# THE GEOMETRIC FINITENESS OBSTRUCTION

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## 0. *Introduction*

The purpose of this paper is to develop a geometric approach to Wall's finiteness obstruction. We will do this for equivariant $CW$-complexes. The main advantage will be that we can derive all the formal properties of the equivariant finiteness obstruction easily from this geometric description. Namely, the obstruction property, homotopy invariance, the sum and product formulas, and the restriction formula can be stated and proved in a simple manner. Also a characterization of the finiteness obstruction by a universal property is quickly available. This geometric approach is similar to the treatment of Whitehead torsion by Cohen in [**3**].

In the first section we define a functor $Wa^G$ from the category of $G$-spaces to the category of abelian groups. We assign to a finitely dominated $G$-$CW$-complex $X$ an element $w^G(X) \in Wa^G(X)$ called its finiteness obstruction. The finiteness obstruction vanishes if and only if $X$ is $G$-homotopic to a finite $G$-$CW$-complex and satisfies a sum formula and is homotopy invariant.

The notion of a universal functorial additive invariant is introduced in the second section where its existence and uniqueness are proved. Product and restriction formulas for the universal additive invariant are obtained by abstract nonsense.

We define equivariant Euler characteristics in the third section generalizing the notion of the Euler characteristic of a finite $CW$-complex.

The goal of the fourth section is to prove that the equivariant Euler characteristic and finiteness obstruction determine the universal functorial additive invariant for finite, respectively finitely dominated, $G$-$CW$-complexes.

The fifth section contains some algebraic computations of $Wa^G$ in terms of reduced projective class groups of certain integral group rings. In the non-equivariant case Wall's algebraic approach and our geometric one agree.

Finally, in the sixth section, the results of the second and fourth sections are used to state an abstract product formula, a restriction formula, and a diagonal product formula.

We make some remarks about the simple-homotopy approach to the finiteness obstruction due to Ferry. The treatment by Ferry in [**7**] is extended by Kwasik in [**14**] to the equivariant case. In § 6 we construct geometrically an injection $I(Y)$: $W_a^G(Y) \to Wh^G(Y \times S^1)$ into the equivariant Whitehead group of $Y \times S^1$ sending our geometric finiteness obstruction to that of Kwasik.

A compact Lie group is denoted by $G$.

## 1. *The geometric finiteness obstruction*

A *$G$-$CW$-complex* $X$ is a $G$-space with a filtration $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \dots$ such that $X_{n+1}$ is obtained from $X_n$ by attaching equivariant $(n+1)$-dimensional

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cells $G/H \times D^{n+1}$, namely, $X_{n+1}$ is the $G$-push-out of

$$\sum_{i \in I} G/H_i \times D^{n+1} \longleftarrow \sum_{i \in I} G/H_i \times S^n \xrightarrow{\sum_{i \in I} q_i} X_n.$$

We equip $X = UX_n$ with the weak topology with respect to the filtration (see Illman [**10**]).

We call $X_n$ the *n-skeleton* of $X$ and $q_i$ the *attaching map* of the cell $G/H_i \times D^{n+1}$. For $G = \{1\}$ the notion of a $G$-$CW$-complex agrees with the one of a $CW$-complex. From now on we write $G$-complex instead of $G$-$CW$-complex.

A $G$-complex $X$ is *finite* if $X$ is built from the empty set by attaching a finite number of cells. We call a $G$-complex $X$ *finitely dominated* if $X$ is a homotopy retract of a finite $G$-complex, that is, there exists a finite $G$-complex $Y$ and $G$-maps $r$: $Y \to X$ and $i$: $X \to Y$ such that $r \circ i$ is $G$-homotopic to the identity: $r \circ i \simeq_G \mathrm{ID}$.

The goal of this section is to construct a functor $Wa^G$ from the category of $G$-spaces into the category of abelian groups, and an assignment $w^G$ associating to a $G$-space $X$ having the homotopy type of a finitely dominated $G$-complex an element $w^G(X)$ in $Wa^G(X)$ such that the following theorem is valid.

THEOREM 1.1. (a) *Homotopy invariance.*

(i) *If $f$: $X \to Y$ is a homotopy equivalence of $G$-spaces of the homotopy type of a finitely-dominated $G$-complex, then $f_*$: $Wa^G(X) \to Wa^G(Y)$ sends $w^G(X)$ to $w^G(Y)$.*

(ii) *If $f$ and $g$: $X \to Y$ are $G$-homotopic, then $f_* = g_*$.*

(b) *Obstruction property. Let $X$ be a $G$-space of the homotopy type of a finitely dominated $G$-complex. Then $X$ is $G$-homotopy equivalent to a finite $G$-complex if and only if $w^G(X)$ vanishes.*

(c) *Additivity. If the following diagram of $G$-spaces having the homotopy type of finitely dominated $G$-complexes is a $G$-push-out and $k$ a $G$-cofibration then $w^G(X) = j_{1*}(w^G(X_1)) + j_{2*}(w^G(X_2)) - j_{0*}(w^G(X_0))$:*

$$\begin{tikzcd}
X_0 \arrow[r, hook, "k"] \arrow[d] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\
X_2 \arrow[r, "j_2"'] & X
\end{tikzcd}$$

Given a $G$-space $Y$, we want to define $Wa^G(Y)$ as the set of equivalence classes of an equivalence relation $\sim$ defined for $G$-maps $f$: $X \to Y$ with $X$ of the homotopy type of a finitely dominated $G$-complex. We call $f_0$: $X_0 \to Y$ and $f_4$: $X_4 \to Y$ *equivalent*, $f_0 \sim f_4$, if there exists a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i_0"'] \arrow[drr, "f_0"'] & X_1 \arrow[r, "\simeq", "j_1"'] \arrow[dr, "f_1"] & X_2 \arrow[d, "f_2"] & X_3 \arrow[l, "\simeq"', "j_2"] \arrow[dl, "f_3"'] & X_4 \arrow[l, hook', "i_3"] \arrow[dll, "f_4"] \\
& & Y & &
\end{tikzcd}$$

such that $j_1$ and $j_2$ are homotopy equivalences and $i_0$ and $i_3$ are inclusions of subcomplexes for which $X_1$, respectively $X_3$, is obtained from $X_0$, respectively $X_4$, by attaching a finite number of cells. One should notice that $X_1$, $X_2$, $X_3$ have the homotopy type of a finitely dominated $G$-complex since the same is true for $X_0$ and $X_4$ by assumption. Obviously $\sim$ is symmetric and reflexive. The main part of the construction of $(Wa^G, w^G)$ and the proof of Theorem 1.1 is the verification that $\sim$ is transitive.

We will symbolize a diagram

$$\begin{tikzcd} X_n \arrow[rr, "k"] \arrow[dr, "f_n"'] & & X_{n+1} \arrow[dl, "f_{n+1}"] \\ & Y & \end{tikzcd}$$

by $\subset$, respectively $\rightarrow$, if it is commutative and $k$ is the inclusion of a subcomplex such that $X_{n+1}$ is obtained from $X_n$ by attaching a finite number of cells, respectively $k$ is a homotopy equivalence. Hence the diagram defining $\sim$ corresponds to the sequence $\subset \rightarrow \leftarrow \supset$. To prove transitivity we have to show that a diagram $\subset \rightarrow \leftarrow \supset \subset \rightarrow \leftarrow \supset$ can be reduced to $\subset \rightarrow \leftarrow \supset$ without changing the ends. For this purpose we will introduce some operations we are allowed to do with diagrams given by a sequence of symbols $\subset, \supset, \rightarrow, \leftarrow$.

