Course Syllabus for Math 287: Algebraic L-Theory and Surgery

Course Description Let X be a finite complex. When does X has the homotopy type of a smooth manifold? First, X must satisfy Poincare duality. Second, there should be a vector bundle T_X on X which plays the role of the tangent bundle. If the dimension of X is divisible by 4, then the signature of the intersection form on the cohomology $H^*(X)$ should be given as a certain characteristic class of T_X (the Hirzebruch signature formula). Amazingly, if X is simply connected and has dimension $4k > 4$, then these conditions are sufficient to guarantee that X is homotopy equivalent to a smooth manifold. In this course, we will study this theorem and some of its generalizations: to manifolds which are not assumed to be smooth, to manifolds which are not assumed to be simply connected, and to manifolds whose dimension is not assumed to be divisible by four. In order to obtain a classification theorem in these settings, we will develop the subject of *algebraic L-theory*, which can be regarded as an elaborate generalization of the classical Witt groups of quadratic forms.

Meeting Time MWF at 1.

Office Hours Thursdays 2-3, or by appointment.

Texts There is no textbook required for this class. Useful references include Ranicki's "Algebraic L-Theory and Topological Manifolds" and Wall's text "Surgery on Compact Manifolds."

Course Website http://www.math.harvard.edu/~lurie/287.html

Prerequisites Familiarity with the machinery of modern algebraic topology (simplicial sets, spectra, ...). The first part of the course (where we develop the subject of L-theory) will require a high tolerance for abstraction, while the second part (where we apply the theory to classify manifolds) will be more concrete.

Topics likely to be covered

- Construction of L-theory spectra.
- Witt groups of quadratic forms.
- Techniques of surgery in algebraic and geometric settings.
- Poincare duality spaces and Spivak normal fibrations.
- L-theory orientations and generalized signature formulas.
- Classification of high-dimensional manifolds, up to h-cobordism.

Grading Undergraduates or graduate students wishing to take this course for a grade should speak with the instructor.

Introduction (Lecture 1)

February 2, 2011

In this course, we will be concerned with variations on the following:

Question 1. Let X be a CW complex. When does there exist a homotopy equivalence $X \simeq M$, where M is a compact smooth manifold?

In other words, what is special about the homotopy type of a compact smooth manifold M? One special feature is obvious:

Fact 2. *Compact manifolds satisfy Poincare duality.*

Let us assume for simplicity that X is simply connected. If M is a compact smooth manifold homotopy equivalent to X, M is also simply connected and therefore orientable. A choice of orientation determines a fundamental homology class $[M] \in \mathrm{H}_n(M; \mathbf{Z})$, where n denotes the dimension of M. If $f : M \to X$ is a homotopy equivalence, then $[X] = f_*[X]$ is an element of $\mathrm{H}_n(X; \mathbf{Z})$ with the following property: for every integer q, the operation of cap product with $[X]$ induces an isomorphism

$$\mathrm{H}^q(X; \mathbf{Z}) \to \mathrm{H}_{n-q}(X; \mathbf{Z}).$$

This motivates the following definition:

Definition 3. Let X be a simply connected CW complex. We say that X is a *simply connected Poincare complex of dimension n* if there exists a homology class $[X] \in \mathrm{H}_n(X; \mathbf{Z})$ such that cap product with $[X]$ induces isomorphisms

$$\mathrm{H}^q(X; \mathbf{Z}) \to \mathrm{H}_{n-q}(X; \mathbf{Z})$$

for every integer q. In this case, we say that $[X]$ is a *fundamental class* of X.

Example 4. Any compact smooth manifold of dimension n is a Poincare complex of dimension n.

Remark 5. Taking $q = 0$ in Definition 3 and using our assumption that X is connected, we obtain an isomorphism

$$\mathbf{Z} \simeq \mathrm{H}^0(X; \mathbf{Z}) \to \mathrm{H}_n(X; \mathbf{Z})$$

given by $1 \mapsto [X]$. It follows that $\mathrm{H}_n(X; \mathbf{Z})$ must be a free abelian group of rank 1, and $[X]$ must be a generator of $\mathrm{H}_n(X; \mathbf{Z})$. Consequently, the fundamental class of X is well-defined up to sign.

Remark 6. If X is a simply connected Poincare complex of dimension n, then we have

$$\mathrm{H}_q(X; \mathbf{Z}) \simeq \mathrm{H}^{n-q}(X; \mathbf{Z}) \simeq 0$$

for $q > n$. In particular, n is uniquely determined by X: it is the largest degree of a nonvanishing homology group of X.

Remark 7. Using the fact that $\mathrm{H}^m(X; \mathbf{Z})$ vanishes for $m > n$ and that it is free when $m = n$, one can show that X is homotopy equivalent to a CW complex whose cells have dimension $\leq n$. However, we will not need this fact and we will not require that X itself have this property.

We can now give a partial answer to Question 1: if X is to be homotopy equivalent to a compact smooth manifold, then X must be a Poincare complex. We can therefore refine Question 1 (in the simply connected case) as follows:

Question 8. Let X be a simply connected Poincare complex of dimension n. When does there exist a homotopy equivalence $X \simeq M$, where M is a smooth manifold of dimension n?

To address Question 8, we make another observation: if M is a smooth manifold of dimension n, then M has a *tangent bundle* T_M, which is a real vector bundle of rank n over M. Moreover, the tangent bundle of M is closely connected with our discussion of Poincare duality.

We begin by considering the *normal bundle* of M. Choose an embedding $i : M \to \mathbb{R}^k$ for some large integer k. Let N_M denote the normal bundle to this embedding. By choosing a tubular neighborhood of M in $\mathbb{R}^k$, we can identify N_M with an open subset of $\mathbb{R}^k$. The Thom space $T(N_M)$ is given by the one-point compactification of N_M, given by $N_M \cup \{*\}$. We have a Thom-Pontryagin collapse map

$$c : S^k = \mathbb{R}^k \cup \{\infty\} \to T(N_M),$$

given by $c(v) = \begin{cases} v & \text{if } v \in N \\ * & \text{otherwise.} \end{cases}$ which determines an element $[c] \in \pi_k(T(N_M), \infty)$. Since M is simply connected, the normal bundle N is oriented. Choosing an orientation, we obtain a Thom isomorphism $\mathrm{H}_k(T(N_M), *; \mathbf{Z}) \simeq \mathrm{H}_n(M; \mathbf{Z})$. Composing with the Hurewicz map, we obtain a homomorphism

$$\pi_k(T(N_M), *) \to \mathrm{H}_k(T(N_M), *; \mathbf{Z}) \simeq \mathrm{H}_n(M; \mathbf{Z}).$$

The image of $[c]$ under this composite map is a fundamental homology class $[M]$. (with respect to the orientation determined by the choice of orientation on N_M).

The vector bundle N_M is not unique: it depends on a choice of embedding $M \hookrightarrow \mathbb{R}^k$. However, the *spectrum* $\Sigma^{\infty - k} T(N_M)$ is uniquely determined. We have a canonical exact sequence of vector bundles

$$T_M \to T_{\mathbb{R}^n}|M \to N_M.$$

Choosing a splitting of this exact sequence, we obtain a direct sum decomposition $N_M \oplus T_M \simeq \underline{R}^k$, where $\underline{R}^k$ denotes the trivial vector bundle of rank k. It follows that the spectrum $\Sigma^{\infty-k} T(N_M)$ can be identified with the *Thom spectrum* M^{-T_M} of the virtual vector bundle $-T_M$ on M. The element $[c] \in \pi_k(T(N_M), \infty)$ determines an element $\eta_M \in \pi_0 M^{-T_M}$, which is independent of the choice of embedding i.

We can summarize the above discussion as follows:

Fact 9. *Let M be a simply connected smooth manifold of dimension n. Then there exists a vector bundle ζ on M of dimension n (namely, the tangent bundle T_M) and a class $\eta_M \in \pi_0 M^{-\zeta}$ such that the image of η_M under the composite map*

$$\pi_0(M^{-\zeta}) \to \mathrm{H}_0(M^{-\zeta}; \mathbf{Z}) \simeq \mathrm{H}_n(M; \mathbf{Z})$$

is a fundamental class of M. Here the second map is the Thom isomorphism (determined by a choice of orientation of ζ).

This gives us another necessary condition that a simply connected CW complex X must satisfy if X is to be homotopy equivalent to a manifold of dimension n: there must exist a vector bundle ζ on X and a homotopy class $\eta_X \in \pi_0 X^{-\zeta}$ whose image $[X] \in \mathrm{H}_n(X; \mathbf{Z})$ exhibits X as a Poincare complex of dimension n. We may therefore refine our question yet again:

Question 10. Let X be a simply connected Poincare complex of dimension n. Suppose we are given a vector bundle ζ of dimension n on X and a homotopy class $\eta_X \in \pi_0 X^{-\zeta}$ whose image in $\mathrm{H}_n(X; \mathbf{Z})$ is a fundamental homology class for X. Does there exist a smooth manifold M of dimension n and a homotopy equivalence $f : M \to X$ such that $f^*\zeta$ is (stably) isomorphic to T_M and $f^*\eta_X = \eta_M \in \pi_0 M^{-T_M}$?

We now give the answer to Question 10 in the simplest case. Assume that X is a simply connected Poincare complex of dimension $n = 4k$. In this case, we have a symmetric bilinear form

$$\langle , \rangle : \mathrm{H}^{2k}(X; \mathbb{R}) \times \mathrm{H}^{2k}(X; \mathbb{R}) \to \mathrm{H}^{4k}(X; \mathbb{R}) \xrightarrow{[X]} \mathbb{R}$$

and Poincare duality ensures that this form is nondegenerate. We may therefore choose an orthogonal basis $(x_1, \ldots, x_a, y_1, \ldots, y_b)$ for $\mathrm{H}^{2k}(X; \mathbb{R})$ satisfying $\langle x_i, x_i \rangle = 1$ and $\langle y_i, y_i \rangle = -1$. The difference $a - b$ is called the *signature* of X, and will be denoted by σ_X. Note that the sign of σ_X depends on a choice of fundamental class for X.

If M is a compact smooth manifold of dimension $n = 4k$, then the signature of M is given by the *Hirzebruch signature formula*. Namely, there is a formula

$$\sigma_M = L(p_1(T_M), p_2(T_M), \ldots, p_k(T_M))[M].$$

Here $L(p_1(T_M), p_2(T_M), \ldots, p_k(T_M))$ denotes some polynomial in the Pontryagin classes $p_i(T_M)$ (note that the right hand side of this formula also depends up to sign on our choice of orientation of M). For example, when $n = 4$ we have $\sigma_M = \frac{p_1(T_M)}{3}[M]$, and when $n = 8$ we have

$$\sigma_M = \frac{7p_2(T_M) - p_1(T_M)^2}{45}[M].$$

Remark 11. If we choose a connection on the manifold M, then we can use Chern-Weil theory to obtain explicit differential forms representing the Pontryagin classes of the tangent bundle T_M. Consequently, the signature of M can be computed by integrating over M an explicitly given n-form on M. We can therefore regard the Hirzebruch signature formula as saying that there is a *purely local* formula for the signature, which is defined *a priori* as a global invariant of M.

Remark 12. Here is a very rough heuristic justification for why there should exist a Hirzebruch signature formula. If X is a Poincare complex of dimension $4k$, then the signature σ_X is defined because we can define an intersection form using Poincare duality. If X is a manifold, the Poincare duality is satisfied for a "local" reason, so we might expect to obtain a "local" formula for σ_X. Later in this course, we will prove the Hirzebruch signature formula by making this heuristic more precise.

This gives us one further condition that a triple, $(X, \zeta, \nu_X \in \pi_0 X^{-\zeta})$ must satisfy to obtain an affirmative answer to Question 10. Namely, we must have

$$\sigma_X = L(p_1(\zeta), p_2(\zeta), \ldots, p_k(\zeta))[X].$$

Simply-connected surgery provides a converse in high dimensions:

Theorem 13 (Browder, Novikov?)**.** *Let X be a simply connected Poincare complex of dimension $4k > 4$, let ζ be a vector bundle (of rank $4k$) on X, and let $\eta_X \in \pi_0 X^{-\zeta}$ be such that the image of η_X in $\mathrm{H}_{4k}(X; \mathbf{Z})$ is a fundamental class. Then Question 10 has an affirmative answer if and only if $\sigma_X = L(p_1(\zeta), p_2(\zeta), \ldots, p_k(\zeta))[X]$: that is, if and only if X satisfies the Hirzebruch signature formula.*

Theorem 13 is a prototype for the type of result we would like to obtain in this class. We will pursue a number of variations:

(*a*) We can contemplate Question 1 for Poincare complexes X which are not assumed to be simply connected.

(*b*) Question 1 concerns the *existence* of a manifold M in the homotopy type of X. If the answer is affirmative, one can further ask if M is unique.

Let us briefly describe how problem (b) can be attacked. Suppose that we are given a Poincare complex X and a pair of homotopy equivalences $f : X \to M$, $g : X \to M'$, where M and M' are compact manifolds of dimension n. We can then consider the "double mapping cylinder" $Y = M \coprod_{X \times \{0\}} (X \times [0,1]) \coprod_{X \times \{1\}} M'$. The pair $(Y, M \amalg M')$ satisfies a relative version of Poincare duality. This suggests that we might look for an $(n+1)$-manifold B with boundary $M \amalg M'$ and a homotopy equivalence $(B, M \amalg M') \to (Y, M \amalg M')$ which restricts to the identity map on M and M'. If we can solve this problem, then B is an *h-cobordism* from M to M': that is, a bordism from M to M' such that the inclusions $M \hookrightarrow B \hookleftarrow M'$ are homotopy equivalences. If $n \geq 5$ and M is simply connected, then the h-cobordism theorem guarantees the existence of a diffeomorphism $B \simeq M \times [0,1]$, which in particular gives a diffeomorphism $M \simeq M'$.

In summary, the problem of deciding whether M is unique can be regarded as another of roughly the same type as Question 1. This motivates considering two more types of variations of Question 1:

(c) Rather than considering the *absolute* case of a Poincare complex X, we should consider the problem of proving that a pair of spaces $(X, \partial X)$ is homotopy equivalent to a manifold with boundary.

(d) We should not restrict our attention to the case of a fixed dimension n: a lot of information about the classification of manifolds of dimension n can be obtained by thinking about manifolds of dimension $> n$. In particular, we should not restrict our attention to the case where n is a multiple of four. (However, we will retain the assumption that $n > 4$: this is the domain of *high-dimensional topology* where techniques of surgery work well).

If M is not simply connected, then an h-cobordism from M to M' does not generally guarantee that M and M' are diffeomorphic: one encounters an algebraic obstruction called the *Whitehead torsion*. This is an interesting story, but not one we will discuss in this class: we will be content to give the classification of manifolds in a homotopy type *up to h-cobordism*.

In fact, we will do more. Suppose that M and M' are as above, and that Y has the homotopy type of an h-cobordism B from M to M'. We might then ask a higher-order uniqueness question: to what extent is the bordism B uniquely determined? To ask these questions in an organized way, it is convenient to introduce the *structure space* $S(X)$ of a Poincare complex X. This is a space whose connected components are given by manifolds M with a homotopy equivalence $M \to X$, up to h-cobordism. Question 1 is the question of whether or not $S(X)$ is nonempty, and the uniqueness problem amounts to the question of whether or not $S(X)$ is connected. Better still, we might try to discuss the entire homotopy type of $S(X)$.

(e) We can ask an analogue of Question 1 for manifolds equipped with various structure. Suppose, for example, that we wanted to find a spin manifold in the homotopy type of the Poincare complex X. The collection of h-cobordism classes of such manifolds can be described as connected components of a slightly different structure space $S_{\mathrm{Spin}}(X)$. By forgetting spin structures, we obtain a map of structure spaces $\theta : S_{\mathrm{Spin}}(X) \to S(X)$. Giving a spin structure on a manifold M is equivalent to giving a spin structure on its tangent bundle T_M: that is, to reducing the structure group of M from the orthogonal group $\emptyset(n)$ to the spin group $\mathrm{Spin}(N)$. Consequently, the homotopy fibers of the map θ are easy to describe. Consequently, the problem of determining the homotopy type of $S_{\mathrm{Spin}}(X)$ can be reduced to the problem of determining the homotopy type of $S(X)$.

What we have denoted by $S(X)$ should really be denoted $S_{\mathrm{Sm}}(X)$, the *smooth* structure space, because in the above discussion we required all manifolds to be smooth. We can also define a *topological* structure space $S_{\mathrm{Top}}(X)$ by considering topological manifolds with a homotopy equivalence to X. By forgetting smooth structures, we obtain a map of structures spaces $S_{\mathrm{Sm}}(X) \to S_{\mathrm{Top}}(X)$. The relationship between $S_{\mathrm{Sm}}(X)$ and $S_{\mathrm{Top}}(X)$ is similar to the relationship between $S_{\mathrm{Spin}}(X)$ and $S_{\mathrm{Sm}}(X)$: according to smoothing theory, for topological manifolds M of dimension > 4, giving a smooth structure on M is equivalent giving a vector bundle structure on the topological tangent bundle T_M. In other words, to classify smooth manifolds in the homotopy type of X we can proceed by first classifying the topological manifolds in the homotopy type of X and then studying the problem of smoothing them, where the second step reduces to a purely homotopy-theoretic problem. Put another way, there is a good homotopy-theoretic understanding of the homotopy fibers of the map $S_{\mathrm{Sm}}(X) \to S_{\mathrm{Top}}(X)$.

However, there is a much more compelling reason to work with topological manifolds rather than smooth manifolds: the topological versions of these questions often have nicer answers. For example, there is only one topological manifold in the homotopy type of the n-sphere S^n (by the generalized Poincare conjecture), but this topological manifold admits many different smooth structures (exotic spheres). The ultimate algebraic description of structure spaces which we obtain will be cleanest in the topological category. For example, $S_{\mathrm{Sm}}(X)$ is just a space, but we will later see that $S_{\mathrm{Top}}(X)$ is an infinite loop space (if nonempty). A concrete consequence of this is that if we fix a topological manifold M, then the collection of h-cobordism classes of manifolds in with a homotopy equivalence to M has the structure of an abelian group.

Our ultimate goal in this course is to obtain a purely homotopy theoretic description of the structure space $S(X)$ of a Poincare complex X. Though we are not yet ready to formulate this description precisely, let us assert that it has the same basic form as the statement of Theorem 13. Namely, we will associate to X a certain invariant σ, called the *visible symmetric signature of* X. We will then show that finding a manifold in the homotopy type of X amounts to verifying a "local formula" for this invariant, generalizing the Hirzebruch signature formula (see Remark 11). The fine print is that this invariant is not an integer, but something more sophisticated. Explaining exactly what that "something" is will require us to develop the apparatus of *algebraic L-theory*. That is our objective for the first half of this course. In the second half, we will return to the theory of manifolds, using the algebraic apparatus to prove a very general version of Theorem 13.

Categorical Background (Lecture 2)

February 2, 2011

In the last lecture, we stated the main theorem of simply-connected surgery (at least for manifolds of dimension $4m$), which highlights the importance of the *signature* σ_X as an invariant of an (oriented) Poincare complex. Let us begin with a few remarks about how this invariant is defined.

Let V be a finite dimensional vector space over the real numbers and let $q : V \to \mathbb{R}$ be a nondegenerate quadratic form on V, with associated bilinear form $(,) : V \times V \to \mathbb{R}$. We can always choose an orthogonal basis $\{x_1, \ldots, x_a, y_1, \ldots, y_b\}$ for V satisfying

$$(x_i, x_i) = 1 \qquad (y_i, y_i) = -1.$$

The sum $a + b$ is the *dimension* of the vector space V, and the difference $a - b$ is called the *signature* of q, and denoted $\sigma(q)$.

Theorem 1 (Sylvester)**.** *Let V be a finite dimensional vector space over $\mathbb{R}$ equipped with a nondegenerate quadratic form $q : V \to \mathbb{R}$. Then the signature $\sigma(q)$ is well-defined: that is, it does not depend on the choice of orthogonal basis for V. Moreover, if V' is another finite dimensional vector space over $\mathbb{R}$ with a quadratic form $q' : V' \to \mathbb{R}$, then there exists an isometry $(V, q) \simeq (V', q')$ if and only if $\dim V = \dim V'$ and $\sigma(q) = \sigma(q')$.*

The notion of a vector space V with a quadratic form makes sense over an arbitrary field k. We have emphasized the case $k = \mathbb{R}$ for two reasons: first, the classification of quadratic forms over $\mathbb{R}$ is particularly simple (because of Theorem 1). Second, information about the intersection form on the middle cohomology $\mathrm{H}^{2m}(X; \mathbb{R})$ of a simply connected Poincare complex of dimension $4m$ plays an important role in determining whether or not X is homotopy equivalent to a manifold. However, quadratic forms over other fields are also of geometric interest. For example, if X is a simply connected Poincare complex of dimension $4m + 2 > 4$, then the problem of finding a manifold in the homotopy type of X turns out to depend on more subtle properties of quadratic forms over the field $\mathbf{F}_2$. We would ultimately like to treat manifolds of all dimensions in a uniform way, which will require us to somehow interpolate between the fields $k = \mathbb{R}$ and $k = \mathbf{F}_2$. We can do this by considering quadratic forms over the integers, or over more general rings.

The theory of quadratic forms over an arbitrary ring R can be very complicated. For example, the classification of quadratic forms over the field $\mathbf{Q}$ of rational numbers is a nontrivial achievement in number theory (the Hasse-Minkowski theorem). In general, we cannot detect whether two quadratic forms are isomorphic by means of simple integer invariants, as in Theorem 1. However, there are more elaborate theories that are designed to generalize the dimension and signature to other contexts:

	K-theory	L-theory
input:	projective module	module with quadratic form
for $\mathbb{R}$-vector spaces	dimension	signature
classical version	Grothendieck group K_0	Witt group
invariant of manifolds	Euler characteristic	signature
local-global principle	Gauss-Bonnet theorem	Hirzebruch signature formula

We are ultimately interested in understanding the right hand column of this table. But let us first spend some time discussing the left column, which is perhaps more familiar. We begin by recalling the definition of the Grothendieck group $K_0(A)$ of an associative ring A. The collection of isomorphism classes of finitely generated projective A-modules forms a commutative monoid under the formation of direct sums. The group completion of this monoid is denoted by $K_0(A)$, and called the (0th) K-group of A. Put another way, the K-group $K_0(A)$ is the abelian group generated by symbols $[P]$, where P ranges over all projective left A-modules of finite rank, subject to the relations given by

$$[P \oplus P'] = [P] + [P'].$$

Remark 2. If A is a field, then all finitely generated A-modules are automatically projective, and are determined up to isomorphism by their rank (in other words, by their dimension as A-vector spaces). Consequently, there is a canonical isomorphism $K_0(A) \simeq \mathbf{Z}$, which assigns to each A-module P its dimension $\dim_A(P)$. Consequently, we can think of the K-group $K_0(A)$ as a device which allows us to generalize the notion of dimension to the case of modules over arbitrary rings.

Let us now consider the following question:

Question 3. What is K-theory an invariant of?

We give several answers, beginning with the obvious.

(a) K-theory is an invariant of rings.

However, we can immediately improve on (a). Note that the Grothendieck group $K_0(A)$ is defined purely in terms of the category of finitely generated projective modules over A. In particular, Morita equivalent rings (such as matrix rings $M_n(A)$) have the same K-theory as A. We can therefore improve on our first answer:

(b) K-theory is an invariant of additive categories.

For our purposes, it will be convenient to give a variation on this answer. Recall that we are ultimately interested in assigning geometric invariants to manifolds and other Poincare complexes. We can obtain some by considering algebraic invariants to vector spaces. Let k be a field. For any finite CW complex X, we can consider the *Betti numbers*

$$b_i = \dim_k \mathrm{H}_i(X; k).$$

In general, these invariants depend on the field k. However, the Euler characteristic

$$\chi(X) = \sum_i (-1)^i b_i = \sum_i (-1)^i \dim_k \mathrm{H}_i(X; k)$$

does not depend on k. This invariant $\chi(X)$ is not generally not the dimension of a vector space (for example, it can be negative). Instead, we should think of it as an invariant of a *chain complex* of vector spaces: the singular chain complex $C_*(X; k)$.

More generally, we would like to say that for any ring A, the chain complex $C_*(X; A)$ determines a class in the Grothendieck group $K_0(A)$. (Of course, we know what this class should be: namely, $\chi(X)[A] \in K_0(A)$. Ignore this for the moment.) We generally cannot define this class to be the alternating sum

$$\sum_i (-1)^i [\mathrm{H}_i(X; A)],$$

because the individual A-modules $\mathrm{H}_i(X; A)$ need not be projective.

To show that $C_*(X; A)$ determines a well-defined class in the group $K_0(A)$, it is convenient to describe $K_0(A)$ in a different way. Rather than representing K-theory classes by projective modules over A, we take as representatives *chain complexes* of modules over A. Of course, we do not want to allow arbitrary chain complexes. In order to obtain reasonable invariants, we should restrict our attention to chain complexes which are *perfect*: that is, which are quasi-isomorphic to bounded chain complexes of finite projective modules. If X is a finite CW complex, then the singular chain complex $C_*(X; A)$ is always perfect: for example, it is quasi-isomorphic to the chain complex which computes the cellular homology of X.

Let **Perf** denote the category whose objects are bounded chain complexes of finite projective modules. For every perfect complex $M_\bullet$, we can find a quasi-isomorphism $P_\bullet \to M_\bullet$, where $P_\bullet \in \mathbf{Perf}$. The chain complex $P_\bullet$ need not be unique. However, it is unique up to chain homotopy equivalence. We can obtain a stronger uniqueness result by passing to the homotopy category. Let **hPerf** be the category with the same objects as **Perf**, but whose morphisms are given by *chain homotopy classes* of chain maps. The category **hPerf** is called the *perfect derived category of A*. Every perfect complex of A-modules determines an object of **hPerf**, which is well-defined up to isomorphism. Moreover, **hPerf** is an example of a *triangulated category*: that is, there is a notion of *distinguished triangle*

$$P'_\bullet \to P_\bullet \to P''_\bullet \to P'_\bullet[1]$$

in **hPerf**. Let $K'_0(A)$ denote the abelian group generated by symbols $[P_\bullet]$, where $P_\bullet$ is an object of **hPerf**, with relations $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ for every distinguished triangle

$$P'_\bullet \to P_\bullet \to P''_\bullet \to P'_\bullet[1].$$

Every finitely generated projective A-module can be regarded as a perfect complex which is concentrated in degree zero, and this construction determines a map of abelian groups $K_0(A) \to K'_0(A)$. One can show that this map is an isomorphism. In other words, one can define the Grothendieck group using (perfect) chain complexes of modules, rather than individual modules. With this definition, it is easy to see that $C_*(X; A)$ determines a well-defined K-theory class, for every finite CW complex X. This suggests a different answer to Question 3:

(c) K-theory is an invariant of triangulated categories.

For purposes of this course, answer (c) is a little bit misleading. The passage from the category **Perf** to its homotopy category **hPerf** has upsides and downsides. It has the virtue of allowing us to treat quasi-isomorphic chain complexes as if they are actually isomorphic. Unfortunately, it also loses a lot of information, because it has the effect of identifying chain homotopic morphisms without remembering any information about the chain homotopy. For our purposes, we will need to work with an intermediate object $\mathcal{D}^{\mathrm{perf}}(A)$, which will allow us to treat quasi-isomorphisms between chain complexes as if they were isomorphisms while retaining information about chain homotopies. The fine print is that unlike **Perf** and **hPerf**, $\mathcal{D}^{\mathrm{perf}}(A)$ is not a category: rather it is a more general object called an ∞-category.

We are now ready to give our final answer to Question 3:

(d) K-theory is an invariant of stable ∞-categories.

Our goal in this lecture and the next is to explain the meaning of this statement. To do so will require a long digression.

Definition 4. Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is a simplicial set $\mathrm{N}(\mathcal{C})$ whose n-simplices are given by composable sequences of morphisms

$$C_0 \to C_1 \to \cdots \to C_n$$

in $\mathcal{C}$.

If $\mathcal{C}$ is a category, then $\mathcal{C}$ can be recovered (up to canonical isomorphism) from its nerve $\mathrm{N}(\mathcal{C})$. The objects of $\mathcal{C}$ are in bijection with the 0-simplices of $\mathrm{N}(\mathcal{C})$. If X and Y are objects in $\mathcal{C}$, then $\mathrm{Map}_{\mathcal{C}}(X, Y)$ can be identified with the set of 1-simplices of $\mathrm{N}(\mathcal{C})$ joining X with Y. Given a pair of composable morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, there is a unique 2-simplex of $\mathrm{N}(\mathcal{C})$ having 0th face g and 2nd face f, and the composition $g \circ f$ is the 1st face of this 2-simplex:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed] & & Z. \end{tikzcd}$$

We can summarize this discussion as follows: the construction $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})$ determines a fully faithful embedding from the category of (small) categories into the category of simplicial sets. The essential image is described by the following claim:

Fact 5. *Let S be a simplicial set. Then S is isomorphic to the nerve of a category if and only if the following condition is satisfied:*

($*$) *For every pair of integers $0 < i < n$, every map $f_0 : \Lambda^n_i \to S$ extends uniquely to an n-simplex $f : \Delta^n \to S$.*

Here Λ^n_i denotes the ith horn: the simplicial subset of Δ^n obtained by removing the interior and the face opposite the ith vertex.

