ j_0 = u \circ u_1$ . Then the canonical map

$$\operatorname{colim}_{\mathcal{J}} \pi_i(\mathbf{E}(j)) \xrightarrow{\cong} \pi_i(\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}(j))$$

is bijective for all  $i \in \mathbb{Z}$ .

*Proof of Theorem 5.1.* Let  $\mathbf{E}$  be a  $\mathcal{J}$ -diagram in  $\operatorname{func}_c(\operatorname{Add-Cat}, \operatorname{Spectra})$ . We have by definition the equalities

$$\begin{aligned} \mathbf{Z}(\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}) &= \operatorname{hocolim}_{\mathcal{J}} (\mathbf{Z}\mathbf{E}); \\ \mathbf{Z}_{\pm}(\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}) &= \operatorname{hocolim}_{\mathcal{J}} (\mathbf{Z}_{\pm}\mathbf{E}). \end{aligned}$$

We obtain from Lemma 5.2 (ii) and (v) natural weak homotopy equivalences

$$\begin{aligned} \operatorname{hocolim}_{\mathcal{J}} (N_{\pm}\mathbf{E}) &\xrightarrow{\simeq} \mathbf{N}_{\pm}(\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}); \\ (\operatorname{hocolim}_{\mathcal{J}} (\mathbf{E} \wedge (S^1)_+)) &\xrightarrow{\simeq} (\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}) \wedge (S^1)_+, \end{aligned}$$

and thus by Lemma 5.2 (i) a natural weak homotopy equivalence

$$\operatorname{hocolim}_{\mathcal{J}} (\mathbf{B}_r\mathbf{E}) \xrightarrow{\simeq} \mathbf{B}_r(\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}).$$

As  $\mathcal{J}$  takes values in functors satisfying Condition 1.1, the maps  $\mathcal{A}(j): \mathbf{E}(j) \wedge (S^1)_+ \rightarrow \mathbf{Z}\mathbf{E}(j)$  are natural in  $j \in \mathcal{J}$ . In this way  $\mathbf{L}\mathbf{E}(j)$  also becomes a functor in  $j$ , and further applications of Lemma 5.2 show that the induced map

$$\operatorname{hocolim}_{\mathcal{J}} \mathbf{L}\mathbf{E} \xrightarrow{\simeq} \mathbf{L}(\operatorname{hocolim}_{\mathcal{J}} \mathbf{E})$$

is a weak equivalence. We obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim}_{\mathcal{J}} \mathbf{E} & \xrightarrow{\operatorname{id}} & (\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}) \\ \downarrow \operatorname{hocolim}_{\mathcal{J}} \mathbf{s} & & \downarrow \mathbf{s} \\ \operatorname{hocolim}_{\mathcal{J}} \Omega \mathbf{L}\mathbf{E} & \xrightarrow{\simeq} & \Omega \mathbf{L}(\operatorname{hocolim}_{\mathcal{J}} \mathbf{E}) \end{array}$$

with weak homotopy equivalences as vertical arrows.

Iterating this construction leads to a commutative diagram with weak homotopy equivalences as vertical arrows.

$$\begin{array}{ccccccc} \operatorname{hocolim}_{\mathcal{J}} \mathbf{E} & \xrightarrow{\operatorname{hocolim}_{\mathcal{J}} \mathbf{s}[0]} & \operatorname{hocolim}_{\mathcal{J}} \mathbf{E}[1] & \xrightarrow{\operatorname{hocolim}_{\mathcal{J}} \mathbf{s}[1]} & \operatorname{hocolim}_{\mathcal{J}} \mathbf{E}[2] & \xrightarrow{\operatorname{hocolim}_{\mathcal{J}} \mathbf{s}[2]} & \dots \\ \downarrow \operatorname{id} & & \downarrow \simeq & & \downarrow \simeq & & \\ \operatorname{hocolim}_{\mathcal{J}} \mathbf{E} & \xrightarrow{\mathbf{s}[0]} & (\operatorname{hocolim}_{\mathcal{J}} \mathbf{E})[1] & \xrightarrow{\mathbf{s}[1]} & (\operatorname{hocolim}_{\mathcal{J}} \mathbf{E})[2] & \xrightarrow{\mathbf{s}[2]} & \dots \end{array}$$

After an application of Lemma 5.2 (iv) the induced map on homotopy colimits is the desired map  $\gamma(\mathbf{E})$ .  $\square$

## 6. THE BASS-HELLER-SWAN DECOMPOSITION FOR ADDITIVE CATEGORIES WITH AUTOMORPHISMS

In [19, Theorem 0.4] the following twisted Bass-Heller-Swan decomposition is proved for the connective  $K$ -theory spectrum.

**Theorem 6.1** (The Bass-Heller-Swan decomposition for connective  $K$ -theory of additive categories). *Let  $\mathcal{A}$  be an additive category which is idempotent complete. Let  $\Phi: \mathcal{A} \rightarrow \mathcal{A}$  be an automorphism of additive categories.*

(i) Then there is a weak equivalence of spectra, natural in  $(\mathcal{A}, \Phi)$ ,

$$\mathbf{a} \vee \mathbf{b}_+ \vee \mathbf{b}_- : \mathbf{T}_{\mathbf{K}(\Phi^{-1})} \vee \mathbf{NK}(\mathcal{A}_\Phi[t]) \vee \mathbf{NK}(\mathcal{A}_\Phi[t^{-1}]) \xrightarrow{\sim} \mathbf{K}(\mathcal{A}_\Phi[t, t^{-1}]);$$

(ii) There exist weak homotopy equivalences of spectra, natural in  $(\mathcal{A}, \Phi)$ ,

$$\begin{aligned} \Omega \mathbf{NK}(\mathcal{A}_\Phi[t]) &\xleftarrow{\sim} \mathbf{E}(\mathcal{A}, \Phi); \\ \mathbf{K}(\mathcal{A}) \vee \mathbf{E}(\mathcal{A}, \Phi) &\xrightarrow{\sim} \mathbf{K}(\text{Nil}(\mathcal{A}, \Phi)). \end{aligned}$$

Here  $\mathbf{T}_{\mathbf{K}(\Phi^{-1})}$  is the mapping torus of the map  $\mathbf{K}(\Phi^{-1}) : \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A})$ , that is, the pushout of

$$\mathbf{K}(\mathcal{A}) \wedge [0, 1]_+ \xleftarrow{\text{id} \wedge \text{incl}} \mathbf{K}(\mathcal{A}) \wedge \{0, 1\}_+ \xrightarrow{\mathbf{K}(\Phi^{-1}) \vee \text{id}} \mathbf{K}(\mathcal{A}).$$

The spectrum  $\mathbf{NK}(\mathcal{A}_\Phi[t^{\pm 1}])$  is by definition the homotopy fiber of the map

$$\mathbf{K}(\mathcal{A}_\Phi[t^{\pm 1}]) \rightarrow \mathbf{K}(\mathcal{A})$$

induced by evaluation at 0. The category  $\text{Nil}(\mathcal{A}, \Phi)$  is the exact category of  $\Phi$ -nilpotent endomorphisms of  $\mathcal{A}$  whose objects are morphisms  $f : \Phi(A) \rightarrow A$ , with  $A \in \text{ob}(\mathcal{A})$  which are nilpotent in a suitable sense. For more details of the construction of the spectra and maps appearing the result above, we refer to [19, Theorem 0.1]. In that paper it is also claimed that Theorem 6.1 implies by the delooping construction of this paper in a formal manner the following non-connective version, where the maps  $\mathbf{a}^\infty$ ,  $\mathbf{b}_+^\infty$ , and  $\mathbf{b}_-^\infty$  are defined completely analogous to the maps  $\mathbf{a}$ ,  $\mathbf{b}_+$ ,  $\mathbf{b}_-$ , but now for  $\mathbf{K}^\infty$  instead of  $\mathbf{K}$ .