(1) $\supset \subset \Rightarrow \subset \supset$. The sequence $\supset \subset$ stands for

$$\begin{tikzcd} X_n \arrow[dr, "f_n"'] & X_{n+1} \arrow[l, hook', "i_n"'] \arrow[r, hook, "i_{n+1}"] \arrow[d, "f_{n+1}"] & X_{n+2} \arrow[dl, "f_{n+2}"] \\ & Y & \end{tikzcd}$$

If $f'_{n+1}$: $X'_{n+1} \rightarrow Y$ is defined by the push-out, one gets $\subset \supset$ by

$$\begin{tikzcd} X_n \arrow[r, hook, "i'_n"] \arrow[dr, "f_n"'] & X'_{n+1} \arrow[d, "f'_{n+1}"] & X_{n+2} \arrow[l, hook', "i'_{n+1}"'] \arrow[dl, "f_{n+2}"] \\ & Y & \end{tikzcd}$$

(2) $\supset \rightarrow \Rightarrow \rightarrow \supset$ and $\leftarrow \subset \Rightarrow \subset \leftarrow$. This is analogous to (1).

(3) $\leftarrow \rightarrow \Rightarrow \rightarrow \leftarrow$. Glue the mapping cylinders together.

(4) $\rightarrow \subset \Rightarrow \leftarrow \subset \rightarrow \leftarrow$ and $\supset \leftarrow \Rightarrow \rightarrow \leftarrow \supset \rightarrow$. Consider the diagram

$$\begin{tikzcd} X_n \arrow[r, "j_n", "\simeq"'] \arrow[dr, "f_n"'] & X_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "f_{n+1}"] & X_{n+2} \arrow[dl, "f_{n+2}"] \\ & Y & \end{tikzcd}$$

Let $v$ be a homotopy inverse to $j_n$. Since $i_{n+1}$ is an equivariant cofibration, we can choose a homotopy $h$: $X_{n+2} \times I \rightarrow Y$ with $h_0 \mid X_{n+1} = f_n \circ v$ and $h_1 = f_{n+2}$. This yields the commutative diagram:

$$\begin{tikzcd} X_n \arrow[drr, "f_n"'] & X_{n+1} \arrow[l, "v"'] \arrow[r, hook, "i_{n+1}"] \arrow[dr, "f_n \circ v"] & X_{n+2} \arrow[r, hook, "\simeq", "k_0"'] \arrow[d, "h_0"] & X_{n+2} \times I \arrow[dl, "h"] & X_{n+2} \arrow[l, hook', "\simeq"', "k_1"] \arrow[dll, "f_{n+2}"] \\ & & Y & & \end{tikzcd}$$

(5) $\to\to \Rightarrow \to$ and $\leftarrow\leftarrow \Rightarrow \leftarrow$, $\subset\subset \Rightarrow \subset$ and $\supset\supset \Rightarrow \supset$.

Now we get using the operations (1) to (5):

$$\begin{array}{ll}
\subset \to \leftarrow \supset \subset \to \leftarrow \supset & \Rightarrow \\
\subset \to \leftarrow \subset \supset \to \leftarrow \supset & \Rightarrow \\
\subset \to \subset \leftarrow \to \supset \leftarrow \supset & \Rightarrow \\
\subset \leftarrow \subset \to \leftarrow \leftarrow \to \to \leftarrow \supset \to \supset & \Rightarrow \\
\subset \subset \leftarrow \to \leftarrow \to \leftarrow \to \supset \supset & \Rightarrow \\
\subset \to \leftarrow \leftarrow \to \to \leftarrow \supset & \Rightarrow \\
\subset \to \leftarrow \to \leftarrow \supset & \Rightarrow \\
\subset \to \to \leftarrow \leftarrow \supset & \Rightarrow \\
\subset \to \leftarrow \supset. &
\end{array}$$

This finishes the proof that $\sim$ is an equivalence relation. Hence we can define $Wa^G(Y)$ as the set of equivalence classes. The topological sum induces an addition on $Wa^G(Y)$ by

$$[f_0\colon X_0 \to Y] + [f_1\colon X_1 \to Y] := [f_0 + f_1\colon X_0 + X_1 \to Y].$$

The inclusion of the empty set defines a zero element. Given an element $[f\colon X \to Y]$, we can construct an inverse element in the following way, because $X$ has the homotopy type of a finitely dominated $G$-complex.

Choose a finite $G$-complex $Z$ and $G$-maps $r\colon Z \to X$ and $i\colon X \to Z$ with $r \circ i \simeq_G \mathrm{ID}$. Let $C_i$, respectively $C_r$, be the mapping cylinder of $i$, respectively $r$. Construct a map $F\colon C_i \to X$ with $F \mid X = \mathrm{ID}_X$ and $F \mid Z = r$. Then an inverse for $[f]$ is given by the composition

$$C_i \cup_X C_i \xrightarrow{F \cup_X F} X \xrightarrow{f} Y.$$

Namely, let

$$g\colon C_r \cup_Z C_i \cup_X C_i \to X$$

be an extension of $\mathrm{ID} + F \cup_X F\colon X + C_i \cup_X C_i \to X$, and let

$$h\colon Z \to C_r \cup_Z C_i \cup_X C_i$$

be a homotopy equivalence. This yields the following commutative diagram:

$$\begin{tikzcd}
X + C_i \cup_X C_i \arrow[r, hook] \arrow[dr, "f + f \circ F \cup_X F"'] & C_r \cup_Z C_i \cup_X C_i \arrow[d, "f \circ g"] & Z \arrow[l, "h"', "\simeq"] \arrow[dl, "f \circ g \circ h"'] & \varnothing \arrow[l, hook'] \arrow[dll] \\
& Y & &
\end{tikzcd}$$

This implies that $[f] + [f \circ F \cup_X F] = [\varnothing \to Y] = 0$. Hence $Wa^G(Y)$ is an abelian group. A $G$-map $f\colon Y \to Y'$ induces a homomorphism of abelian groups $f_*\colon Wa^G(Y) \to Wa^G(Y')$ by composition. So $Wa^G$ is a functor from the category of $G$-spaces into the category of abelian groups.

DEFINITION 1.2. Let $X$ be a $G$-space of the homotopy type of a finitely dominated $G$-complex. Define its *finiteness obstruction* $w^G(X) \in Wa^G(X)$ by the class of the identity map of $X$.

*Proof of Theorem* 1.1. (a) The verification of the homotopy invariance is trivial.

(b) Obstruction property. Let $X$ be a $G$-space having the homotopy type of a finitely dominated $G$-complex with $w^G(X) = 0$. Hence there are a $G$-space $Y$, a $G$-map $r$: $Y \to X$, and a $G$-homotopy equivalence $Y \to Z$ into a finite $G$-complex $Z$ such that $Y$ is obtained from $X$ by attaching finitely many cells and $r \circ i = \mathrm{ID}$ is valid for the inclusion $i$: $X \hookrightarrow Y$. The mapping cylinder $C_r$ is built up from the mapping cylinder $C_i$ by attaching a finite number of cells. Choose a $G$-homotopy equivalence $g$: $C_i \to Z$. Consider the push-out

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, "g"'] & C_r \arrow[d, "g'"] \\ Z \arrow[r, hook] & Z' \end{tikzcd}$$

Since $g$ is a $G$-homotopy equivalence, the same is true for $g'$ (see Whitehead [**21**, p. 26]). Hence $X$ is homotopy equivalent to the finite $G$-complex $Z'$.