Example 6. When $i = 1$ and $n = 2$, condition ($*$) says that every pair of "composable" edges f and g determine a unique 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed] & & Z. \end{tikzcd}$$

Recall that a simplicial set S is a *Kan complex* if it satisfies the following variant of ($*$):

($*'$) For every pair of integers $0 \leq i \leq n$, every map $f_0 : \Lambda^n_i \to S$ extends to an n-simplex $f : \Delta^n \to S$.

Conditions ($*$) and ($*'$) look similar, but neither implies the other. Condition ($*$) requires that we can uniquely fill any *inner* horn (that is, a horn Λ^n_i with $0 < i < n$), but says nothing about the extremal cases $i = 0$ and $i = n$. Condition ($*'$) requires that we can fill every horn Λ^n_i, but does not require the filler to be unique. However, these two conditions admit a common generalization:

Definition 7. An ∞-*category* is a simplicial set S satisfying the following condition:

($*''$) For every pair of integers $0 < i < n$, every map $f_0 : \Lambda^n_i \to S$ extends to an n-simplex $f : \Delta^n \to S$.

Remark 8. In the literature, ∞-categories are often referred to as *quasi-categories* or *weak Kan complexes.*

Example 9. If $\mathcal{C}$ is a category, then its nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category. Since passage to the nerve loses no information about a category, this construction allows us to view the usual definition of a category as a special case of the notion of ∞-category.

Example 10. Any Kan complex is an ∞-category. In particular, if X is a topological space, then the singular complex $\mathrm{Sing}(X)$ (whose n-simplices are given by continuous maps from a topological n-simplex into X) is an ∞-category. Since the singular complex $\mathrm{Sing}(X)$ determines X up to weak homotopy equivalence, not much information is lost by the construction $X \mapsto \mathrm{Sing}(X)$. Consequently, for many purposes, we can think of ∞-categories as a generalization of topological spaces.

We will typically use the symbol $\mathcal{C}$ to denote an ∞-category. We will refer to the 0-simplices of $\mathcal{C}$ as its *objects* and the 1-simplices of $\mathcal{C}$ as its *morphisms*. In the simplest case ($i = 1$ and $n = 2$), the horn-filling condition $(*'')$ asserts that for every pair of "composable" morphisms $f : X \to Y$ and $g : Y \to Z$, we can find a 2-simplex σ :

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{C}$. Here we can think of h as a *composition* of f and g, and we will write $h = g \circ f$. A word of warning is in order: condition $(*'')$ does not require that σ is unique, so there may be several choices for the composition h. However, one can show that h is unique up to a suitable notion of homotopy. This turns out to be good enough for many purposes: Definition 7 provides a robust generalization of classical category theory. Many of the useful concepts from classical category theory (limits and colimits, adjoint functors, etcetera) can be generalized to the setting of ∞-categories.

Stable ∞-Categories (Lecture 3)

February 2, 2011

In the last lecture, we introduced the definition of an ∞-*category* as a generalization of the usual notion of category. This definition is one way of formalizing the notion of a higher category in which all k-morphisms are invertible for $k > 1$. Many other approaches are possible. The following is probably more intuitive:

Definition 1. A *topological category* is a category $\mathcal{C}$ together with a topology on the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for every pair of objects X and Y, such that the composition maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ are continuous. (In other words, a category which is *enriched* over the category of topological spaces.) If $\mathcal{C}$ is a topological category containing a pair of objects X and Y, we will denote $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ when we wish to emphasize that we are thinking of it as a topological space.

Remark 2. To accommodate certain examples, it is convenient to modify Definition 1 by working with *compactly generated* topological spaces rather than topological spaces. That is, we require that each $\mathrm{Map}_{\mathcal{C}}(X, Y)$ be compactly generated, and require that composition is given by continuous maps

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$

where the product is taken in the category of compactly generated topological spaces. This is a technical point which may be safely ignored.

The theory of ∞-categories is closely related to the theory of topological categories.

Construction 3 (Sketch)**.** Let $\mathcal{C}$ be a topological category. We define a simplicial set $\mathrm{N}^t(\mathcal{C})$, the *homotopy coherent nerve* of $\mathcal{C}$, as follows:

- The 0-simplices of $\mathrm{N}^t(\mathcal{C})$ are the objects of $\mathcal{C}$.
- The 1-simplices of $\mathrm{N}^t(\mathcal{C})$ are morphisms $f : X \to Y$ in $\mathcal{C}$.
- The 2-simplices of $\mathrm{N}^t(\mathcal{C})$ are given by (noncommuting) diagrams

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

  in $\mathcal{C}$, together with a choice of path from h to $g \circ f$ in $\mathrm{Map}_{\mathcal{C}}(X, Z)$.
- $\cdots$

Example 4. Let $\mathcal{C}$ be an ordinary category. We can regard $\mathcal{C}$ as a topological category by endowing each mapping set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ with the discrete topology.

It turns out that for any topological category $\mathcal{C}$, the homotopy coherent nerve $\mathrm{N}^t(\mathcal{C})$ is an ∞-category. Moreover, there is a sort of converse: every ∞-category is equivalent to $\mathrm{N}^t(\mathcal{C})$ for some topological category $\mathcal{C}$, and the topological category $\mathcal{C}$ is essentially unique (up to a suitable notion of weak homotopy equivalence). In other words, the construction $\mathcal{C} \mapsto \mathrm{N}^t(\mathcal{C})$ determines an equivalence between the theory of topological categories and the theory of ∞-categories. From this point forward, we will work at an informal level and freely mix these two notions. For example, if S is an ∞-category containing a pair of 0-simplices x and y, we will use $\mathrm{Map}_S(x, y)$ to denote a mapping space between x and y, when viewed as objects of a topological category whose homotopy coherent nerve is equivalent to S.

Remark 5. To any topological category $\mathcal{C}$ (and therefore to any ∞-category) we can associate an ordinary category $\mathrm{h}\mathcal{C}$, called the *homotopy category* of $\mathcal{C}$. It has the same objects, with morphisms given by $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$.

Example 6. The collection of finite CW complexes forms a topological category: for any pair of finite CW complexes X and Y, we can endow the set of continuous maps $\mathrm{Hom}(X, Y)$ with the compact-open topology. If we use the convention of Remark 2, then this generalizes to arbitrary CW complexes. We will denote the homotopy coherent nerve of this (larger) topological category by $\mathcal{S}$, and refer to it as the *∞-category of spaces.*

Here is an example of greater interest to us:

Example 7 (Sketch)**.** Let A be an associative ring. There is an ∞-category $\mathcal{D}^{\mathrm{perf}}(A)$ which may be described as follows:

- The 0-simplices of $\mathcal{D}^{\mathrm{perf}}(A)$ are given by bounded chain complexes of finitely generated projective left A-modules
$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to \cdots$$
- A 1-simplex of $\mathcal{D}^{\mathrm{perf}}(A)$ consists of a pair of chain complexes $P_\bullet$ and $Q_\bullet$, together with a map of chain complexes $f : P_\bullet \to Q_\bullet$.
- A 2-simplex of $\mathcal{D}^{\mathrm{perf}}(A)$ consists of a (not necessarily commutative) diagram of chain complexes
$$\begin{tikzcd} & Q_\bullet \arrow[dr, "g"] & \\ P_\bullet \arrow[ur, "f"] \arrow[rr, "h"] & & R_\bullet, \end{tikzcd}$$
together with a chain homotopy from h to $g \circ f$.
- Higher dimensional simplices are defined using higher-order chain homotopies.

The homotopy category of $\mathcal{D}^{\mathrm{perf}}(A)$ is equivalent to the category **hPerf** of the previous lecture.

Our next goal is to axiomatize some of the special features enjoyed by ∞-categories of the form $\mathcal{D}^{\mathrm{perf}}(A)$. First, we need to introduce a bit of terminology. Let $\mathcal{C}$ be an ∞-category and $\mathcal{I}$ an ordinary category. It makes sense to speak of *functors* from $\mathcal{I}$ to $\mathcal{C}$: these are given by maps of simplicial sets $\mathrm{N}(\mathcal{I}) \to \mathcal{C}$. We can use this notion to make sense of commutative diagrams in $\mathcal{C}$. For example, a square diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$
in $\mathcal{C}$ is just a map of simplicial sets $\Delta^1 \times \Delta^1 \to \mathcal{C}$.

Definition 8. Let $\mathcal{C}$ be an ∞-category. We will say that an object $0 \in \mathcal{C}$ is a *zero object* if, for every object $X \in \mathcal{C}$, the mapping spaces

$$\mathrm{Map}_{\mathcal{C}}(0, X) \qquad \mathrm{Map}_{\mathcal{C}}(X, 0)$$

are contractible. We will say that $\mathcal{C}$ is *pointed* if it admits a zero object.

If $\mathcal{C}$ is a pointed ∞-category, then for any pair of objects X and Y there is a *zero map* from X to Y, given by the composition

$$X \to 0 \to Y$$

where 0 is a zero object of $\mathcal{C}$. This map is well-defined up to a contractible space of choices.

Definition 9. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ consists of a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{C}$, together with a path from $g \circ f$ to the zero map in $\mathrm{Map}_{\mathcal{C}}(X, Z)$ (in other words, a *nullhomotopy* of $g \circ f$). More formally: a triangle in $\mathcal{C}$ is a square diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

where 0 is a zero object of $\mathcal{C}$.

Suppose we are given a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in a pointed ∞-category $\mathcal{C}$. We will say that this triangle is a *fiber sequence* if, for every object $C \in \mathcal{C}$, the associated sequence of topological spaces

$$\mathrm{Map}_{\mathcal{C}}(C, X) \to \mathrm{Map}_{\mathcal{C}}(C, Y) \to \mathrm{Map}_{\mathcal{C}}(C, Z)$$

is a homotopy fiber sequence. In this case, X is determined (up to equivalence) by g. We will say that X is the *fiber of* g and write $X = \mathrm{fib}(g)$.

Dually, we say that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a *cofiber sequence* if, for every object $C \in \mathcal{C}$, the associated sequence of topological spaces

$$\mathrm{Map}_{\mathcal{C}}(Z, C) \to \mathrm{Map}_{\mathcal{C}}(Y, C) \to \mathrm{Map}_{\mathcal{C}}(X, C)$$

is a homotopy fiber sequence. In this case, Z is determined (up to equivalence) by f. We will say that Z is the *cofiber of* g and write $Z = \mathrm{cofib}(g)$.

Definition 10. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *stable* if the following conditions are satisfied:

(1) $\mathcal{C}$ is pointed: that is, there is a zero object of $\mathcal{C}$.

(2) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a fiber and a cofiber.

(3) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Example 11. There is a stable ∞-category Sp whose objects are spectra. The homotopy category hSp is the classical *stable homotopy category*.

Example 12. The ∞-category $\mathcal{D}^{\mathrm{perf}}(A)$ considered above is stable.

Let $\mathcal{C}$ be a stable ∞-category. We define an abelian group $K_0(\mathcal{C})$ as follows: $K_0(\mathcal{C})$ is obtained from the free abelian group generated by symbols $[X]$, where X is an object of $\mathcal{C}$, subject to the following relation: if there is a fiber sequence

$$X \to Y \to Z,$$

then $[Y] = [X] + [Z]$.

Remark 13. Let $\mathcal{C}$ be a stable ∞-category. One can show that the homotopy category of $\mathcal{C}$ is triangulated. Moreover, the K-group $K_0(\mathcal{C})$ depends only on the homotopy category of $\mathcal{C}$, viewed as a triangulated category. However, when discussing more sophisticated invariants of $\mathcal{C}$ (like higher K-groups or L-groups) it is better not to pass to the homotopy category.

Example 14. If A is a ring, then the K-group $K_0(\mathcal{D}^{\mathrm{perf}}(A))$ is canonically isomorphic to the group $K_0(A)$ defined in the last lecture. In particular, if A is a field, then there is a canonical isomorphism $K_0(\mathcal{D}^{\mathrm{perf}}(A)) \simeq \mathbf{Z}$. If $P_\bullet$ is an object of $\mathcal{D}^{\mathrm{perf}}(A)$, then $[P_\bullet] \in K_0(\mathcal{D}^{\mathrm{perf}}(A)) \simeq \mathbf{Z}$ can be identified with the Euler characteristic

$$\sum_i (-1)^i \dim_A \mathrm{H}_i(P_\bullet)$$

of the chain complex $P_\bullet$.

In general, if we are given an object X of a stable ∞-category $\mathcal{C}$, then we can view the class $[X] \in K_0(\mathcal{C})$ as a kind of "generalized Euler characteristic" of X. It is in general not an integer, but an element of the abelian group $K_0(\mathcal{C})$ which depends on $\mathcal{C}$. We can think of the construction $\mathcal{C} \mapsto K_0(\mathcal{C})$ as a kind of *categorificatied* Euler characteristic:

Input	Invariant
vector space V	dimension $\dim(V) \in \mathbf{Z}$
complex of vector spaces	Euler characteristic $\chi \in \mathbf{Z}$
stable ∞-category $\mathcal{C}$	abelian group $K_0(\mathcal{C})$
object of a stable ∞-category $\mathcal{C}$	K-theory class $[X] \in K_0(\mathcal{C})$

However, these are not the invariants we really want to study in this class. Recall that if V is a finite dimensional vector space over $\mathbb{R}$ with a nondegenerate quadratic form q, then V is determined (up to isometry) by two invariants: the dimension of V and the signature of V. It is the latter that we would really like to generalize. Let us briefly indicate the form that this generalization will take:

Input	Invariant
nondegenerate quadratic space (V, q) over $\mathbb{R}$	signature $\sigma(V, q) \in \mathbf{Z}$
Poincare chain complex (V_*, q) over $\mathbb{R}$	signature σ of middle homology
stable ∞-category $\mathcal{C}$ with quadratic functor Q	L-group $L_0(\mathcal{C}, Q)$
Object $X \in \mathcal{C}$ satisfying Poincare duality	$[X] \in L_0(\mathcal{C}, Q)$

We will start making sense of some of these words in the next lecture.

Quadratic Functors (Lecture 4)

February 2, 2011

In this lecture, we will introduce the notion of a *quadratic functor* Q on a stable ∞-category $\mathcal{C}$, and define the L-group $L_0(\mathcal{C}, Q)$. We begin with a short review of the classical theory of quadratic forms.

Definition 1. Let M and A be abelian groups. An *A-valued bilinear form* on M is a map

$$b : M \times M \to A$$

such that, for each $x \in M$, the maps $y \mapsto b(x, y)$ and $y \mapsto b(y, x)$ are abelian group homomorphisms from M into A. We will say that b is *symmetric* if $b(x, y) = b(y, x)$.

An *inhomogeneous A-valued quadratic form on M* is a map $q : M \to A$ such that $q(0) = 0$ and the function $b(x, y) = q(x + y) - q(x) - q(y)$ is a bilinear form. We will say that q is a *quadratic form* if, in addition, we have $q(nx) = n^2 q(x)$ for every integer n and every $x \in M$.

The theory of quadratic forms and bilinear forms are closely connected. If q is an inhomogeneous quadratic form on an abelian group M, then the function $b(x, y) = q(x+y) - q(x) - q(y)$ is a symmetric bilinear form. If multiplication by 2 is invertible on A, we can almost recover q from the bilinear form b: namely, we have $q(x) = \frac{1}{2} b(x, x) + l(x)$ for some group homomomorphism $l : M \to A$. In particular, the construction $b \mapsto \frac{1}{2} b(x, x)$ determines a bijective correspondence between symmetric bilinear forms and quadratic forms (whenever multiplication by 2 is invertible on A).

Our next goal is to *categorify* some of these ideas: that is, to make sense of the algebraic structures described above when the notion of module is replaced by some sort of category (in our case, stable ∞-categories). Let us begin by drawing up a table of analogies:

Classical Story	Categorified Story
abelian group	stable ∞-category
$\mathbf{Z}$	∞-category Sp of spectra
abelian group homomorphism	exact functor
(symmetric) bilinear form	(symmetric) bilinear functor
inhomogeneous quadratic form	quadratic functor

We now introduce some of the relevant definitions.

Definition 2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. We say that F is *exact* if it carries zero objects to zero objects and fiber sequences to fiber sequences.

Let Sp denote the ∞-category of spectra.

Definition 3. Let $\mathcal{C}$ be a stable ∞-category. A *bilinear functor* on $\mathcal{C}$ is a functor

$$B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$$

with the following property: for every object $C \in \mathcal{C}$, the functors

$$D \mapsto B(C, D) \qquad D \mapsto B(D, C)$$

are exact functors from $\mathcal{C}^{op}$ to Sp.

The collection of bilinear functors on $\mathcal{C}$ is evidently acted on by the symmetric group Σ_2 on two letters (by permuting the arguments). A *symmetric bilinear functor* is a homotopy fixed point for this action.

Let $\mathcal{C}$ be a stable ∞-category containing an object X. For every object Y, the sequence of mapping spaces $\{\mathrm{Map}_{\mathcal{C}}(Y, \Sigma^n X)\}_{n \geq 0}$ constitutes a spectrum (that is, each is homotopy equivalent to the loop space on the next). We will denote this spectrum by $\mathrm{Mor}_{\mathcal{C}}(Y, X)$. The construction $Y \mapsto \mathrm{Mor}_{\mathcal{C}}(Y, X)$ determines an (exact) functor from $\mathcal{C}^{op}$ to Sp. We will say that a functor $F : \mathcal{C}^{op} \to \mathrm{Sp}$ is *representable* if it arises in this way.

Suppose that $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ is a symmetric bilinear functor. We will say that B is *representable* if, for all $X \in \mathcal{C}$, the functor $Y \mapsto B(X, Y)$ is representable. In this case, we write $B(X, Y) = \mathrm{Mor}_{\mathcal{C}}(Y, \mathbb{D}X)$ for some object $\mathbb{D}X$ in $\mathcal{C}$, which is determined up to contractible ambiguity. The construction $X \mapsto \mathbb{D}X$ determines a functor from $\mathcal{C}$ to $\mathcal{C}^{op}$. For each $X \in \mathcal{C}$, the identity map $\mathrm{id}_{\mathbb{D}X}$ determines point in the zeroth space of $B(X, \mathbb{D}X) \simeq B(\mathbb{D}X, X)$, and therefore a morphism $e_X : X \to \mathbb{D}^2 X$. We will say that B is *nondegenerate* if it is representable and the canonical map e_X is an equivalence for every $X \in \mathcal{C}$.

Example 4. Let $\mathcal{C}$ be the ∞-category of spectra, and let $\wedge$ denote the smash product functor. The functor $B(X, Y) = \mathrm{Mor}_{\mathrm{Sp}}(X \wedge Y, S)$ determines a symmetric bilinear functor on $\mathcal{C}$. This symmetric bilinear functor is representable, and the corresponding functor $\mathbb{D} : \mathcal{C} \to \mathcal{C}$ is *Spanier-Whitehead duality.* If we restrict our attention to the full subcategory of $\mathcal{C}$ spanned by the *finite* spectra, then B becomes nondegenerate.

Definition 5. Let $\mathcal{C}$ be a stable ∞-category. We say that a functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ is *reduced* if Q carries zero objects to zero objects. In this case, Q carries also carries zero morphisms to zero morphisms.

Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor. If $X, Y \in \mathcal{C}$, we obtain maps

$$Q(X) \oplus Q(Y) \to Q(X \oplus Y) \to Q(X) \oplus Q(Y)$$

where the composition is given by applying Q to the matrix

$$\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix}$$

If Q is reduced, this map is the identity so that $Q(X) \oplus Q(Y)$ is a summand of $Q(X \oplus Y)$; that is, we have a direct sum decomposition $Q(X \oplus Y) \simeq Q(X) \oplus Q(Y) \oplus B(X, Y)$, for some functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$. We will refer to B as the *polarization of* Q. Note that B is manifestly symmetric in its arguments.

Suppose we are given a reduced functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ with polarization B. For every object $X \in \mathcal{C}$, the codiagonal map $X \oplus X \to X$ induces a map $Q(X) \to Q(X \oplus X)$. Projecting onto the component $B(X, X)$, we obtain a map $Q(X) \to B(X, X)$. This construction is evidently Σ_2-invariant, and gives a map $Q(X) \to B(X, X)^{h\Sigma_2}$ (here $B(X, X)^{h\Sigma_2}$ denotes the homotopy fixed point spectrum for the action of Σ_2 on $B(X, X)$).

Definition 6. Let $\mathcal{C}$ be a stable ∞-category and let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a functor. We will say that Q is *quadratic* if the following conditions are satisfied:

(1) The functor Q is reduced.

(2) The polarization B of Q is bilinear.

(3) The functor $X \mapsto \mathrm{fib}(Q(X) \to B(X,X)^{h\Sigma_2})$ is exact.

Example 7. Let $\mathcal{C}$ be a stable ∞-category and let B be a symmetric bilinear functor on $\mathcal{C}$. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be given by the formula $Q(X) = B(X,X)^{h\Sigma_2}$, and let B' be the polarization of Q. A simple calculation gives

$$B'(X,Y) = (B(X,Y) \oplus B(Y,X))^{h\Sigma_2} \simeq B(X,Y),$$

and that the canonical map $Q(X) \to B'(X,X)^{h\Sigma_2}$ is an equivalence. Consequently, Q is a quadratic functor.

Example 8. Let $\mathcal{C}$ be a stable ∞-category and let B be a symmetric bilinear functor on $\mathcal{C}$. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be given by the formula $Q(X) = B(X,X)_{h\Sigma_2}$, the *homotopy coinvariants* for the action of Σ_2 on $B(X,X)$, and let B' be the polarization of Q. A simple calculation gives $B'(X,Y) = (B(X,Y) \oplus B(Y,X))_{h\Sigma_2} \simeq B(X,Y)$. Moreover, the canonical map

$$Q(X) \to B'(X,Y)^{h\Sigma_2}$$

can be identified with the *norm map*

$$B(X,X)_{h\Sigma_2} \to B(X,X)^{h\Sigma_2}.$$

The cofiber of this map is the Tate cohomology spectrum $B(X,X)^{t\Sigma_2}$. The functor $X \mapsto B(X,X)^{t\Sigma_2}$ is an exact functor of X, so that Q is a quadratic functor.

Remark 9. Let $\mathcal{C}$ be a stable ∞-category and let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor with polarization B. Using the diagonal map $X \to X \oplus X$ instead of the codiagonal in the preceding discussion, we obtain a canonical map $B(X,X)_{h\Sigma_2} \to Q(X)$. The composition

$$B(X,X)_{h\Sigma_2} \to Q(X) \to B(X,X)^{h\Sigma_2}$$

is given by the norm map (averaging with respect to the action of Σ_2). If the homotopy groups of the spectrum $B(X,X)$ are uniquely 2-divisible, then this norm map is a homotopy equivalence of spectra. It follows in this case that we obtain a direct sum decomposition

$$Q(X) \simeq B(X,X)^{h\Sigma_2} \oplus L(X) \simeq B(X,X)_{h\Sigma_2} \oplus L(X)$$

for some reduced functor $L : \mathcal{C}^{op} \to \mathrm{Sp}$ with trivial polarization. Then Q is quadratic if and only if B is bilinear and L is exact. We can informally summarize the situation as follows: if we work in the setting where 2 is invertible (for example, if multiplication by 2 induces an isomorphism from each object of $\mathcal{C}$ to itself), then every quadratic functor on $\mathcal{C}$ decomposes uniquely as the sum of an exact functor and a functor of the form $B(X,X)^{h\Sigma_2}$, where B is a symmetric bilinear functor on $\mathcal{C}$.

Remark 10. Our definition of quadratic functor is a special case of a much more general notion which arises in Goodwillie's *calculus of functors*.

Remark 11. Let $\mathcal{C}$ be a stable ∞-category. Suppose we are given a fiber sequence

$$Q_0(X) \to Q(X) \to B(X,X)^{h\Sigma_2}$$

for some symmetric bilinear functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$. If Q_0 is exact, then a simple calculation shows that the polarizaton of Q is given by

$$F(X,Y) = (B(X,Y) \oplus B(Y,X))^{h\Sigma_2} \simeq B(X,Y),$$

so that Q is quadratic. In other words, a functor Q is quadratic if and only if is arises an extension of $B(X,X)^{h\Sigma_2}$ by an exact functor, for some symmetric bilinear functor B.

L Groups (Lecture 5)

February 2, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. The polarization B of Q is a symmetric bilinear functor on $\mathcal{C}$. We will say that Q is *nondegenerate* if B is nondegenerate: that is, if there is an equivalence of ∞-categories $\mathbb{D}_Q : \mathcal{C}^{op} \to \mathcal{C}$ such that $B(X, Y) = \mathrm{Mor}_{\mathcal{C}}(X, \mathbb{D}_Q Y)$.

Assume now Q is a nondegenerate quadratic functor on a symmetric monoidal ∞-category $\mathcal{C}$. Our objective in this lecture is to define an abelian group $L_0(\mathcal{C}, Q)$, which we will call the *L-group* of the pair $(\mathcal{C}, Q)$.

Definition 1. Let Q be nondegenerate quadratic functor on a stable ∞-category $\mathcal{C}$. A *quadratic object* of $(\mathcal{C}, Q)$ is a pair (X, q), where $X \in \mathcal{C}$ and q is a point of the 0th space $\Omega^\infty Q(X)$. In this case, q determines a point in the zeroth space of $B(X, X)^{h\Sigma_2}$, hence a map $X \to \mathbb{D}_Q X$. We will say that (X, q) is a *Poincare object* if this map is invertible.

We can describe the intuition behind Definition 1 as follows: we think of Q as a functor which assigns to each object $X \in \mathcal{C}$ a "spectrum of quadratic forms on X". A quadratic object of $(\mathcal{C}, Q)$ can then be thought of as an object of $\mathcal{C}$ equipped with a some type of quadratic form (whose exact nature depends on Q), and a Poincare object of $(\mathcal{C}, Q)$ as an object of $\mathcal{C}$ equipped with a nondegenerate quadratic form.

Example 2. Here is the motivating example. Fix an integer $n \geq 0$. Let $B : \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})^{op} \times \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})^{op} \to \mathrm{Sp}$ be the bilinear functor given informally by the formula

$$(P_\bullet, Q_\bullet) \mapsto \mathrm{Mor}_{\mathcal{D}^{\mathrm{perf}}(\mathbf{Z})}(P_\bullet \otimes Q_\bullet, \mathbf{Z}[-n])$$

(here $\mathbf{Z}[-n]$ denotes the chain complex consisting of the single abelian group $\mathbf{Z}$, concentrated in homological degree $-n$). Then B is a symmetric bilinear functor; let $Q : \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})^{op} \to \mathrm{Sp}$ be the quadratic functor given by $Q(P_\bullet) = B(P_\bullet, P_\bullet)^{h\Sigma_2}$.

Let M be a compact oriented manifold of dimension n. We can identify the singular cochain complex $C^*(M; \mathbf{Z})$ with an object of $\mathcal{D}^{\mathrm{perf}}(\mathbf{Z})$. The intersection pairing

$$C^*(M; \mathbf{Z}) \otimes C^*(M; \mathbf{Z}) \to C^*(M; \mathbf{Z}) \xrightarrow{[M]} \mathbf{Z}[-n]$$

determines a point $q_M \in \Omega^\infty Q(C^*(M; \mathbf{Z}))$. Poincare duality is equivalent to the assertion that the pair $(C^*(M; \mathbf{Z}), q_M)$ is a Poincare object of $(\mathcal{D}^{\mathrm{perf}}(\mathbf{Z}), Q)$.

Example 3. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. The space $\Omega^\infty Q(0)$ is contractible; let q denote any point of this contractible space. Then the pair $(0, q)$ is a Poincare object of $(\mathcal{C}, Q)$.

Example 4. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. Suppose we are given quadratic objects (X, q) and (X', q') of $(\mathcal{C}, Q)$. Let $q \oplus q'$ denote the image of (q, q') under the map $Q(X) \oplus Q(X') \to Q(X \oplus X')$. The pair $(X \oplus X', q \oplus q')$ is another quadratic object of $(\mathcal{C}, Q)$, which we call the *sum* of (X, q) and (X', q') and denote by $(X, q) \oplus (X', q')$. Note that if (X, q) and (X', q') are Poincare objects, then $(X \oplus X', q \oplus q')$ is also a Poincare object.