**Theorem 6.2** (The Bass-Heller-Swan decomposition for non-connective  $K$ -theory of additive categories). *Let  $\mathcal{A}$  be an additive category. Let  $\Phi : \mathcal{A} \rightarrow \mathcal{A}$  be an automorphism of additive categories.*

(i) There exists a weak homotopy equivalence of spectra, natural in  $(\mathcal{A}, \Phi)$ ,

$$\mathbf{a}^\infty \vee \mathbf{b}_+^\infty \vee \mathbf{b}_-^\infty : \mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})} \vee \mathbf{NK}^\infty(\mathcal{A}_\Phi[t]) \vee \mathbf{NK}^\infty(\mathcal{A}_\Phi[t^{-1}]) \xrightarrow{\sim} \mathbf{K}^\infty(\mathcal{A}_\Phi[t, t^{-1}]);$$

(ii) There exist weak homotopy equivalences of spectra, natural in  $(\mathcal{A}, \Phi)$ ,

$$\begin{aligned} \Omega \mathbf{NK}^\infty(\mathcal{A}_\Phi[t]) &\xleftarrow{\sim} \mathbf{E}^\infty(\mathcal{A}, \Phi); \\ \mathbf{K}^\infty(\mathcal{A}) \vee \mathbf{E}^\infty(\mathcal{A}, \Phi) &\xrightarrow{\sim} \mathbf{K}_{\text{Nil}}^\infty(\mathcal{A}, \Phi), \end{aligned}$$

where  $\mathbf{K}_{\text{Nil}}^\infty(\mathcal{A}, \Phi)$  is a specific spectrum whose connective cover is  $\mathbf{K}(\text{Nil}(\mathcal{A}, \Phi))$ .

To deduce Theorem 6.2 from Theorem 6.1, we will think of all functors appearing as functors in the pair  $(\mathcal{A}, \Phi)$  and apply a variation of the delooping construction to these functors. In particular, the spectrum  $\mathbf{K}_{\text{Nil}}^\infty(\mathcal{A}, \Phi)$  is obtained by delooping the functor

$$(\mathcal{A}, \Phi) \mapsto \mathbf{K}(\text{Nil}(\mathcal{A}, \Phi)).$$

The next remark formalizes how to do deloop functors in  $(\mathcal{A}, \Phi)$ .

**Remark 6.3** (Extension of the delooping construction to diagrams). Let  $\mathcal{C}$  be a fixed small category which will become an index category. Let  $\text{Add-Cat}^\mathcal{C}$  be the category of  $\mathcal{C}$ -diagrams in  $\text{Add-Cat}$ , i.e., objects are covariant functors  $\mathcal{C} \rightarrow \text{Add-Cat}$  and morphisms are natural transformations of these.

Our delooping construction can be extended from  $\text{Add-Cat}$  to  $\text{Add-Cat}^\mathcal{C}$  as follows, provided that the obvious version of Condition 1.1 which was originally stated for  $\mathbf{E} : \text{Add-Cat} \rightarrow \text{Spectra}$ , holds now for  $\mathbf{E} : \text{Add-Cat}^\mathcal{C} \rightarrow \text{Spectra}$ .

The functors  $z[t]$ ,  $z[t^{-1}]$ ,  $z[t, t^{-1}]$  from  $\text{Add-Cat}$  to  $\text{Add-Cat}$ , extend to functors  $z[t]^\mathcal{C}$ ,  $z[t^{-1}]^\mathcal{C}$ ,  $z[t, t^{-1}]^\mathcal{C} : \text{Add-Cat}^\mathcal{C} \rightarrow \text{Add-Cat}^\mathcal{C}$  by composition. Analogously the natural transformations  $i_0$ ,  $i_\pm$ ,  $j_\pm$  and  $\text{ev}_0^\pm$ , originally defined for  $\text{Add-Cat}$ , do extend to natural transformations of functors  $\text{Add-Cat}^\mathcal{C} \rightarrow \text{Add-Cat}^\mathcal{C}$ . Now all



functors defined for a covariant functor  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  in Notation 1.3 make still sense if we start with a functors  $\mathbf{E}: \text{Add-Cat}^c \rightarrow \text{Spectra}$  and end up with functors  $\text{Add-Cat}^c \rightarrow \text{Spectra}$ , where it is to be understood that everything is compatible with Condition 1.1. Moreover, the notion of a  $c$ -contracted functor, the construction of  $\mathbf{E}[\infty]$ , Lemma 2.2, Theorem 3.4, Corollary 3.7 and Theorem 5.1 carry over word by word if we replace  $\text{Add-Cat}$  by  $\text{Add-Cat}^c$  everywhere. From the definitions we also conclude:

**Lemma 6.4.** *Let  $G: \text{Add-Cat}^c \rightarrow \text{Add-Cat}$  be a functor. Suppose that there are natural isomorphisms in a commutative diagram*

$$\begin{array}{ccc} G \circ z[t]^c & \xrightarrow{\cong} & z[t] \circ G \\ \downarrow G(j_+) & & \downarrow j_+ \\ G \circ z[t^{-1}]^c & \xrightarrow{\cong} & z[t^{-1}] \circ G; \\ \uparrow G(j_-) & & \uparrow j_- \\ G \circ z[t, t^{-1}]^c & \xrightarrow{\cong} & z[t, t^{-1}] \circ G \end{array}$$

Let  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  be a functor respecting Condition 1.1.

Then  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  is  $c$ -contracted if and only if  $\mathbf{E} \circ G: \text{Add-Cat}^c \rightarrow \text{Spectra}$  is  $c$ -contracted, and we have a natural isomorphism

$$\mathbf{E}[\infty] \circ G \cong (\mathbf{E} \circ G)[\infty].$$

We will always be interested in the case where  $\mathcal{C}$  is the groupoid with one object and  $\mathbb{Z}$  as automorphism group of this object. We will write  $\text{Add-Cat}_t$  for  $\text{Add-Cat}^c$  in this case. Then objects in  $\text{Add-Cat}_t$  are pairs  $(\mathcal{A}, \Phi)$  consisting of a small additive category  $\mathcal{A}$  and an automorphism  $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$  and a morphism  $F: (\mathcal{A}_0, \Phi_0) \rightarrow (\mathcal{A}_1, \Phi_1)$  is a functor of additive categories  $F: \mathcal{A}_0 \rightarrow \mathcal{A}_1$  satisfying  $\Phi_1 \circ F = F \circ \Phi_0$ .

Our main examples for functors  $G$  as appearing in Lemma 6.4 will be the functors

$$\begin{aligned} z_t[s^{\pm 1}]: \text{Add-Cat}_t &\rightarrow \text{Spectra}, & (\mathcal{A}, \Phi) &\mapsto \mathcal{A}_\Phi[s^{\pm 1}], \\ z_t[s, s^{-1}]: \text{Add-Cat}_t &\rightarrow \text{Spectra}, & (\mathcal{A}, \Phi) &\mapsto \mathcal{A}_\Phi[s, s^{-1}]. \end{aligned}$$

(The subscript “ $t$ ” stands for “twisted”, since these functors are the obvious twisted generalizations of the functors  $z[t^{\pm 1}]$  and  $z[t, t^{-1}]$  from Section 1, with the variable  $t$  replaced by  $s$  for the sake of readability.)

Let us go back to the situation of Section 1 where we were given a functor  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  satisfying Condition 1.1. Replacing  $z[t^{\pm 1}]$  and  $z[t, t^{-1}]$  by their twisted versions throughout, we may define the twisted versions

$$\mathbf{B}^t \mathbf{E}, \mathbf{N}_\pm^t \mathbf{E}, \mathbf{Z}_\pm^t \mathbf{E}, \mathbf{Z} \mathbf{E}: \text{Add-Cat}_t \rightarrow \text{Spectra}$$

of the corresponding functors appearing in Section 1. The role of the functor  $\mathbf{E} \wedge (S^1)_+$  is now taken by the functor

$$\mathbf{T}^t \mathbf{E}(\mathcal{A}, \Phi) = \mathbf{T}_{\mathbf{E}(\Phi^{-1})}$$

given by the mapping torus of the map  $\mathbf{E}(\Phi^{-1}): \mathbf{E}(\mathcal{A}) \rightarrow \mathbf{E}(\mathcal{A})$ . Condition 1.1 implies in this setting that there is a natural transformation

$$\mathbf{a}^t: \mathbf{T}^t \mathbf{E} \rightarrow \mathbf{Z}^t \mathbf{E}.$$

(It is induced by the natural transformation

$$\text{id}_A \cdot t: \Phi^{-1}(A) \rightarrow A$$

between the functors  $\Phi^{-1} \circ i$  and  $i$ , where  $i: \mathcal{A} \rightarrow \mathcal{A}_\Phi[t, t^{-1}]$  is the canonical inclusion.)