(c) Additivity. Consider the push-out

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"'] & X \end{tikzcd}$$

Choose a finite $G$-complex $Z$ and $r$: $Z \to X_0$ and $i$: $X_0 \to Z$ with $r \circ i \simeq_G \mathrm{ID}$. If $F$: $C_i \to X_0$ is a map with $F \mid X_0 = \mathrm{ID}$ and $F \mid Z = r$, an inverse for $[j_0]$ in $Wa^G(X)$ is given by $[j_0 \circ F \cup_{X_0} F]$. To prove that

$$[j_1] + [j_2] + [j_0 \circ F \cup_{X_0} F] = [\mathrm{ID}_X]$$

in $Wa^G(X)$, construct a commutative diagram with a homotopy equivalence $h$:

$$\begin{tikzcd} X_1 + C_i \cup_{X_0} C_i + X_2 \arrow[r, hook] \arrow[dr, "j_1 + j_0 \circ F \cup_X F + j_2"'] & X_1 \cup_{X_0} C_r \cup_Z C_i \cup_{X_0} C_i \cup_Z C_r \cup_{X_0} X_2 \arrow[d] & X_1 \cup_{X_0} X_2 \arrow[l, "h"'] \arrow[dl, "j_1 \cup_{j_0} j_2"] \\ & X & \end{tikzcd}$$

## 2. *Universal functorial additive invariants*

In this section the notion of a universal functorial additive invariant is developed. Later this is used to characterize the finiteness obstruction by a simple universal property.

Let $C$ be a small full subcategory of the category of $G$-spaces containing $\varnothing$ and $\{\mathrm{pt}\}$. We assume that $C$ is closed under $G$-equivalences and $G$-push-outs, that is, if $X$ is an object in $C$ and $Y$ a $G$-space equivalent to $X$, then $Y$ also belongs to $C$, and if $X$ is the push-out of

$$X_1 \longleftarrow X_0 \xhookrightarrow{k} X_1$$

with $k$ a $G$-cofibration and $X_0$, $X_1$, $X_2$ are objects in $C$, then $X$ is an object in $C$.

DEFINITION 2.1. A *functorial additive invariant* $(B, b)$ for $C$ consists of a functor $B$ from $C$ into the category of abelian groups and an assignment $b$ associating to an object $X$ in $C$ an element $b(X) \in B(X)$ such that the following conditions are fulfilled.

(a) Homotopy invariance.

(i) If $f$: $X \to Y$ is a $G$-homotopy equivalence in $C$, then $B(f)(b(X)) = b(Y)$.

(ii) $f \simeq_G g \Rightarrow B(f) = B(g)$.

(b) Additivity. Given a $G$-push-out in $C$ with $k$ a $G$-cofibration

$$\begin{tikzcd} X_0 \arrow[r, hook, "k"] \arrow[d] \arrow[dr, "j_0"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"'] & X \end{tikzcd}$$

the following formula is valid:

$$b(X) = B(j_1)(b(X_1)) + B(j_2)(b(X_2)) - B(j_0)(b(X_0)).$$

(c) $b(\varnothing) = 0$.

Because of Theorem 1.1 the pair $(Wa^G, w^G)$ is a functorial additive invariant if $C$ is the category of $G$-spaces having the homotopy type of a finitely dominated $G$-complex. In § 4, $(Wa^G, w^G)$ is characterized by the following universal property.

DEFINITION 2.2. A functorial additive invariant $(U, u)$ for $C$ is *universal* if there exists for any functorial additive invariant $(B, b)$ of $C$ a natural transformation $F$: $U \to B$ uniquely determined by the property that $F(X)(u(X)) = b(X)$ is valid for all objects $X$ in $C$.

This notion is a generalization of the well-known notion of an additive invariant.

DEFINITION 2.3. An *additive invariant* $(B, b)$ for $C$ consists of an abelian group $B$ and an assignment sending an object $X$ to $b(X) \in B$ with $b(\varnothing) = 0$, such that $b(X) = b(Y)$ holds for $G$-homotopy equivalent objects $X$ and $Y$ and the sum formula

$$b(X) = b(X_1) + b(X_2) - b(X_0)$$

is valid for a $G$-push out as above.

We call an additive invariant $(U, u)$ *universal* if for any additive invariant $(B, b)$ there is a homomorphism $F$: $U \to B$ uniquely determined by $F(u(X)) = b(X)$ for all objects $X$.

In other words, an additive invariant is a functorial additive invariant $(B, b)$ where $B$ is a constant functor. In particular, each additive invariant can be regarded as a functorial additive invariant. Given a functorial additive invariant $(B, b)$, we can define an additive invariant $(\hat{B}, \hat{b})$ by $\hat{B} := B(\{\mathrm{pt}\})$ and $\hat{b}(X) := B(X \to \{\mathrm{pt}\})(b(X))$.

PROPOSITION 2.4. (a) *There exists a universal functorial additive invariant unique up to natural equivalence.*

(b) *There exists a universal additive invariant unique up to isomorphism. It is given by $(\hat{U}, \hat{u})$ for the universal functorial additive invariant $(U, u)$.*

*Proof.* (a) The uniqueness is a direct consequence of the universal property. It remains to construct a universal functorial additive invariant $(U, u)$.

Given an object $Y$ in $C$, define $U(Y)$ as the quotient of the free abelian group generated by the $G$-homotopy classes $[f]$ of $G$-maps $f: X \to Y$ in $C$ and the subgroup generated by elements:

$[f] - [g]$, if there exists a $G$-equivalence $h$ with $f \circ h \simeq_G g$;

$[f] - [f_1] - [f_2] + [f_0]$, if there exist representatives $f_0, f_1, f_2, f$ and a $G$-push-out with $k$ a $G$-cofibration

$$\begin{tikzcd} X_0 \arrow[r, hook, "k"] \arrow[d] \arrow[drr, "f_0"] & X_1 \arrow[d] \arrow[dr, "f_1"] & \\ X_2 \arrow[r] \arrow[rr, bend right, "f_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

A $G$-map $g: Y \to Z$ induces $U(g): U(Y) \to U(Z)$ by composition. We assign to an object $X$ in $C$ the element $u(X) \in U(X)$ represented by the identity.

(b) This is left to the reader.

For an object $Y$ let $C(Y)$ be the category of morphisms over $Y$ in $C$. The universal additive invariant for $C(Y)$ is given by $U(Y)$ and $(f: X \to Y) \to U(f)(u(X))$. Hence $(U, u)$ can be described by the universal additive invariants for all $C(Y)$.

Table 1 gives the universal functorial additive invariant $(U, u)$ and the universal additive invariant $(\hat{U}, \hat{u})$ for three categories $C$. Here $\pi$ is a finite group. For a set $M$ let $\mathbb{Z}(M)$ be the free abelian group generated by $M$, and let (1) in $\mathbb{Z}(M)$ be the element $\sum_{x \in M} x$.