If $\mathcal{C}$ is a stable ∞-category equipped with a nondegenerate quadratic functor, then the collection of homotopy equivalence classes of Poincare objects forms a commutative monoid with respect to the addition of Example 4; the unit for this addition is the zero Poincare object given in Example 3. However, this monoid is evidently not a group: if $(X, q) \oplus (X', q') \simeq 0$, then we must have $X \simeq X' \simeq 0$. We will correct this problem by introducing a suitable equivalence relation on Poincare objects.

Definition 5. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$, and suppose we are given Poincare objects (X, q) and (X', q'). A *cobordism* from (X, q) to (X', q') consists of the following data:

(*i*) An object $L \in \mathcal{C}$ equipped with maps $\alpha : L \to X$ and $\alpha' : L \to X'$.

(*ii*) A path p joining the images of q and q' in the space $\Omega^\infty Q(L)$.

Moreover, this data must satisfy the following nondegeneracy condition:

(*iii*) The diagram

$$\begin{tikzcd} X \arrow[d] & L \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] & X' \arrow[d] \\ \mathbb{D}_Q(X) \arrow[r, "\mathbb{D}_Q(\alpha)"] & \mathbb{D}_Q(L) & \mathbb{D}_Q(X') \arrow[l, "\mathbb{D}_Q(\alpha')"'] \end{tikzcd}$$

commutes up to a homotopy determined by the path p. It follows that the composition

$$\mathrm{fib}(\alpha) \to L \xrightarrow{\alpha'} X' \to \mathbb{D}_Q(X') \to \mathbb{D}_Q(L)$$

is canonically nullhomotopic, so we obtain a map of fibers

$$u : \mathrm{fib}(\alpha) \to \mathrm{fib}(\mathbb{D}_Q(\alpha'))$$

or, more informally, a map $u : \Omega X/L \to \mathbb{D}_Q(X'/L)$. We require that u is invertible.

We will say that a pair of Poincare objects (X, q) and (X', q') are *cobordant* if there is a cobordism from (X, q) to (X', q').

Example 6. Let M and M' be compact oriented n-manifolds, and let $(C^*(M; \mathbf{Z}), q_M)$, $(C^*(M'; \mathbf{Z}), q_{M'})$ be the Poincare objects of $\mathcal{D}^{\mathrm{perf}}(\mathbf{Z})$ described in Example 2. Suppose that B is an (oriented) bordism from M to M', and let $L = C^*(B; \mathbf{Z})$ be the singular cochain complex of B. Then we have restriction maps $\alpha : L \to C^*(M; \mathbf{Z})$ and $\alpha' : L \to C^*(M'; \mathbf{Z})$. Moreover, the images of q_M and $q_{M'}$ in $\Omega^\infty Q(L)$ are joined by a canonical path, because the difference of fundamental homology classes $[M] - [M']$ in B is given as the boundary of the fundamental homology class of B. This path exhibits L as a cobordism from the Poincare object $(C^*(M; \mathbf{Z}), q_M)$ to the Poincare object $(C^*(M'; \mathbf{Z}), q_{M'})$: unwinding the definitions, this amounts to verifying that cap product with the fundamental class of B induces isomorphisms

$$\mathrm{H}^m(B, M; \mathbf{Z}) \to \mathrm{H}_{n+1-m}(B, M'; \mathbf{Z})$$

(which is a form of Poincare duality for manifolds with boundary).

Example 7. An important special case of Definition 5 occurs when (X, q) is the zero Poincare object. In this case, a cobordism from (X, q) to (X', q') is given by a map $\beta : L \to X'$ and a nullhomotopy of the image of q' in $Q(L)$, satisfying a nondegeneracy condition which requires that the induced map $u : L \to \mathrm{fib}(\mathbb{D}_Q(\beta)) \simeq \mathbb{D}_Q \,\mathrm{cofib}(\beta) = \mathbb{D}_Q X'/L$ is an equivalence. In this case, we will say that L is a *Lagrangian* in (X', q) (this terminology is slightly abusive: the condition of being a Lagrangian depends not only on L, but also on the map β and the choice of nullhomotopy).

Example 8. In the situation of Definition 5, suppose that (X, q) and (X', q') are both zero Poincare objects. Then a cobordism from (X, q) to (X', q') can be identified with an object $L \in \mathcal{C}$ together with a point $p \in \Omega^{\infty+1} Q(L)$ which induces an equivalence $L \to \Omega \mathbb{D}_Q(L)$. In other words, a cobordism from (X, q) to (X', q') can be identified with a Poincare object of $\mathcal{C}$ with respect to the shifted quadratic functor ΩQ.

Proposition 9. *Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. The relation of cobordism is an equivalence relation on the collection of Poincare objects of $(\mathcal{C}, Q)$.*

Proof. We first show that cobordism is reflexive. Let (X, q) be a Poincare object of $(\mathcal{C}, Q)$. Take $L = X$ and let $\alpha : L \to X$ and $\alpha' : L \to X$ be the identity maps. Let p be the constant path between the images of q in $\Omega^\infty Q(L)$. Then (L, α, α', p) is a cobordism from (X, q) to itself.

We next show that cobordism is symmetric. Let (X, q) and (X', q') be Poincare objects of $(\mathcal{C}, Q)$, and suppose we are given a diagram

$$X \overset{\alpha}{\leftarrow} L \overset{\alpha'}{\rightarrow} X'$$

in $\mathcal{C}$ and a path joining the images of q and q' in $\Omega^\infty Q(L)$. We claim that if this data is a cobordism from (X, q) to (X', q'), then it is also a cobordism from (X', q') to (X, q). Condition (iii) of Definition 5 guarantees that the canonical map

$$u : \mathrm{fib}(\alpha) \to \mathrm{fib}(\mathbb{D}_Q(\alpha')) \simeq \mathbb{D}_Q \, \mathrm{cofib}(\alpha')$$

is an equivalence. We wish to show that the canonical map

$$v : \mathrm{fib}(\alpha') \to \mathrm{fib}(\mathbb{D}_Q(\alpha)) \simeq \mathbb{D}_Q \, \mathrm{cofib}(\alpha)$$

is also an equivalence, or equivalently that

$$\Sigma(v) : \mathrm{cofib}(\alpha') \to \mathbb{D}_Q \, \mathrm{fib}(\alpha)$$

is an equivalence. For this, one shows that $\Sigma(v)$ agrees with $\mathbb{D}_Q(u)$ up to a sign.

We now show that cobordism is transitive. Suppose we are given a triple of Poincare objects (X, q), (X', q'), and (X'', q''), together with a diagram

$$X \overset{\alpha}{\leftarrow} L \overset{\alpha'}{\rightarrow} X' \overset{\beta}{\leftarrow} L' \overset{\beta'}{\rightarrow} X'',$$

a path p joining the image of q and q' in $\Omega^\infty Q(L)$, and a path p' joining the images of q' and q'' in $\Omega^\infty Q(L')$. Let S denote the fiber product $L \times_{X'} L'$. We have evident maps $X \overset{\gamma}{\leftarrow} S \overset{\gamma'}{\rightarrow} X''$ so that the concatentation of p and p' determines a path between the images of q and q'' in the space $\Omega^\infty Q(S)$. We claim that this path exhibits S as a cobordism from (X, q) to (X'', q''). To prove this, we must show that the induced map $u : \mathrm{fib}(\gamma) \to \mathrm{fib}(\mathbb{D}(\gamma'))$ is invertible. It now suffices to observe that this map fits into a diagram of fiber sequences

$$\begin{tikzcd}
\mathrm{fib}(\beta) \arrow[r] \arrow[d] & \mathrm{fib}(\gamma) \arrow[r] \arrow[d, "u"] & \mathrm{fib}(\alpha) \arrow[d] \\
\mathrm{fib}(\mathbb{D}_Q(\beta')) \arrow[r] & \mathrm{fib}(\mathbb{D}_Q(\gamma')) \arrow[r] & \mathrm{fib}(\mathbb{D}_Q(\alpha'))
\end{tikzcd}$$

where the left and right vertical maps are invertible by virtue of our assumptions that we have cobordisms from (X', q') to (X'', q'') and (X, q) to (X', q'), respectively. □

Definition 10. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. We let $L_0(\mathcal{C}, Q)$ denote the set of cobordism classes of Poincare objects of $(\mathcal{C}, Q)$.

The direct sum operation on Poincare objects descends to give an addition on the set $L_0(\mathcal{C}, Q)$ (since there is a corresponding direct sum operation on cobordisms themselves), making $L_0(\mathcal{C}, Q)$ into a commutative monoid. In fact, $L_0(\mathcal{C}, Q)$ is an abelian group. Suppose that (X, q) is a Poincare object of $(\mathcal{C}, Q)$. Since $\pi_0 Q(X)$ is an abelian group, we can choose a point $-q \in \Omega^\infty Q(X)$ which is inverse to q up to homotopy. Note that the pair $(X, -q)$ is also a Poincare object of $(\mathcal{C}, Q)$, which is determined up to (noncanonical) homotopy equivalence by (X, q). We claim that this Poincare object is an inverse to (X, q) in $L_0(\mathcal{C}, Q)$:

Proposition 11. *In the above situation, we have* $(X, q) \oplus (X, -q) = 0$ *in* $L_0(\mathcal{C}, Q)$*. That is, there is a cobordism from* $(X \oplus X, q \oplus -q)$ *to the zero Poincare object.*

Proof. By Example 7, we must show that there exists a Lagrangian $\beta : L \to X \oplus X$. For this, we take $L = X$ and β to be the diagonal map, and choose any path from the sum $(q + -q) \in \Omega^\infty Q(X)$ to the base point. The requisite nondegeneracy condition follows from our assumption that q induces an equivalence $X \to \mathbb{D}_Q X$. $\square$

By virtue of the above result, we are now justified in referring to $L_0(\mathcal{C}, Q)$ as the 0*th* L*-group* of the pair $(\mathcal{C}, Q)$.

Remark 12. Let M be a compact oriented manifold of dimension n, and let $(C^*(M; \mathbf{Z}), q_M)$ as in Example 6. Then $(C^*(M; \mathbf{Z}), q_M)$ determines an element of $L_0(\mathcal{D}^{\mathrm{perf}}(\mathbf{Z}), Q)$, and this element is an incarnation of the *signature* of the manifold M. (In fact, when n is divisible by 4 one can show that $L_0(\mathcal{D}^{\mathrm{perf}}(\mathbf{Z}), Q)$ is isomorphic to $\mathbf{Z}$ and this invariant is precisely the signature). We will ultimately need a more refined version of the signature invariant in order to describe the surgery classification of manifolds. However, this more refined invariant will have the same basic flavor: it will live in a group $L_0(\mathcal{C}, Q)$ for some quadratic functor on a stable ∞-category $\mathcal{C}$, and the invariant associated to M will be some avatar of the stable homotopy type of M, equipped with its intersection product.

L-theory Spaces (Lecture 6)

February 3, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$, which we regard as fixed throughout this lecture. We let $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ denote the polarization of Q, and $\mathbb{D} : \mathcal{C}^{op} \to \mathcal{C}$ the corresponding duality functor. In the last lecture, we introduce the notion of a *cobordism* between two Poincare objects (X, q) and (X', q') of $\mathcal{C}$. We saw that cobordism is an equivalence relation and defined $L_0(\mathcal{C}, Q)$ to be the set of equivalence classes.

In this lecture, we would like to refine the invariant $L_0(\mathcal{C}, Q)$. We will accomplish this by defining an *L-theory space* $L(\mathcal{C}, Q)$, with $\pi_0 L(\mathcal{C}, Q) = L_0(\mathcal{C}, Q)$.

We first describe an approximation to this L-theory space. We let $\mathrm{Poinc}(\mathcal{C}, Q)$ denote a *classifying space for Poincare objects of* $\mathcal{C}$. That is, $\mathrm{Poinc}(\mathcal{C}, Q)$ is an ∞-category whose objects are Poincare objects (X, q) of $\mathcal{C}$, where a morphism from (X, q) to (X', q') is an isomorphism $\alpha : X \to X'$ together with a path joining q to the image of q' in the space $\Omega^\infty Q(X)$. $\mathrm{Poinc}(\mathcal{C}, Q)$ is an ∞-category in which every morphism is invertible and therefore a Kan complex. We will simply refer to $\mathrm{Poinc}(\mathcal{C}, Q)$ as a *space*. It is not the space we are looking for, because cobordant Poincare objects need not lie in the same connected component of $\mathrm{Poinc}(\mathcal{C}, Q)$.

Notation 1. Fix an integer $n \geq 0$. We let $\mathcal{F}_n$ denote the collection of nonempty subsets of the set $\{0, 1, \ldots, n\}$. We regard $\mathcal{F}_n$ as a partially ordered set with respect to inclusions. (It may be helpful to think of $\mathcal{F}_n$ as the partially ordered set of faces of the standard n-simplex Δ^n.) Note that $\mathcal{F}_n$ has a largest element, given by the set $[n] = \{0, \ldots, n\}$.

Let $\mathcal{C}_{[n]}$ denote the ∞-category of functors $\mathrm{Fun}(\mathcal{F}_n^{op}, \mathcal{C})$ from $\mathcal{F}_n^{op}$ into $\mathcal{C}$. We define a functor

$$Q_{[n]} : \mathcal{C}_{[n]}^{op} \to \mathrm{Sp}$$

by the formula $Q_{[n]}(X) = \varprojlim_{S \in \mathcal{F}_n} Q(X(S))$.

Using the fact that Q is quadratic, it follows immediately that $Q_{[n]}$ is a quadratic functor. The polarization of $Q_{[n]}$ is the functor $B_{[n]}$ given by

$$B_{[n]}(X, X') = \varprojlim_{S \in \mathcal{F}_n} B(X(S), X'(S)).$$

Proposition 2. *The bilinear functor $B_{[n]}$ is representable. Its associated duality functor $\mathbb{D}_{[n]}$ is described by the formula*

$$(\mathbb{D}_{[n]} X)(S) = \varprojlim_{T \subseteq S} \mathbb{D}(X(T)).$$

Proof. For simplicity let us assume the existence of $\mathbb{D}_{[n]}$ and show that it is characterized by the above formula (the existence is proven in essentially the same way). We will show that for each object $C \in \mathcal{C}$, there is a canonical homotopy equivalence of spectra

$$\mathrm{Mor}_{\mathcal{C}}(C, (\mathbb{D}_{[n]} X)(S)) \simeq \varprojlim_{T \subseteq S} \mathrm{Mor}_{\mathcal{C}}(C, \mathbb{D}(X(T))).$$

Let $Y : \mathcal{F}_n^{op} \to \mathcal{C}$ be given by the formula

$$Y(T) = \begin{cases} C & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{C}}(C, (\mathbb{D}_{[n]}X)(S)) &\simeq \mathrm{Mor}_{\mathcal{C}_{[n]}}(Y, \mathbb{D}_{[n]}X) \\
&\simeq B_{[n]}(Y, X) \\
&\simeq \varprojlim_{T} B(Y(T), X(T)) \\
&\simeq \varprojlim_{T \subseteq S} B(C, X(T)) \\
&\simeq \varprojlim_{T \subseteq S} \mathrm{Mor}_{\mathcal{C}}(C, \mathbb{D}X(T)) \\
&\simeq \mathrm{Mor}_{\mathcal{C}}(C, \varprojlim_{T \subseteq S} X(T)).
\end{aligned}$$

$\square$

Proposition 3. *The bilinear functor $B_{[n]}$ is nondegenerate.*

Proof. We must show that the canonical map $\mathrm{id} \to \mathbb{D}_n^2$ is an equivalence from $\mathcal{C}_{[n]}$ to itself. Fix an object $X \in \mathcal{C}_{[n]}$. We compute

$$\begin{aligned}
(\mathbb{D}_{[n]}^2 X)(S) &\simeq \varprojlim_{T \subseteq S} \mathbb{D}((\mathbb{D}_{[n]}X)(T)) \\
&\simeq \mathbb{D} \varinjlim_{T \subseteq S} (\mathbb{D}_{[n]}X)(T) \\
&\simeq \mathbb{D} \varinjlim_{T \subseteq S} \varprojlim_{U \subseteq T} \mathbb{D}X(U) \\
&\simeq \mathbb{D}^2 \varprojlim_{T \subseteq S} \varinjlim_{U \subseteq T} X(U) \\
&\simeq \varprojlim_{T \subseteq S} \varinjlim_{U \subseteq T} X(U).
\end{aligned}$$

We wish to show that the canonical map

$$X(S) \to \varprojlim_{T \subseteq S} \varinjlim_{U \subseteq T} X(U)$$

is an equivalence. Let $\mathcal{P}$ be the collection of all subsets of S. We define a cubical diagram $Y : \mathcal{P} \to \mathcal{C}$ by the formula

$$Y(T) = \begin{cases} X(S) & \text{if } T = \emptyset \\ \varinjlim_{\emptyset \neq U \subseteq T} X(U) & \text{otherwise.} \end{cases}$$

We wish to show that Y is a homotopy limit cube in $\mathcal{C}$. Because $\mathcal{C}$ is stable, this is equivalent to the condition that Y is a homotopy colimit cube, which follows from unwinding the definitions. For example, when S has two elements $\{s\}$ and $\{t\}$, then Y is the diagram

$$\begin{tikzcd}
X(S) \arrow[r] \arrow[d] & X(\{s\}) \arrow[d] \\
X(\{t\}) \arrow[r] & X(\{s\}) \coprod_{X(S)} X(\{t\}).
\end{tikzcd}$$

$\square$

Example 4. Let $n = 1$. Then an object X of $\mathcal{C}_{[1]}$ consists of a diagram

$$X(\{0\}) \leftarrow X([1]) \rightarrow X(\{1\})$$

in $\mathcal{C}$. The spectrum $Q_{[2]}(X)$ is given by the homotopy fiber product

$$Q(X(\{0\})) \times_{Q(X([1]))} Q(X(\{1\})).$$

In particular, we can identify a point of $\Omega^\infty Q_{[2]}(X)$ with a point $q_0 \in \Omega^\infty Q(X(\{0\}))$, a point $q_1 \in \Omega^\infty Q(X(\{1\}))$, and a path joining their images in $\Omega^\infty Q(X([1]))$. Such a point determines an equivalence $X \to \mathbb{D}_2 X$ if and only if the following three conditions are satisfied:

- q_0 induces an equivalence $v_0 : X(\{0\}) \to \mathbb{D}X(\{0\})$
- q_1 induces an equivalence $X(\{1\}) \to \mathbb{D}X(\{1\})$
- The induced map $v : X([1]) \to \mathbb{D}X(\{0\}) \times_{\mathbb{D}X([1])} \mathbb{D}X(\{1\})$ is an equivalence.

Note that v fits into a commutative diagram of fiber sequences

$$\begin{tikzcd}
\mathrm{fib}(X([1]) \to X(\{0\})) \arrow[r] \arrow[d, "u"] & X([1]) \arrow[r] \arrow[d, "v"] & X(\{0\}) \arrow[d, "v_0"] \\
\mathrm{fib}(\mathbb{D}X(\{1\}) \to \mathbb{D}X([1])) \arrow[r] & \mathbb{D}X(\{0\}) \times_{\mathbb{D}X([1])} \mathbb{D}X(\{1\}) \arrow[r] & \mathbb{D}X(\{0\})
\end{tikzcd}$$

where u is the map appearing in the previous lecture. If v_0 is an isomorphism, then v is an isomorphism if and only if u is an isomorphism.

We can summarize the situation as follows: giving a Poincare object of $\mathcal{C}_{[1]}$ is equivalent to giving a pair of Poincare objects of $\mathcal{C} = \mathcal{C}_{[0]}$, together with a cobordism between them.

The construction $[n] \mapsto \mathcal{C}_{[n]}$ is contravariantly functorial in the finite set $[n] = \{0, \ldots, n\}$. Given a map of finite sets $f : [m] \to [n]$, there is an induced functor $f^* : \mathcal{C}_{[n]} \to \mathcal{C}_{[m]}$, given by $(f^*X)(S) = X(f(S))$. Note that there is a canonical map

$$Q_{[n]}(X) = \varprojlim_{S \subseteq [n]} Q(X(S)) \to \varprojlim_{T \subseteq [m]} Q(X(f(T))) \simeq Q_{[m]}(f^*X).$$

In particular, every quadratic object (X, q) of $\mathcal{C}_{[n]}$ determines a quadratic object (f^*X, f^*q) of $\mathcal{C}_{[m]}$.

Proposition 5. *In the situation above, if (X, q) is a Poincare object of $\mathcal{C}_{[n]}$, then (f^*X, f^*q) is a Poincare object of $\mathcal{C}_{[m]}$.*

Proof. Fix a nonempty set $S \subseteq [m]$; we wish to show that q induces an isomorphism

$$(f^*X)(S) \to \varprojlim_{T \subseteq S} \mathbb{D}(f^*X)(T).$$

We can rewrite this map as a composition

$$X(f(S)) \xrightarrow{\phi} \varprojlim_{U \subseteq f(S)} \mathbb{D}X(U) \xrightarrow{\psi} \varprojlim_{T \subseteq S} \mathbb{D}X(f(T)).$$

Here the map ϕ is an isomorphism if q is nondegenerate, and ψ is an isomorphism by a cofinality argument. $\square$

For each $n \geq 0$, let $\mathrm{Poinc}(\mathcal{C}, Q)_n$ denote a classifying space for Poincare objects of $(\mathcal{C}_{[n]}, Q_{[n]})$. It follows from the preceding result that a map of finite sets $f : [m] \to [n]$ induces a map of classifying spaces $\mathrm{Poinc}(\mathcal{C}, Q)_n \to \mathrm{Poinc}(\mathcal{C}, Q)_m$. Restricting our attention to order-preserving maps f, we see that $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ has the structure of a *simplicial space*.

Definition 6. We define $L(\mathcal{C}, Q)$ to be classifying space of the simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$. We will refer to $L(\mathcal{C}, Q)$ as the *L-theory space* of $(\mathcal{C}, Q)$.

Remark 7. The set $\pi_0 L(\mathcal{C}, Q)$ can be identified with the quotient of $\pi_0 \mathrm{Poinc}(\mathcal{C}, Q)$ by the equivalence relation generated by the image of $\pi_0 \mathrm{Poinc}(\mathcal{C}, Q)_1$ in $\pi_0 \mathrm{Poinc}(\mathcal{C}, Q) \times \pi_0 \mathrm{Poinc}(\mathcal{C}, Q)$. Using Example 4, we see that this is exactly the relation of cobordism defined in the previous lecture. It follows that we have a canonical isomorphism

$$\pi_0 L(\mathcal{C}, Q) \simeq L_0(\mathcal{C}, Q).$$

All of the constructions of this lecture are compatible with the formation of direct sums of Poincare objects. It follows that the L-theory space $L(\mathcal{C}, Q)$ inherits a monoid structure, which is commutative and associative up to coherent homotopy: that is, $L(\mathcal{C}, Q)$ is an E_∞-space. Moreover, we saw in the last lecture that the induced monoid structure on $\pi_0 L(\mathcal{C}, Q) \simeq L_0(\mathcal{C}, Q)$ is actually an abelian group structure. In other words, $L(\mathcal{C}, Q)$ is a *grouplike E_∞-space*, and therefore an infinite loop space.

Remark 8. We will later construct a *nonconnective* delooping of $L(\mathcal{C}, Q)$.

Definition 9. For $n \geq 0$, we let $L_n(\mathcal{C}, Q)$ denote the homotopy group $\pi_n L(\mathcal{C}, Q)$. We will refer to $L_n(\mathcal{C}, Q)$ as the *nth L-group of* $(\mathcal{C}, Q)$.

We will return to the study of these higher L-groups in the next lecture.

Simplicial Spaces (Lecture 7)

February 6, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. In the last lecture, we defined an L-theory space $L(\mathcal{C}, Q)$, whose path components comprise the abelian group $L_0(\mathcal{C}, Q)$ of Lecture 5. We would like to understand the homotopy type $L(\mathcal{C}, Q)$ better. For example, we might ask for an interpretation of the higher homotopy groups $L_n(\mathcal{C}, Q) = \pi_n L(\mathcal{C}, Q)$.

By definition, $L(\mathcal{C}, Q)$ is given to us as the geometric realization of a simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$. In general, it is not easy to describe the homotopy groups of a geometric realization even if the homotopy groups of the indiviual terms are well-understood (for example, it is hard to describe the homotopy groups of the geometric realization of the simplicial set $\partial\,\Delta^3$).

For a general simplicial space $X_\bullet$, there are two face maps $d_0, d_1 : X_1 \to X_0$ which induce a map $\pi_0(X_1) \to \pi_0 X_0 \times \pi_0 X_0$. The image of this map is a relation R on $\pi_0 X_0$, and the quotient of $\pi_0 X_0$ by the equivalence relation generated by R can be identified with $\pi_0 |X_\bullet|$. However, in our case R is the relation of cobordism of Poincare objects, which is already an equivalence relation. This is a special feature of $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ which makes the homotopy group $\pi_0 L(\mathcal{C}, Q)$ easier to compute. We would like to generalize this observation.

We begin by introducing some notation.

Definition 1. Let $\mathbf{\Delta}$ denote the category of *combinatorial simplices*: that is, nonempty finite linearly ordered sets of the form $\{0, \ldots, n\}$. In this lecture, we will identify the objects of $\mathbf{\Delta}$ with the corresponding simplicial sets $\Delta^0, \Delta^1, \cdots$. A *simplicial space* is a functor from $\mathbf{\Delta}^{op}$ to the ∞-category of spaces. If X is a simplicial space, we will denote the individual spaces of X by $X(\Delta^0)$, $X(\Delta^1)$, and so forth.

If X is a simplicial space, then X determines a functor from the ordinary category of simplicial sets into the ∞-category of spaces, given by

$$K \mapsto \varprojlim_{\sigma : \Delta^n \to K} X(\Delta^n).$$

We will denote this functor by $K \mapsto X(K)$. (More abstractly: we regard X as a functor defined on all simplicial sets, rather than just standard simplices, by taking a right Kan extension.)

Remark 2. We can identify $X(K)$ with the space of maps from K to X in the ∞-category of simplicial spaces (where we regard K as a simplicial space by endowing it with the discrete topology in each degree).

Definition 3. Let $f : X \to Y$ be a map of simplicial spaces. We will say that f is a *Kan fibration* if the following condition is satisfied: for $0 \leq i \leq n$, the map

$$X(\Delta^n) \to X(\Lambda^n_i) \times_{Y(\Lambda^n_i)} Y(\Delta^n)$$

is surjective on connected components (here the fiber product denotes a homotopy fiber product). We will say that f is a *trivial Kan fibration* if, for each $n \geq 0$, the map

$$X(\Delta^n) \to X(\partial\,\Delta^n) \times_{Y(\partial\,\Delta^n)} Y(\Delta^n).$$

We will say that a simplicial space X *satisfies the Kan condition* if the map $X \to *$ is a Kan fibration, where $*$ denotes the constant simplicial space with value equal to a single point.

Remark 4. When we restrict our attention to simplicial sets (which we regard as a special case of simplicial spaces), Definition 3 recovers the usual notion of Kan fibration, trivial Kan fibration, and Kan complex.

If X is a simplicial space satisfying the Kan condition, then the surjectivity of the map $\pi_0 X(\Delta^2) \to \pi_0 X(\Lambda^2_1)$ guarantees that the image of $\pi_0 X(\Delta^1)$ is an equivalence relation on $\pi_0 X(\Delta^0)$. Since we know that the latter condition holds for $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$, we are naturally led to conjecture the following:

Theorem 5. *The simplicial space* $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ *satisfies the Kan condition.*

We will prove Theorem 5 later this week. The remainder of this lecture is devoted to exploring some consequences of Theorem 5.

Recall that if $f : X \to Y$ is a trivial Kan fibration of simplicial sets, then f induces a homotopy equivalence of geometric realizations $|X| \to |Y|$. This generalizes to simplicial spaces:

Proposition 6. *Let* $f : X \to Y$ *be a trivial Kan fibration of simplicial spaces. Then the induced map* $|X| \to |Y|$ *is a homotopy equivalence.*

Proof. The category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets is a model for the ∞-category of spaces. We may therefore choose a simplicial object $\overline{X}$ of the category of simplicial sets representing X, and a simplicial object $\overline{Y}$ of the category of simplicial sets representing Y, such that f is modelled by a map of bisimplicial sets $\overline{f} : \overline{X} \to \overline{Y}$. Without loss of generality, we may assume that $\overline{X}$ and $\overline{Y}$ are Reedy fibrant and that $\overline{f}$ is a Reedy fibration. Then each of the maps

$$X(\Delta^n) \to X(\partial \Delta^n) \times_{Y(\partial \Delta^n)} Y(\Delta^n)$$

is modelled by a Kan fibration of simplicial sets

$$\overline{X}(\Delta^n) \to \overline{X}(\partial \Delta^n) \times_{\overline{Y}(\partial \Delta^n)} \overline{Y}(\Delta^n)$$

Our assumption on f guarantees that this map is surjective on connected components. Since it is a Kan fibration, it is surjective on simplices of every dimension. In other words, we deduce that for each $m \geq 0$, the map of simplicial sets

$$\overline{X}_m \to \overline{Y}_m$$

is a trivial Kan fibration. In particular, the map of bisimplicial sets $\overline{f} : \overline{X} \to \overline{Y}$ is a levelwise homotopy equivalence in the "horizontal" direction, and so induces a homotopy equivalence after geometric realization. □

Proposition 7. *Let* Y *be a simplicial space. Then there exists a simplicial set* X *and a trivial Kan fibration* $f : X \to Y$.