In these terms, the natural transformation from Theorem 6.1 (i) is just given by the twisted version of the Bass-Heller-Swan map

$$\mathbf{BHS}^t: \mathbf{T}^t \mathbf{E} \vee \mathbf{N}_+^t \mathbf{E} \vee \mathbf{N}_-^t \mathbf{E} \rightarrow \mathbf{Z}^t \mathbf{E}$$

applied to  $\mathbf{E} = \mathbf{K}$ .

Next we want to apply the delooping construction to the Nil-groups.

**Lemma 6.5.** *The functor  $(\mathcal{A}, \Phi) \mapsto \mathbf{K}(\text{Nil}(\mathcal{A}, \Phi))$  is 1-contracted.*

*Proof.* The functors

$$\begin{aligned} (\mathcal{A}, \Phi) &\mapsto \mathbf{K}(\text{Nil}(\text{Idem } \mathcal{A}, \text{Idem } \Phi)), \\ (\mathcal{A}, \Phi) &\mapsto \mathbf{K}(\text{Idem } \mathcal{A}) \vee \Omega \mathbf{N} \mathbf{K}((\text{Idem } \mathcal{A})_{\text{Idem } \Phi}[t^{-1}]) \end{aligned}$$

are, by Theorem 6.1 (ii), naturally equivalent. In the second functor, the first summand is 0-contracted by Theorem 2.7. The second summand is 0-contracted by Theorem 2.4 and Lemma 6.4, noticing that  $\Omega$  decreases the degree of contraction by one. From Lemma 2.2 (ii) it follows that second functor is 0-contracted. Hence the first functor is 0-contracted, too.

There is a natural splitting

$$\mathbf{K}(\text{Nil}(\mathcal{A}, \Phi)) \simeq \mathbf{K}(\mathcal{A}) \vee \widetilde{\text{Nil}}(\mathcal{A}, \Phi)$$

induced by the obvious projection  $\text{Nil}(\mathcal{A}, \Phi) \rightarrow \mathcal{A}$  and its section  $A \mapsto (A, 0)$ . Next we show that the map induced by the inclusion

$$\widetilde{\text{Nil}}(\mathcal{A}, \Phi) \rightarrow \widetilde{\text{Nil}}(\text{Idem } \mathcal{A}, \text{Idem } \Phi)$$

is an equivalence of spectra.

Denote  $\widetilde{\text{Nil}}_i := \pi_i \widetilde{\text{Nil}}$ . In the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_i(\mathcal{A}) & \longrightarrow & K_i(\text{Nil}(\mathcal{A}, \Phi)) & \longrightarrow & \widetilde{\text{Nil}}_i(\mathcal{A}, \Phi) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_i(\text{Idem } \mathcal{A}) & \longrightarrow & K_i(\text{Nil}(\text{Idem } \mathcal{A}, \text{Idem } \Phi)) & \longrightarrow & \widetilde{\text{Nil}}_i(\text{Idem } \mathcal{A}, \text{Idem } \Phi) \longrightarrow 0 \end{array}$$

the left and middle vertical arrows are bijections for  $i \geq 1$  and injections for  $i = 0$ , by cofinality. Since both rows are split exact and the splittings are compatible with the vertical arrows, also the right arrow is bijective for  $i \geq 1$  and injective for  $i = 0$ .

We are left to show surjectivity for  $\widetilde{\text{Nil}}_0$ . So let an element of  $\widetilde{\text{Nil}}_0(\text{Idem } \mathcal{A}, \text{Idem } \Phi)$  be represented by  $((A, p), \phi)$ . Then it is also represented by  $((A, p), \phi) \oplus ((A, 1-p), 0)$  which clearly has a preimage in  $\widetilde{\text{Nil}}_0(\mathcal{A}, \Phi)$ .

Now the following diagram commutes:

$$\begin{array}{ccc} \mathbf{K}(\text{Nil}(\mathcal{A}, \Phi)) & \xrightarrow{\simeq} & \mathbf{K}(\mathcal{A}_\Phi[t]) \vee \widetilde{\text{Nil}}(\mathcal{A}, \Phi) \\ \downarrow & & \downarrow \\ \mathbf{K}(\text{Nil}(\text{Idem } \mathcal{A}, \text{Idem } \Phi)) & \xrightarrow{\simeq} & \mathbf{K}((\text{Idem } \mathcal{A})_{\text{Idem } \Phi}[t]) \vee \widetilde{\text{Nil}}(\text{Idem } \mathcal{A}, \text{Idem } \Phi) \end{array}$$

Thinking of all terms as functors in  $(\mathcal{A}, \Phi)$ , we know that the lower left term is 0-contracted. It follows from Lemma 2.2 (ii) that  $\widetilde{\text{Nil}}(\mathcal{A}, \Phi)$  is 0-contracted. Moreover the functor  $(\mathcal{A}, \Phi) \mapsto \mathbf{K}(\mathcal{A}_\Phi[t])$  is 1-contracted by Theorem 2.4 and Lemma 6.4. Applying again 2.2 (ii) proves the claim.  $\square$

Hence we may apply the delooping construction to the functor

$$(\mathcal{A}, \Phi) \mapsto \mathbf{K}(\text{Nil}(\mathcal{A}, \Phi))$$

to obtain a new functor  $\mathbf{K}_{\text{Nil}}^\infty(\mathcal{A}, \Phi)$ . It follows from cofinality and Lemma 3.6 that the map induced by the inclusion

$$\mathbf{K}_{\text{Nil}}^\infty(\mathcal{A}, \Phi) \rightarrow \mathbf{K}_{\text{Nil}}^\infty(\text{Idem } \mathcal{A}, \text{Idem } \Phi)$$

is a weak equivalence.

*Proof of Theorem 6.2.* (i) As  $\mathbf{K}$  satisfies Condition 1.1, the same is true for  $\mathbf{T}^t \mathbf{K}$  and  $\mathbf{N}_\pm^t \mathbf{E}$  and hence for their wedge. So we may apply the delooping construction to the transformation  $\mathbf{BHS}^t$ ; using compatibility with the smash product, we get from Theorem 6.1 (i) a natural homotopy equivalence

$$(6.6) \quad \mathbf{BHS}^t[\infty]: (\mathbf{T}^t \mathbf{K})[\infty] \vee (\mathbf{N}_+^t \mathbf{K})[\infty] \vee (\mathbf{N}_-^t \mathbf{K})[\infty] \xrightarrow{\sim} (\mathbf{Z}^t \mathbf{K})[\infty].$$

An application of Lemma 3.6 to the functors  $z_t[s, s^{-1}]$  and  $z_t[s^{\pm 1}]$  implies that

$$(6.7) \quad (\mathbf{Z}^t \mathbf{K})[\infty] \cong \mathbf{Z}^t \mathbf{K}^\infty, \quad (\mathbf{N}_\pm^t \mathbf{K})[\infty] \cong \mathbf{N}_\pm^t \mathbf{K}^\infty.$$

By definition, the mapping torus is a homotopy pushout; the compatibility of the delooping construction with homotopy colimits (Theorem 5.1) implies that the canonical transformation

$$\alpha: \mathbf{T}^t \mathbf{K}^\infty \rightarrow (\mathbf{T}^t \mathbf{K})[\infty]$$

is a weak equivalence. Thus, from (6.6) we obtain a natural homotopy equivalence

$$\mathbf{a}[\infty] \circ \alpha \vee \mathbf{b}_+^\infty \vee \mathbf{b}_-^\infty: \mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})} \vee \mathbf{NK}^\infty(\mathcal{A}_\Phi[t]) \vee \mathbf{NK}^\infty(\mathcal{A}_\Phi[t^{-1}]) \xrightarrow{\sim} \mathbf{K}^\infty(\mathcal{A}_\Phi[t, t^{-1}]);$$