TABLE 1

| $C$ | finite sets | finite $\pi$-sets | finite $CW$-complexes |
|---|---|---|---|
| $U(X)$ | $\mathbb{Z}(X)$ | $\mathbb{Z}$(Orbits of $X$) | $H_0(X)$ |
| $u(X)$ | (1) | (1) | componentwise Euler-characteristic |
| $\hat{U}$ | $\mathbb{Z}$ | Burnside ring $A(\pi)$ | $\mathbb{Z}$ |
| $\hat{u}$ | cardinality | $[X]$ | Euler-characteristic |

The universal additive invariant for $C$ as the category of $G$-spaces having the homotopy type of a finite $G$-complex has been computed by tom Dieck

[4, p. 98 ff]. We can re-prove this result by applying Proposition 2.4(b) to Theorem 4.1.

Now we will show that we can derive the existence of a product and restriction formula by abstract nonsense.

Let $G$ and $G'$ be compact Lie-groups and $C$, respectively $C'$ respectively $D$, be subcategories of the category of $G$-spaces, respectively $G'$-spaces respectively $G \times G'$-spaces, as defined above such that $C \times C'$ is contained in $D$. Let $(U, u)$, respectively $(U', u')$ respectively $(V, v)$, be the universal functorial additive invariant for $C$, respectively $C'$ respectively $D$. Given an object $Y'$ in $C'$, we denote by $T(Y')$ the abelian group of natural transformations from $U(?)$ to $V(? \times Y')$. A map $f\colon Y' \to Y''$ in $C'$ induces a homomorphism $T(Y') \to T(Y'')$ by composition with $V(\mathrm{ID} \times f)$. As $(V(? \times Y'), v(? \times Y'))$ is a functorial additive invariant for $C$, there exists a natural transformation $t(Y')\colon U(?) \to V(? \times Y')$ uniquely determined by the property that $t(Y')(Y)$ sends $u(Y)$ to $v(Y \times Y')$ for all objects $Y$ in $C$. Then $(T, t)$ is a functorial additive invariant for $C'$, so that there exists exactly one natural transformation $F\colon U \to T$ with $F(Y')(u(Y')) = t(Y')$ for all objects $Y'$ in $C'$. This can be interpreted as a natural pairing

$$P(Y, Y')\colon\ U(Y) \otimes U'(Y') \to V(Y \times Y')$$

uniquely determined by the property that $P(Y, Y')$ sends $u(Y) \otimes u'(Y')$ to $v(Y \times Y')$.

Hence one gets a product formula. Because of Proposition 2.4(b) this also yields a product formula for the universal additive invariant.

Let $H$ be a closed subgroup of $G$ and $C$, respectively $D$, a subcategory of the category of $G$-spaces, respectively $H$-spaces, as defined above. Assume for each object $Y$ of $C$ that its restriction $\mathrm{res}(Y)$ lies in $D$. The universal functorial additive invariant of $C$, respectively $D$, is denoted by $(U, u)$, respectively $(V, v)$. The restriction defines a functor $\mathrm{res}\colon C \to D$. Since $(V \circ \mathrm{res}, v \circ \mathrm{res})$ is a functorial additive invariant for $C$, there exists exactly one natural transformation $R\colon U \to V \circ \mathrm{res}$ such that $R(Y)$ sends $u(Y)$ to $v(\mathrm{res}(Y))$ for all objects $Y$ in $C$. Hence one gets a restriction formula. This also yields a restriction formula for the universal additive invariant. Applying this to a finite group $\pi$ and the trivial subgroup and $C$ as the category of finite $\pi$-sets and $D$ as the category of finite sets, one gets the homomorphism $A(\pi) \to \mathbb{Z}$ associating to a finite $\pi$-set its cardinality.

Let $C$ be a subcategory of $G$-spaces as defined before. Given two $G$-spaces $Y$ and $Y'$ we equip $Y \times Y'$ with the diagonal $G$-action. Assume that $C$ is closed under the product. Combining the product and restriction formula we get a diagonal product formula. Namely, if $(U, u)$ is the universal functorial additive invariant for $C$, there is a natural pairing

$$P(Y, Y')\colon\ U(Y) \otimes U(Y') \to U(Y \times Y')$$

uniquely determined by

$$P(Y, Y')(u(Y) \otimes u(Y')) = u(Y \times Y').$$

Because of Proposition 2.4(b) this also yields a diagonal product formula $\hat{U} \otimes \hat{U} \to \hat{U}$ for the universal additive invariant $\hat{U}$ sending $\hat{u}(Y) \otimes \hat{u}(Y')$ to $\hat{u}(Y \times Y')$. Hence $\hat{U}$ is a commutative ring with unit $u(\{\mathrm{pt}\})$ and $U$ becomes a functor into the category of $\hat{U}$-modules by the diagonal product formula. If $C$ is the category of finite $\pi$-sets for a finite group $\pi$, the universal additive invariant

$A(\pi)$ is just the Burnside ring of $\pi$ (see tom Dieck [**4**, p. 1 ff]). For $C$ as the category of $G$-spaces having the homotopy type of a finite $G$-complex the ring structure of the universal additive invariant is computed in [**4**, p. 101 ff].

Finally we mention that the notion of a functorial additive invariant can be introduced in more general situations than above and can also be applied to modules and chain complexes. For example, the projective class group of a ring is the universal additive invariant for the category of finitely generated projective modules.

## 3. *Equivariant Euler characteristics*

In this section we introduce the notion of an equivariant Euler characteristic. They appear in the next section in the universal functorial additive invariant of finite, respectively finitely dominated, $G$-complexes.

Define a functor $A^G$ from the category of $G$-spaces to the category of abelian groups in which the equivariant Euler characteristic lives. Given a $G$-space $X$, denote by $\{G/?\to X\}$ the set of all $G$-maps $G/H\to X$ for all closed subgroups $H$. Call $x\colon G/H\to X$ and $y\colon G/K\to X$ equivalent, $x\sim y$, if there is a $G$-isomorphism $f\colon G/H\to G/K$ with $y\circ f\simeq_G x$. Obviously $\sim$ is an equivalence relation. Let $\{G/?\to X\}/\sim$ be the set of equivalence classes.

DEFINITION 3.1. Define $A^G(X)$ as the free abelian group generated by the set $\{G/?\to X\}/\sim$. A map $f\colon X\to Y$ induces a homomorphism

$$A^G(f)\colon A^G(X)\to A^G(Y)$$

by composition.

A $G$-map $x\colon G/H\to X$ is the same as a point $x=x(1H)$ in the $H$-fixed point set $X^H=\{x\in X\mid hx=x \text{ for all } h\in H\}$. Denote by $WH$ the Weyl group $NH/H$ of $H$. It acts on $X^H$ by $gH$, $x\mapsto gx$ and hence on $\pi_0(X^H)$. For a $G$-map $x\colon G/H\to X$ we write $V(x)$ for the component of $X^H$ containing $x(1H)$, and $[V(x)]$ for its class in $\pi_0(X^H)/WH$. Let $C(G)$ be a complete system of representatives for the conjugacy classes of closed subgroups of $G$.

LEMMA 3.2. *There is a natural bijection*

$$\{G/?\to X\}/\sim\;\to \sum_{C(G)} \pi_0(X^H)/WH$$

*sending the class of* $x\colon G/H\to X$ *to* $[V(x)]\in\pi_0(X^H)/WH$.