Proof. We successively build n-skeletal simplicial sets $\mathrm{sk}^n X$ and maps $\mathrm{sk}^n X \to Y$ such that the maps

$$\mathrm{sk}^n X(\Delta^m) \to (\mathrm{sk}^n X)(\partial \Delta^m) \times_{Y(\partial \Delta^m)} Y(\Delta^m)$$

are surjective on connective components for $m \leq n$. Assume that $\mathrm{sk}^{n-1} X$ has already been constructed. Let S be the set of connected components of the fiber product

$$(\mathrm{sk}^{n-1} X)(\partial \Delta^n) \times_{Y(\partial \Delta^n)} Y(\Delta^n).$$

and let $\mathrm{sk}^n X$ be the simplicial set obtained from $\mathrm{sk}^{n-1} X$ by adjoining one nondegenerate n-simplex for every element of S (with the obvious attaching maps). There is an evident map of simplicial spaces $\mathrm{sk}^n X \to Y$ having the desired properties. □

Let Y be a simplicial space satisfying the Kan condition, and suppose that we wish to describe the homotopy groups of the geometric realization $|Y|$. Choose a trivial Kan fibration $X \to Y$, where X is a simplicial set. Then the map $|X| \to |Y|$ is a homotopy equivalence, so the homotopy groups of $|Y|$ are the same as the homotopy groups of $|X|$. Moreover, since $X \to Y$ is a Kan fibration, the simplicial set X also satisfies the Kan condition: that is, it is a Kan complex in the usual sense. Let us fix a base point x of $X(\Delta^0)$ (which determines a base point in $Y(\Delta^0)$) and compute all homotopy groups with respect to that base point. If K is a simplicial set with a simplicial subset K_0, let $Y(K, K_0)$ denote the homotopy fiber of the map $Y(K) \to Y(K_0)$ (over the point determined by the base point), and define $X(K, K_0)$ similarly.

Because X is a Kan complex, there is a simple combinatorial recipe for extracting the homotopy groups $\pi_n|X|$. Let us recall how this goes. Every class in $\pi_n|X|$ is represented by a point $\eta \in X(\Delta^n, \partial \Delta^n)$. Let $K \subseteq \partial \Delta^{n+1}$ be the subset obtained by removing the interiors of two faces, so that we have a canonical bijection

$$X(\partial \Delta^{n+1}, K) \to X(\Delta^n, \partial \Delta^n) \times X(\Delta^n, \partial \Delta^n).$$

A pair of elements $\eta, \eta' \in X(\Delta^n, \partial \Delta^n)$ determine the same element in $\pi_n|X|$ if and only if the corresponding element of $X(\partial \Delta^{n+1}, K)$ can be lifted to $X(\Delta^{n+1}, K)$.

Since $X \to Y$ is a trivial Kan fibration, the map

$$\phi : X(\Delta^n, \partial \Delta^n) \to Y(\Delta^n, \partial \Delta^n).$$

is surjective on connected components: that is, every element of $\pi_0 Y(\Delta^n, \partial \Delta^n)$ comes from a point $\eta \in X(\Delta^n, \partial \Delta^n)$. Suppose we are given a pair of points of $Y(\Delta^n, \partial \Delta^n)$, given by the images of elements $\eta, \eta' \in X(\Delta^n, \partial \Delta^n)$. This pair of points determines a point $\zeta \in X(\partial \Delta^{n+1}, K)$ having image $\zeta_0 \in Y(\partial \Delta^{n+1}, K)$. Since the map

$$X(\Delta^{n+1}) \to X(\partial \Delta^{n+1}) \times_{Y(\partial \Delta^{n+1})} Y(\Delta^{n+1})$$

is surjective on connected components, we deduce that ζ_0 lifts to a point of $Y(\Delta^{n+1})$ if and only if ζ lifts to a point of $X(\Delta^{n+1})$. We have proven the following:

Proposition 8. *Let Y be a simplicial space satisfying the Kan condition, and choose a base point $y \in Y(\Delta^0)$ (so that we can regard Y as a simplicial pointed space). Then $\pi_n|Y|$ can be identified with the quotient of the set $\pi_0 Y(\Delta^n, \partial \Delta^n)$ by the following equivalence relation: two homotopy classes $[\eta], [\eta'] \in \pi_0 Y(\Delta^n, \partial \Delta^n)$ represent the same class in $\pi_n|Y|$ if and only if the corresponding point of $Y(\partial \Delta^{n+1}, K)$ lifts to a point of $Y(\Delta^{n+1})$.*

Let us now apply this analysis to the case of interest, where Y is the simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$. Unwinding the definitions, we see that $Y(\Delta^n, \partial \Delta^n)$ is a classifying space for Poincare objects (X, q) of $\mathcal{C}_{[n]}$ (using the notation of the previous lecture) such that $X(S) \simeq 0$ for all proper subsets $S \subsetneq [n]$. In this case, X is determined by a single object $C = X([n]) \in \mathcal{C}$. Moreover, we have

$$Q_{[n]}(X) = \varprojlim_S Q(X(S)) = \varprojlim_S \begin{cases} Q(C) & \text{if } S = [n] \\ 0 & \text{otherwise.} \end{cases}$$

The relevant diagram is parametrizes by partially ordered set of faces of an n-simplex, taking the value 0 on every proper face. Consequently, the limit in question is given by $\Omega^n Q(C)$. We can summarize our analysis as follows:

($*$) Let $Y = \mathrm{Poinc}(\mathcal{C}, Q)_\bullet$. Then $Y(\Delta^n, \partial \Delta^n)$ is a classifying space for Poincare objects of $(\mathcal{C}, \Omega^n Q)$.

Now suppose we are given two Poincare objects for $(\mathcal{C}, \Omega^n Q)$. They determine a point of $Y(\partial \Delta^{n+1})$: that is, a functor from the partially ordered set of all nonempty *proper* subsets of $[n+1]$ into $\mathcal{C}$. Moreover, this functor vanishes identically except on two subsets of $[n+1]$ of cardinality n. Unwinding the definitions, we see that lifting this data to a Poincare object of $\mathcal{C}_{[n+1]}$ is equivalent to specifying a *cobordism* betwee the corresponding Poincare object of $(\mathcal{C}, \Omega^n Q)$. We have proven the following:

Theorem 9. *The abelian group $L_n(\mathcal{C}, Q) = \pi_n L(\mathcal{C}, Q)$ is canonically isomorphic to $L_0(\mathcal{C}, \Omega^n Q)$.*

We close with a result that will be needed in the next lecture:

Proposition 10. *Let $X \to Y \xrightarrow{u} Z$ be a fiber sequence of simplicial spaces. Suppose that u is a Kan fibration. Then*

$$|X| \to |Y| \to |Z|$$

is a fiber sequence of spaces.

Proof. Choose a trivial Kan fibration $f : Z' \to Z$, where Z' is a simplicial set (and choose a base point of Z' lying over the chosen base point of Z). Now choose a trivial Kan fibration $g : Y' \to Y \times_Z Z'$, where Y' is a simplicial set. The canonical map $Y' \to Y$ is a composition of g with a pullback of f, and therefore a trivial Kan fibration. Let X' be the fiber of the map of simplicial sets $Y' \to Z'$. The canonical map $X' \to X$ is a pullback of g and therefore a trivial Kan fibration. We have a commutative diagram of fiber sequences

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

where the vertical maps are trivial Kan fibrations, and therefore induce homotopy equivalences after geometric realization. It will therefore suffice to prove that the sequence of spaces

$$|X'| \to |Y'| \to |Z'|$$

is a fiber sequence. Since these are simplicial sets, it suffices to prove that the map $Y' \to Z'$ is a Kan fibration. This map is given by the composition of g (a trivial Kan fibraation) with the projection map $u' : Y \times_Z Z' \to Z'$. It will therefore suffice to show that u' is a Kan fibration (of simplicial spaces). This is clear, since u' is a pullback of the Kan fibration u. $\square$

Localization (Lecture 8)

February 9, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. In the last lecture, we asserted without proof that the simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ satisfies the Kan condition. Our goal in this lecture is to formulate a generalization of this assertion, which we will prove in the next lecture.

We begin with some generalities. Let $\mathcal{J}$ be an ∞-category. We say that $\mathcal{J}$ is *filtered* if it satisfies the following conditions:

- $\mathcal{J}$ is nonempty.
- For every pair of objects $X, Y \in \mathcal{J}$, there is a third object $Z \in \mathcal{J}$ and a pair of maps $X \to Z \leftarrow Y$.
- For every pair of objects $X, Y \in \mathcal{J}$ and every map of spaces $S^n \to \mathrm{Map}_{\mathcal{J}}(X, Y)$, there is a map $g : Y \to Z$ such that the composite map $S^n \to \mathrm{Map}_{\mathcal{J}}(X, Y) \to \mathrm{Map}_{\mathcal{J}}(X, Z)$ is nullhomotopic.

Let $\mathcal{C}$ be an ∞-category. By a *filtered diagram* in $\mathcal{C}$ we will refer to a functor $\mathcal{J}^{op} \to \mathcal{C}$, where $\mathcal{J}$ is a filtered ∞-category. We will denote a filtered diagram in $\mathcal{C}$ by (X_α), where each X_α is an object of $\mathcal{C}$. The collection of filtered diagrams in $\mathcal{C}$ can be organized into an ∞-category $\mathrm{Pro}(\mathcal{C})$, where morphism spaces are given by

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}((X_\alpha), (Y_\beta)) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Map}_{\mathcal{C}}(X_\alpha, Y_\beta).$$

We refer to the objects of $\mathrm{Pro}(\mathcal{C})$ as *Pro-objects* of $\mathcal{C}$.

We will identify $\mathcal{C}$ with a full subcategory of $\mathrm{Pro}(\mathcal{C})$ (each object $X \in \mathcal{C}$ determines a filtered diagram (X) indexed by the one-point ∞-category $*$). For every filtered diagram (X_α), we can identify the corresponding object of $\mathrm{Pro}(\mathcal{C})$ with the (homotopy) limit $\varprojlim X_\alpha$ in $\mathrm{Pro}(\mathcal{C})$. We can think of $\mathrm{Pro}(\mathcal{C})$ as the ∞-category obtained from $\mathcal{C}$ by formally adjoining limits of filtered diagrams. In fact, $\mathrm{Pro}(\mathcal{C})$ has the following universal property: if $\mathcal{D}$ is an ∞-category which admits filtered limits, then the ∞-category of functors $\mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ which preserve filtered limits is equivalent to the ∞-category of functors $\mathcal{C} \to \mathcal{D}$.

Remark 1. It is not necessary to allow arbitrary filtered ∞-categories in the definition of $\mathrm{Pro}(\mathcal{C})$. One can show that every filtered diagram is equivalent (in the ∞-category $\mathrm{Pro}(\mathcal{C})$) to a diagram indexed by a filtered partially ordered set.

Remark 2. If $\mathcal{C}$ is a stable ∞-category, then the ∞-category $\mathrm{Pro}(\mathcal{C})$ is also stable.

Suppose that $\mathcal{C}$ is a stable ∞-category and let $\mathcal{C}_0$ be a stable subcategory of $\mathcal{C}$ (that is, a subcategory closed under the formation of fibers and cofibers). We will say that a map $f : X' \to X$ in $\mathcal{C}$ is a $\mathcal{C}_0$*-equivalence* if the cofiber $\mathrm{cofib}(f)$ belongs to $\mathcal{C}_0$. If we regard X as fixed, then the collection of all $\mathcal{C}_0$-equivalences $f_\alpha : X_\alpha \to X$ forms a filtered ∞-category (in fact, it is an ∞-category which admits finite limits). Consequently, we can regard (X_α) as a Pro-object of $\mathcal{C}$. We will denote this pro-object by $I(X) \in \mathrm{Pro}(\mathcal{C})$.

Definition 3. Let $\mathcal{C}$ be a (small) stable ∞-category and let $\mathcal{C}_0$ be a stable subcategory of $\mathcal{C}$. We let $\mathcal{C} \,/\, \mathcal{C}_0$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by objects of the form $I(X)$, where $X \in \mathcal{C}$.

It is not difficult to see that the construction $X \mapsto I(X)$ commutes with finite limits. From this, one can deduce that $\mathcal{C} / \mathcal{C}_0$ is closed under passing to fibers in $\mathrm{Pro}(\mathcal{C})$. It follows that $\mathcal{C} / \mathcal{C}_0$ is a stable subcategory of $\mathrm{Pro}(\mathcal{C})$.

The following result justifies our notation:

Proposition 4. *Let $\mathcal{D}$ be a stable ∞-category. Then composition with the functor $I : \mathcal{C} \to \mathcal{C} / \mathcal{C}_0$ induces an equivalence from the ∞-category of exact functors $f : \mathcal{C} / \mathcal{C}_0 \to \mathcal{D}$, to the ∞-category of exact functors $F : \mathcal{C} \to \mathcal{D}$ such that $F| \mathcal{C}_0$ is trivial.*

Proof. Note that if $X \in \mathcal{C}_0$, then the filtered ∞-category of $\mathcal{C}_0$-equivalences $X' \to X$ has an initial object (namely, the map $0 \to X$), so that $I(X) \simeq 0$. It follows that for any exact functor $f : \mathcal{C} / \mathcal{C}_0 \to \mathcal{D}$, the composition $F = f \circ I$ is an exact functor from $\mathcal{C}$ to $\mathcal{D}$ which annihilates $\mathcal{C}_0$.

We now produce an inverse to the preceding construction. Embed $\mathcal{D}$ in a stable ∞-category $\overline{\mathcal{D}}$ which admits filtered limits (for example, we can take $\overline{\mathcal{D}} = \mathrm{Pro}(\mathcal{D})$). Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor which annhilates $\mathcal{C}_0$. Then F extends to a functor $\overline{F} : \mathrm{Pro}(\mathcal{C}) \to \overline{\mathcal{D}}$ which commutes with filtered limits. For each $X \in \mathcal{C}$, we have a canonical map $I(X) \to X$ in $\mathrm{Pro}(\mathcal{C})$, hence a map $u : \overline{F}(I(X)) \to F(X)$ in $\overline{\mathcal{D}}$. The cofiber of the map $I(X) \to X$ is a filtered limit of objects of $\mathcal{C}_0$. Since F annihilates $\mathcal{C}_0$ and $\overline{F}$ commutes with filtered limits, we deduce that u is invertible: that is, we can write $F = \overline{F} \circ I$. In particular, $\overline{F}$ carries $I(\mathcal{C}) = \mathcal{C} / \mathcal{C}_0$ into the subcategory $\mathcal{D} \subseteq \overline{\mathcal{D}}$. Let us denote this restricted functor by $f : \mathcal{C} / \mathcal{C}_0 \to \mathcal{D}$; then $F = f \circ I$ as desired. $\square$

Now suppose that $\mathcal{C}$ is equipped with a nondegenerate quadratic functor Q. We can extend Q to a functor $\widehat{Q} : \mathrm{Pro}(\mathcal{C})^{op} \to \mathrm{Sp}$ by the formula

$$\widehat{Q}((X_\alpha)) = \varinjlim_{\alpha} Q(X_\alpha).$$

It is easy to see that $\widehat{Q}$ is a quadratic functor on $\mathrm{Pro}(\mathcal{C})$, whose polarization $\widehat{B}$ is given by the formula

$$\widehat{B}((X_\alpha), (Y_\beta)) = \varinjlim_{\alpha, \beta} B(X_\alpha, Y_\beta)$$

where B denotes the polarization of Q.

Definition 5. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. Let B be the polarization of Q and let $\mathbb{D}$ denote the corresponding duality functor. We will say that Q is *compatible* with a stable subcategory $\mathcal{C}_0$ if the duality functor $\mathbb{D}$ carries $\mathcal{C}_0$ to itself. In this case, $Q' = Q| \mathcal{C}_0^{op}$ is a nondegenerate quadratic functor on $\mathcal{C}_0$, having polarization $B' = B| \mathcal{C}_0^{op} \times \mathcal{C}_0^{op}$ and duality functor $\mathbb{D}' = \mathbb{D}| \mathcal{C}_0$.

In the above situation, the composition

$$\mathcal{C} \xrightarrow{\mathbb{D}} \mathcal{C}^{op} \to (\mathcal{C} / \mathcal{C}_0)^{op}$$

annihilates the subcategory $\mathcal{C}_0$, and therefore factors (in an essentially unique way) as a composition

$$\mathcal{C} \to \mathcal{C} / \mathcal{C}_0 \xrightarrow{\mathbb{D}''} (\mathcal{C} / \mathcal{C}_0)^{op}.$$

Using the fact that $\mathbb{D}$ has order 2, we deduce easily that $\mathbb{D}''$ also has order 2; in particular, it is a contravariant equivalence of $\mathcal{C} / \mathcal{C}_0$ with itself.

Proposition 6. *In the above situation, the quadratic functor $\widehat{Q}$ on $\mathrm{Pro}(\mathcal{C})$ restricts to a nondegenerate quadratic functor Q'' on $\mathcal{C} / \mathcal{C}_0$, whose duality functor is given by $\mathbb{D}''$.*

Proof. Let X be an object of $\mathcal{C}$ and write $I(X) = (X_\alpha)$. Note that if $Z \in \mathcal{C}_0$, then any map $I(X) \to Z$ in $\mathrm{Pro}(\mathcal{C})$ is nullhomotopic: such a map must factor through some X_α, but the fiber F of the induced map $X_\alpha \to Z$ also belongs to the filtered system (X_α) (and composition $F \to X_\alpha \to Z$ is nullhomotopic).

Let $Y \in \mathcal{C}$ and write $I(Y) = (Y_\beta)$. The above argument shows that

$$\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), Y_\beta) \to \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), Y)$$

for each index β, so that $\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), I(Y)) \simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), Y)$. We therefore obtain a canonical homotopy equivalence

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}'' I(Y)) &\simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), I(\mathbb{D}Y)) \\
&\simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}Y) \\
&\simeq \varinjlim_{\alpha} \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(X_\alpha, \mathbb{D}Y) \\
&\simeq \varinjlim_{\alpha} B(X_\alpha, Y).
\end{aligned}$$

For every index β, the cofiber Z of the map $Y_\beta \to Y$ belongs to $\mathcal{C}_0$, so that $\mathbb{D}(Z) \in \mathcal{C}_0$. It follows that

$$\varinjlim_{\alpha} B(X_\alpha, Z) \simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}(Z)) \simeq 0,$$

so that

$$\varinjlim_{\alpha} B(X_\alpha, Y) \simeq \varinjlim_{\alpha} B(X_\alpha, Y_\beta).$$

Passing to the limit over β, we get

$$\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}'' I(Y)) \simeq \varinjlim_{\alpha} B(X_\alpha, Y) \simeq \varinjlim_{\alpha, \beta} B(X_\alpha, Y_\beta) = \widehat{B}(I(X), I(Y)),$$

so that $\mathbb{D}''$ is the duality functor associated to the bilinear pairing $\widehat{B}$ restricted to $\mathcal{C} / \mathcal{C}_0$. □

Let (X, q) be a quadratic object of $\mathcal{C}$. Then q determines a point $q'' \in \Omega^\infty \widehat{Q}(I(X))$, so that $(I(X), q'')$ can be viewed as a quadratic object of $(\mathcal{C} / \mathcal{C}_0, Q'')$. We have just verified that the functor $I : \mathcal{C} \to \mathcal{C} / \mathcal{C}_0$ interchanges the duality functors induced by Q and Q'' respectively. It follows that if (X, q) is a Poincare object of $(\mathcal{C}, Q)$, then $(I(X), q'')$ is a Poincare object of $(\mathcal{C} / \mathcal{C}_0, Q'')$. This construction determines a map of classifying spaces

$$\mathrm{Poinc}(\mathcal{C}, Q) \to \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')$$

whose fiber (over the zero object) can be identified with $\mathrm{Poinc}(\mathcal{C}_0, Q')$. Applying the same reasoning to the ∞-categories $\mathcal{C}_{[n]}$ for $n \geq 0$, we obtain a fiber sequence of simplicial spaces

$$\mathrm{Poinc}(\mathcal{C}_0, Q')_\bullet \to \mathrm{Poinc}(\mathcal{C}, Q)_\bullet \xrightarrow{\phi} \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet$$

We will prove the following result in the next lecture:

Theorem 7. *Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q, and let $\mathcal{C}_0$ be a stable subcategory of $\mathcal{C}$ which is closed under duality. Then the map $\phi : \mathrm{Poinc}(\mathcal{C}, Q)_\bullet \to \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet$ is a Kan fibration of simplicial spaces.*

Corollary 8. *Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. Then the simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ satisfies the Kan condition.*

Proof. Apply Theorem 7 in the special case $\mathcal{C}_0 = \mathcal{C}$. □

Corollary 9. *In the situation of Theorem 7, we have a fiber sequence of L-theory spaces*

$$L(\mathcal{C}_0, Q') \to L(\mathcal{C}, Q) \to L(\mathcal{C} \,/\, \mathcal{C}_0, Q''),$$

and therefore a long exact sequence of abelian groups

$$\cdots \to L_1(\mathcal{C} \,/\, \mathcal{C}_0, Q'') \to L_0(\mathcal{C}_0, Q') \to L_0(\mathcal{C}, Q) \to L_0(\mathcal{C} \,/\, \mathcal{C}_0, Q'').$$

Proof. Combine Theorem 7 with the result stated at the end of the previous lecture. □

Proof of the Kan Property (Lecture 9)

February 11, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. Let B be the polarization of Q and $\mathbb{D}$ the associated duality functor. Our goal in this lecture is to prove the following:

Theorem 1. *Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a stable subcategory which is closed under $\mathbb{D}$. Then the canonical map $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet \to \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet$ is a Kan fibration of simplicial spaces (here Q'' is the quadratic functor on $\mathcal{C} / \mathcal{C}_0$ defined in the previous lecture).*

Before giving the proof, we embark on two digressions.

Notation 2. We let $\mathrm{Lagr}(\mathcal{C}, Q)$ denote the full subcategory of $\mathcal{C}_{[1]}$ spanned by those diagrams

$$X_0 \leftarrow X_{01} \to X_1$$

where $X_1 = 0$, and endow $\mathrm{Lagr}(\mathcal{C}, Q)$ with the quadratic functor Q_{Lagr} given by the restriction of $Q_{[1]}$.

Lemma 3. *We have $L_0(\mathrm{Lagr}(\mathcal{C}, Q), Q_{\mathrm{Lagr}}) = 0$.*

Remark 4. Lemma 3 can be regarded as an algebraic analogue of the following topological fact: for any topological space X with a base point $x \in X$, the space of paths in X ending in x is contractible.

Proof. Fix a Poincare object (X, q) of $\mathrm{Lagr}(\mathcal{C}, Q)$, so that X is given by a diagram

$$X_0 \leftarrow X_{01} \to 0$$

Let $L \in \mathrm{Lagr}(\mathcal{C}, Q)$ be the diagram

$$X_{01} \leftarrow X_{01} \to 0.$$

There is a canonical map $L \to X$. Note that $Q_{\mathrm{Lagr}}(L) = Q(0) \times_{Q(X_{01})} Q(X_{01}) \simeq 0$, so that the restriction of q to L is canonically nullhomotopic. A simple calculation shows that L is a Lagrangian in X, so that (X, q) represents $0 \in L_0(\mathrm{Lagr}(\mathcal{C}, Q), Q_{\mathrm{Lagr}})$. $\square$

We now discuss the classification of *quadratic objects* of $(\mathcal{C}, Q)$ which are not necessarily Poincare. Let us begin with an example for motivation. Let $\mathcal{C} = \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})$, let B be the symmetric bilinear functor associated to the duality functor

$$P_\bullet \mapsto \mathrm{Hom}(P_\bullet, \mathbf{Z}[-n])$$

and let $Q(X) = B(X, X)^{h\Sigma_2}$ be the associated quadratic functor. Let M be a compact oriented n-manifold with boundary. Then we have an intersection form

$$C^*(M, \partial M) \otimes C^*(M, \partial M) \xrightarrow{[M]} \mathbf{Z}[-n]$$

which determines a quadratic object $(C^*(M, \partial M), q_M)$ of $\mathcal{C}$. If M is a closed manifold, then this object is Poincare. In general it is not: the point q_M determines a map $C^*(M, \partial M) \to \mathbb{D}C^*(M, \partial M) \simeq C^*(M)$, which can be identified with the standard inclusion of relative cochains into cochains. The cofiber of this map is $C^*(\partial M)$. Note that this cofiber is itself described as a complex of cochains on a manifold (of dimension $n-1$ rather than n), and so determines a Poincare object of $(\mathcal{C}, \Sigma Q)$. We will now show that this is a general phenomenon.

So far, we have not used the full strength of our assumption that Q is a *quadratic* functor. The next result remedies this:

Proposition 5. *Suppose we are given a fiber sequence*

$$X' \to X \to X''$$

in $\mathcal{C}$. *Then* $Q(X'')$ *is the total homotopy fiber of the diagram*

$$\begin{tikzcd} Q(X) \arrow[r] \arrow[d] & Q(X') \arrow[d] \\ B(X', X) \arrow[r] & B(X', X') \end{tikzcd}$$

In other words, $Q(X'')$ *is equivalent to the fiber of the map*

$$Q(X') \times_{B(X',X')} B(X', X).$$

Remark 6. Let us state this result more informally. Think of $Q(X)$ as a spectrum parametrizing "quadratic forms" on X. The Proposition addresses the following question: given a quadratic form on X, when does it descend to $X'' = X/X'$? An obvious necessary condition is that it should vanish on X'. This implies that the associated bilinear form vanishes on X'. But we need a bit more: namely, to know that X' lies in the kernel of the associated bilinear form.

Proof. We wish to show that for every quadratic functor Q on $\mathcal{C}$, the triangle

$$Q(X'') \to Q(X) \to Q(X') \times_{B(X',X')} B(X, X')$$

is a fiber sequence. We have a fiber sequence of functors

$$Q_0(Z) \to Q(Z) \to B(Z, Z)^{h\Sigma_2}.$$

It therefore suffices to prove the result after replacing Q by Q_0 or $B(Z, Z)^{h\Sigma_2}$. In the first case, the polarization of Q_0 vanishes and the desired result follows from the fact that Q_0 is exact. Let us therefore assume that $Q(Z) = B(Z, Z)^{h\Sigma_2}$. Since B is exact in each variable, we can identify $B(X'', X'')$ with the total homotopy fiber of the diagram

$$\begin{tikzcd} B(X, X) \arrow[r] \arrow[d] & B(X, X') \arrow[d] \\ B(X', X) \arrow[r] & B(X', X'). \end{tikzcd}$$

In other words, $B(X'', X'')$ is the fiber of the map

$$B(X, X) \to B(X, X') \times_{B(X',X')} B(X', X) \simeq (B(X, X') \times B(X', X)) \times_{B(X',X') \times B(X',X')} B(X', X').$$

Taking homotopy fixed points with respect to Σ_2, we get a fiber sequence

$$Q(X'') \to Q(X) \to B(X, X') \times_{B(X',X')} Q(X').$$

$\square$

Let us rewrite the fiber sequence of Proposition 5 as

$$Q(X') \times_{B(X',X')} B(X',X) \to \Sigma Q(X'') \to \Sigma Q(X).$$

Suppose that (Y,q) is a quadratic object of $(\mathcal{C},Q)$. There is a canonical point $\eta \in \Omega^\infty B(Y,\mathbb{D}Y)$ (corresponding to the identity map from Y to itself). Then q determines a point of $\Omega^\infty B(Y,Y)$, which is the image of η under the map $B(Y,\mathbb{D}Y) \to B(Y,Y)$ for some essentially unique map $u : Y \to \mathbb{D}Y$. Form a fiber sequence $Y \xrightarrow{u} \mathbb{D}(Y) \to \mathbb{D}(Y)/Y$. Then the pair (q,η) determines a point of $\Omega^\infty(Q(Y) \times_{B(Y,Y)} B(Y,\mathbb{D}Y)$. According to the above analysis, this is the 0th space of the fiber of the map of spectra

$$\Sigma Q(\mathbb{D}(Y)/Y) \to \Sigma Q(\mathbb{D}Y).$$

In particular, q determines a quadratic object of $(\mathcal{C},\Sigma Q)$, which we will denote by $(\mathbb{D}Y/Y,\overline{q})$.