It remains to show that the map  $\mathbf{a}[\infty] \circ \alpha$  defined in this way agree with the map  $\mathbf{a}^\infty$ , that is, the map

$$\mathbf{a}^t: \mathbf{T}^t \mathbf{E} \rightarrow \mathbf{Z}^t \mathbf{E}$$

for  $\mathbf{E} = \mathbf{K}^\infty$  as a functor which satisfies Condition 1.1. In fact, the diagram

$$\begin{array}{ccc} \mathbf{T}^t(\Omega \mathbf{L} \mathbf{E}) & \xrightarrow{\cong} & \Omega \mathbf{L} \mathbf{T}^t \mathbf{E} \\ \downarrow \mathbf{a}_{\mathbf{L} \mathbf{E}} & & \downarrow \mathbf{L} \mathbf{a}_{\mathbf{E}} \\ \mathbf{Z}^t(\Omega \mathbf{L} \mathbf{E}) & \xrightarrow{\cong} & \Omega \mathbf{L} \mathbf{Z}^t \mathbf{E} \end{array}$$

with the canonical horizontal arrows is commutative. Iterating the construction shows that

$$\begin{array}{ccc} \mathbf{T}^t(\mathbf{E}[\infty]) & \xrightarrow{\cong} & (\mathbf{T}^t \mathbf{E})[\infty] \\ \downarrow \mathbf{a}_{\mathbf{E}[\infty]} & & \downarrow \mathbf{a}_{\mathbf{E}[\infty]} \\ \mathbf{Z}^t(\mathbf{E}[\infty]) & \xrightarrow{\cong} & (\mathbf{Z}^t \mathbf{E})[\infty] \end{array}$$

is also commutative. The lower horizontal isomorphism is (6.7); comparing with the proof of Theorem 5.1 shows that the upper horizontal map agrees with  $\alpha$ . This implies the claim.

(ii) Given  $(A, f)$  in  $\text{Nil}(\mathcal{A}, \Phi)$ , denote by  $\chi(A, f)$  the following 1-dimensional chain complex in  $\mathcal{A}_\Phi[t]$ :

$$\Phi(A) \xrightarrow{t-f} A$$

This leads to a functor  $\chi: \text{Nil}(\mathcal{A}, \Phi) \rightarrow \text{Ch}(\mathcal{A}_\Phi[t])$  into the category of bounded chain complexes; this functor induces a map

$$\mathbf{K}(\chi): \mathbf{K}(\text{Nil}(\mathcal{A}, \Phi)) \rightarrow \mathbf{K}_{\text{Ch}}(\mathcal{A}_\Phi[t])$$

where we abbreviate  $\mathbf{K}_{\text{Ch}} := \mathbf{K} \circ \text{Ch}$ . In [19, Section 8] it is shown that if  $\mathcal{A}$  is idempotent complete, then  $\mathbf{K}(\chi)$  is part of a homotopy fiber sequence

$$(6.8) \quad \mathbf{K}(\text{Nil}(\mathcal{A}, \Phi)) \rightarrow \mathbf{K}_{\text{Ch}}(\mathcal{A}_\Phi[t^{-1}]) \rightarrow \mathbf{K}_{\text{Ch}}(\mathcal{A}_\Phi[t, t^{-1}]).$$

(Note that  $\mathbf{K}_{\text{Ch}}$  is naturally equivalent to  $\mathbf{K}$  by the Gillet-Waldhausen Theorem [19, Theorem 5.1].)

Now each of the terms in this sequence, as a functor in  $(\mathcal{A}, \Phi)$ , satisfies Condition 1.1 and the maps in the sequence are compatible transformations. Applying the delooping construction to each of the terms leads to a sequence

$$(6.9) \quad \mathbf{K}_{\text{Nil}}^\infty(\mathcal{A}, \Phi) \rightarrow \mathbf{K}_{\text{Ch}}^\infty(\mathcal{A}_\Phi[t]) \rightarrow \mathbf{K}_{\text{Ch}}^\infty(\mathcal{A}_\Phi[t, t^{-1}]).$$

Note that by the Gillet-Waldhausen Theorem and Lemma 6.4, the middle and right terms are naturally homotopy equivalent to  $\mathbf{K}^\infty(\mathcal{A}_\Phi[t])$  and  $\mathbf{K}^\infty(\mathcal{A}_\Phi[t, t^{-1}])$ , respectively.

We will show that this sequence is a fibration sequence for any  $(\mathcal{A}, \Phi)$ . To do this, consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{K} \circ \text{Nil} & \longrightarrow & \mathbf{K}_{\text{Ch}} \circ \mathbf{Z}_+^t & \longrightarrow & \mathbf{K}_{\text{Ch}} \circ \mathbf{Z}^t \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{K} \circ \text{Nil} \circ \text{Idem}_t & \longrightarrow & \mathbf{K}_{\text{Ch}} \circ \mathbf{Z}_+^t \circ \text{Idem}_t & \longrightarrow & \mathbf{K}_{\text{Ch}} \circ \mathbf{Z}^t \circ \text{Idem}_t \end{array}$$

of functors whose top line at the object  $(\mathcal{A}, \Phi)$  is just (6.8), and where we define  $\text{Idem}_t(\mathcal{A}, \Phi) := (\text{Idem}(\mathcal{A}), \text{Idem}(\Phi))$ . Notice that the bottom line of this diagram, at any object  $(\mathcal{A}, \Phi)$ , is a fibration sequence.

We claim that all the functors in this diagram are 1-contracted. In fact, we showed in Lemma 6.5 (and its proof) that the left terms are 1-contracted. The middle and right upper terms are 1-contracted as the functor  $\mathbf{K} \simeq \mathbf{K}_{\text{Ch}}$  is 1-contracted.

At the object  $(\mathcal{A}, \Phi)$ , the middle vertical arrow is given by the map induced by the inclusion

$$\mathbf{K}_{\text{Ch}}(\mathcal{A}_\Phi[t]) \rightarrow \mathbf{K}_{\text{Ch}}((\text{Idem} \mathcal{A})_{\text{Idem}(\Phi)}[t]).$$

In particular, by cofinality, it induces an isomorphism in degrees  $\geq 1$ . When precomposed with  $\mathbf{Z}_+$ , it becomes

$$(6.10) \quad \mathbf{K}_{\text{Ch}}((\mathcal{A}[s])_\Phi[t]) \rightarrow \mathbf{K}_{\text{Ch}}((\text{Idem}(\mathcal{A}[s]))_{\text{Idem}(\Phi)}[t]).$$

Now the idempotent completion of  $(\mathcal{A}[s])_\Phi[t]$  is also an idempotent completion of  $(\text{Idem}(\mathcal{A}[s]))_{\text{Idem}(\Phi)}[t]$ . As  $\pi_n \mathbf{K}$  is invariant under idempotent completions, we see that (6.10) induces isomorphisms in homotopy groups of degree  $\geq 1$ .

The same argument works with  $\mathbf{Z}_+$  replaced by  $\mathbf{Z}_-$  and  $\mathbf{Z}$ . We conclude that the restricted Bass-Heller-Swan maps for the upper and the lower middle terms in the diagram are isomorphic in degree  $\geq 1$ . Smashing with  $(S^1)_+$  preserves connectivity; so the unrestricted Bass-Heller-Swan maps for the middle terms are also isomorphic in degrees  $\geq 2$ . Thus, to show that the lower middle term is 1-contracted, we are left to show split injectivity of the restricted Bass-Heller-Swan map in degree 0.