Thus $A^G(X)$ can be written as

$$\bigoplus_{C(G)}\ \bigoplus_{\pi_0(X^H)/WH} \mathbb{Z}.$$

If $\pi$ is a finite group, the additive group of the Burnside ring $A(\pi)$ is just $A^\pi(\{\mathrm{pt}\})$. For the trivial group $\{1\}$ we can identify $A^{\{1\}}(X)$ with the singular homology $H_0(X)$.

Sometimes an element $f\in A^G(X)$ is written as a function from $\{G/?\to X\}/\sim$ to the integers. For $V\in\pi_0(X^H)$ let $I(V)\subset WH$ be the isotropy group of $V\in\pi_0(X^H)$ under the $WH$-operation. The operation of $WH$ on $X^H$ induces an operation of $I(V)$ on $V$. Let $V^{>H}$ be $\{v\in V^H\mid$ there exists $h\in G$, $h\notin H$ with

$hv = v\}$. Given a pair $(Y, B)$ of spaces such that $H_*(Y, B)$ is finitely generated, define $\chi(Y, B)$ as $\sum_{n=0}^{\infty}(-1)^n \mathrm{rk}(H_n(Y, B))$.

DEFINITION 3.3. Let $X$ be a $G$-space of the homotopy type of a finitely dominated $G$-complex. Define its *equivariant Euler characteristic* $\chi^G(X) \in A^G(X)$ by

$$\chi^G(X)(x\colon G/H \to X) = \chi(V(x)/I(V(x)), V(x)^{>H}/I(V(x))).$$

Since $X$ is finitely dominated, the homology of

$$(V(x)/I(V(x)), V(x)^{>H}/I(V(x)))$$

is finitely generated, so that this definition makes sense. If $X$ is a finite $G$-complex, $\chi^G(X)(x\colon G/H \to X)$ can be computed by counting equivariant cells. Namely, $\chi^G(X)(x\colon G/H \to X)$ is $\sum_{n=0}^{\infty}(-1)^n \beta(n, x)$ with $\beta(n, x)$ the number of $n$-dimensional cells in the relative $CW$-complex $(V(x)/I(V(x)), V(x)^{>H}/I(V(x)))$. The number of free $n$-dimensional cells $I(V(x)) \times D^n$ in $V(x)\backslash V(x)^{>H}$ is also $\beta(n, x)$.

PROPOSITION 3.4. *The pair $(A^G, \chi^G)$ is a functorial additive invariant for the category $C$ of $G$-spaces having the homotopy type of a finitely dominated $G$-complex.*

*Proof.* The homotopy invariance (see Definition 2.1) is obviously fulfilled, so that it remains to verify additivity. Using mapping cylinders and homotopy invariance one shows that it suffices to regard finitely dominated $G$-complexes $X_0, X_1, X_2, X$ with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Let $j_k\colon X_k \to X$ be the inclusion for $k = 0, 1, 2$ and $x\colon G/H \to X$ a $G$-map. We have assigned to $x$ a space $V(x) \subset X^H$ with $I(V(x))$-action for $I(V(x)) \subset WH$. Because of the relations

$$\begin{aligned}
&(V(x) \cap X_1) \cup (V(x) \cap X_2) = V(x),\\
&(V(x) \cap X_1) \cap (V(x) \cap X_2) = V(x) \cap X_0,\\
&(V(x)^{>H} \cap X_1) \cup (V(x)^{>H} \cap X_2) = V(x)^{>H},
\end{aligned}$$

and

$$(V(x)^{>H} \cap X_1) \cap (V(x)^{>H} \cap X_2) = V(x)^{>H} \cap X_0,$$

one obtains

$$\begin{aligned}
&\chi(V(x)/I(V(x)), V(x)^{>H}/I(V(x)))\\
&\qquad = \chi(V(x) \cap X_1/I(V(x)), V(x)^{>H} \cap X_1/I(V(x)))\\
&\qquad\quad + \chi(V(x) \cap X_2/I(V(x)), V(x)^{>H} \cap X_2/I(V(x)))\\
&\qquad\quad - \chi(V(x) \cap X_0/I(V(x)), V(x)^{>H} \cap X_0/I(V(x))).
\end{aligned}$$

Hence it suffices to prove for $k = 0, 1, 2$, that

$$\chi(V(x) \cap X_k/I(V(x)), V(x)^{>H} \cap X_k/I(V(x)))$$

is $\sum \chi^G(X_k)(y\colon G/H \to X_k)$ where the sum is taken over all $y \in \{G/? \to X_k\}/\sim$

with $j_k \circ y = x$ in $\{G/? \to X\}/\sim$. Choose a complete system of representatives $y_i$: $G/H \to X_k$ for $i = 1, \ldots, r$ for these classes in $\{G/? \to X_k\}/\sim$. Now one verifies that $V(x) \cap X_k / I(V(x))$ is the topological sum $\sum_{i=1}^{r} V(y_i)/I(V(y_i))$. This finishes the proof.

## 4. *The universal property of the Euler characteristic and the finiteness obstruction*

In this section a characterization of the equivariant Euler characteristic and the finiteness obstruction by a universal property is given in:

THEOREM 4.1. (a) *The pair* $(A^G, \chi^G)$ *is the universal functorial additive invariant for the category* $C$ *of* $G$*-spaces having the homotopy type of a finite* $G$*-complex.*

(b) *The pair* $(A^G \oplus Wa^G, (\chi^G, w^G))$ *is the universal functorial additive invariant for the category* $C$ *of* $G$*-spaces having the homotopy type of a finitely dominated* $G$*-complex.*

*Proof.* (a) Because of Proposition 3.4 it remains to verify the universal property. Given an arbitrary functorial additive invariant $(B, b)$, define a natural transformation $F$: $A^G \to B$: for a $G$-space $X$ and $\eta \in A^G(X)$ represented by a function $\eta$: $\{G/? \to X\}/\sim \to \mathbb{Z}$ the homomorphism $F(X)$: $A^G(X) \to B(X)$ sends $\eta$ to $\sum \eta(x: G/H \to X) \cdot B(x)(b(G/H))$ where the sum is taken over $\{G/? \to X\}/\sim$.

Each element $\eta \in A^G(X)$ can be written as a sum over $\{G/? \to X\}/\sim$ by

$$\sum \eta(x: G/H \to X) \cdot A^G(x: G/H \to X)(\chi^G(G/H)).$$

Hence the natural transformation $F$ is uniquely determined by the property that $F(G/H)$: $A^G(G/H) \to B(G/H)$ maps $\chi^G(G/H)$ to $b(G/H)$. It remains to prove that for any finite $G$-complex $X$ the relation $F(X)(\chi^G(X)) = b(X)$ is valid. Use induction over the number of cells. If $X$ consists only of one cell, $X$ is a homogeneous space $G/H$ and the relation holds by definition of $F$.

Suppose that the assertion is true for $X$ and that $Y$ is obtained from $X$ by attaching one equivariant cell. Namely, we have the $G$-push out

$$\begin{tikzcd} G/H \times S^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ G/H \times D^{n+1} \arrow[r] & Y \end{tikzcd}$$

Applying additivity to this $G$-push out and homotopy invariance to $G/H \to G/H \times D^{n+1}$ for both $(A^G, \chi^G)$ and $(B, b)$ one proves the assertion for $Y$.