The point $\overline{q}$ determines a map

$$v : \mathbb{D}Y/Y \to \Sigma\mathbb{D}(\mathbb{D}Y/Y) \simeq \Sigma\operatorname{fib}(\mathbb{D}^2(Y) \to \mathbb{D}Y) \simeq \Sigma\operatorname{fib}(u) \simeq \operatorname{cofib}(u) = \mathbb{D}Y/Y.$$

Unwinding the definitions, one sees that this is the identity map (up to a sign). Consequently, $(\mathbb{D}(Y)/Y,\overline{q})$ is a Poincare object of $(\mathcal{C},\Sigma Q)$. We have a canonical map $w : \mathbb{D}(Y) \to \mathbb{D}(Y)/Y$, and a canonical nullhomotopy of the image of $\overline{q}$ in $Q(\mathbb{D}(Y))$. This data determines a map

$$\mathbb{D}(Y) \to (\Sigma\mathbb{D})(\operatorname{cofib}(W)) \simeq \Sigma\mathbb{D}\Sigma Y \simeq \Sigma\Omega\mathbb{D}(Y) \simeq \mathbb{D}(Y),$$

which is also the identity map (up to sign). Consequently, we can regard $\mathbb{D}(Y)$ as a Lagrangian in $(\mathbb{D}(Y)/Y,\overline{q})$.

We can summarize our discussion as follows:

(∗) Given a quadratic object (Y,q) of $(\mathcal{C},Q)$, we can construct a Poincare object $(\mathbb{D}(Y)/Y,\overline{q})$ of $(\mathcal{C},\Sigma Q)$ and a Lagrangian $\mathbb{D}(Y)$ in $(\mathbb{D}(Y)/Y,\overline{q})$.

Remark 7. The converse to (∗) is true as well: given a Poincare object $(Z,\overline{q})$ in $(\mathcal{C},Q)$ and a Lagrangian $f : L \to Z$, we can equip the fiber $\operatorname{fib}(f)$ with the structure of a quadratic object of $(\mathcal{C},Q)$.

Let us now suppose that $\mathcal{C}_0 \subseteq \mathcal{C}$ is a stable subcategory which is closed under duality. Suppose we are given a quadratic object (Y,q) of $(\mathcal{C},Q)$ whose image in $\mathcal{C}/\mathcal{C}_0$ is a Poincare object. Then the canonical map $u : Y \to \mathbb{D}(Y)$ becomes invertible in $\mathcal{C}/\mathcal{C}_0$, so the cofiber $\operatorname{cofib}(u) = \mathbb{D}(Y)/Y$ belongs to $\mathcal{C}_0$. Consequently, the above construction produces a Poincare object $(\mathbb{D}(Y)/Y,\overline{q})$ of $(\mathcal{C}_0,\Sigma Q')$ (where $Q' = Q|\,\mathcal{C}_0$). Suppose that this Poincare object is nullcobordant: that is, we can choose a Lagrangian $L \to \mathbb{D}(Y)/Y$ in $(\mathcal{C}_0,\Sigma Q')$. Then $(\mathbb{D}(Y)/Y,\overline{q})$ has *two* Lagrangians in the ∞-category $\mathcal{C}$: L and $\mathbb{D}(Y)$. Each of these provides a cobordism of $(\mathbb{D}(Y)/Y,\overline{q})$ with the zero object. We have seen that cobordisms can be composed: if we compose these cobordisms, we obtain a cobordism $L \times_{\mathbb{D}(Y)/Y} \mathbb{D}(Y)$ from the zero Poincare object of $(\mathcal{C},\Sigma Q)$ to itself. Such a cobordism can be regarded as a Poincare object (Y',q') of $(\mathcal{C},\Omega\Sigma Q) = (\mathcal{C},Q)$. Moreover, since L vanishes in $\mathcal{C}/\mathcal{C}_0$, we note that (Y',q') and (Y,q) determine the same quadratic object of $\mathcal{C}/\mathcal{C}_0$.

Remark 8. Let (Y_0,q_0) be any Poincare object of $\mathcal{C}/\mathcal{C}_0$. By construction, we can always lift (Y_0,q_0) to a quadratic object (Y,q) of $\mathcal{C}$. The above discussion shows that we can adjust (Y,q) to be a Poincare object of $\mathcal{C}$ if and only if a certain obstruction in $L_0(\mathcal{C}_0,\Sigma Q')$ vanishes. One we have proven the theorem, we can identify this obstruction with the image of (Y_0,q_0) under the boundary map

$$\pi_1 L(\mathcal{C}/\mathcal{C}_0,\Sigma Q'') \to \pi_0 L(\mathcal{C}_0,\Sigma Q')$$

determined by the fiber sequence of spaces

$$L(\mathcal{C}_0,\Sigma Q') \to L(\mathcal{C},\Sigma Q) \to L(\mathcal{C}/\mathcal{C}_0,\Sigma Q'').$$

We now turn to the proof of Theorem 1. We must show that every point η of

$$\mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet(\Delta^n) \times_{\mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet(\Lambda^n_i)} \mathrm{Poinc}(\mathcal{C}, Q)_\bullet(\Lambda^n_i)$$

can be lifted (up to homotopy) to a point of $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet(\Delta^n)$. The point η determines a Poincare object (X_0, q_0) of $(\mathcal{C} / \mathcal{C}_0)_{[n]}$. Here we think of X_0 as a contravariant functor from the collection of nonempty subsets of $\{0, \ldots, n\}$ into $\mathcal{C} / \mathcal{C}_0$. Let $\sigma = [n] = \{0, \ldots, n\}$ and let $\tau = [n] - \{i\} \subseteq \sigma$. Then η determines objects $X(S) \in \mathcal{C}$ lifting $X_0(S)$ for $S \notin \{\sigma, \tau\}$, together with a point of

$$q_1 \in \Omega^\infty \varprojlim_{S \notin \{\sigma, \tau\}} Q(X(S))$$

which is nondegenerate and compatible with q_0. Using the construction of $\mathcal{C} / \mathcal{C}_0$ and Q'', we can extend X to a functor defined on all nonempty subsets of $[n]$ and q_1 to a point $q \in \Omega^\infty Q_{[n]}(X)$ (compatible with q_0). What is not clear is that (X, q) is a Poincare object of $(\mathcal{C}_{[n]}, Q_{[n]})$. The point q determines a map $X \to \mathbb{D}_{[n]}(X)$, which may fail to be invertible when evaluated at σ and τ.

Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}_{[n]}$ spanned by those functors Z such that $Z(S) \simeq 0$ for $S \notin \{\sigma, \tau\}$, and $Z(\sigma), Z(\tau) \in \mathcal{C}_0$. We can identify objects of $\mathcal{D}$ with morphisms $Z(\sigma) \to Z(\tau)$ in $\mathcal{C}_0$, and so have an equivalence of ∞-categories $\mathcal{D} \simeq \mathrm{Lagr}(\mathcal{C}_0, Q')$. When $n = 1$, the quadratic functor $Q_{[n]} | \mathcal{D}^{op}$ is precisely the quadratic functor Q_{Lagr} appearing in Lemma 3. In the general case, a simple calculation gives $Q_{[n]} | \mathcal{D}^{op} \simeq \Omega^{n-1} Q_{\mathrm{Lagr}}$.

Let Y denote the cofiber of the map $u : X \to \mathbb{D}_{[n]}(X)$, so that Y has the structure of a Poincare object of $(\mathcal{D}, (\Sigma Q_{[n]}) | \mathcal{D}^{op}) = (\mathrm{Lagr}(\mathcal{C}, Q), \Omega^{n-2} Q_{\mathrm{Lagr}})$. Invoking Lemma 3, we deduce that every Poincare object of $(\mathcal{D}, (\Sigma Q_{[n]}) | \mathcal{D})$ is nullcobordant. In particular, Y is nullcobordant. Choosing a Lagrangian in Y, we obtain a procedure for modifying (X, q) to obtain a Poincare object of $(\mathcal{C}_{[n]}, Q_{[n]})$, which gives the desired lift of η.

L-Theory of Rings and Ring Spectra (Lecture 10)

February 14, 2011

Let R be an associative ring. Recall that we earlier introduced the ∞-category $\mathcal{D}^{\text{perf}}(R)$ whose objects can be identified with bounded complexes of finite projective R-modules.

Definition 1. An *involution* on R is a map $\sigma : R \to R$ satisfying the following conditions:

- $\sigma(a+b) = \sigma(a) + \sigma(b)$
- $\sigma(ab) = \sigma(b)\sigma(a)$
- $\sigma\sigma(a) = a$.

If the ring R is equipped with an involution, then any left R-module M can be regarded as a right R-module, via the formula

$$xa = \sigma(a)x.$$

If M is a left R-module, then the R-linear dual $\mathrm{Hom}_R(M, R)$ has the structure of a right R-module. If R is equipped with an involution σ, we can use σ to regard $\mathrm{Hom}_R(M, R)$ as a left module again. Concretely, the left R-module structure is given by the formula

$$(a\lambda)(x) = \lambda(x)\sigma(a)$$

for $a \in R$, $x \in M$, and $\lambda \in \mathrm{Hom}_R(M, R)$.

Let $P_\bullet$ be a bounded chain complex of finitely generated projective left R-modules. We let $\mathbb{D}(P_\bullet)$ denote the chain complex obtained by applying R-linear duality termwise. Using the involution σ on R, we can regard $\mathbb{D}(P_\bullet)$ is also a bounded chain complex of finitely generated projective left R-modules. The construction

$$P_\bullet \mapsto \mathbb{D}(P_\bullet)$$

determines a (contravariant) equivalence of the ∞-category $\mathcal{D}^{\text{perf}}(R)$ with itself. Let B denote the bilinear functor $\mathcal{D}^{\text{perf}}(R)^{op} \times \mathcal{D}^{\text{perf}}(R)^{op} \to \mathrm{Sp}$ given by the formula $B(P_\bullet, Q_\bullet) = \mathrm{Mor}_{\mathcal{D}^{\text{perf}}(R)}(P_\bullet, \mathbb{D}(Q_\bullet))$. The condition $\sigma^2 = \mathrm{id}$ implies that the bilinear functor B is symmetric.

Let us now describe a generalization of the above construction. The ∞-category Sp of spectra is *symmetric monoidal*: that is, there is an operation $\wedge : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$, called the *smash product*, which is commutative and associative up to coherent homotopy. It therefore makes sense to consider associative algebra objects of Sp: that is, spectra R equipped with a multiplication map

$$R \wedge R \to R$$

which are associative (and unital) up to coherent homotopy. We will refer to such an algebra as an *A_∞-ring*.

If R is an A_∞-ring, we can define an ∞-category LMod_R of *left R-module spectra*. The objects of LMod_R are spectra M equipped with an action map

$$R \wedge M \to M$$

satisfying the usual transitivity property, up to coherent homotopy.

Let $\mathrm{LMod}_R^{\mathrm{fp}}$ denote the smallest stable subcategory of LMod_R which contains the R-module R. Let $\mathrm{LMod}_R^{\mathrm{perf}}$ be the smallest subcategory of LMod_R which contains R and is closed under the formation of direct summands. We say that a left R-module M is *finitely presented* if it belongs to $\mathrm{LMod}_R^{\mathrm{fp}}$, and *perfect* if it belongs to $\mathrm{LMod}_R^{\mathrm{perf}}$.

We say that an A_∞-ring spectrum R is *discrete* if $\pi_i R \simeq 0$ for $i \neq 0$. In this case, R is determined (up to canonical homotopy equivalence) by $\pi_0 R$, which is an ordinary associative ring. Moreover, there is a canonical equivalence of ∞-categories

$$\mathrm{LMod}_R^{\mathrm{perf}} \simeq \mathcal{D}^{\mathrm{perf}}(\pi_0 R):$$

in other words, we can identify (perfect) R-module spectra with (perfect) chain complexes of ordinary modules over the ordinary associative ring $\pi_0 R$.

The collection of all A_∞-rings is organized into an ∞-category. This ∞-category is acted on by the group Σ_2, where the nontrivial element of Σ_2 sends each A_∞-ring R to the same spectrum equipped with the opposite multiplication; we will denote this A_∞-ring by R^{op}. We can identify left R-module spectra with right R^{op}-module spectra. In particular, if M is a left R-module spectrum, then $\mathrm{Mor}(M,R)$ admits a right R-module structure, and so has the structure of a left module over R^{op}. This construction determines a contravariant equivalence of $\mathrm{LMod}_R^{\mathrm{perf}}$ with $\mathrm{LMod}_{R^{op}}^{\mathrm{perf}}$, which restricts to a contravariant equivalence of $\mathrm{LMod}_R^{\mathrm{fp}}$ with $\mathrm{LMod}_{R^{op}}^{\mathrm{fp}}$.

By an *A_∞-ring with involution*, we will mean a homotopy fixed point for the action of Σ_2 on the ∞-category of A_∞-rings. If R is an A_∞-ring with involution, then the construction $M \mapsto \mathrm{Mor}(M,R)$ determines a duality equivalence

$$\mathbb{D} : \mathrm{LMod}_R^{\mathrm{perf},op} \to \mathrm{LMod}_R^{\mathrm{perf}}.$$

This is classified by a symmetric bilinear functor B on $\mathrm{LMod}_R^{\mathrm{perf}}$. Note that this functor carries $\mathrm{LMod}_R^{\mathrm{fp}}$ to itself.

Remark 2. The class of stable ∞-categories and symmetric bilinear functors constructed above is quite general.

- Let $\mathcal{C}$ be any stable ∞-category containing an object X. Then the spectrum $\mathrm{Mor}_{\mathcal{C}}(X,X)$ is an A_∞-ring spectrum R. Moreover, the construction $M \mapsto X \wedge_R M$ determines a fully faithful embedding $\mathrm{LMod}_R^{\mathrm{fp}} \to \mathcal{C}$, carrying R to X. The essential image of this functor is the smallest stable subcategory of $\mathcal{C}$ containing X. If $\mathcal{C}$ is idempotent complete, then this functor extends to a map $\mathrm{LMod}_R^{\mathrm{perf}} \to \mathcal{C}$.

- Let R and R' be A_∞-rings. Let

$$B : (\mathrm{LMod}_R^{\mathrm{fp}})^{op} \times (\mathrm{LMod}_{R'}^{\mathrm{fp}})^{op} \to \mathrm{Sp}$$

  be a bilinear functor. Regard R and R' as left modules over themselves. Then R is a right R-module in $\mathrm{LMod}_R^{\mathrm{fp}}$, and similarly for R'. Since B is contravariant, we deduce that $B(R,R')$ is a spectrum with commuting left actions of R and R'; this endows $B(R,R')$ with the structure of a left module over $R \wedge R'$. Let us denote this module by P. We can recover B from P: it is given by

$$B(M,N) = \mathrm{Mor}_{\mathrm{LMod}_{R\wedge R'}}(M \wedge N, P).$$

- Let R, R', and P be as above, and suppose we are given a point $\eta \in \Omega^\infty P$, corresponding to a map of spectra $S \to P$ where S is the sphere spectrum. Then η determines a map of left R-modules $u_\eta : R \to P$. Suppose that u_η is an isomorphism. Then u_η endows R with the structure of a left R'-module, which commutes with the left R-action of R on itself. All endomorphisms of R as a left R-module are given by the right action of R on itself. Consequently, the left action of R' on P is encoded by a map of A_∞-rings $\sigma : R' \to R^{op}$.

- Now suppose that $R = R'$, and let B be the bilinear functor determined by a left $R \wedge R$-module P. Promoting B to a *symmetric* bilinear functor is equivalent to giving an action of the symmetric group Σ_2 on P, which permutes the two R-actions on P. Suppose that, in addition, we have a point $\overline{\eta} \in \Omega^\infty P^{h\Sigma_2}$ which satisfies the condition above (that is, $\overline{\eta}$ induces an isomorphism $R \to P$). Then $\overline{\eta}$ determines a map $\sigma : R \to R^{op}$, as above. Using the fact that $\overline{\eta}$ is a Σ_2-homotopy fixed point, we see that σ is an involution on R.

Definition 3. Let R be an A_∞-ring with involution σ. We let

$$Q^{\mathrm{s}}_\sigma : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$$

$$Q^{\mathrm{q}}_\sigma : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$$

be the quadratic functors given by

$$Q^{\mathrm{s}}_\sigma(M) = B(M,M)^{h\Sigma_2} \qquad Q^{\mathrm{q}}_\sigma(M) = B(M,M)_{h\Sigma_2}$$

For every integer n, we let

$$L^{\mathrm{s}}_n(R) = L_0(\mathrm{LMod}^{\mathrm{fp}}_R, \Omega^n Q^{\mathrm{s}}_\sigma) \qquad L^{\mathrm{q}}_n(R) = L_0(\mathrm{LMod}^{\mathrm{fp}}_R, \Omega^n Q^{\mathrm{q}}_\sigma).$$

We will refer to the groups $L^{\mathrm{s}}_*(R)$ as the *symmetric L-groups of* R, and $L^{\mathrm{q}}_*(R)$ as the *quadratic L-groups of* R. Note that these groups depend not only on the A_∞-ring R, but also on the involution σ.

Warning 4. This notation is not standard.

Variant 5. Let R be an associative ring with involution. Then we can choose a discrete A_∞-ring $\overline{R}$ with involution, such that $R \simeq \pi_0 \overline{R}$. (Concretely, $\overline{R}$ is given by the *Eilenberg-MacLane spectrum* HR associated to R.) We let $L^{\mathrm{q}}_*(R) = L^{\mathrm{q}}_*(\overline{R})$ and $L^{\mathrm{s}}_*(R) = L^{\mathrm{s}}_*(\overline{R})$.

Variant 6. In the above definition, we can replace $\mathrm{LMod}^{\mathrm{fp}}_R$ with the larger ∞-category $\mathrm{LMod}^{\mathrm{perf}}_R$ of perfect R-modules. Sometimes, this makes no difference (for example, if $R = \mathbf{Z}$), but in general it leads to different L-groups. These are sometimes called the *projective* (symmetric and quadratic) L-groups of R.

Remark 7. Let R be an associative ring with involution σ and let M be a free R-module of finite rank. Then $B(M,M)$ can be identified with the abelian group $\mathrm{Hom}_R(M, \mathrm{Hom}_R(M,R))$ of bilinear forms on M: that is, maps $b : M \times M \to R$ which are additive in each varaible and satisfy

$$b(ax, a'x') = ab(x,x')\sigma(a').$$

We say that b is *symmetric* if $b(x,y) = \sigma b(y,x)$. Promoting M to a quadratic object of $(\mathcal{D}^{\mathrm{fp}}(R), Q^{\mathrm{s}}_\sigma)$ is equivalent to choosing a symmetric bilinear form b on M. In this case, the pair (M,b) is a Poincare object of $(\mathcal{D}^{\mathrm{fp}}(R), Q^{\mathrm{s}}_\sigma)$ if and only if b is nondegenerate (that is, b induces an isomorphism $M \to \mathrm{Hom}_R(M,R)$). We can summarize our analysis as follows: the symmetric L-theory of an associative ring R with involution is closely related to the theory of R-modules equipped with symmetric bilinear forms. In particular, every symmetric bilinear form on R^n determines a class in $L^{\mathrm{s}}_0(R)$.

Remark 8. Let R be a commutative ring, which we regard as an associative ring with involution where the involution is given by id_R. Let M be a free R-module of finite rank.

A *quadratic form* $q : M \to R$ is a map satisfying the following conditions:

(i) $q(ax) = a^2 q(x)$

(ii) $q(x+y) - q(x) - q(y)$ is a bilinear map $M \times M \to R$.

Every bilinear form $b : M \times M \to R$ determines a quadratic form $q : M \to R$ by the formula $q(x) = b(x,x)$. Moreover, every quadratic form arises in this way: if we choose a basis $x_1, \ldots, x_n$ for M over R, then we can define a bilinear form b by the formula

$$b(x_i, x_j) = \begin{cases} q(x_i) & \text{if } i = j \\ q(x_i + x_j) - q(x_i) - q(x_j) & \text{if } i < j \\ 0 & \text{if } i > j. \end{cases}$$

Note that if $\epsilon : M \times M \to R$ is any other bilinear form and we set $b'(x,y) = b(x,y) + \epsilon(x,y) - \epsilon(y,x)$, then b' and b determine the same quadratic form on M. Conversely, suppose that b' and b determine the same quadratic form on M, and define a bilinear form ϵ by the formula

$$\epsilon(x_i, x_j) = \begin{cases} b'(x_i, x_j) - b(x_i, x_j) & \text{if } i < j \\ 0 & \text{if } i \geq j. \end{cases}$$

A simple calculation gives $b'(x,y) = b(x,y) + \epsilon(x,y) - \epsilon(y,x)$. We can summarize this discussion as follows: the abelian group of quadratic forms on M is given by the 0th homology of the group Σ_2 acting on the abelian group of all bilinear forms on M. This homology group can be identified with $\pi_0 B(M,M)_{h\Sigma_2}$. Consequently, up to homotopy, equipping M with the structure of a quadratic object of $(\mathcal{D}^{\mathrm{fp}}(R), Q^{\mathrm{q}}_{\sigma})$ is equivalent to choosing a quadratic form q on M. The pair (M,q) is a Poincare object if and only if the polarization $q(x+y) - q(x) - q(y)$ is a nondegenerate symmetric bilinear form on M.

We can summarize the above discussion as follows: the quadratic L-theory of a commutative ring R (with the identity involution) is closely related to the theory of quadratic forms on R-modules. In particular, every nondegenerate quadratic form on R^n determines a class in $L^{\mathrm{q}}_0(R)$.

Remark 9. Let R be an A_∞-ring with involution. The norm map

$$B(M,M)_{h\Sigma_2} \to B(M,M)^{h\Sigma_2}$$

determines a map of quadratic functors $Q^{\mathrm{q}}_{\sigma} \to Q^{\mathrm{s}}_{\sigma}$, which induces an isomorphism after polarization. This construction determines maps of L-groups

$$L^{\mathrm{q}}_*(R) \to L^{\mathrm{s}}_*(R).$$

These maps are isomorphisms if 2 is invertible in $\pi_0 R$, but not in general. For example, these groups are different in the case $R = \mathbf{Z}$.

Algebraic Surgery (Lecture 11)

February 17, 2011

Before getting to the main topic of this lecture, let us pick up a few loose ends from the previous lecture. Recall that for every A_∞-ring R with involution, we defined stable ∞-categories $\mathrm{LMod}_R^{\mathrm{perf}}$ and $\mathrm{LMod}_R^{\mathrm{fp}}$ which are equipped with quadratic functors Q and Q^{symm}. We ask the following question: how general are these examples, among all pairs $(\mathcal{C}, Q)$ where $\mathcal{C}$ is a stable ∞-category and Q a nondegenerate quadratic functor on $\mathcal{C}$?

- Let $\mathcal{C}$ be any stable ∞-category containing an object X. Then the spectrum $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is an A_∞-ring spectrum R. Moreover, the construction $M \mapsto X \wedge_R M$ determines a fully faithful embedding $\mathrm{LMod}_R^{\mathrm{fp}} \to \mathcal{C}$, carrying R to X. The essential image of this functor is the smallest stable subcategory of $\mathcal{C}$ containing X. If $\mathcal{C}$ is idempotent complete, then this functor extends to a fully faithful embedding $\mathrm{LMod}_R^{\mathrm{perf}} \to \mathcal{C}$.
- Suppose now that $\mathcal{C}$ is equipped with a symmetric bilinear functor B, and let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be given by the formula $Q(X) = B(X, X)^{h\Sigma_2}$. Let us assume that B is nondegenerate, and denote the associated duality functor by $\mathbb{D}$. The functor $\mathbb{D}$ is a contravariant equivalence from $\mathcal{C}$ to itself. Consequently, for any object $X \in \mathcal{C}$ we have a canonical equivalence of A_∞-rings $\mathrm{Mor}_{\mathcal{C}}(X, X) \simeq \mathrm{Mor}_{\mathcal{C}}(\mathbb{D}(X), \mathbb{D}(X))^{op}$. If (X, q) is a Poincare object of $\mathcal{C}$, then this equivalence determines an involution σ on the A_∞-ring $R = \mathrm{Mor}_{\mathcal{C}}(X, X)$.

We can summarize the above discussion as follows: a pair $(\mathcal{C}, Q)$ is of the form $(\mathrm{LMod}_R^{\mathrm{fp}}, Q)$ if and only if $Q(X) \simeq B(X, X)^{h\Sigma_2}$ for $X \in \mathcal{C}$, and there exists a Poincare object (M, q) in $\mathcal{C}$ such that M generates $\mathcal{C}$ as a stable ∞-category.

Example 1. Let R be an associative ring, and let $\mathcal{D}^{\mathrm{fp}}(R)$ be the ∞-category of bounded chain complexes of finitely generated free modules. We can regard R as an object of $\mathcal{D}^{\mathrm{fp}}(R)$. Let $A = \mathrm{Mor}(R, R)$. The homotopy groups of A are then given by the formula

$$\pi_i A = \mathrm{Ext}_R^{-i}(R, R) = \begin{cases} R & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

In other words, we can identify A with the discrete A_∞-ring corresponding to R. Following the above outline, we obtain a fully faithful embedding $\mathrm{LMod}_A^{\mathrm{fp}} \to \mathcal{D}^{\mathrm{fp}}(R)$. Since $\mathcal{D}^{\mathrm{fp}}(R)$ is generated by R (under taking fibers and cofibers), we see that this fully faithful embedding is an equivalence.

Our goal in the next few lectures is to obtain a *concrete* description of the quadratic L-theory for a ring R with involution (and, more generally, for an A_∞-ring with involution). The obstacle we have to overcome is this: by definition, elements of L_0 (R) are represented by arbitrary finite complexes of free R-modules $P_\bullet$, equipped with a quadratic form represented by a cycle q in $\mathrm{Hom}(P_\bullet \otimes P_\bullet, R)_{h\Sigma_2}$. We saw in the last lecture that there is a concrete description of what it means to give have a cycle, at least in the special case where R is a commutative ring with trivial involution and $P_\bullet$ is concentrated in a single degree. In this lecture, we describe a mechanism which can be used to show that an arbitrary Poincare object $(P_\bullet, q)$ is cobordant (and therefore represents the same L-theory class) to a Poincare object whose underlying chain complex is concentrated in a single degree. The cobordism itself will be constructed using the method of *surgery*.

Let us begin in a general setting. Let $\mathcal{C}$ be a stable ∞-category, $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ a nondegenerate quadratic functor with polarization B, and $\mathbb{D}$ the associated duality functor. Suppose we are given a fiber sequence

$$X' \xrightarrow{\alpha} X \to X/X'$$

in $\mathcal{C}$. Let $q \in \Omega^\infty Q(X)$, and suppose that we are given a nullhomotopy of $q|X' \in \Omega^\infty Q(X')$. We have seen that this is generally not enough information to allow us to descend q to a point of $\Omega^\infty Q(X/X')$, because q may have nontrivial image $b \in \Omega^\infty B(X', X)$. Note however that q does have trivial image in $\Omega^\infty B(X', X')$. In other words, the composite map

$$X' \xrightarrow{\alpha} X \to \mathbb{D}(X) \xrightarrow{\mathbb{D}(\alpha)} \mathbb{D}(X')$$

is canonically nullhomotopic. We therefore obtain a triangle

$$X' \xrightarrow{\alpha} X \xrightarrow{\beta} \mathbb{D}(X)$$

in the stable ∞-category $\mathcal{C}$. In general, this triangle is not a fiber sequence. Its failure to be a fiber sequence can be measured by taking *homology*: that is, by extracting the object

$$\mathrm{cofib}(X \to \mathrm{fib}(\beta)) \simeq \mathrm{fib}(\mathrm{cofib}(\alpha) \to \mathbb{D}(X))$$

of $\mathcal{C}$ (which vanishes if and only if the sequence above is a fiber sequence). Let us denote this homology object by X_α. This is abusive: it depends not only on α, but on a choice of nullhomotopy of $q|X'$.

Let us write $X_\alpha = \mathrm{fib}(\beta)/X$. We have seen that there is a fiber sequence

$$Q(X_\alpha) \to Q(\mathrm{fib}(\beta)) \to Q(X') \times_{B(X',X')} B(\mathrm{fib}(\beta), X').$$

The point q determines a point of $\Omega^\infty B(X, X')$, classifying the map $\beta : X \to \mathbb{D}X'$. By construction, this map is canonically nullhomotopy after composition with the map $\mathrm{fib}(\beta) \to X$. Consequently, the restriction $q|\mathrm{fib}(\beta)$ has trivial image in $\Omega^\infty(Q(X') \times_{B(X',X')} B(\mathrm{fib}(\beta), X'))$, and therefore lifts to a point $q_\alpha \in \Omega^\infty Q(X_\alpha)$. We may therefore view (X_α, q_α) as another quadratic object of $(\mathcal{C}, Q)$. We say that (X_α, q_α) is obtained from (X, q) via *surgery on α*.