This map is given by

$$\begin{aligned} \pi_0 \mathbf{K}((\text{Idem}(\mathcal{A})_\Phi[t]) \oplus \pi_0 \mathbf{K}(\text{Idem}(\mathcal{A}[s])_\Phi[t]) \oplus \pi_0 \mathbf{K}(\text{Idem}(\mathcal{A}[s^{-1}])_\Phi[t]) \\ \rightarrow \pi_0 \mathbf{K}((\text{Idem}(\mathcal{A}[s, s^{-1}])_\Phi[t]). \end{aligned}$$

Split injectivity holds as  $\pi_0 \mathbf{K}(\mathcal{A}) \cong \pi_0 \mathbf{K}(\mathcal{A}_\Phi[t])$  for any  $(\mathcal{A}, \Phi)$  (as in the proof of Theorem 2.4), and  $\mathbf{K}_{\text{Idem}}$  is 0-contracted.

Thus the lower middle term is 1-contracted and the middle vertical map induces an isomorphism in degrees  $\geq 1$ . The corresponding statements hold for the right terms in the diagram, by the very same arguments.

Applying the delooping construction to the whole diagram, we obtain a new diagram whose top line, at  $(\mathcal{A}, \Phi)$  is given by (6.9), and whose bottom line is still a fiber sequence by Theorem 5.1. By Lemma 3.6 all the vertical maps are weak

homotopy equivalences. We conclude that the upper line is a fibration sequence. This is what we claimed, for the upper line, at the object  $(\mathcal{A}, \Phi)$ , is just (6.9).

In [19, Section 3] it is shown that part (ii) of Theorem 6.1 follows formally from (6.8). The same arguments apply to prove that part (ii) of Theorem 6.2 follows from (6.9).  $\square$

**Remark 6.11** (Schlichting's non-connective  $K$ -theory spectrum for exact categories). Notice that  $\text{Nil}(\mathcal{A}, \Phi)$  is an exact category whose exact structure does not come from the structure of an additive category. Schlichting [24] has defined non-connective  $K$ -theory for exact categories. It is very likely that Schlichting's non-connected  $K$ -theory applied to the exact category  $\text{Nil}(\mathcal{A}, \Phi)$  is weakly homotopy equivalent to our non-connective version  $\mathbf{K}_{\text{Nil}}^\infty(\mathcal{A}, \Phi)$  in a natural way. This would follow from Corollary 3.7 if Schlichting's non-connective  $K$ -theory of  $\text{Nil}(\mathcal{A}, \Phi)$  is  $\infty$ -contracted, or, equivalently, has a Bass-Heller-Swan decomposition.

It is conceivable that the twisted Bass-Heller-Swan decomposition for connective  $K$ -theory, which is described in Theorem 6.1 and whose proof is given in [19], can be extended directly to the non-connective setting described in Theorem 6.2 using Schlichting's non-connective version of  $K$ -theory for exact categories. This would require that fundamental results such as the Fibration and Approximation Theorem of Waldhausen for connective  $K$ -theory can be established in Schlichting's non-connective setting.

The main advantage of Schlichting's construction of non-connective  $K$ -theory is that it applies to all exact categories, while our construction has the charm of being elementary and possessing an accessible universal property.

## 7. FILTERED COLIMITS

Suppose that the small category  $\mathcal{J}$  is filtered, i.e., for any two objects  $j_0$  and  $j_1$  there exists a morphism  $u: j \rightarrow j'$  in  $\mathcal{J}$  with the following property: There exist morphisms from both  $j_0$  and  $j_1$  to  $j$ , and for any two morphisms  $u_0: j_0 \rightarrow j$  and  $u_1: j_1 \rightarrow j$  we have  $u \circ j_0 = u \circ j_1$ . Given a functor  $\mathcal{A}: \mathcal{J} \rightarrow \text{Add-Cat}$ , its *colimit*  $\text{colim } \mathcal{A}$  in the category of small categories exists and is in a natural way an additive category.

(We do not need an explicit description, but one can see that

$$\text{ob}(\text{colim } \mathcal{A}) = \text{colim}(\text{ob}(\mathcal{A}))$$

and that the abelian group of morphisms from  $A$  to  $B$  is given by

$$(\text{colim } \mathcal{A})(A, B) = \text{colim}_{j \in \mathcal{J}} \left( \bigoplus_{A_j, B_j} \mathcal{A}(j)(A_j, B_j) \right)$$

where the coproducts range over all objects  $A_j, B_j \in \mathcal{A}(j)$  projecting to  $A$  resp.  $B$ .)

For a functor  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$ , we say that  $\mathbf{E}$  *commutes with filtered colimits* if for any  $\mathcal{A}: \mathcal{J} \rightarrow \text{Add-Cat}$  the canonical map

$$\text{hocolim } \mathbf{E} \circ \mathcal{A} \rightarrow \mathbf{E}(\text{colim } \mathcal{A})$$

is a weak homotopy equivalence. By Lemma 5.2 (vi) this is equivalent to saying that

$$\text{colim } \pi_* \mathbf{E} \circ \mathcal{A} \xrightarrow{\cong} \pi_* \mathbf{E}(\text{colim } \mathcal{A}).$$

**Proposition 7.1.** *Suppose that  $\mathbf{E}$  commutes with filtered colimits and satisfies Condition 1.1. Then  $\mathbf{E}[\infty]$  commutes with filtered colimits.*

Quillen's (or Waldhausen's) connective  $K$ -theory spectrum  $\mathbf{K}$  commutes with filtered colimits [21, (9) on page 20]. Letting  $K_i^\infty(\mathcal{A}) := \pi_i(\mathbf{K}^\infty(\mathcal{A}))$ , we conclude:

**Corollary 7.2.** *If  $\mathcal{J}$  is filtered and  $\mathcal{A}: \mathcal{J} \rightarrow \text{Add-Cat}$  is a covariant functor, then the canonical homomorphism*

$$\text{colim}_{\mathcal{J}} K_i^{\infty}(\mathcal{A}) \rightarrow K_i^{\infty}(\text{colim}_{\mathcal{J}} \mathcal{A})$$

*is bijective for all  $i \in \mathbb{Z}$ .*

Schlichting proves compatibility of negative  $K$ -theory with colimits in [25, Corollary 5], cf. Section 7.

*Proof of Proposition 7.1.* Let  $\mathcal{A}: \mathcal{J} \rightarrow \text{Add-Cat}$ . It follows from the definition of the categories  $\mathcal{A}[t^{\pm 1}]$  and  $\mathcal{A}[t, t^{-1}]$  that the canonical functors

$$\text{colim}_j(\mathcal{A}(j)[t^{\pm 1}]) \rightarrow (\text{colim } \mathcal{A})[t^{\pm 1}], \quad \text{colim}_j(\mathcal{A}(j)[t, t^{-1}]) \rightarrow (\text{colim } \mathcal{A})[t, t^{-1}]$$

are isomorphisms. It follows that the functors  $\mathbf{Z}_{\pm} \mathbf{E}$  and  $\mathbf{Z} \mathbf{E}$  commute with filtered colimits, too.

Lemma 5.2 implies then that for  $\mathbf{F} \in \{\mathbf{N}_{\pm}, \mathbf{B}_r, \mathbf{B}, \mathbf{L}\}$ ,  $\mathbf{F} \mathbf{E}$  commutes with filtered colimits. The square

$$\begin{array}{ccc} \text{hocolim } \mathbf{E} \circ \mathcal{A} & \xrightarrow{\text{hocolim } \mathbf{s}} & \text{hocolim } \Omega \mathbf{L} \mathbf{E} \circ \mathcal{A} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbf{E}(\text{colim } \mathcal{A}) & \xrightarrow{\mathbf{s}} & \Omega \mathbf{L} \mathbf{E}(\text{colim } \mathcal{A}) \end{array}$$

with canonical vertical arrows is commutative. Iterating this construction and applying Lemma 5.2 (iv), we see that

$$\text{hocolim } \mathbf{E}[\infty] \circ \mathcal{A} \rightarrow \mathbf{E}[\infty](\text{colim } \mathcal{A})$$

is a weak equivalence. □

**Remark 7.3.** In [1, Theorem 1.8 (i) on page 43] it is shown that the Farrell-Jones Isomorphism Conjecture is inherited under filtered colimits of groups (with not necessarily injective structure maps), but only for rings as coefficients. The same statement remains true if one allows coefficients in additive categories, as stated in [5, Corollary 0.8]. The proof of [1, Theorem 1.8 (i) on page 43] in the  $K$ -theory case carries over directly as soon as one has Theorem 7.2 available, it is needed in the extension of [5, Lemma 6.2 on page 61] to additive categories. The analog of Theorem 7.2 for  $L$ -theory is obvious since the  $L$ -groups of an additive category with involution can be defined elementwise instead of referring to the homotopy groups of a spectrum.