(b) Let $(B, b)$ be any functorial additive invariant for $C$ and $F$: $A^G \oplus Wa^G \to B$ be a natural transformation with $F(X)(\chi^G(X), w^G(X)) = b(X)$ for all objects $X$ in $C$. Then $F$ is already determined by $(B, b)$. Namely, let $Y$ be any object in $C$ and $\eta$: $\{G/? \to Y\}/\sim \to \mathbb{Z}$ be a function representing $\eta \in A^G(Y)$ and $f$: $X \to Y$ a $G$-map with $X$ of the homotopy type of a finitely dominated $G$-complex representing $[f] \in Wa^G(Y)$. Consider the following computation. The sums are taken over $\{G/? \to Y\}/\sim$, respectively $\{G/? \to X\}/\sim$.

$$
\begin{aligned}
&F(Y)(\eta, [f]) \\
&\quad = F(Y)(\eta, 0) - F(Y)(A^G(f)(\chi^G(X)), 0) + F(Y)(A^G(f)(\chi^G(X)), [f]) \\
&\quad = \sum \eta(y\colon G/H \to Y) \cdot F(Y)(A^G(y)(\chi^G(G/H)), 0) \\
&\qquad - F(Y)(A^G(f)(\sum \chi^G(X)(x\colon G/H \to X) \cdot A^G(x)(\chi^G(G/H))), 0) \\
&\qquad + F(Y) \circ (A^G(f) \oplus Wa^G(f))(\chi^G(X), w^G(X)) \\
&\qquad = \sum \eta(y\colon G/H \to Y) \cdot B(y)(b(G/H)) \\
&\qquad - \sum \chi^G(X)(x\colon G/H \to X) \cdot B(f \circ x)(b(G/H)) + B(f)(b(X)).
\end{aligned}
$$

This shows uniqueness.

To prove the existence of $F$ define $F$ just by the formula above. Namely, $F(Y)\colon A^G(Y) \oplus Wa^G(Y) \to B(Y)$ sends $(\eta, [f])$ to the element

$$
\begin{aligned}
&\sum \eta(y\colon G/H \to Y) \cdot B(y)(b(G/H)) \\
&\quad - \sum \chi^G(X)(x\colon G/H \to X) \cdot B(f \circ x)(b(G/H)) + B(f)(b(X)).
\end{aligned}
$$

The verification that $F$ is well defined is left to the reader. One has to check that this is compatible with the equivalence relation appearing in the definition of $Wa^G$. Obviously $F(Y)$ sends $(\chi^G(Y), w^G(Y))$ to $b(Y)$.

## 5. *Computations of the obstruction group*

Firstly $Wa^G$ is computed for $G$ as the trivial group $\{1\}$, written briefly as 1. Then $w^1$ is related to Wall's finiteness obstruction.

Let $X$ be a connected finitely dominated $CW$-complex with universal covering $\tilde{X}$. If $C(\tilde{X})$ is the cellular $\mathbb{Z}[\pi_1(X)]$-chain complex, we can choose a finitely generated projective $\mathbb{Z}[\pi_1(X)]$-chain complex $P$ which is homotopy equivalent to $C(\tilde{X})$. The finiteness obstruction $[X] \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of Wall is defined by $\sum_{n=0}^{\infty} (-1)^n [P_n]$ (see Wall [20, p. 138]).

If $Y$ is a topological space, define $\tilde{K}_0(\mathbb{Z}[\pi(Y)])$ by

$$\bigoplus_{V \in \pi_0(Y)} \tilde{K}(\mathbb{Z}[\pi_1(V)]).$$

A map $f\colon Y \to Z$ induces

$$\pi_0(f)\colon \pi_0(Y) \to \pi_0(Z) \quad \text{and} \quad \pi_1(f \mid V)\colon \pi_1(V) \to \pi_1(W)$$

for $V \in \pi_0(Y)$, $W \in \pi_0(Z)$, with $\pi_0(f)(V) = W$. This yields a homomorphism

$$f_*\colon \tilde{K}_0(\mathbb{Z}[\pi(Y)]) \to \tilde{K}_0(\mathbb{Z}[\pi(Z)]),$$

so that $\tilde{K}_0(\mathbb{Z}[\pi(?)])$ becomes a functor from the category of topological spaces into the category of abelian groups.

Given a not necessarily connected finitely dominated $CW$-complex $X$, define its Wall obstruction

$$[X] \in \tilde{K}_0(\mathbb{Z}[\pi(X)]) = \bigoplus_{V \in \pi_0(X)} \tilde{K}_0(\mathbb{Z}[\pi_1(V)])$$

by the collection of the finiteness obstructions $[V] \in \tilde{K}_0(\mathbb{Z}[\pi_1(V)])$ for each component $V$ of $X$.

PROPOSITION 5.1. (a) *Let $X$ be a finitely dominated CW-complex. Then $X$ is homotopic to a finite CW-complex if and only if $[X] \in \tilde{K}_0(\mathbb{Z}[\pi(X)])$ vanishes.*

(b) *The pair $(\tilde{K}_0(\mathbb{Z}[\pi(?)]), [?])$ is a functorial additive invariant for the category of finitely dominated CW-complexes.*

*Proof.* (a) This follows from [**19**, p. 66; **20**].

(b) Additivity is proved by Siebenmann in [**18**].

Given a topological space $Y$ with $\pi_1(Y, y)$ finitely represented, define a homomorphism $F(Y)$: $Wa^1(Y) \to \tilde{K}_0(\mathbb{Z}[\pi(Y)])$ by $[f: X \to Y] \mapsto f_*([X])$.

THEOREM 5.2. *This induces a natural equivalence*

$$F: Wa^1 \xrightarrow{\cong} \tilde{K}_0(\mathbb{Z}[\pi(?)])$$

*such that for a finitely dominated CW-complex $X$ the relation $F(X)(w^1(X)) = [X]$ holds.*

*Proof.* Given a topological space $Y$ one has to show that $F(Y)$ is bijective.

(a) Injectivity. Let $X$ be a finitely dominated $CW$-complex and $f$: $X \to Y$ a map such that $F(Y)$ sends $[f] \in Wa^1(Y)$ to zero. By attaching finitely many cells one can extend $f$ to a map $g$: $Z \to Y$ such that $\pi_0(g)$: $\pi_0(Z) \to \pi_0(Y)$ is bijective and $\pi_1(g \mid V)$: $\pi_1(V) \to \pi_1(W)$ an isomorphism for all $V \in \pi_0(Z)$ and $W \in \pi_0(Y)$ with $\pi_0(g)(V) = W$. Hence the homomorphism

$$g_*: \tilde{K}_0(\mathbb{Z}[\pi(Z)]) \to \tilde{K}_0(\mathbb{Z}[\pi(Y)])$$

is bijective, so that $[Z] \in \tilde{K}(\mathbb{Z}[\pi(Z)])$ vanishes because

$$0 = F(Y)([f]) = F(Y)([g]) = g_*([Z]).$$

Proposition 5.1 implies the existence of a finite $CW$-complex $Z'$ and a homotopy equivalence $h$: $Z' \to Z$.