The point q_α determines a map from X_α to its dual. This map can be described more explicitly as follows. Note that if we are given a triangle

$$Y' \to Y \to Y'',$$

in $\mathcal{C}$, then we can dualize to obtain a new triangle

$$\mathbb{D}(Y'') \to \mathbb{D}(Y) \to \mathbb{D}(Y')$$

in $\mathcal{C}$. The process of extracting homology is self-dual (rather, the two descriptions of homology given above are dual to one another). The map $X_\alpha \to \mathbb{D}X_\alpha$ is given by the map on homology induced by a map of triangles

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \mathbb{D}(X') \arrow[d] \\
\mathbb{D}^2(X') \arrow[r] & \mathbb{D}(X) \arrow[r] & \mathbb{D}(X').
\end{tikzcd}$$

Here the outer maps vertical maps are isomorphisms and the middle map is induced by q. Consequently, the cofiber of the map $X_\alpha \to \mathbb{D}(X_\alpha)$ is given by the homology of the triangle

$$0 \to \mathrm{cofib}(X \to \mathbb{D}(X)) \to 0,$$

which is the same as the cofiber of the map $X \to \mathbb{D}(X)$. In particular, one cofiber vanishes if and only if the other does. We have proven:

Proposition 2. *Let (X, q) be a Poincare object of $\mathcal{C}$. Suppose we are given a map $\alpha : X' \to X$ and a nullhomotopy of $q|X'$, and let (X_α, q_α) be obtained by surgery along α. Then (X_α, q_α) is also a Poincare object of $\mathcal{C}$.*

In fact, we can say more. By construction, q_α and q have the same restriction to $L = \mathrm{fib}(X \to \mathbb{D}(X'))$. The identification of these restrictions determines a map

$$X' \simeq \mathrm{fib}(L \to X_\alpha) \to \mathbb{D}\,\mathrm{cofib}(L \to X) = \mathbb{D}(\mathbb{D}(X')) \simeq X'.$$

Unwinding the definitions, one shows that this map is the identity up to a sign. Consequently, the Poincare object (X, q) and (X_α, q_α) are cobordant, and determine the same element of the abelian group $L_0(\mathcal{C}, Q)$.

Remark 3. With some additional effort, one can show that *all* cobordisms arise via this construction. That is, every Poincare object cobordant to (X, q) has the form (X_α, q_α), for some map $\alpha : X' \to X$ and some nullhomotopy of $q|X'$.

Surgery Below the Middle Dimension (Lecture 12)

February 17, 2011

In the previous lecture, we discussed the process of *surgery*. If $\mathcal{C}$ is a stable ∞-category equipped with a nondegenerate functor Q and we are given a quadratic object (X, q), a map $\alpha : X' \to X$, and a nullhomotopy of $q|X'$, then we can construct a new quadratic (X_α, q_α) by "surgery along α". Our goal in this lecture is to show how this construction can be used to simplify a quadratic object.

First, we need to review a bit of terminology. Let X be a spectrum and $n \in \mathbf{Z}$ an integer. We say that X is *n-connective* if the homotopy groups $\pi_i X$ vanish for $i < n$. We say that X is *connective* if X is 0-connective: that is, if the negative homotopy groups of X are trivial. The collection of connective spectra is stable under smash products and homotopy colimits.

Let R be an A_∞-ring. Suppose M and N are right and left R-module spectra, respectively. Then $M \wedge_R N$ is the given by the geometric realization of a simplicial spectrum, whose nth term is an iterated smash product

$$M \wedge R \wedge \cdots \wedge R \wedge N.$$

If M, R, and N are connective, then $M \wedge_R N$ is also connective.

For every perfect left module M over R, we let $\mathbb{D}(M)$ denote the mapping spectrum $\mathrm{Mor}_R(M, R)$. This is a perfect right module over R (which we can identify with a left R-module in the special case where R has an involution). We will say that M has *projective amplitude* $\leq n$ if $\mathbb{D}(M)$ is $(-n)$-connective. Let N be any other left R-module. We then have a canonical equivalence

$$\mathrm{Mor}_R(M, N) \simeq \mathbb{D}(M) \wedge_R N.$$

If M has projective amplitude $\leq n$ and both R and N are connective, then the above discussion shows that $\mathbb{D}(M) \wedge_R N$ is $(-n)$-connective. In particular, if M has projective amplitude ≤ 0, then $\mathbb{D}(M) \wedge_R N$ is connective.

Proposition 1. *Let R be a connective A_∞-ring. Let M be a perfect left R-module which is connective and of projective amplitude ≤ 0. Then M is a direct summand of R^n for some n. In this case, we will say that M is projective.*

Proof. Using the fact that M is perfect and that the negative homotopy groups $\pi_i M$ vanish, one can show that $\pi_0 M$ is a finitely generated R-module. We can therefore choose a map $R^n \to M$ which is surjective on π_0. We have a fiber sequence

$$M' \to R^n \to M,$$

where M' is also projective. This gives another fiber sequence of spectra

$$\mathrm{Mor}_R(M, M') \to \mathrm{Mor}_R(M, R^n) \to \mathrm{Mor}_R(M, M).$$

Since M' is connective and M is of projective amplitude ≤ 0, the mapping spectrum $\mathrm{Mor}_R(M, M')$ is connective. Using the long exact sequence of homotopy groups, we see that the map $\pi_0 \mathrm{Mor}_R(M, R^n) \to \pi_0 \mathrm{Mor}_R(M, M)$ is surjective. This means that there is a map $\phi : M \to R^n$ lifting the identity map $\mathrm{id}_M : M \to M$, so that M is a direct summand of R^n. $\square$

Variant 2. In the situation of Proposition 1, if M is k-connective and of projective amplitude $\leq k$, then the same argument shows that M is a summand of $\Sigma^k R^n$ for some n.

Let us now suppose that R is a connective A_∞-ring equipped with an involution σ, and let Q^{quad} be the quadratic functor on $\text{LMod}_R^{\text{fp}}$ given by the formula $Q^{\text{quad}}(M) = \text{Mor}_{R-R}(M \wedge M, R)_{h\Sigma_2}$. We would like to study the L-groups $L_n^{\text{quad}}(R)$. We begin by studying the case where n is even, so we can write $n = -2k$. Then $L_n^{\text{quad}}(R) = L_0(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$.

Suppose we are given a quadratic object (M, q) (so that q lies in the zeroth space of the spectrum $\Sigma^{2k}(\mathbb{D}(M) \wedge_R \mathbb{D}(M))_{h\Sigma_2}$). Note that if $M = \Sigma^m R$, then $\Sigma^{2k} Q^{\text{quad}}(M)$ can be identified with the homotopy coinvariants of Σ_2 acting on the spectrum $\Sigma^{2k-2m}(R)$ (through a mixture of the involution σ on R and a permutation of suspension coordinates). If $m < k$, then the homotopy groups of this spectrumvanish for $i \leq 0$. This remains true after passing to homotopy coinvariants: that is, the spectrum $\Sigma^{2k} Q^{\text{quad}}(M)$ is connected (even simply connected), so there exists a nullhomotopy of q.

Now let (M, q) be an arbitrary quadratic object of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$. Suppose we are given a homotopy class $\eta \in \pi_m M$. Then η determines a map of R-module spectra $\alpha : \Sigma^m R \to M$. If $m < k$, then the above calculation shows that $q | \Sigma^m R$ is automatically nullhomotopic. Choosing a nullhomotopy, we can perform surgery to obtain a new quadratic object (M_α, q_α). Let us study the homotopy groups of resulting object.

Consider the map $\beta : M \to \Sigma^{2k} \mathbb{D}(\Sigma^m R) = \Sigma^{2k-m} R$ determined by our choice of nullhomotopy of $q | \Sigma^m R$. We have an exact sequence

$$\pi_{i+1} \Sigma^{2k-m} R \to \pi_i \operatorname{fib}(\beta) \to \pi_i M \to \pi_i \Sigma^{2k-m} R.$$

Since R is connective, the homotopy group $\pi_i \Sigma^{2k-m} R = \pi_{i+m-2k} R$ vanishes if $i + m < 2k$. Similarly, $\pi_{i+1} \Sigma^{2k-m} R \simeq 0$ if $i + m + 1 < 2k$. If $m < k$, then we conclude that $\pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i \leq k$.

We have another exact sequence

$$\pi_i \Sigma^m R \to \pi_i \operatorname{fib}(\beta) \to \pi_i M_\alpha \to \pi_{i-1} \Sigma^m R.$$

Since R is connective, we conclude that $\pi_i M_\alpha \simeq \pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i < m$, and that $\pi_m M_\alpha$ is the quotient of $\pi_m \operatorname{fib}(\beta) \simeq \pi_m M$ by the submodule generated by η. We can summarize our discussion as follows:

Lemma 3. *Let (M, q) be a quadratic object of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$. Suppose that we are given a class $\eta \in \pi_m M$, where $m < k$. Then (M, q) is cobordant to another Poincare object (M', q'), where $\pi_i M' \simeq \pi_i M$ for $i < m$, and $\pi_m M'$ is the quotient of $\pi_m M$ by the submodule generated by η.*

Let (M, q) be as in Lemma 3, and assume that M is nonzero. Since R is connective, there exists a smallest integer m such that $\pi_m M$ is nonzero. Moreover, the finiteness of M guarantees that $\pi_m M$ is a finitely generated module over $\pi_0 R$. If $m < k$, we can apply Lemma 3 repeatedly to obtain another quadratic object (M', q'), with $\pi_i M' \simeq 0$ for $i \leq m$. Applying *this* argument repeatedly, we obtain the following result:

Proposition 4. *Let (M, q) be a Poincare object of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$. Then (M, q) is cobordant to a Poincare object (M', q') with $\pi_i M' \simeq 0$ for $i < k$.*

The Poincare object (M', q') of Proposition 7 is k-connective. Moreover, since it is Poincare, we have a canonical equivalence $M' \simeq \Sigma^{2k} \mathbb{D} M'$, so that $\mathbb{D} M'$ is $(-k)$-connective: that is, M' has projective amplitude $\leq k$. Applying Variant 2, we obtain the following:

Proposition 5. *Let R be a connective A_∞-ring with involution. Then every Poincare object (M, q) of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$ is cobordant to a Poincare object (M', q'), where M' is a direct summand of $\Sigma^k R^n$ for some n.*

In other words, every class in $L_{-2k}(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$ can be represented by a Poincare object (M, q) which is concentrated in the "middle dimension" k.

Remark 6. Exactly the same analysis applies if we replace $\mathrm{LMod}_R^{\mathrm{fp}}$ by $\mathrm{LMod}_R^{\mathrm{perf}}$. Note that it is important that we are working with quadratic L-theory rather than symmetric L-theory.

We now carry out the analogous discussion for L-groups in odd degrees. Let (M, q) be a quadratic object of $(\mathrm{LMod}_R^{\mathrm{fp}}, \Sigma^{2k+1} Q^{\mathrm{quad}})$. Note that if $M = \Sigma^m R$ for $m \leq k$, then $\Sigma^{2k+1} Q^{\mathrm{quad}}(M) \simeq (\Sigma^{2k+1-2m} R)_{h\Sigma_2}$ is connected for $k \leq m$. Consequently, if (M, q) is a general Poincare object we can always do surgery on any class $\eta \in \pi_m M$ for $m \leq k$, to produce a new quadratic object (M_α, q_α). Let us see what effect this surgery has on the homotopy groups of M.

Choose a class $\eta \in \pi_m M$, so that the quadratic form q on M determines a map

$$\beta : M \to \Sigma^{2k+1} \mathbb{D}(M) \to \Sigma^{2k+1} \mathbb{D}(\Sigma^m R) \simeq \Sigma^{2k+1-m} R.$$

We therefore have a long exact sequence of homotopy groups

$$\pi_{i+1} \Sigma^{2k+1-m} R \to \pi_i \operatorname{fib}(\beta) \to \pi_i M \to \pi_i \Sigma^{2k+1-m} R.$$

Consequently, $\pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i < 2k - m$. In particular, if $m < k$, then the homotopy groups of $\operatorname{fib}(\beta)$ agree with the homotopy groups of M below m. Let us assume that $m < k$. We have a fiber sequence

$$\Sigma^m R \to \operatorname{fib}(\beta) \to M_\alpha,$$

so $\pi_i M_\alpha \simeq \pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i < m$, while $\pi_m M_\alpha$ is obtained from $\pi_m \operatorname{fib}(\beta) \simeq \pi_m M$ by killing the class η.

Arguing as before, we obtain the following result:

Proposition 7. *Let (M, q) be a Poincare object of $(\mathrm{LMod}_R^{\mathrm{fp}}, \Sigma^{2k+1} Q^{\mathrm{quad}})$. Then (M, q) is cobordant to a Poincare object (M', q') with $\pi_i M' \simeq 0$ for $i < k$.*

Unfortunately, this conclusion is not quite as strong. Since M' is Poincare, it is isomorphic to $\Sigma^{2k+1} \mathbb{D}(M')$, so that $\mathbb{D}(M')$ is $(-k-1)$-connective. That is, we know that M' is k-connective and has projective amplitude $\leq k+1$. This is not enough to conclude that M' is the suspension of a projective R-module: rather, we can ensure that it is concentrated in two degrees (in a sense which we will take up in the next lecture).

Remark 8. If (M, q) is a quadratic object of $(\mathrm{LMod}_R^{\mathrm{fp}}, \Sigma^{2k+1}, Q^{\mathrm{quad}})$, we can always do surgery on a class $\eta \in \pi_k M$. The problem is that this surgery does not always simplify matters: it has the effect of killing the class η, but might introduce new elements of π_m.

L-Groups of Fields (Lecture 13)

February 23, 2011

Our goal in this section is to carry out some calculuations of L-groups in simple cases. We begin with the following observation:

Proposition 1. *Let R be an associative ring with involution. Then the L-groups of R (symmetric or quadratic) are 4-periodic. That is, there are canonical isomorphisms*

$$L^s_n(R) \simeq L^s_{n+4}(R) \qquad L^q_n(R) \simeq L^q_{n+4}(R)$$

Proof. Let $\mathcal{C} = \mathrm{LMod}^{\mathrm{fp}}_R$. Since $\mathcal{C}$ is a stable ∞-category, the suspension functor is an equivalence from $\mathcal{C}$ to itself. Let B be the symmetric bilinear functor given by $B(M,N) = \mathrm{Mor}_{R-R}(M \wedge N, R)$. Then $B(\Sigma M, \Sigma N) \simeq \Sigma^{-2} B(M,N)$. Here $\Sigma^{-2} B$ is also a symmetric bilinear functor, where the symmetric group Σ_2 acts on B and also on the desuspenion functor Σ^{-2} by permuting the suspension coordinates. Because the "swap" map on the sphere $S^2 = S^1 \wedge S^1$ reverses orientation, this second action is nontrivial: it acts by a sign. However, the square of this action is trivial. Consequently, we have a equivalence $B(\Sigma^2 M, \Sigma^2 N) \simeq \Sigma^{-4} B(M,N)$, compatible with the action of Σ_2 (where Σ_2 does not act on the desuspension functor Σ^{-4}). Consequently, the double suspension map $\mathcal{C} \to \mathcal{C}$ determines equivalences

$$(\mathcal{C}, Q^s) \simeq (\mathcal{C}, \Sigma^{-4} Q^s) \qquad (\mathcal{C}, Q^q) \simeq (\mathcal{C}, \Sigma^{-4} Q^q).$$

□

Remark 2. Suppose that $2 = 0$ in R. Then we can ignore signs. The proof of Proposition 1 then shows that the L-groups of R are 2-periodic.

Let us now restrict our attention to the case where R is a (commutative) field k, equipped with the trivial involution. Note that if the characteristic of k is different from 2, then there is no difference between symmetric and quadratic L-theory. We will confine our attention to quadratic L-theory in what follows.

Proposition 3. *Let k be a field. Then the odd-dimensional quadratic L-groups $L^q_{-2m-1}(k)$ are trivial.*

Proof. Let (V,q) be a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_k, \Sigma^{2m+1} Q^q)$. We wish to show that (V,q) is nullcobordant. In the last lecture, we saw that we can reduce to the case where V is k-connective. The nondegeneracy of q gives an isomorphism $V \simeq \Sigma^{2m+1} \mathbb{D}(V)$. Since we are working over a field, this has concrete consequences: for every integer i, $\pi_i V$ is the k-linear dual of $\pi_{2m+1-i}(V)$. In particular, the homotopy groups $\pi_i V$ vanish for $i \notin \{m, m+1\}$. Let $W = \pi_m V$ so that $W^\vee \simeq \pi_{m+1} V$. Let $W[m] \in \mathrm{LMod}^{\mathrm{fp}}_k$ denote the module given by W, placed in degree k. Since k is a field, W is free as a k-module. We may therefore construct a map

$$\alpha : W[m] \to V$$

which induces the identity map

$$W \simeq \pi_m W[m] \to \pi_m V \simeq W.$$

Note that $\Sigma^{2m+1} Q^q(W[k]) \simeq (W \otimes_k W)[1]_{h\Sigma_2}$ is connected, so $q|W[m]$ is automatically nullhomotopic. Any choice of nullhomotopy exhibits $W[m]$ as a Lagrangian in V. □

Proposition 4. *Let k be a field of characteristic different from 2. Then the L-groups $L^q_{-4m-2}(k)$ are trivial. (If k has characteristic 2, then $L^q_{-4m-2}(k) \simeq L^q_0(k)$ by Remark 2.)*

Proof. Let (M, q) be a Poincare object of $(\mathrm{LMod}_k^{\mathrm{fp}}, \Sigma^{4m+2}Q^q)$. The results of the last lecture show that we can assume that $M = V[2m+1]$ for some vector space V over k. Let $B(V, V)$ denote the k-vector space of symmetric bilinear forms on V (regarded as a spectrum concentrated in a single degree). Then $\Sigma^{4m+2}Q^q(M) = \Sigma^{4m+2}(\Sigma^{-4m-2}B(V,V))_{h\Sigma_2}$. Here we can ignore the distinction between invariants and coinvariants (since 2 is invertible in k). However, we cannot ignore the fact that Σ_2 acts nontrivially on the suspension coordinates. The upshot is that $\Sigma^{4m+2}Q^q(M)$ is the Eilenberg-MacLane spectrum corresponding to the vector space of *skew-symmetric* bilinear forms $b : V \times V \to k$. Since (M, q) is a Poincare object, the corresponding skew-symmetric form is nondegenerate. It follows from elementary linear algebra that the dimension of V must be even, and that V admits a subspace $L \subseteq V$ of such that $b|(L \times L)$ is trivial $\dim(V) = 2\dim(L)$. Then L is a Lagrangian in V, so that (M, q) is nullcobordant.

Here is a slight variant on the above argument: if $V \neq 0$, then by skew-symmetry the bilinear form b vanishes on the one-dimensional subspace generated by any nonzero element $v \in V$. We can therefore perform surgery to reduce the dimension of V. Repeat until $V \simeq 0$.) □

In view of Propositions 1, 3, and 4, the calculation of the (quadratic) L-groups of fields reduces to the problem of understanding the group $L^q_0(k)$. This is an interesting classical invariant.

Definition 5. Let k be a field. A *quadratic space* over k is a pair (V, q), where V is a finite-dimensional vector space over k and $q : V \to k$ is a quadratic form. That is, q satisfies

$$q(ax) = a^2 q(x) \qquad q(x+y) = q(x) + q(y) + b(x,y)$$

for some bilinear form $b : V \times V \to k$. We say that q is *nondegenerate* if b is nondegenerate.

Example 6. Let k be any field. There is a quadratic space $H = (k^2, q)$ over k, where q is given by the formula $q(a, b) = ab$. We refer to H as the *hyperbolic plane*.

There is an evident direct sum operation on quadratic spaces: given a pair of quadratic spaces (V, q) and (V', q'), we define $(V, q) \oplus (V', q')$ to be $(V \oplus V', q \oplus q')$, where $q \oplus q' : V \oplus V' \to k$ is given by the formula

$$(q \oplus q')(v, v') = q(v) + q'(v').$$

Remark 7. Let (V, q) be a nondegenerate quadratic space over a field k. Suppose we are given a nonzero element $x \in V$ such that $q(x) = 0$. Since the associated bilinear form b is nondegenerate, we can choose $y \in V$ with $b(x, y) = 1$. Note that $b(x, x) = q(2x) - q(x) - q(x) = 2q(x) = 0$. It follows that $q(y + ax) = q(y) + ab(y, x) + q(ax) = q(y) + a$. In particular, $q(y - q(y)x) = 0$. Replacing y by $y - q(y)x$, we can reduce to the case where $q(y) = 0$. Then if V_0 denotes the subspace of V generated by x and y, then we have an isomorphism $(V_0, q|V_0) \simeq H$. In particular, q is nondegenerate on V_0 and we therefore have a decomposition $(V, q) \simeq H \oplus (V_1, q|V_1)$, where V_1 is the orthogonal complement of V_0.

More generally, if we are given a subspace $W \subseteq V$ of dimension a such that $q|W = 0$, we can apply this argument repeatedly to obtain a decomposition $(V, q) \simeq H^{\oplus a} \oplus (V', q')$.

Definition 8. Let k be a field. We say that two nondegenerate quadratic spaces (V, q) and (V', q') are *stably equivalent* if $(V, q) \oplus H^{\oplus a}$ is isomorphic to $(V', q') \oplus H^{\oplus b}$ for some integers a and b. The collection of stable equivalences classes of nondegenerate quadratic spaces over k is called the *Witt group* of k. We will denote it by $W(k)$ (not to be confused with the *ring of Witt vectors over* k).

The set $W(k)$ evidently has the structure of a commutative monoid under direct sum. In fact, this monoid structure is a group: for any nondegenerate quadratic space (V, q) where V has dimension d, the sum $(V, q) \oplus (V, -q)$ has an isotopic subspace of dimension d (the image of V under the diagonal map $V \to V \oplus V$) and is therefore isomorphic to $H^{\oplus d}$ by Remark 7.

Remark 9. Let (V, q) be any nondegenerate quadratic space over k. Using Remark 7 repeatedly, we deduce that (V, q) is isomorphic to a direct sum $(V', q') \oplus H^{\oplus d}$ for some integer d, where (V', q') is *anisotropic*: that is, q' does not vanish on any nonzero element of V'. Consequently, every class in the Witt group $W(k)$ can be represented by an anisotropic quadratic space (V, q). In fact, this representative is *unique* up to isomorphism. This is a consequence of the Witt cancellation theorem, which asserts that if we have an isomorphism of nondegenerate quadratic spaces

$$(V, q) \oplus (V'', q'') \simeq (V', q') \oplus (V'', q''),$$

then (V, q) and (V', q') must already be isomorphic.

Let (V, q) be a nondegenerate quadratic space over a field k. Viewing V as a chain complex over k concentrated in degree zero, we can think of (V, q) as a Poincare object of $(\mathrm{LMod}_k^{\mathrm{fp}}, Q^q)$. This construction determines a map $W(k) \to L_0^q(k)$.

Proposition 10. *Let k be a field. Then the map $\phi : W(k) \to L_0^q(k)$ is an isomorphism of abelian groups.*

Proof. We have already seen that ϕ is surjective (using surgery below the middle dimension). Let us show that ϕ is injective. Let (V, q) be a quadratic space over k, and suppose that there exists a Lagrangian in V (as a Poincare object of $(\mathrm{LMod}_k^{\mathrm{fp}}, Q^q)$. Denoting this Lagrangian by L, we have a fiber sequence of spectra

$$L \xrightarrow{\alpha} V \to \mathrm{cofib}(\alpha)$$

which is self-dual (with the duality on V determined by q). In particular, we have a self-dual short exact sequence of vector spaces

$$0 \to (\mathrm{Im}\, \pi_0 L \to V) \to V \to (\mathrm{Im}\, V \to \pi_0 \,\mathrm{cofib}(\alpha)) \to 0.$$

The self-duality implies that the dimensions of the outer two vector spaces are the same, so that the dimension of V is twice as large as the dimension of $W = \mathrm{Im}(\pi_0 L \to V)$. The map $W \to V$ factors through L, so $q|W = 0$. Using Remark 7, we deduce that V is isomorphic to a direct sum of hyperbolic planes so that (V, q) is equivalent to zero in the Witt group $W(k)$. □

Example 11. Let $k = \mathbf{F}_2$ be the finite field with two elements. Let (V, q) be a nondegenerate quadratic space over k. Then the dimension of V must be even (since the symmetric bilinear form b is also a nondegenerate skew-symmetric bilinear form). Suppose that V is anisotropic: then $q(v) = 1$ for every nonzero element $v \in V$. It follows that if $v, w \in W$ are distinct and nonzero, then $b(v, w) = q(v + w) - q(v) - q(w) = 1$. If $u, v, w \in W$ are linearly independent, we get

$$1 = b(u, v + w) = b(u, v) + b(u, w) = 0.$$

Thus any nontrivial anisotropic quadratic space must be of dimension 2. There is such a space (V, q): take $V = \mathbf{F}_2 \oplus \mathbf{F}_2$, and q to be given by the formula

$$q(a, b) = a^2 + ab + b^2.$$

It follows from the Witt cancellation theorem that (V, q) determines a nontrivial element of $W(k)$ (this can also be deduced by evaluating some of the invariants introduced below). We therefore have an isomorphism $W(k) \simeq \mathbf{Z}/2\mathbf{Z}$.

To any nondegenerate quadratic space (V, q) over $k = \mathbf{F}_2$, we can associate an invariant in the group $W(k = \mathbf{Z}/2\mathbf{Z}$. This is called the *Arf invariant* of (V, q). It can be described concretely as follows: the Arf invariant of q is 0 if q takes the value 0 more often than 1 (that is, if the set $q^{-1}\{0\} \subseteq V$ is larger than the set $q^{-1}\{1\} \subseteq V$), and takes the value 1 otherwise. A more conceptual description of this invariant is given below.

Example 12. Let k be an algebraically closed field. Any two nondegenerate quadratic spaces (V, q) over k of the same dimension are isomorphic. It follows that $W(k)$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ if the characteristic of k is different from 2, and is trivial if the characteristic of k is equal to 2 (since any nondegenerate quadratic space must be even dimension in the latter case).

Example 13. Let k be the field of real numbers (or any real-closed field). Then Sylvester's invariance of signature theorem gives an isomorphism of abelian groups $W(k) \simeq \mathbf{Z}$, which carries a nondegenerate quadratic space (V, q) to the signature $\sigma(q) \in \mathbf{Z}$.

Remark 14. Let k be a field of characteristic $\neq 2$. For every nonzero element $a \in k$, we have a nondegenerate quadratic form $q : k \to k$ given by $x \mapsto ax^2$. We denote the image of this element in $W(k)$ by $\langle a \rangle$. These elements generate $W(k)$ under addition, because any nondegenerate quadratic space (V, q) has an orthogonal basis. It is possible to explicitly write down a set of relations between these generators, and thereby obtain a presentation for $W(k)$.

We now describe some invariants that can help get a handle on the structure of a Witt ring $W(k)$. We first note that since the hyperbolic plane H has dimension 2, every element of $W(k)$ has a well-defined dimension modulo 2. This yields a group homomorphism

$$d : W(k) \to \mathbf{Z}/2\mathbf{Z}.$$

The map d is surjective when k has characteristic different from 2, and is the zero map when k has characteristic 2. Let I denote the kernel of d.

Suppose we are given an element of $I \subseteq W(k)$. We can represent this element by a nondegenerate quadratic space (V, q) of even dimension over k. We define the *Clifford algebra* $\mathrm{Cl}(V, q)$ to be the quotient of the free associative k-algebra on V by the relations

$$x^2 = q(x)$$

for $x \in V$. This Clifford algebra has a canonical $\mathbf{Z}/2\mathbf{Z}$-grading

$$\mathrm{Cl}(V, q) \simeq \mathrm{Cl}_0(V, q) \oplus \mathrm{Cl}_1(V, q),$$

where we take the elements of V to have degree 1. If $V \neq 0$, then one can show that the center of $\mathrm{Cl}_0(V, q)$ is a rank 2 étale extension of k: that is, it is either isomorphic to $k \times k$ or to a separable quadratic extension field k' of k. This extension of k determines a map $\mathrm{Gal}(\overline{k}/k) \to \mathbf{Z}/2\mathbf{Z}$, (which is the zero map if and only if the center is isomorphic to $k \times k$). The formation of this invariant determines a group homomorphism

$$\psi : I \to \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k); \mathbf{Z}/2\mathbf{Z}).$$

The image of a quadratic space (V, q) under this map is called the *discriminant* of (V, q). When the characteristic of k is different from 2, we can identify the discriminant with an element in $k^\times/(k^\times)^2$ (using Kummer theory). When k has characteristic 2, we can identify the discriminant with an element of the cokernel of the map

$$k \xrightarrow{x \mapsto x - x^2} k$$

(using Artin-Schreier theory). When $k = \mathbf{F}_2$, we recover the Arf invariant discussed above.