## 8. HOMOTOPY $K$ -THEORY

Let  $\mathbf{E}: \text{Add-Cat} \rightarrow \text{Spectra}$  be a (covariant) functor. Denote by  $F_+: \text{Add-Cat} \rightarrow \text{Add-Cat}$  the functor sending  $\mathcal{A}$  to  $\mathcal{A}[t]$ . Denote by  $\mathbf{F}_+ \mathbf{E}: \text{Add-Cat} \rightarrow \text{Add-Cat}$  the composite  $\mathbf{E} \circ F_+$ . The natural inclusion  $i_+: \mathcal{A} \rightarrow \mathcal{A}[t]$ , which sends a morphism  $f: A \rightarrow B$  to  $f \cdot t^0: A \rightarrow B$ , induces a natural transformation  $\mathbf{i}_+: \mathbf{E} \rightarrow \mathbf{F}_+ \mathbf{E}$  of functors  $\text{Add-Cat} \rightarrow \text{Spectra}$ . Define inductively the functor  $\mathbf{F}_+^n: \text{Add-Cat} \rightarrow \text{Spectra}$  by  $\mathbf{F}_+^n \mathbf{E} := \mathbf{F}_+(\mathbf{F}_+^{n-1} \mathbf{E})$  starting with  $\mathbf{F}_+^0 = \mathbf{E}$ . Define inductively  $\mathbf{i}_+^n: \mathbf{F}_+^{n-1} \mathbf{E} \rightarrow \mathbf{F}_+^n \mathbf{E}$  by  $\mathbf{i}_+^n = \mathbf{i}_+(\mathbf{i}_+^{n-1})$  starting with  $\mathbf{i}_+^1 := \mathbf{i}_+$ . Thus we obtain a sequence of transformations of functors  $\text{Add-Cat} \rightarrow \text{Spectra}$

$$\mathbf{E} = \mathbf{F}_+^0 \mathbf{E} \xrightarrow{\mathbf{i}_+^1} \mathbf{F}_+^1 \mathbf{E} \xrightarrow{\mathbf{i}_+^2} \mathbf{F}_+^2 \mathbf{E} \xrightarrow{\mathbf{i}_+^3} \dots$$

**Definition 8.1** (Homotopy stabilization **HE**). Define the homotopy stabilization

$$\mathbf{HE}: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$$

of  $\mathbf{E}: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$  to be the homotopy colimit of the sequence above. Let

$$\mathbf{h}: \mathbf{E} \rightarrow \mathbf{HE}$$

be given by the zero-th structure map of the homotopy colimit. We call  $\mathbf{E}$  homotopy stable if  $\mathbf{h}$  is a weak equivalence.

This construction has the following basic properties. Let  $\text{ev}_0^+: \mathcal{A}_\Phi[t] \rightarrow \mathcal{A}$  be the functor sending  $\sum_{i \geq 0} f_i \cdot t^i$  to  $f_0$ .

**Lemma 8.2.** Let  $\mathbf{E}: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$  be a covariant functor.

- (i)  $\mathbf{HE}$  is homotopy stable;
- (ii) Suppose that  $\mathbf{E}$  is homotopy stable. Let  $\mathcal{A}$  be any additive category with an automorphism  $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ . Then the maps

$$\begin{aligned} \mathbf{E}(\text{ev}_0^+): \mathbf{E}(\mathcal{A}_\Phi[t]) &\xrightarrow{\cong} \mathbf{E}(\mathcal{A}); \\ \mathbf{E}(\mathbf{i}_+^+): \mathbf{E}(\mathcal{A}) &\xrightarrow{\cong} \mathbf{E}(\mathcal{A}_\Phi[t]), \end{aligned}$$

are weak homotopy equivalences.

*Proof.* (i) This follows from the definitions since the obvious map from the homotopy colimit of

$$\mathbf{F}_+^0 \mathbf{E} \xrightarrow{\mathbf{i}_+^1} \mathbf{F}_+^1 \mathbf{E} \xrightarrow{\mathbf{i}_+^2} \mathbf{F}_+^2 \mathbf{E} \xrightarrow{\mathbf{i}_+^3} \dots$$

to

$$\mathbf{F}_+^1 \mathbf{E} \xrightarrow{\mathbf{i}_+^2} \mathbf{F}_+^2 \mathbf{E} \xrightarrow{\mathbf{i}_+^3} \mathbf{F}_+^3 \mathbf{E} \xrightarrow{\mathbf{i}_+^4} \dots$$

given by applying  $\mathbf{i}_+$  in each degree is a weak homotopy equivalence.

(ii) Consider an additive category  $\mathcal{A}$  with an automorphism  $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ . Define a functor  $j_s: \mathcal{A}_\Phi[t] \rightarrow (\mathcal{A}_\Phi[t])[s]$  by sending  $\sum_{i \geq 0} f_i \cdot t^i$  to  $\sum_{i \geq 0} (f_i \cdot t^i) \cdot s^i$ . Let  $\text{ev}_{s=0}$  and  $\text{ev}_{s=1}$  respectively be the functors  $(\mathcal{A}_\Phi[t])[s] \rightarrow \mathcal{A}_\Phi[t]$  sending a morphism  $\sum_{i \geq 0} (\sum_{j_i \geq 0} f_{j_i, i} \cdot t^{j_i}) \cdot s^i: A \rightarrow B$  to  $\sum_{j_0 \geq 0} f_{j_0, 0} \cdot t^{j_0}: A \rightarrow B$  and  $\sum_{i \geq 0} \sum_{j_i \geq 0} f_{j_i, i} \cdot t^{j_i}: A \rightarrow B$  respectively. Recall that  $\mathbf{i}_+: \mathcal{A} \rightarrow \mathcal{A}_\Phi[t]$  is the obvious inclusion. Then

$$\text{ev}_{s=0} \circ j_s = j_+ \circ \text{ev}_0^+, \quad \text{ev}_{s=1} \circ j_s = \text{id}_{\mathcal{A}_\Phi[t]}.$$

The composite of both  $\text{ev}_{s=0}$  and  $\text{ev}_{s=1}$  with the canonical inclusion  $k_+: \mathcal{A}_\Phi[t] \rightarrow (\mathcal{A}_\Phi[t])[s]$  is the identity. Since  $\mathbf{E}$  is homotopy stable by assumption, the map  $\mathbf{E}(k_+): \mathbf{E}(\mathcal{A}_\Phi[t]) \rightarrow \mathbf{E}((\mathcal{A}_\Phi[t])[s])$  is a weak equivalence. Hence the functors  $\text{ev}_{s=0}$  and  $\text{ev}_{s=1}$  induce same homomorphism after applying  $\pi_i \circ \mathbf{E}$  for  $i \in \mathbb{Z}$ . This implies that the composite

$$\pi_i(\mathbf{E}(\mathcal{A}_\Phi[t])) \xrightarrow{\pi_i(\mathbf{E}(\text{ev}_0^+))} \pi_i(\mathbf{E}(\mathcal{A})) \xrightarrow{\pi_i(\mathbf{E}(\mathbf{i}_+))} \pi_i(\mathbf{E}(\mathcal{A}_\Phi[t]))$$

is the identity. Since  $\text{ev}_0^+ \circ \mathbf{i}_+ = \text{id}_{\mathcal{A}}$ , also the composite

$$\pi_i(\mathbf{E}(\mathcal{A})) \xrightarrow{\pi_i(\mathbf{E}(\mathbf{i}_+))} \pi_i(\mathbf{E}(\mathcal{A}_\Phi[t])) \xrightarrow{\pi_i(\mathbf{E}(\text{ev}_0^+))} \pi_i(\mathbf{E}(\mathcal{A}))$$

is the identity. Hence  $\mathbf{E}(\text{ev}_0^+): \mathbf{E}(\mathcal{A}_\Phi[t]) \rightarrow \mathbf{E}(\mathcal{A})$  is a weak homotopy equivalence.  $\square$