Because of the following diagram $[f]$ vanishes in $Wa^1(Y)$:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[drr, "f"'] & Z \arrow[dr, "g"] & & Z' \arrow[ll, "h"'] \arrow[dl, "g\circ h"'] & 0 \arrow[l, hook'] \arrow[dll] \\
& & Y & &
\end{tikzcd}$$

(b) Surjectivity. Since $K_0(\mathbb{Z}[\pi(Y)])$ is the direct sum $\bigoplus_{V \in \pi_0(Y)} K_0(\mathbb{Z}[\pi_1(V)])$, we can assume without loss of generality that $Y$ is connected. Choose a finite connected complex $Y_1$ and a map $g$: $Y_1 \to Y$ inducing isomorphisms on the fundamental groups. As

$$g_*: \tilde{K}_0(\mathbb{Z}[\pi_1(Y_1)]) \to \tilde{K}_0(\mathbb{Z}[\pi_1(Y)])$$

is an isomorphism, it suffices to show that $F(Y_1)$ is an epimorphism. Given $\eta \in \tilde{K}_0(\mathbb{Z}[\pi_1(Y_1)])$, a retraction $r$: $Y_2 \to Y_1$ with $Y_2$ a finitely dominated $CW$-complex and $r_*([Y_2]) = \eta$ can easily be constructed (see Wall [**19**]). Therefore $F(Y_1)([r]) = \eta$.

There is also an unreduced version for Wall's finiteness obstruction $[X] \in K_0(\mathbb{Z}[\pi(X)])$. For any group $\pi$ the unreduced projective class group $K_0(\mathbb{Z}[\pi])$ splits as $\tilde{K}_0(\mathbb{Z}[\pi]) \oplus \mathbb{Z}$. This induces a natural isomorphism

$$K_0(\mathbb{Z}[\pi(X)]) \to \tilde{K}_0(\mathbb{Z}[\pi(X)]) \oplus A^1(X)$$

sending $[X] \in K_0(\mathbb{Z}[\pi(X)])$ to $([X], \chi^1(X)) \in \tilde{K}_0(\mathbb{Z}[\pi(X)]) \oplus A^1(X)$ for a finitely dominated $CW$-complex $X$.

COROLLARY 5.3. *There is a natural equivalence $F$: $Wa^1 \oplus A^1 \xrightarrow{\cong} K_0(\mathbb{Z}[\pi(?)])$. Given a finitely dominated $CW$-complex $X$, the homomorphism $F(X)$ sends $(w^1(X), \chi^1(X))$ to $[X]$.*

Because of Theorem 4.1, $K_0(\mathbb{Z}[\pi(?)])$ with Wall's finiteness obstruction is the universal functorial additive invariant for the category of finitely dominated $CW$-complexes.

For a $G$-space $Y$ there is a natural homomorphism $Q(Y)$: $Wa^G(Y) \to Wa^1(Y/G)$ sending $[f\colon X \to Y]$ to $[f/G\colon X/G \to Y/G]$.

THEOREM 5.4. *The natural homomorphism $Q(Y)$: $Wa^G(Y) \to Wa^1(Y/G)$ sends $w^G(Y)$ to $w^1(Y/G)$ if $Y$ is a finitely dominated $G$-complex. If $Y$ is free then $Q(Y)$ is an isomorphism.*

*Proof.* For a free $G$-complex $Y$ an inverse map $Wa^1(Y/G) \to Wa^G(Y)$ is given by the pull-back construction. It sends $[f\colon X \to Y/G]$ to $[\bar{f}\colon \bar{X} \to \bar{Y}]$ where $\bar{f}$ is obtained from the pull-back construction applied to $f$ and the principal $G$-bundle $Y \to Y/G$.

Combining Corollary 5.3 and Theorem 5.4 one gets a natural isomorphism

$$Wa^G(Y) \xrightarrow{\cong} \tilde{K}_0(\mathbb{Z}[\pi(Y/G)])$$

for a free finitely dominated $G$-complex $Y$. It sends $w^G(Y)$ to $[Y/G]$.

Using induction over the orbit bundles one can show the following splitting theorem.

THEOREM 5.5. *If $Y$ is a $G$-space of finite orbit type, there exists a natural isomorphism*

$$Wa^G(Y) \xrightarrow{\cong} \bigoplus_{\{G/?\to Y\}/\sim} \tilde{K}_0(\mathbb{Z}[\pi_1(EI(V(x))) \times_{I(V(x))} V(x)]).$$

The definition of $V(x)$ and $I(V(x))$ for $x\colon G/H \to Y$ was given in § 3 and $EI(V(x))$ is the classifying bundle of the Lie group $I(V(x)) \subset WH$. This result is proved algebraically by the author in **[15]** for a discrete group $G$. The arguments given there can be generalized to arbitrary compact Lie groups and yield an analogous statement for the equivariant Whitehead group defined by Illman in **[10]**. The splitting theorem for equivariant Whitehead groups is also proved algebraically by Illman **[11]** and geometrically by Hauschild **[9]** and Kunihiko **[13]** and under some restrictions also by Baglivo **[2]**. See also **[1]**.

## 6. *Product and restriction formulas*

In this section the existence of a product and restriction formula is easily derived from the universal property of the finiteness obstruction. In contrast to the geometric approach it is difficult to give an algebraic description. One reason for this is the complicated structure of the splitting Theorem 5.5 and the bad behaviour of the orbit bundles under restriction to subgroups.

Applying the remarks of § 2 about product and restriction formulas to Theorem 4.1 we get:

THEOREM 6.1. *Product formula. Let $G$ and $G'$ be compact Lie groups and $Y$, respectively $Y'$, a finitely dominated $G$-complex, respectively $G'$-complex. There exists a natural pairing*

$$P(Y, Y'): (A^G(Y) \oplus Wa^G(Y)) \oplus (A^{G'}(Y') \oplus Wa^{G'}(Y')) \\ \to A^{G\times G'}(Y \times Y') \oplus Wa^{G\times G'}(Y \times Y')$$

*uniquely determined by the property that $P(Y, Y')$ sends*

$$(\chi^G(Y), w^G(Y)) \oplus (\chi^{G'}(Y'), w^{G'}(Y')) \quad \text{to} \quad (\chi^{G\times G'}(Y \times Y'), w^{G\times G'}(Y \times Y')).$$

THEOREM 6.2. *Restriction formula. Let $H$ be a closed subgroup of $G$ and $Y$ a finitely dominated $G$-complex. There exists a natural homomorphism*

$$R(Y): A^G(Y) \oplus Wa^G(Y) \to A^H(\mathrm{res}(Y)) \oplus Wa^H(\mathrm{res}(Y))$$

*uniquely determined by*

$$R(Y)(\chi^G(Y), w^G(Y)) = (\chi^H(\mathrm{res}(Y)), w^H(\mathrm{res}(Y))).$$

THEOREM 6.3. *Diagonal product formula. Let $X$ and $Y$ be finitely dominated $G$-complexes. There exists a natural pairing*

$$P(X, Y): (A^G(X) \oplus Wa^G(X)) \otimes (A^G(Y) \oplus Wa^G(Y)) \\ \to A^G(X \times Y) \oplus Wa^G(X \times Y)$$

*uniquely determined by*

$$P(X, Y)((\chi^G(X), w^G(X)) \otimes (\chi^G(Y), w^G(Y))) = (\chi^G(X \times Y), w^G(X \times Y))$$

*with $G$ acting diagonally on $X \times Y$.*

To get these formulas we have always worked with $G$-spaces having the homotopy type of a finitely dominated $G$-complex and not only with finitely dominated $G$-complexes. The restriction to a subgroup $H$ of a $G$-complex does not have the structure of an $H$-complex canonically but has the homotopy type of an $H$-complex.