Let $J \subseteq W(k)$ denote the kernel of the map ψ defined above. Elements of J can be represented by quadratic spaces (V, q) of even dimension such that the center of $\mathrm{Cl}_0(V, q)$ splits as a product $k \times k$. It follows that $\mathrm{Cl}_0(V, q)$ itself splits as a product of two factors. One can show that each of these factors is a central simple algebra over k, and determines a 2-torsion element in the Brauer group of k. Let us assume that k has characteristic different from 2, so we can identify $\mathbf{Z}/2\mathbf{Z}$ with the subset $\{1, -1\} \subseteq k^\times$. Extracting these Brauer invariants gives a homomorphism

$$J \to \mathrm{H}^2(\mathrm{Gal}(\overline{k}/k); \mathbf{Z}/2\mathbf{Z}).$$

This pattern continues. If the characteristic of k is different from 2, then the Witt group $W(k)$ actually has the structure of a *ring* (given symmetric bilinear forms on vector spaces V and W, we obtain a symmetric bilinear form on $V \otimes W$). The map $d : W(k) \to \mathbf{Z}/2\mathbf{Z}$ is a ring homomorphism, so that $I \subseteq W(k)$ is an ideal. The following is a deep result of Voevodsky:

Theorem 15 (The Milnor Conjecture)**.** *If k is a field of characteristic different from 2, there are canonical isomorphisms*

$$I^m/I^{m+1} \simeq \mathrm{H}^m(\mathrm{Gal}(\overline{k}/k); \mathbf{Z}/2\mathbf{Z})$$

The Algebraic π-π Theorem (Lecture 14)

February 25, 2011

In the last few lectures, we did some concrete calculations of the quadratic L-groups of discrete rings.

Our goal in this lecture is to prove the following result:

Theorem 1. *Let $f : R \to R'$ be a map of connective A_∞-rings with involution, and suppose that f induces an isomorphism of associative rings $\pi_0 R \to \pi_0 R'$. Then, for every integer n, f induces an isomorphism $L^q_n(R) \to L^q_n(R')$.*

Remark 2. Let $f : R \to R'$ be a map of A_∞-rings with involution, and let $M \in \mathrm{LMod}^{\mathrm{fp}}_R$. Let B and B' denote the bilinear functors on $\mathrm{LMod}^{\mathrm{fp}}_R$ and $\mathrm{LMod}^{\mathrm{fp}}_{R'}$ determined by the involutions, so that B and B' are given by the formulas

$$B(M, N) = \mathrm{Mor}_{R-R}(M \wedge N, R) \qquad B'(M', N') = \mathrm{Mor}_{R'-R'}(M' \wedge N', R').$$

For every pair of objects $M, N \in \mathrm{LMod}^{\mathrm{fp}}_R$, we have a canonical map

$$B(M, N) = \mathrm{Mor}_{R-R}(M \wedge N, R) \to \mathrm{Mor}_{R'-R'}(R' \wedge_R M \wedge N \wedge_R R', R') \simeq B'(R' \wedge_R M, R' \wedge_R N),$$

which is compatible with the Σ_2-action. Consequently, every quadratic object (M, q) of $(\mathrm{LMod}^{\mathrm{fp}}_R, \Sigma^{-n} Q^q)$ determines a quadratic object $(R' \wedge_R M, q')$ of $(\mathrm{LMod}^{\mathrm{fp}}_{R'}, \Sigma^{-n} Q^q)$. This construction determines the map $L^q_n(R) \to L^q_n(R')$ which is the subject of Theorem 1.

Theorem 1 is a version of Ranicki's *algebraic π-π theorem*, which is named for a geometric result of Wall which says something to the following effect: if $(X, \partial X)$ is a Poincare pair such that ∂X is a manifold, both X and ∂X are connected, and the inclusion $\pi_1 \partial X \to \pi_1 X$ is an isomorphism, then there there are no "surgery obstructions" to finding a homotopy equivalence $(X, \partial X) \simeq (M, \partial X)$, where M is a manifold with boundary ∂X: the only thing you need is a suitable candidate for the tangent bundle of X. We will later show that these obstructions lie in the homotopy fiber of a map between the L-theory spectra of certain A_∞-rings, whose ring of connected components are given by the group algebras of $\pi_1 X$ and $\pi_1 \partial X$, respectively.

In the formulation of Theorem 1, we may regard the associative ring $\pi_0 R \simeq \pi_0 R'$ as a *discrete* A_∞-ring with involution (here we engage in our standard abuse of notation, identifying a ring with the corresponding Eilenberg-MacLane spectrum). We have a commutative diagram of A_∞-rings with involution

$$\begin{tikzcd}
R \arrow[rr] \arrow[dr] & & R' \arrow[dl] \\
& \pi_0 R' &
\end{tikzcd}$$

and therefore a commutative diagram of abelian groups

$$\begin{tikzcd}
L^q_n(R) \arrow[rr] \arrow[dr] & & L^q_n(R') \arrow[dl] \\
& L^q_n(\pi_0 R'). &
\end{tikzcd}$$

It will therefore suffice to show that the vertical maps are isomorphisms. Consequently, in Theorem 1, we may assume without loss of generality that the ring R' is discrete. We will use this assumption in the following form: the map $f : R \to R'$ induces a surjection $\pi_1 R \to \pi_1 R'$, so that the fiber $\mathrm{fib}(f)$ is a connected spectrum.

We now turn to the proof of Theorem 1. Our first step is to show that the map $L_n(R) \to L_n(R')$ is *surjective*. In other words, we claim that if (M', q') is a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{R'}, \Sigma^{-n} Q^q)$, then (M', q') is cobordant to a Poincare object which can be lifted to a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{R}, \Sigma^{-n} Q^q)$. In fact, we only need to construct a lifting as a quadratic object, by virtue of the following:

Proposition 3. *Let (M, q) be a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{R}, \Sigma^{-n} Q^q)$, and let (M', q') be the induced quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{R'}, \Sigma^{-n} Q^q)$ (so that $M' = R' \wedge_R M$). Then (M, q) is a Poincare object if and only if (M', q') is a Poincare object.*

Proof. The "only if" direction is obvious (and requires no assumptions on the map $f : R \to R'$. For the converse, note that q determines a map $u : M \to \Sigma^{-n}\mathbb{D}(M)$ and q' determines a map $u' : M' \to \Sigma^{-n}\mathbb{D}(M')$, given by applying the functor $\bullet \mapsto R' \wedge_R \bullet$ to u. Suppose (M, q) is not a Poincare object, so that $\mathrm{fib}(u)$ is nonzero. Since $\mathrm{fib}(u)$ is finitely presented as an R-module (and R is a connective ring), there exists a smallest integer k such that $\pi_k \mathrm{fib}(u) \neq 0$. Since R' is connective, we obtain an isomorphism

$$\pi_k(\mathrm{fib}(u')) \simeq \pi_k(R' \wedge_R \mathrm{fib}(u)) \simeq (\pi_0 R') \otimes_{\pi_0 R} (\pi_k \mathrm{fib}(u)).$$

Since $\pi_0 R \simeq \pi_0 R'$, we conclude that $\pi_k \mathrm{fib}(u') \neq 0$, so that (M', q') is not a Poincare object. □

Let us now return to the proof that $L^q_n(R) \to L^q_n(R')$ is surjective. We first treat the case where $n = -2k$ is even. Let (M', q') be an arbitrary Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{R'}, \Sigma^{2k} Q^q)$. We have seen (via surgery below the middle dimension) that we can assume (replacing (M', q') by a cobordant Poincare object if necessary) that $M' = \Sigma^k P$, where P is a projective R'-module: that is, a direct summand of R'^a for some $a \geq 0$. The assumption that $M' \in \mathrm{LMod}^{\mathrm{fp}}_{R}$ gives a little more: it implies that P is *stably free*, in the sense that $P \oplus R'^{a'} \simeq R'^{a''}$ for some integers $a', a'' \geq 0$. Performing surgery on M' along the zero map $\Sigma^{k-1} R'^{a'} \to M'$ has the effect of replacing M' by $M' \oplus \Sigma^k R'^{2a'}$. We may therefore suppose that P is free: that is, $M' \simeq \Sigma^k R'^a$ for some integer $a \geq 0$. Then we can obviously lift M' to an R-module M, given by $M = \Sigma^k R^a$. We can identify the map $\Sigma^{2k} Q^q(M) \to \Sigma^{2k} Q^q(M')$ with a map of homotopy coinvariant spectra

$$\theta : (R^{a^2})_{h\Sigma_2} \to (R'^{a^2})_{h\Sigma_2}.$$

The condition that $\pi_0 R \to \pi_0 R'$ is an isomorphism implies that θ is an isomorphism on π_0, so we can lift (M', q') to a quadratic object (M, q). This quadratic object is automatically Poincare, by Proposition 3.

Now suppose that $n = -2k - 1$ is odd. We will need to work a little bit harder. Let (M', q') be a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{R'}, \Sigma^{2k+1} Q^q)$. Using surgery below the middle dimension, we may assume that M' is k-connective (and therefore, by duality, of projective amplitude $\leq k+1$). It will therefore suffice to prove the following:

Lemma 4. *Let (M', q') be a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{R'}, \Sigma^{2k+1} Q^q)$ which is k-connective and of projective amplitude $\leq k+1$. Then (M', q') can be lifted to a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{R}, \Sigma^{2k+1} Q^q)$ (automatically Poincare if (M, q) is Poincare, by Proposition 3.*

To prove Lemma **??**, we note that $\pi_k M'$ is a finitely generated module over $\pi_0 R'$, so we can choose a map $u : \Sigma^k R'^a \to M'$ which is surjective on π_0, which fits into a fiber sequence

$$K \to \Sigma^k R'^a \to M'.$$

Since u is surjective on π_0, we deduce that K is k-connective. We have a fiber sequence

$$\mathbb{D}(M') \to \Sigma^{-k} R'^a \to \mathbb{D}(K).$$

Since $\mathbb{D}(M')$ is $(-k-1)$-connective, the associated long exact sequence shows that $\mathbb{D}(K)$ is $(-k)$-connective: that is, K has projective amplitude $\leq k$. It follows that $K \simeq \Sigma^k P$ for some projective R-module P, and we can write M' as the cofiber of a map

$$\phi : \Sigma^k P \to \Sigma^k R'^a.$$

Since $P \in \mathrm{LMod}^{\mathrm{fp}}_{R'}$, it is stably free. We may therefore replace ϕ by its direct sum with finitely many copies of the identity map $\mathrm{id} : \Sigma^k R' \to \Sigma^k R'$, we may reduce to the case where P is free, so that M' is the cofiber of a map

$$\phi : \Sigma^k R'^b \to \Sigma^k R'^a.$$

The set of homotopy classes of such maps can be identified with the set of matrices $M_{a,b}(\pi_0 R')$ with coefficients in $\pi_0 R'$. Since the map $\pi_0 R \to \pi_0 R'$ is an isomorphism, the map ϕ admits a lifting (which is unique up to homotopy) to a map $\phi_0 : \Sigma^k R^b \to \Sigma^k R^a$. Let $M = \mathrm{cofib}(\phi_0)$, so that $M' \simeq R' \wedge_R M$.

To complete the proof of surjectivity, we need to show that we can find a point $q \in \Omega^\infty(\Sigma^{2k+1} Q^q(M))$ lifting q' (then (M, q) will automatically be a Poincare object, by Proposition 3). For this, it suffices to show that the canonical map

$$\pi_{-2k-1} Q^q(M) \to \pi_{-2k-1} Q^q(M')$$

is surjective. Let F denote the homotopy fiber of the map of spectra $Q^q(M) \to Q^q(M')$. By the long exact seuqence in homotopy, it will suffice to show that $\pi_{-2k-2} F$ is trivial. In fact, we claim that F is $(-2k-1)$-connective. To prove this, we introduce the following notation. For every pair $N, N' \in \mathrm{LMod}^{\mathrm{fp}}_R$, let $U(N, N')$ denote the fiber of the map of spectra $B(N, N') \to B'(R' \wedge_R N, R' \wedge_R N')$. Then F is given by the homotopy coinvariants for the action of Σ_2 on $U(M, M)$. It will therefore suffice to show that $U(M, M)$ is $(-2k-1)$-connective. We have a fiber sequence

$$\Sigma^k R^b \to \Sigma^k R^a \to M,$$

hence a fiber sequence

$$U(M, M) \to U(M, \Sigma^k R)^a \to U(M, \Sigma^k R)^b.$$

It will therefore suffice to show that $U(M, \Sigma^k R)$ is $(-2k)$-connective. Since we also have a fiber sequence

$$U(M, \Sigma^k R) \to U(\Sigma^k R, \Sigma^k R)^a \to U(\Sigma^k R, \Sigma^k R)^b,$$

it will suffice to show that $U(\Sigma^k R, \Sigma^k R)$ is $(-2k+1)$-connective. Equivalently, we must show that $U(R, R)$ is 1-connective. Unwinding the definitions, this is equivalent to the requirement that the map $f : R \to R'$ is surjective on π_1 and bijective on π_0 (which we arranged above). This completes the proof that f induces a surjection $L^q_n(R) \to L^q_n(R')$.

The proof of injectivity will require a bit more effort. Suppose that we are given a class in $L^q_n(R)$ whose image in $L^q_n(R')$ vanishes. We can represent this class by a Poincare object (M, q) of $(\mathrm{LMod}^{\mathrm{fp}}_R, \Sigma^{-n} Q^q)$; we wish to show that (M, q) is nullcobordant. Write $n = -2k$ or $n = -2k-1$. Using surgery, we may assume that M is k-connective.

Let (M', q') denote the quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{R'}, \Sigma^{-n} Q^q)$ determined by (M, q). Then (M', q') is nullcobordant: we can therefore choose a Lagrangian $\alpha : L \to M'$. Let N' be the fiber of α. We have two nullhomotopies of $q'|N'$: one from our nullhomotopy of $q'|L$, and one from a nullhomotopy of the composite map $N' \to L \to M'$. These two nullhomotopies determine a point of $p' \in \Omega^\infty \Sigma^{-n-1} Q^q(N')$, so that (N', p') is a quadratic object of $\Omega^\infty \Sigma^{-n-1} Q^q(N')$. One can show that (M', q') and its Lagrangian can be recovered from (N', p'), using the construction of Lecture 9. It will therefore suffice to show that (N', p') can be lifted to a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_R, \Sigma^{-n-1} Q^q)$: such a quadratic object will then determine a Lagrangian in (M, q).

There are two cases to consider. We first treat the case where $n = -2k$ is even. Using surgery below the middle dimension, we can assume that N' is $k-1$-connective (note that surgery changes the Lagrangian L but not the underlying Poincare object M'). We have a fiber sequence

$$N' \to L \to M',$$

so that L is also $(k-1)$-connective. Since $L \simeq \Sigma^{2k-1}\mathbb{D}(N)$, the assumption that L is $(k-1)$-connective means that N' has projective amplitude $\leq k$. The desired result now follows from Lemma 4.

Now suppose that $n = -2k-1$ is odd. Then surgery below the middle dimension can be used to ensure that N' is k-connective, so that L is also k-connective. Since $L \simeq \Sigma^{2k}\mathbb{D}(N')$, we deduce that N' has projective amplitude $\leq k$ and is therefore of the form $\Sigma^k P$, for some stably free R'-module P. Performing surgery on (N', p') along a zero map $\Sigma^{k-1}R^a \to N'$, we can reduce to the case where P is free. It follows that we can lift N' to a (free) R-module N, and the proof given of surjectivity in odd degrees shows that we can lift p' to a point $p \in \Omega^\infty \Sigma^{2k} Q^q(N)$.

Odd L-Theory of the Integers (Lecture 15)

February 28, 2011

Our goal now is to compute the quadratic L-groups $L_n^q(\mathbf{Z})$. We have seen that the answers depend only on the congruence class of n modulo 4. We therefore have four calculations to perform. We will compute the odd L-groups of $\mathbf{Z}$ in this lecture, and the even L-groups of $\mathbf{Z}$ in the next lecture.

We begin with the easy part:

Proposition 1. *The quadratic groups $L_n^q(\mathbf{Z})$ vanish when n is congruent to 1 modulo 4.*

Proof. Write $n = -4k + 1$. Let (M, q) be a quadratic object of $(\mathrm{LMod}_{\mathbf{Z}}^{\mathrm{fp}}, \Sigma^{4k-1} Q^q)$. For every integer i, the homotopy group $\pi_i M$ is a finitely generated abelian group. We may therefore write $\pi_i M$ (noncanonically) as a direct sum $F_i \oplus T_i$, where F_i is a finitely generated free abelian group and T_i is torsion.

Let us identify abelian groups with the corresponding Eilenberg-MacLane spectra. Using the fact that the ring $\mathbf{Z}$ has projective dimension 1, we see that M is isomorphic to the direct sum

$$\bigoplus \Sigma^i F_i \oplus \Sigma^i T_i.$$

Moreover, this sum is finite (since M is perfect). The $\mathbf{Z}$-linear dual of M is given by

$$\bigoplus (\Sigma^{-i} \operatorname{Hom}(F_i, \mathbf{Z}) \oplus \Sigma^{-i-1} \operatorname{Ext}(T_i, \mathbf{Z})).$$

Since (M, q) is Poincare, we have an isomorphism $M \simeq \Sigma^{4k-1} \mathbb{D} M$. Passing to homotopy groups (and using the structure theory of finitely generated abelian groups) we obtain isomorphisms

$$F_i \simeq \operatorname{Hom}(F_{4k-1-i}, \mathbf{Z}) \qquad T_i \simeq \operatorname{Ext}(T_{4k-1-i}, \mathbf{Z}).$$

Replacing (M, q) by a cobordant Poincare object if necessary, we can assume that M is $(2k-1)$-connective: that is, the abelian groups F_i and T_i vanish for $i < 2k - 1$. From duality, we deduce that $F_i \simeq 0$ for $i > 2k$ and that $T_i \simeq 0$ for $i \geq 2k$. Let $F = F_{2k-1}$ and $T = T_{2k-1}$, so that $\pi_{2k} M \simeq \operatorname{Hom}(F_k, \mathbf{Z})$ and $T \simeq \operatorname{Ext}(T, \mathbf{Z})$.

Let us now try to kill the torsion group T. Choose a nonzero element $\eta \in T$, classifying a map $\alpha : \Sigma^{2k-1} \mathbf{Z} \to M$. Recall that $\Sigma^{4k-1} Q^q(\Sigma^k \mathbf{Z})$ is a connected spectrum, so that $q | (\Sigma^k \mathbf{Z})$ automatically vanishes and we can therefore do surgery along α.

We have a fiber sequence of spectra

$$\Sigma^{2k-1} \mathbf{Z} \xrightarrow{\alpha} M \to \operatorname{cofib}(\alpha)$$

which determines a long exact sequence of abelian groups

$$0 \to \pi_{2k} M \to \pi_{2k} \operatorname{cofib}(\alpha) \to \mathbf{Z} \xrightarrow{\eta} \pi_{2k-1} M \to \pi_{2k-1} \operatorname{cofib}(\alpha) \to 0$$

It follows that $\pi_{2k-1} \operatorname{cofib}(\alpha)$ is the direct sum $F \oplus T/\eta$, and that $\pi_{2k} \operatorname{cofib}(\alpha)$ is an extension of a finite index subgroup of $m\mathbf{Z} \subseteq \mathbf{Z}$ (where m is the order of η) by $\pi_{2m} M$. Such an extension is automatically split, so that $\pi_{2k} \operatorname{cofib}(\alpha) \simeq \operatorname{Hom}(F, \mathbf{Z}) \oplus m\mathbf{Z}$.

Let M_α denote the result of surgery along α, so that we have a fiber sequence

$$M_\alpha \to \mathrm{cofib}(\alpha) \to \Sigma^{2k}\mathbf{Z}$$

and therefore a long exact sequence of homotopy groups

$$0 \to \pi_{2m} M_\alpha \to \pi_{2m} \mathrm{cofib}(\alpha) \xrightarrow{\phi} \mathbf{Z} \to \pi_{2m-1} M_\alpha \to \pi_{2m-1} \mathrm{cofib}(\alpha) \to 0$$

Note that the restriction of ϕ to $\pi_{2m} M = \mathrm{Hom}(F, \mathbf{Z})$ is dual to the map $\mathbf{Z} \to F$ determined by η. Since η was chosen to be a torsion element, this map is zero. Consequently, ϕ factors as a composition

$$\pi_{2k} \mathrm{cofib}(\alpha) \to m\mathbf{Z} \xrightarrow{\phi_0} \mathbf{Z}.$$

We will need the following:

($*$) The map ϕ_0 vanishes.

Assuming ($*$) for the moment, we deduce that $\pi_{2k-1} M_\alpha$ is an extension of $\pi_{2k-1} \mathrm{cofib}(\alpha)$ by the group $\mathbf{Z}$. In particular, the torsion subgroup of $\pi_{2k-1} M_\alpha$ injects into the torsion subgroup T/η of $\mathrm{cofib}(\alpha)$. We conclude that $\pi_{2k-1} M_\alpha$ has a smaller torsion subgroup than $\pi_{2k-1} M$ (though it has a larger free part).

Repeating this procedure, we can reduce to the case where $T = 0$. Then $\pi_{2k-1} M$ is a free abelian group F, and $\pi_{2k} M \simeq \mathrm{Hom}(F, \mathbf{Z})$ is the dual abelian group. Since F is free, we can choose a map $\Sigma^k F \to M$ which induces the identity map

$$F \simeq \pi_{2k-1} \Sigma^{2k-1} F \to \pi_{2k-1} M \simeq F.$$

Note that $\Sigma^{4k-1} Q^q(\Sigma^{2k-1} F)$ is a connected spectrum, so that $q|\Sigma^k F$ is automatically nullhomotopic. Any choice of nullhomotopy exhibits F as a Lagrangian in (M, q), so that (M, q) is nullcobordant.

It remains to prove ($*$). For this, we can tensor everything with $\mathbf{Q}$. In this case, the map $\alpha_{\mathbf{Q}} : \Sigma^{2k-1}\mathbf{Q} \to M_{\mathbf{Q}}$ is the zero map, so that the surgery data is given by a choice of nullhomotopy of $q|(\Sigma^{2k-1}) = 0$: that is, by an element $p \in \pi_0 \Sigma^{4k-2} Q^q(\Sigma^{2k-1}\mathbf{Q})$. We can identify ϕ_0 with the underlying bilinear form on $\mathbf{Q}$. Since $2k-1$ is odd, this bilinear form is skew-symmetric, and therefore vanishes. □

Remark 2. The above argument works if we replace $\mathbf{Z}$ by any Dedekind ring: for example, the ring of integers in a number field.

The next case is more difficult.

Proposition 3. *The quadratic groups $L^q_n(\mathbf{Z})$ vanish when n is congruent to 3 modulo 4.*

Proof. Without loss of generality, $n = -1$: that is, we are studying the L-groups of the pair $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma Q^q)$. Let (M, q) be a Poincare object. Arguing as in the proof of Proposition 1, we can use surgery below the middle dimension to assume that

$$\pi_* M \simeq \begin{cases} F \oplus T & \text{if } * = 0 \\ \mathrm{Hom}(F, \mathbf{Z}) & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

where F is a free abelian group and T is a finite abelian group. These isomorphisms determine a map $\alpha : F \to M$, which is well-defined up to homotopy. Performing surgery along α, we can reduce to the case where $F = 0$. Then $M \simeq T$ can be identified with a finite abelian group, concentrated in degree zero.

Let A be any finite abelian group. Let us spell out what it means to endow A with the structure of a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma Q^q)$. Choose a free abelian group Λ and a surjection $\Lambda \to A$, with kernel Λ_0. We then have a fiber sequence

$$Q^q(A) \to Q^q(\Lambda) \to Q^q(\Lambda_0) \times_{B(\Lambda_0, \Lambda_0)} B(\Lambda_0, \Lambda).$$

The spectra $B(\Lambda_0, \Lambda)$ and $B(\Lambda_0, \Lambda_0)$. Since $\pi_{-1} Q^q(\Lambda) \simeq 0$, we deduce that $\pi_{-1} Q^q(A)$ can be identified with the cokernel of the map of abelian groups

$$\pi_0 Q^q(\Lambda) \to \pi_0(Q^q(\Lambda_0) \times_{B(\Lambda_0,\Lambda_0)} B(\Lambda, \Lambda_0)).$$

We have a fiber sequence of spectra

$$Q^q(\Lambda_0) \times_{B(\Lambda_0,\Lambda_0)} B(\Lambda, \Lambda_0) \to Q^q(\Lambda_0) \times B(\Lambda, \Lambda_0) \to B(\Lambda_0, \Lambda_0)$$

Since $\pi_1 B(\Lambda_0, \Lambda_0) \simeq 0$, we can identify $\pi_0(Q^q(\Lambda_0) \times_{B(\Lambda_0,\Lambda_0)} B(\Lambda, \Lambda_0))$ with the fiber product of abelian groups

$$\pi_0 Q^q(\Lambda_0) \times_{\pi_0 B(\Lambda_0,\Lambda_0)} \pi_0 B(\Lambda, \Lambda_0).$$

Recall that $\pi_0 Q^q(\Lambda_0)$ can be identified with the abelian group of quadratic forms $f : \Lambda_0 \to \mathbf{Z}$. Every such quadratic form determines symmetric bilinear form $b : \Lambda_0 \times \Lambda_0 \to \mathbf{Z}$, hence a map $b_{\mathbf{Q}} : \Lambda \times \Lambda \to \mathbf{Q}$. We can identify $\pi_0 Q^q(\Lambda_0) \times_{\pi_0 B(\Lambda_0,\Lambda_0)} \pi_0 B(\Lambda, \Lambda_0)$ with the subgroup of $\pi_0 Q^q(\Lambda_0)$ consisting of those quadratic forms f such that b is integral on $\Lambda \times \Lambda_0$. We have proven the following:

$(*)$ If we are given an exact sequence of abelian groups

$$0 \to \Lambda_0 \to \Lambda \to A \to 0$$

as above, then $\pi_{-1} Q^q(A)$ can be identified with the quotient X/X_0, where X is the abelian group of quadratic forms on Λ_0 whose associated bilinear form is integral on $\Lambda_0 \times \Lambda$, and X_0 is the abelian group of quadratic forms on Λ.

Note that every element of X/X_0 determines a well-defined quadratic form $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$. Conversely, suppose that $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$ is any quadratic form. Write $A = \bigoplus (\mathbf{Z}/n_i\mathbf{Z}) x_i$, and take $\Lambda = \bigoplus \mathbf{Z}\overline{x}_i$ and $\Lambda_0 = \bigoplus (n_i \mathbf{Z}) \overline{x}_i$. Choose rational numbers $\delta_i \in \mathbf{Q}$ lifting $\epsilon(x_i) \in \mathbf{Q}/\mathbf{Z}$, and rational numbers $\beta_{i,j} \in \mathbf{Q}$ lifting $\epsilon(x_i + x_j) - \epsilon(x_i) - \epsilon(x_j) \in \mathbf{Q}/\mathbf{Z}$ for $i < j$. Define a quadratic form $q : \Lambda \to \mathbf{Q}$ so that $q(\overline{x}_i) = \delta_i$ and $q(\overline{x}_i + \overline{x}_j) = q(\overline{x}_i) + q(\overline{x}_j) + \beta_{i,j}$. Let b be the associated bilinear form. By construction, the composite map

$$\Lambda \times \Lambda \xrightarrow{b} \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$$

factors through $A \times A$, and therefore vanishes on $\Lambda_0 \times \Lambda$. Thus b is integral on $\Lambda_0 \times \Lambda$, and therefore on $\Lambda_0 \times \Lambda_0$. We claim that q is integral on Λ_0. Since b is integral on $\Lambda_0 \times \Lambda_0$, it suffices to check on generators: that is, we need $q(n_i \overline{x}_i) \in \mathbf{Z}$. Note that $q(n_i x_i) = n_i^2 \delta_i$ has image $n_i^2 \epsilon(x_i) = \epsilon(n_i x_i) = 0$ in $\mathbf{Q}/\mathbf{Z}$. It follows that every quadratic form $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$ can be lifted to an element of X, which is obviously unique up to an element of X_0. We therefore have the following more invariant description of $\pi_{-1} Q^q(A)$:

$(*')$ Let A be a finite abelian group. Then $\pi_{-1} Q^q(A)$ can be identified with the abelian group of quadratic forms $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$.