**Remark 8.3** (Universal property of **H**). Notice that Lemma 8.2 (i) says that up to weak homotopy equivalence the transformation  $\mathbf{h}: \mathbf{E} \rightarrow \mathbf{HE}$  is universal (from the left) among transformations  $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$  to homotopy stable functors

$\mathbf{F}: \text{Add-Cat} \rightarrow \text{Spectra}$  since we obtain a commutative square whose lower vertical arrow is a weak homotopy equivalence

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\mathbf{h}} & \mathbf{HE} \\ \downarrow \mathbf{f} & & \downarrow \mathbf{Hf} \\ \mathbf{F} & \xrightarrow[\mathbf{h}]{\simeq} & \mathbf{HF} \end{array}$$

Lemma 8.2 (ii) essentially says that homotopy stable automatically implies homotopy stable in the twisted case.

**Definition 8.4** (Homotopy  $K$ -theory). *Define the homotopy  $K$ -theory functors*

$$\mathbf{HK}, \mathbf{HK}^\infty: \text{Add-Cat} \rightarrow \text{Spectra}$$

*to be the homotopy stabilization in the sense of Definition 8.1 of the functors  $\mathbf{K}, \mathbf{K}^\infty: \text{Add-Cat} \rightarrow \text{Spectra}$ .*

**Lemma 8.5.** *The functor  $\mathbf{HK}$  is 1-contracted and there is a weak equivalence*

$$\mathbf{HK}[\infty] \xrightarrow{\simeq} \mathbf{HK}^\infty.$$

*Proof.* As

$$\pi_* \mathbf{HE}(\mathcal{A}) \cong \text{colim}_n \pi_* \mathbf{E}(\mathcal{A}[t_1, \dots, t_n])$$

we conclude that  $\mathbf{HE}$  is  $c$ -contracted provided that  $E$  is  $c$ -contracted. Applying Theorem 2.4 we see that  $\mathbf{HK}$  is 1-contracted. Also  $\mathbf{HK}^\infty$  is  $\infty$ -contracted. The claim now follows from Lemma 3.6.  $\square$

**Lemma 8.6** (Bass-Heller-Swan for homotopy  $K$ -theory). *Let  $\mathcal{A}$  be an additive category with an automorphism  $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ . Then we get for all  $n \in \mathbb{Z}$  a weak homotopy equivalence*

$$\mathbf{a}^\infty: \mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})} \xrightarrow{\simeq} \mathbf{HK}^\infty(\mathcal{A}_\Phi[t, t^{-1}]).$$

*Proof.* We conclude from Theorem 6.2 (i) and the fact that the Bass-Heller-Swan map is compatible with homotopy colimits in the spectrum variable and  $\mathbf{HK}^\infty$  is defined as a homotopy colimit in terms of  $\mathbf{K}^\infty$  that there is a weak equivalence of spectra, natural in  $(\mathcal{A}, \Phi)$ ,

$$\mathbf{a} \vee \mathbf{b}_+ \vee \mathbf{b}_-: \mathbf{T}_{\mathbf{HK}^\infty(\Phi^{-1})} \vee \mathbf{N}\mathbf{HK}^\infty(\mathcal{A}_\Phi[t]) \vee \mathbf{N}\mathbf{HK}^\infty(\mathcal{A}_\Phi[t^{-1}]) \xrightarrow{\simeq} \mathbf{HK}^\infty(\mathcal{A}_\Phi[t, t^{-1}]).$$

Since all the homotopy groups of the terms  $\mathbf{N}\mathbf{HK}^\infty(\mathcal{A}_\Phi[t])$  and  $\mathbf{N}\mathbf{HK}^\infty(\mathcal{A}_\Phi[t^{-1}])$  vanish by Lemma 8.2 (ii), Lemma 8.6 follows.  $\square$

**Remark 8.7** (Identification with Weibel's definition). Weibel has defined a connective version of homotopy  $K$ -theory for a ring  $R$  by a simplicial construction in [28]. This definition was extended, again for rings, by Bartels-Lueck [3] to the non-connective case. It is not hard to check using Remark 8.3, which applies also to the constructions of [28] and of [3] instead of  $\mathbf{H}$ , that  $\pi_i(\mathbf{HK}_{\text{Idem}}(\mathcal{R}))$  and  $\pi_i(\mathbf{HK}^\infty(\mathcal{R}))$  can be identified with the ones in [28] and [3], if  $\mathcal{R}$  is a skeleton of the category of finitely generated free  $R$ -modules.

## 9. THE FARRELL-JONES CONJECTURE FOR HOMOTOPY $K$ -THEORY

The Farrell-Jones Conjecture for (non-connective) homotopy  $K$ -theory has been treated for rings in [3]. Meanwhile it has turned out to be useful to formulate the Farrell-Jones Conjecture for additive categories as coefficients since then one has much better inheritance properties, see for instance [5] and [10]. The Farrell-Jones



Conjecture for (non-connective)  $K$ -theory for additive categories is true for a group  $G$ , if for any additive  $G$ -category  $\mathcal{A}$  the assembly map

$$H_n^G(E_{\mathcal{VC}}(G); \mathbf{K}_{\mathcal{A}}^\infty) \rightarrow H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}^\infty) = K_n(\int_G \mathcal{A})$$

is bijective for all  $n \in \mathbb{Z}$ , where  $E_{\mathcal{VC}}(G)$  is the classifying space for the family of virtually cyclic groups. If one replaces  $\mathbf{K}^\infty: \mathbf{Add-Cat} \rightarrow \mathbf{Spectra}$  by the functor  $\mathbf{HK}^\infty: \mathbf{Spectra} \rightarrow \mathbf{Add-Cat}$  and  $E_{\mathcal{VC}}(G)$  by the classifying space  $E_{\mathcal{Fin}}(G)$  for the family of finite subgroups, one obtains the Farrell-Jones Conjecture for algebraic (non-connective) homotopy  $K$ -theory with coefficients in additive categories. It predicts the bijectivity of

$$H_n^G(E_{\mathcal{Fin}}(G); \mathbf{HK}_{\mathcal{A}}^\infty) \rightarrow H_n^G(\mathrm{pt}; \mathbf{HK}_{\mathcal{A}}^\infty) = KH_n(\int_G \mathcal{A})$$

for all  $n \in \mathbb{Z}$ , where  $KH_n(\mathcal{B})$  denotes  $\pi_n((\mathbf{HK}^\infty(\mathcal{B}))$  for an additive category  $\mathcal{B}$ .

For the following result, denote by  $\mathrm{FJH}(G)$  the statement “The Farrell-Jones Conjecture for algebraic homotopy  $K$ -theory with coefficients in additive categories holds for  $G$ ”.

**Theorem 9.1** (Farrell-Jones Conjecture for homotopy  $K$ -theory).

- (i) Let  $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$  be an extensions of groups. If  $\mathrm{FJH}(Q)$  and  $\mathrm{FJH}(p^{-1}(H))$  for any finite subgroup  $H \subseteq Q$ , then  $\mathrm{FJH}(G)$ ;
- (ii) If  $G$  acts on a tree  $T$  such that  $\mathrm{FJH}(G_x)$  for every stabilizer group  $G_x$  of  $x \in T$ , then  $\mathrm{FJH}(G)$ ;
- (iii) If  $G$  satisfies the Farrell-Jones Conjecture for algebraic  $K$ -theory with coefficients in additive categories, then  $\mathrm{FJH}(G)$ .

*Proof.* (i) This follows from [3, Corollary 4.4].