Now some explanations of these formulas and computations of them are given. Using Corollary 5.3 the product formula in Theorem 6.1 reduces to the product formulas for Wall's finiteness obstruction in **[8, 18]** for $G$ as the trivial group. One can also give an explicit version of the product formula using the algebraic computation of $Wa^G$ in Theorem 5.5. This was stated for $G$ as a discrete group in **[15]**. The arguments given there can be generalized to compact Lie groups without difficulty and can also be applied to the equivariant Whitehead group (see also Illman [**12**]).

The existence of the product formula implies some interesting facts.

COROLLARY 6.4. *Let $X$ be a finitely dominated $G$-complex and $Y$ a connected finite $CW$-complex with vanishing Euler characteristic. Then $X \times Y$ is $G$ homotopic to a finite $G$-complex.*

*Proof.* This follows from Theorem 6.1 for $G' = \{1\}$ because $(\chi^1(Y), w^1(Y)) = 0$ in $A^1(Y) \oplus Wa^1(Y)$.

In particular, this can be applied to $Y$ as the one-dimensional circle $S^1$. The geometric proof of Mather [**17**, p. 93] that $X \times S^1$ is up to homotopy finite if $X$ is a finitely dominated $CW$-complex can be generalized directly to the equivariant case. Let $X$ be a $G$-complex and $K$ a finite $G$-complex and $r$: $K \to X$ and $i$: $X \to K$ $G$-maps with $r \circ i \simeq_G \mathrm{ID}$. Define the mapping torus $T(i \circ r)$ as the space obtained from the mapping cylinder $C_{i \circ r}$ of $i \circ r$ by identifying the top and the bottom using the identity map. Now $C_{i \circ r}$ is homotopic to $C_i \cup_X C_r$ relative to $K + K$ and $C_r \cup_K C_i$ is homotopic to $X \times I$ relative to $X \times \partial I$ because $r \circ i \simeq_G \mathrm{ID}$. This yields a homotopy equivalence $\Phi$: $T(i \circ r) \to X \times S^1$. But $T(i \circ r)$ is a finite $G$-complex.

Mather's idea was used by Ferry [**7**] to develop a simple homotopy approach to Wall's finiteness obstruction. This was extended by Kwasik [**14**] to the equivariant case. Here is a reformulation using our approach to the finiteness obstruction.

Given a $G$-space $Y$, define a homomorphism $\varphi(Y)$: $Wa^G(Y) \to Wh^G(Y \times S^1)$. Let $X$ be a finitely dominated $G$-complex and $f$: $X \to Y$ be a $G$-map representing $[f]$ in $Wa^G(Y)$. Choose a finite $G$-complex $K$ and $G$-maps $r$: $K \to X$ and $i$: $X \to K$ with $r \circ i \simeq_G \mathrm{ID}_X$. Let $\Phi$: $T(i \circ r) \to X \times S^1$ be the $G$-homotopy equivalence above and $\Phi^{-1}$ be a homotopy inverse. If $\theta$: $S^1 \to S^1$ sends $z$ to $z^{-1}$, denote by $C$ the mapping cylinder of the homotopy equivalence

$$\Phi^{-1} \circ (\mathrm{ID} \times \theta) \circ \Phi: \ T(i \circ r) \to T(i \circ r)$$

between finite $G$-complexes. Then the pair $(C, T(i \circ r))$ determines an element in $Wh^G(T(i \circ r))$ called the torsion of $\Phi^1 \circ (\mathrm{ID} \times \theta) \circ \Phi$. Define $\varphi(Y)([f])$ by the image of $(C, T(i \circ r))$ under

$$(f \times \mathrm{ID} \circ \Phi)_*: \ Wh^G(T(i \circ r)) \to Wh^G(Y \times S^1),$$

namely $\varphi(Y)([f])$ is represented by $(C \cup_{f \times \mathrm{ID} \circ \Phi} Y \times S^1, Y \times S^1)$. Using the arguments in [**7**] one proves that $\varphi(Y)$ is a well-defined homomorphism.

The invariant $O_G(X) \in Wh^G(X \times S^1)$ defined by Kwasik [**14**, p. 366] for a finitely dominated $CW$-complex is just $\varphi(X)(w^G(X))$. The statement in [**14**, p. 366] that $X$ is homotopy equivalent to a finite $G$-complex if and only if $O_G(X)$ vanishes is equivalent to the statement that $\varphi(Y)$ is injective.

An algebraic computation of the restriction formula for finite groups is given in [**15**] using Theorem 5.5. It turns out, however, that this cannot be generalized to compact Lie groups directly if the dimension of the subgroup $H$ is smaller than the dimension of $G$. As an illustration consider the following special case.

Let $Y$ be a free connected finitely dominated $G$-complex. Applying the restriction formula to the trivial subgroup one gets a homomorphism

$$Wa^G(Y) \oplus A^G(Y) \to Wa^1(\mathrm{res}(Y)) \oplus A^1(\mathrm{res}(Y))$$

sending $(w^G(Y), \chi^G(Y))$ to $(w^1(\mathrm{res}(Y)), \chi^1(Y))$. Corollary 5.3 and Theorem 5.4 yield a homomorphism $K_0(\mathbb{Z}[\pi_1(Y/G)]) \to K_0(\mathbb{Z}[\pi_1(Y)])$ mapping $[Y/G]$ to $[Y]$ if $[\ ]$ denotes the non-equivariant finiteness obstruction due to Wall. One easily checks that this coincides with the geometric transfer homomorphism $p^!$: $K_0(\mathbb{Z}[\pi_1(Y/G)]) \to K_0(\mathbb{Z}[\pi_1(Y)])$ associated by Ehrlich [6] to the principle $G$-bundle $p$: $Y \to Y/G$ regarded as a fibration

$$G \longrightarrow Y \xrightarrow{p} Y/G.$$

If $G$ is finite, the homomorphism $p_*$: $\pi_1(Y) \to \pi_1(Y/G)$ is injective with finite cokernel. Then restriction with $p_*$ defines a homomorphism

$$K_0(\mathbb{Z}[\pi_1(Y/G)]) \to K_0(\mathbb{Z}[\pi_1(Y)])$$

which turns out to be $p^!$. For an arbitrary compact Lie group $G$ such a simple algebraic computation of $p^!$ is not available. An algebraic description of $p^!$ is stated in [16]. It can be used, for example, to show for connected $G$ that $p^!$ vanishes if $G$ is not isomorphic to $SO(3)^n \times (S^1)^m$ or if $\pi_1(Y/G)$ is finite. Combining the methods of [15] and [16] one can give an algebraic description of the formula of Theorem 6.2 generally. This can also be done for the Whitehead torsion.

The same problems arise for the diagonal product formula of Theorem 6.3 which is completely treated in [15] for $G$ as a discrete group (see also tom Dieck [5]). For example, one can prove for a finite group $\pi$ and a finitely dominated $\pi$-complex $X$ and a free finite $\pi$-complex $Y$ that $X \times Y$ is $\pi$-homotopy equivalent to a finite $\pi$-complex.

The diagonal product formula implies that $(A^G(\{\mathrm{pt}\})$ is a commutative ring with unit and $A^G \oplus Wa^G$ is a functor from the category of finitely dominated $G$-complexes into the category of $A^G(\{\mathrm{pt}\})$-modules. We recall that for a finite group $\pi$ the ring $A^\pi(\{\mathrm{pt}\})$ is just the Burnside ring of $\pi$.

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