(ii) It is easy to check that the arguments in Bartels-Lück [3] carry over from rings to additive categories since they are on the level of equivariant homology theories.

(iii) It follows from [13, Remark 1.6] that the assembly map

$$H_n^G(E_{\mathcal{VC}_I}(G); \mathbf{K}_{\mathcal{A}}^\infty) \rightarrow H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}^\infty) = K_n(\int_G \mathcal{A})$$

is bijective for  $n \in \mathbb{Z}$ , where we have replaced  $\mathcal{VC}$  by the smaller family of subgroups  $\mathcal{VC}_I$  of virtually cyclic subgroups of type  $I$ , i.e., of subgroups which are either finite or admit an epimorphism to  $\mathbb{Z}$  with finite kernel. Since  $\mathbf{HK}^\infty$  is given by a specific homotopy colimit, the assembly map is required to be bijective for all additive  $G$ -categories  $\mathcal{A}$  and is compatible with homotopy colimits in the spectrum variable, we conclude that

$$H_n^G(E_{\mathcal{VC}_I}(G); \mathbf{HK}_{\mathcal{A}}^\infty) \rightarrow H_n^G(\mathrm{pt}; \mathbf{HK}_{\mathcal{A}}^\infty) = KH_n(\int_G \mathcal{A})$$

is bijective for  $n \in \mathbb{Z}$ . In order to replace  $\mathcal{VC}_I$  by  $\mathcal{Fin}$ , we have to show in view of the Transitivity Principle, see for instance [2, Theorem 1.11] or [4, Theorem 1.5], that for any virtually cyclic group  $V$  of type  $I$  the assembly map

$$H_n^G(E_{\mathcal{Fin}}(V); \mathbf{HK}_{\mathcal{A}}^\infty) \rightarrow H_n^G(\mathrm{pt}; \mathbf{HK}_{\mathcal{A}}^\infty) = KH_n(\int_V \mathcal{A})$$

is bijective for all  $n \in \mathbb{Z}$ . This follows for  $V = \mathbb{Z}$  from Lemma 8.6 since the assembly map appearing above can be identified with the map appearing in Lemma 8.6. The general case of an extension  $1 \rightarrow F \rightarrow V \rightarrow \mathbb{Z}$  can be reduced to case  $V = \mathbb{Z}$  by assertion (i).  $\square$

**Remark 9.2** (Wreath products). We can also consider the versions “with finite wreath products”, i.e., we require for a group  $G$  that the Farrell-Jones Conjecture is not only satisfied for  $G$  itself, but for all wreath products of  $G$  with finite groups, see for instance [15]. The advantage of this version is that it is inherited to overgroups of finite index. This follows from the fact that an overgroup  $H$  of finite index of  $G$  can be embedded into a wreath product  $G \wr F$  for a finite group  $F$ , see [14, Section 2.6]. Theorem 9.1 remains true for the version with finite wreath products, where assertion (i) can be reduced to the statement that for an extension  $1 \rightarrow G \rightarrow Q \rightarrow 1$  the Farrell-Jones Conjecture with wreath products holds for  $G$  if it holds for  $K$  and  $Q$ .

**Remark 9.3** (Status of the Farrell-Jones Conjecture for homotopy  $K$ -theory). Because of assertion (i) and (ii) of Theorem 9.1, the class of groups for which the Farrell-Jones Conjecture for homotopy algebraic  $K$ -theory is known is larger than the class for the Farrell-Jones Conjecture for algebraic  $K$ -theory. For instance, elementary amenable groups satisfy the version for homotopy  $K$ -theory, just adapt the argument in [8, Theorem 1.3 (i), Lemma 2.12]. On the other hand, it is not known whether the Farrell-Jones Conjecture for algebraic  $K$ -theory holds for the semi-direct product  $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$ , where  $\mathbb{Z}$  acts by multiplication with 2 on  $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$ .

To summarize, the Farrell-Jones Conjecture for homotopy  $K$ -theory for coefficients in additive categories with finite wreath products has the following properties:

- It is known for elementary amenable groups, hyperbolic, CAT(0)-groups,  $GL_n(R)$  for a ring  $R$  whose underlying abelian group is finitely generated, arithmetic groups over number fields, arithmetic groups over global fields, cocompact lattices in almost connected Lie groups, fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary), and one-relator groups;
- It is closed under taking subgroups;
- It is closed under taking finite direct products and finite free products;
- It is closed under directed colimits (with not necessarily injective) structure maps;
- It is closed under extensions as explained in Remark 9.2;
- It has the tree property, see Theorem 9.1 (ii);
- It is closed under passing to overgroups of finite index.

This follows from the results above and [2], [6], [7], [9], and [23].

**Remark 9.4** (Implications of the homotopy  $K$ -theory version to the  $K$ -theory version). We have already seen above that the Farrell-Jones Conjecture for  $K$ -theory with coefficients in additive categories implies the Farrell-Jones Conjecture for homotopy  $K$ -theory with coefficients in additive categories. Next we discuss some cases, where the Farrell-Jones Conjecture for homotopy  $K$ -theory with coefficients in the ring  $R$  gives implications for the injectivity part of the Farrell-Jones Conjecture for  $K$ -theory with coefficients in the ring  $R$ . These all follow by inspecting for a ring  $R$  the following commutative diagram

$$\begin{array}{ccc}
 H_n^G(E_{\mathcal{VC}}(G); \mathbf{K}_R^\infty) & \longrightarrow & H_n^G(\mathrm{pt}; \mathbf{K}_R^\infty) = K_n(RG) \\
 \uparrow f & & \downarrow KH(\mathbf{h}^\infty) \\
 H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_R^\infty) & & \\
 \downarrow h^\infty & & \\
 H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{HK}_R^\infty) & \longrightarrow & H_n^G(\mathrm{pt}; \mathbf{HK}_R^\infty) = KH_n(RG)
 \end{array}$$

where the two vertical arrows pointing downwards are induced by the transformation  $\mathbf{h}^\infty: \mathbf{K}^\infty \rightarrow \mathbf{HK}^\infty$ , the map  $f$  is induced by the inclusion of families  $\mathcal{F}\text{in} \subseteq \mathcal{VC}$  and the two horizontal arrows are the assembly maps for  $K$ -theory and homotopy  $K$ -theory.

Suppose that  $R$  is regular and the order of any finite subgroup of  $G$  is invertible in  $R$ . Then the two left vertical arrows are known to be bijections. This follows for  $f$  from [17, Proposition 2.6 on page 686] and for  $h^\infty$  from [12, Lemma 4.6] and the fact that  $RH$  is regular for all finite subgroups  $H$  of  $G$  and hence  $K_n(RH) \rightarrow KH_n(RH)$  is bijective for all  $n \in \mathbb{Z}$ . Hence the (split) injectivity of the lower horizontal arrow implies the (split) injectivity of the upper horizontal arrow.

Suppose that  $R$  is regular. Then the two left vertical arrows are rational bijections. This follows for  $f$  from [18, Theorem 0.3]. To show it for  $h^\infty$  it suffices because of [12, Lemma 4.6] to show that  $K_n(RH) \rightarrow KH_n(RH)$  is rationally bijective for each finite group  $H$  and  $n \in \mathbb{Z}$ . By the version of the spectral sequence appearing in [28, 1.3] for non-connective  $K$ -theory, it remains to show that  $N^p K_n(RH)$  vanishes rationally for all  $n \in \mathbb{Z}$ . Since  $R[t]$  is regular if  $R$  is, this boils down to show that  $NK_p(RH)$  is rationally trivial for any regular ring  $R$  and any finite group  $H$ . This reduction can also be shown by proving directly that the structure maps of the system of spectra appearing in the Definition 8.1 of  $\mathbf{HK}$  are weak homotopy equivalences. The proof that  $NK_p(RH)$  is rationally trivial for any regular ring  $R$  and any finite group  $H$  can be found for instance in [18, Theorem 9.4]. Hence the upper horizontal arrow is rationally injective if the lower horizontal arrow is rationally injective.

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