Now suppose we are given a Poincare object (A, ϵ). We will prove that (A, ϵ) is nullbordant using induction on the order of A. Let $a \in A$ be an element having order $n > 1$, and identify a with a map $\alpha : \mathbf{Z} \to A$. To perform surgery along α, we need to choose a nullhomotopy of the restriction of ϵ to $\mathbf{Z}$. Unwinding the definitions, this amounts to choosing a quadratic form $q : \mathbf{Z} \to \mathbf{Q}$ such that $q(1) = \epsilon(a)$. In this case, the cofiber of α has homotopy groups

$$\pi_* \operatorname{cofib}(\alpha) = \begin{cases} A/a & \text{if } i = 0 \\ n\mathbf{Z} & \text{if } i = 1. \end{cases}$$

The result of the surgery is a new $\mathbf{Z}$-module spectrum A_α which fits into a fiber sequence

$$A_\alpha \to \operatorname{cofib}(\alpha) \to \mathbf{Z}.$$

In particular, we have an exact sequence

$$n\mathbf{Z} \xrightarrow{\psi} \mathbf{Z} \to \pi_0 A_\alpha \to A/a \to 0.$$

Unwinding the definitions, we see that ψ is determined by the bilinear form determines by q: in other words, we have $\psi(n) = 2nq(1)$. Note that we are free to adjust our nullhomotopy by adding an integral bilinear form to q: in other words, we are free to adjust $\psi(n)$ by multiples of $2n$. We may therefore arrange that the absolute value of $\psi(n)$ is $\leq n$.

If $\psi(n) = 0$, then the exact sequence above shows that $\pi_0 A_\alpha$ is an extension of A/a by the group of integers $\mathbf{Z}$. It follows that the torsion subgroup of $\pi_0 A_\alpha$ injects into A/a. As in the first step, we can perform surgery on A_α to kill the torsion free-part without changing the torsion part: it follows that (A, ϵ) is cobordant to a pair (A', ϵ') where A' is isomorphic to a subgroup of A/a, and is therefore smaller than A. If $\psi(n) \neq 0$, we have an exact sequence

$$0 \to \mathbf{Z}/\psi(n)\mathbf{Z} \to \pi_0 A_\alpha \to A/a \to 0.$$

It follows that A_α can be identified with a finite abelian group of order $\frac{|\psi(n)|}{n}|A|$. This is smaller than the order of A unless $|\psi(n)| = n$. Since $\psi(n) = 2nq(1)$, we get $q(1) \equiv \frac{1}{2}$ (modulo $\mathbf{Z}$), so that $\epsilon(a) = \frac{1}{2}$.

In the above discussion, we choose a to be an arbitrary nonzero element of A. It follows that we can simplify A by means of surgery unless the quadratic form $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$ takes the value $\frac{1}{2}$ on every nonzero element of A. In this case, $2\epsilon = 0$. Since (A, ϵ) is Poincare, ϵ is nondegenerate on A so that A is annihilated by multiplication by 2. We can therefore identify A with a finite-dimensional vector space over $\mathbf{F}_2$ and ϵ with an anisotropic quadratic form on A with values in $\frac{1}{2}\mathbf{Z}/\mathbf{Z} \simeq \mathbf{F}_2$. We have seen that if $A \neq 0$, then A must be isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, with ϵ given by

$$\epsilon(a) = \begin{cases} 0 \text{ if } a = 0 \\ \frac{1}{2} & \text{if } a \neq 0. \end{cases}$$

We now complete the proof by explicitly exhibiting a nullhomotopy of (A, ϵ). To do this, we will write (A, ϵ) as the "boundary" of a quadratic object: that is, we will choose a lattice Λ_0 equipped with an integral quadratic form q_0 which induces an injection $\Lambda_0 \hookrightarrow \Lambda_0^\vee$, so that $A \simeq \Lambda_0^\vee/\Lambda_0$ and ϵ is the reduction modulo $\mathbf{Z}$ of the induced quadratic form $\Lambda_0^\vee \to \mathbf{Q}$. For this, we take Λ_0 to be the D_4-lattice. That is, Λ_0 is the free abelian group on generators w, x, y, and z, satisfying

$$q(w) = q(x) = q(y) = q(z) = 1$$

$$b(w, x) = b(w, y) = b(w, z) = -1$$

$$b(x, y) = b(x, z) = b(y, z) = 0.$$

The cosets of Λ_0 in its dual (which we identify with a subset of $\Lambda_0 \otimes \mathbf{Q}$) are represented by 0, $\frac{x+y}{2}$, $\frac{x+z}{2}$, and $\frac{y+z}{2}$. Note that

$$q(\frac{x+y}{2}) = q(\frac{x}{2}) + q(\frac{y}{2}) + b(\frac{x}{2}, \frac{y}{2}) = \frac{1}{4} + \frac{1}{4} + 0 = \frac{1}{2},$$

and similarly for the other nontrivial cosets. $\square$

The Even L-groups of $\mathbf{Z}$ (Lecture 16)

March 2, 2011

In this lecture we will compute the quadratic L-groups of $\mathbf{Z}$ in even dimensions. We begin by considering $L^q_{-2}(\mathbf{Z})$. Consider the pair $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma^2 Q^q)$. Note that $\Sigma^2 Q^q(\Sigma\mathbf{Z}) \simeq \Sigma^2(\Sigma^{-2}\mathbf{Z})_{h\Sigma_2}$ can be identified with the homotopy coinvariants of the group Σ_2 acting on the Eilenberg-MacLane spectrum corresponding to $\mathbf{Z}$, where the action is via the sign representation. In particular, we deduce that $\pi_0\Sigma^2 Q^q(\Sigma\mathbf{Z})$ is isomorphic to the group $\mathbf{Z}/2\mathbf{Z}$.

Let (M, q) be a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma^2 Q^q)$. Using surgery below the middle dimension, we can replace (M, q) by a cobordant Poincare object which is concentrated in degree 1: that is, we can assume that $M = \Sigma F$ for some finitely generated free abelian group L (which we identify with the corresponding Eilenberg-MacLane spectrum). Every element $\eta \in L$ determines a map $\Sigma\mathbf{Z} \to M$, so that $q|\Sigma\mathbf{Z}$ determines an element $e(\eta) \in \pi_0(\Sigma^2 Q^q(\Sigma\mathbf{Z})) \simeq \mathbf{Z}/2\mathbf{Z}$.

We can understand the construction $\eta \mapsto e(\eta)$ more explicitly as follows. Consider the surjective ring homomorphism $\mathbf{Z} \to \mathbf{F}_2$, where $\mathbf{F}_2$ denotes the finite field with two elements. Then $\mathbf{F}_2 \wedge_{\mathbf{Z}} M \simeq \Sigma(L/2L)$ inherits the structure of a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{F}_2}, \Sigma^2 Q^q)$. Since $\mathbf{F}_2$ has characteristic 2, we can ignore signs and identify the reduction of q modulo 2 as a quadratic form q_0 on the $\mathbf{F}_2$-vector space $L/2L$. The invariant e is given by the composition

$$L \to L/2L \xrightarrow{q_0} \mathbf{Z}/2\mathbf{Z}.$$

Proposition 1. *The canonical map $L^q_{-2}(\mathbf{Z}) \to L^q_{-2}(\mathbf{F}_2) \simeq W(\mathbf{F}_2) \simeq \mathbf{Z}/2\mathbf{Z}$ is an isomorphism.*

Proof. We first prove injectivity. Let (M, q) be a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma^2 Q^q)$, and assume as above that $M = \Sigma L$ for some finitely generated free abelian group L. Suppose that the image of (M, q) in $W(\mathbf{F}_2)$ is trivial; we wish to show that (M, q) is nullcobordant. We proceed by induction on the rank of L. If this rank is positive, then the quadratic form q_0 on $L/2L$ cannot be anisotropic (since $(M, q) \mapsto 0 \in W(\mathbf{F}_2)$). We can therefore choose a nonzero element $\overline{\eta} \in L/2L$ such that $q_0(\overline{\eta}) = 0$. Lift $\overline{\eta}$ to an element $\eta \in L$. Then η is not divisible by 2 (since $\overline{\eta} \neq 0$). Dividing η by an odd integer if necessary, we may assume that $\mathbf{Z}\eta$ is a direct summand of L. Since $e(\eta) = q_0(\overline{\eta}) = 0$, we can do surgery along η to obtain a new Poincare object (M', q'). Note that the homotopy groups of M' are given by the homology of the chain complex

$$\mathbf{Z}\eta \to L \xrightarrow{\phi} \mathbf{Z}.$$

The indivisibility of η (and nondegeneracy of q) imply that ϕ is surjective, so that the homology of this chain complex is concentrated in a single degree (and is therefore free, by duality). Moreover, the rank of the relevant homology group has dropped by 2, so we can finish using the inductive hypothesis.

It remains to show that the map $L^q_{-2}(\mathbf{Z}) \to W(\mathbf{F}_2)$ is surjective. For this, we just have to show that the nontrivial element of $W(\mathbf{F}_2)$ can be lifted to $L^q_{-2}(\mathbf{Z})$. We can represent this element by the nondegenerate quadratic space (V, q_0), where V is a two-dimensional vector space over $\mathbf{F}_2$ with basis $x, y \in V$, and q_0 is given by the formula $q_0(ax + a'y) = a^2 + aa' + a'^2$. Let $b_0 : V \times V \to \mathbf{F}_2$ be the bilinear form given by

$$b_0(x,x) = b_0(x,y) = b_0(y,y) = 1 \qquad b_0(y,x) = 0.$$

Then q_0 is the quadratic form associated to b_0. Note that b_0 lifts to a bilinear form b on $\mathbf{Z}x \oplus \mathbf{Z}y$, given by

$$b(x,x) = b(x,y) = b(y,y) = 1 \qquad b(y,x) = 0.$$

The associated skew-symmetric bilinear form

$$\epsilon(v,w) = b(v,w) - b(w,v)$$

is nondegenerate (since $\epsilon(x,y) = 1$). Note that b determines a point $q \in \Omega^\infty \Sigma^2 Q^q(\Sigma(\mathbf{Z}x \oplus \mathbf{Z}y))$, and that $(\mathbf{Z}x \oplus \mathbf{Z}y, q)$ is a Poincare object lifting (V, q_0). □

We now compute the group $L_0^q(\mathbf{Z})$. The inclusion $\mathbf{Z} \hookrightarrow \mathbb{R}$ determines a map $\psi : L_0^q(\mathbf{Z}) \to L_0^q(\mathbb{R}) = \mathbf{Z}$.

Proposition 2. *The map ψ is injective. Its image is the subgroup $8\mathbf{Z} \subseteq \mathbf{Z}$.*

Proof. By surgery below the middle dimension, we see that every element of $L_0^q(\mathbf{Z})$ can be represented by a pair (M, q), where M is a free abelian group of finite rank and q is a nondegenerate quadratic form on M. Here q is determined by its associated bilinear form $b : M \times M \to \mathbf{Z}$, given by $b(x,y) = q(x+y) - q(x) - q(y)$. This symmetric bilinear form is *even* (for any element $x \in M$ we have $b(x,x) = q(2x) - q(x) - q(x) = 2q(x)$) and *unimodular* (that is, it induces an isomorphism of abelian groups $M \to \mathrm{Hom}(M, \mathbf{Z})$). Conversely, any even symmetric bilinear form b determines a quadratic form q by the formula $q(x) = \frac{b(x,x)}{2}$, which is nondegenerate if and only if b is unimodular. We will prove the proposition by citing some nontrivial results about the structure of even unimodular lattices. Proofs can be found, for example, in Serre's book "A course in arithmetic."

The facts we need are the following:

- The image of ψ is contained in the subgroup $8\mathbf{Z} \subseteq \mathbf{Z}$. More concretely, we assert that if (M, q) is any even unimodular lattice, then the signature of M is divisible by 8.
- The map ψ is surjective: that is, there exists an even unimodular lattice of signature 8. In fact, there exists a *unique* positive definite even unimodular lattice of rank (and therefore signature) 8, the E_8-lattice. All we need here is the existence. For this, we can give a direct construction. Let L be the free abelian group on generators $e_1, \ldots, e_8$ and h. Equip it with an symmetric bilinear form b so that the generators are orthogonal and $b(e_i, e_i) = 1$, $b(h,h) = -1$. This is an odd unimodular lattice of signature 7. Let v denote the vector $e_1 + e_2 + \ldots + e_8 + 3h$. Note that $b(v,v) = 8(1) + 3^2(-1) = -1$. It follows that L splits as a direct sum $L_0 \oplus \mathbf{Z}v$, where L_0 is a unimodular lattice of rank 8 and signature 8. Note that for every $w \in L$, the integers $b(w,w)$ and $b(w,v)$ are congruent modulo 2 (it suffices to check this on generators, where it is obvious). Thus L_0 is an even unimodular lattice of signature 8.
- The map ψ is injective. Suppose we are given an even unimodular lattice (M, q) of signature zero. Let n be the rank of M; note that n must be even, since M is nondegenerate modulo 2. Let H denote the hyperbolic plane: that is, $(\mathbf{Z}^2, q_0)$ where q_0 is the quadratic form given by $q_0(a,b) = ab$. Then $H^{\oplus \frac{n}{2}}$ and (M, q) are indefinite even unimodular lattices of the same rank and signature. It follows from the theory of quadratic forms that $H^{\oplus \frac{n}{2}}$ and (M, q) are isomorphic. Consequently, to prove that (M, q) is nullcobordant, it suffices to show that H is nullcobordant, which is obvious.

□

Combining the above results with the 4-fold periodicity of $L_*^q(\mathbf{Z})$, we obtain the following:

Theorem 3. *The quadratic L-groups of $\mathbf{Z}$ are given by*

$$L_n^q(\mathbf{Z}) = \begin{cases} 8\mathbf{Z} & \text{if } n = 4k \text{ (signature)} \\ 0 & \text{if } n = 4k+1 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 4k+2 \text{ (Kervaire invariant)} \\ 0 & \text{if } n = 4k+3. \end{cases}$$

Let us conclude by saying a few words about the symmetric L-groups $L^s_*(\mathbf{Z})$. The norm map $Q^q \to Q^s$ induces a map

$$\phi : L^q_*(\mathbf{Z}) \to L^s_*(\mathbf{Z}).$$

Note that for any pair of spectra X and Y with actions of the group Σ_2, there is a canonical map

$$X_{h\Sigma_2} \wedge Y^{h\Sigma_2} \to (X \wedge Y)_{h\Sigma_2}.$$

Consequently, if (M, q) is a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, Q^s)$, then $M \otimes E_8$ inherits the structure of a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, Q^q)$, where E_8 denotes the E_8 lattice. This construction determines a map of L-groups

$$\psi : L^s_*(\mathbf{Z}) \to L^q_*(\mathbf{Z}).$$

The composite map $\phi \circ \psi : L^s_*(\mathbf{Z}) \to L^s_*(\mathbf{Z})$ is also given by tensoring with the E_8 lattice: this time, viewed as a lattice equipped with a symmetric bilinear form. Our construction of the E_8 lattice above shows that E_8 is stably isomorphic to the lattice $\mathbf{Z}^8$ with orthonormal basis $e_1, \ldots, e_8$: in fact, they become isomorphic after taking the direct sum with the unimodular lattice of rank 1 and signature -1. It follows that $\phi \circ \psi$ is given by multiplication by 8. In particular, ψ induces an injection $L^s_*(\mathbf{Z})[\frac{1}{2}] \to L^q_*(\mathbf{Z})[\frac{1}{2}]$ whose image is a summand of $L^q_*[\frac{1}{2}]$. This map is surjective in degree 0 (it hits the E_8 lattice by construction, which is a generator for $L^q_0(\mathbf{Z})$). By periodicity, it is surjective in degree $4k$ for every integer k. It is therefore surjective in all degrees (since $L^q_n(\mathbf{Z})[\frac{1}{2}] \simeq 0$ when n is not divisible by 4, by Theorem 3). It follows that ψ is an isomorphism after inverting 2. This proves:

Proposition 4. *The map ϕ induces an isomorphism $L^q_*(\mathbf{Z})[\frac{1}{2}] \to L^s_*(\mathbf{Z})[\frac{1}{2}]$. In particular, we have*

$$L^s_n(\mathbf{Z})[\frac{1}{2}] \simeq \begin{cases} \mathbf{Z}[\frac{1}{2}] & \text{if } n = 4k \\ 0 & \text{otherwise.} \end{cases}$$

With more effort, it is possible to compute the symmetric L-groups of $\mathbf{Z}$ precisely. The answer is given by

$$L^s_n(\mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{if } n = 4k \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 4k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Polyhedra and PL Manifolds (Lecture 17)

February 27, 2011

In this lecture, we will review the notion of a *piecewise linear manifold* (which we will typically abbreviate as *PL manifold*). More information can be found in the lecture notes of my MIT course 18.937.

Definition 1. Let K be a subset of a Euclidean space $\mathbb{R}^n$. We will say that K is a *linear simplex* if it can be written as the convex hull of a finite subset $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ which are independent in the sense that if $\sum c_i x_i = 0 \in \mathbb{R}^n$ and $\sum c_i = 0 \in \mathbb{R}$ imply that each c_i vanishes.

We will say that K is a *polyhedron* if, for every point $x \in K$, there exists a finite number of linear simplices $\sigma_i \subseteq K$ such that the union $\bigcup_i \sigma_i$ contains a neighborhood of X.

Remark 2. Any open subset of a polyhedron in $\mathbb{R}^n$ is again a polyhedron.

Remark 3. Every polyhedron $K \subseteq \mathbb{R}^n$ admits a *triangulation*: that is, we can find a collection of linear simplices $S = \{\sigma_i \subseteq K\}$ with the following properties:

(1) Any face of a simplex belonging to S also belongs to S.

(2) Any nonempty intersection of any two simplices of S is a face of each.

(3) The union of the simplices σ_i is K.

Definition 4. Let $K \subseteq \mathbb{R}^n$ be a polyhedron. We will say that a map $f : K \to \mathbb{R}^m$ is *linear* if it is the restriction of an affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will say that f is *piecewise linear* (PL) if there exists a triangulation $\{\sigma_i \subseteq K\}$ such that each of the restrictions $f|\sigma_i$ is linear.

If $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ are polyhedra, we say that a map $f : K \to L$ is piecewise linear if the underlying map $f : K \to \mathbb{R}^m$ is piecewise linear.

Remark 5. Let $f : K \to L$ be a piecewise linear homeomorphism between polyhedra. Then the inverse map $f^{-1} : L \to K$ is again piecewise linear. To see this, choose any triangulation of K such that the restriction of f to each simplex of the triangulation is linear. Taking the image under f, we obtain a triangulation of L such that the restriction of f^{-1} to each simplex is linear.

Remark 6. The collection of all polyhedra can be organized into a category, where the morphisms are given by piecewise linear maps. This allows us to think about polyhedra *abstractly*, without reference to an embedding into a Euclidean space: a pair of polyhedra $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ can be isomorphic even if $n \neq m$.

Remark 7. Let K be a polyhedron. The following conditions are equivalent:

(1) As a topological space, K is compact.

(2) K admits a triangulation having finitely many simplices.

(3) Every triangulation of K has only finitely many simplices.

If these conditions are satisfied, we say that K is a *finite polyhedron*.

Definition 8. Let M be a polyhedron. We will say that M is a *piecewise linear manifold* (of dimension n) if, for every point $x \in M$, there exists an open neighborhood $U \subseteq M$ containing x and a piecewise linear homeomorphism $U \simeq \mathbb{R}^n$.

Remark 9. If M is a PL manifold of dimension n, then the underlying topological space of M is an n-manifold. We can think of a PL manifold as a topological manifold equipped with some additional structure. There are many ways to describe this additional structure. For example, let $\mathcal{O}_M$ denote the sheaf of continuous real-valued functions on M. A PL structure on M determines a subsheaf $\mathcal{O}_M^{PL}$, which assigns to each open set $U \subseteq M$ the collection of *piecewise linear* continuous functions $U \to \mathbb{R}$. As a polyhedron, M is determined by its underlying topological space together with the sheaf $\mathcal{O}_M^{PL}$, up to PL homeomorphism.

Let K be a polyhedron containing a vertex x, and choose a triangulation of K containing x as a vertex of the triangulation. The *star* of x is the union of those simplices of the triangulation which contain x. The *link* of x consists of those simplices belonging to the star of x which do not contain x. We denote the link of x by $\mathrm{lk}(x)$.

As a subset of K, the link $\mathrm{lk}(x)$ of x depends on the choice of triangulation of K. However, one can show that as an abstract polyhedron, $\mathrm{lk}(x)$ is independent of the triangulation up to piecewise linear homeomorphism. Moreover, $\mathrm{lk}(x)$ depends only on a neighborhood of x in K.

If $K = \mathbb{R}^n$ and $x \in K$ is the origin, then the link $\mathrm{lk}(x)$ can be identified with the sphere S^{n-1} (which can be regarded as a polyhedron via the realization $S^{n-1} \simeq \partial\,\Delta^n$). It follows that if K is any piecewise linear n-manifold, then the link $\mathrm{lk}(x)$ is equivalent to S^{n-1} for every point $x \in K$. Conversely, if K is any polyhedron such that every link in K is an $(n-1)$-sphere, then K is a piecewise linear n-manifold. To see this, we observe that for each $x \in K$, if we choose a triangulation of K containing x as a vertex, then the star of x can be identified with the cone on $\mathrm{lk}(x)$. If $\mathrm{lk}(x) \simeq S^{n-1}$, then the star of x is a piecewise linear (closed) disk, so that x has a neighborhood which admits a piecewise linear homeomorphism to the open disk in $\mathbb{R}^n$.

This argument proves the following:

Proposition 10. *Let K be a polyhedron. The following conditions are equivalent:*

(*i*) *For each $x \in K$, the link $\mathrm{lk}(x)$ is a piecewise linear $(n-1)$-sphere.*

(*ii*) *K is a piecewise linear n-manifold.*

Remark 11. Very roughly speaking, we can think of a piecewise linear manifold M as a topological manifold equipped with a triangulation. However, this is not quite accurate, since a polyhedron does not come equipped with a particular triangulation. Instead, we should think of M as equipped with a distinguished class of triangulations, which is stable under passing to finer and finer subdivisions.

Warning 12. Let K be a polyhedron whose underlying topological space is an n-manifold. Then K need not be a piecewise linear n-manifold: it is generally not possible to choose local charts for K in a piecewise linear fashion.

To get a feel for the sort of problems which might arise, consider the criterion of Proposition 10. To prove that K is a piecewise linear n-manifold, we need to show that for each $x \in K$, the link $\mathrm{lk}(x)$ is a (piecewise-linear) $(n-1)$-sphere. Using the fact that K is a topological manifold, we deduce that $\mathrm{H}_*(K, K - \{x\}; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ in degree n and zero elsewhere; this is equivalent to the assertion that $\mathrm{lk}(x)$ has the homology of an $(n-1)$-sphere. Of course, this does not imply that $\mathrm{lk}(x)$ is itself a sphere. A famous counterexample is due to Poincare: if we let I denote the binary icosahedral group, regarded as a subgroup of $\mathrm{SU}(2) \simeq S^3$, then the quotient $P = \mathrm{SU}(2)/I$ is a homology sphere which is not a sphere (since it is not simply connected).

The suspension ΣP is a 4-dimensional polyhedron whose link is isomorphic to P at precisely two points, which we will denote by x and y. However, ΣP is not a topological manifold. To see this, we note that the point x does not contain arbitrarily small neighborhoods U such that $U - \{x\}$ is simply connected. In other words, the failure of ΣP to be a manifold can be detected by computing the *local fundamental group* *of* $P - \{x\}$ *near* x (which turns out to be isomorphic to the fundamental group of P). However, if we apply the suspension functor again, the same considerations do not apply: the space ΣP is simply connected (by van Kampen's theorem). Surprisingly enough, it turns out to be a manifold:

Theorem 13 (Cannon-Edwards)**.** *Let P be a topological n-manifold which is a homology sphere. Then the double suspension $\Sigma^2 P$ is homeomorphic to an $(n+2)$-sphere.*

In particular, if we take P to be the Poincare homology sphere, then there is a homeomorphism $\Sigma^2 P \simeq S^5$. However, $\Sigma^2 P$ is not a piecewise linear manifold: it contains two points whose links are given by ΣP, which is not even a topological 4-manifold (let alone a piecewise linear 4-sphere).

The upshot of Warning 12 is that a topological manifold M (such as the 5-sphere) admits triangulations which are badly behaved, in the sense that the underlying polyhedron is not locally equivalent to Euclidean space.

Let us now review the relationship between smooth and PL manifolds.

Definition 14. Let K be a polyhedron and M a smooth manifold. We say that a map $f : K \to M$ *piecewise differentiable* (PD) if there exists a triangulation of K such that the restriction of f to each simplex is smooth. We will say that f is a *PD homeomorphism* if f is piecewise differentiable, a homeomorphism, and the restriction of f to each simplex has injective differential at each point. In this case, we say that f is a *Whitehead triangulation of M*.

The problems of smoothing and triangulating manifold can be formulated as follows:

(i) Given a smooth manifold M, does there exist a piecewise linear manifold N and a PD homeomorphism $N \to M$?

(ii) Given a piecewise linear manifold N, does there exist a smooth manifold M and a PD homeomorphism $N \to M$?

Question (i) is addressed by the following theorem of Whitehead.

Theorem 15 (Whitehead)**.** *Let M be a smooth manifold. Then M admits a Whitehead triangulation. That is, there is exists a polyhedron K and a PD homeomorphism $f : K \to M$. Moreover, K is automatically a PL manifold, and is uniquely determined up to PL homeomorphism. (In fact, one can say more: K is uniquely determined up a contractible space of choices. We will return to this point in a future lecture.)*

Problem (ii) is more subtle. In general, piecewise linear manifolds cannot be smoothed (Kervaire) and can admit inequivalent smoothings (Milnor's exotic spheres give examples of smooth manifolds which are not diffeomorphic, but whose underlying PL manifolds are PL homeomorphic). However, the difference between smooth and PL manifolds is governed by an *h-principle*. Given a PL manifold N, the problem of finding a PD homeomorphism to a smooth manifold can be rephrased as a homotopy lifting problem

$$\begin{tikzcd} & BO(n) \arrow[d] \\ N \arrow[r] \arrow[ur, dashed] & BPL(n). \end{tikzcd}$$

(We will return to this point in more detail later, when we discuss microbundles.) In other words, the problem of finding a smooth structure on a PL manifold is equivalent to the problem of finding a suitable candidate for its tangent bundle.

Constructible Sheaves (Lecture 18)

March 7, 2011

In this lecture, we describe the theory of *constructible sheaves* on a polyhedron. First, we summarize a few ideas we will need from piecewise linear topology.

Definition 1. Let K be a polyhedron equipped with a triangulation T. For every simplex $\sigma \in T$, we define $\mathrm{lk}(\sigma)$ to be the union of those simplices $\tau \in T$ such that $\tau \cap \sigma = \emptyset$ and such that $\tau \cup \sigma$ is contained in a simplex of T. The set $\mathrm{lk}(\sigma)$ is called the *link* of σ.

We will need the following:

Fact 2. *Let M be a piecewise linear n manifold with boundary equipped with a trianguation T. If σ is a k-simplex of M which does not belong to ∂M, then $\mathrm{lk}_\sigma(M)$ is PL homeomorphic to a sphere $S^{n-1-k} \simeq \partial \Delta^{n-k}$. If $\sigma \subseteq \partial M$, then $\mathrm{lk}(\sigma)$ is PL homeomorphic to a disk D^{n-1-k} (with ∂D^{n-1-k} being the link of σ in ∂M).*

Definition 3. Let X be a polyhedron equipped with a triangulation T. We regard T as a partially ordered set (with respect to inclusions of simplices). Let $\mathcal{C}$ be an ∞-category. We will let $\mathrm{Shv}_T(X; \mathcal{C})$ denote the ∞-category of functors from T to $\mathcal{C}$. We will refer to $\mathrm{Shv}_T(X; \mathcal{C})$ as the *∞-category of $\mathcal{C}$-valued T-constructible sheaves on X.*

For the remainder of this lecture, we will fix a polyhedron X and a stable ∞-category $\mathcal{C}$. Given a triangulation T of X, we let say that an object $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$ is *compactly supported* if $\mathcal{F}(\tau) = 0$ for all but finitely many simplices $\tau \in T$. Let $\mathrm{Shv}^c_T(X; \mathcal{C})$ denote the full subcategory of $\mathrm{Shv}_T(K; \mathcal{C})$ spanned by the compactly supported objects.

Example 4. Let $\tau_0 \in T$ and $C \in \mathcal{C}$, and define $\mathcal{F}^{\tau_0, C} : T \to \mathcal{C}$ by the formula

$$\mathcal{F}^{\tau_0, C}(\tau) = \begin{cases} C & \text{if } \tau_0 \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}^{\tau_0, C}$ is a compactly supported T-constructible sheaf on X. It has the following universal property: for every $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$, there is a canonical homotopy equivalence

$$\mathrm{Mor}_{\mathrm{Shv}_T(X;\mathcal{C})}(\mathcal{F}^{\tau_0, C}, \mathcal{F}) \simeq \mathrm{Mor}_{\mathcal{C}}(C, \mathcal{F}(\tau_0)).$$

Remark 5. The stable ∞-category $\mathrm{Shv}^c_T(X; \mathcal{C})$ is generated (under the formation of fibers and cofibers) by objects of the form $\mathcal{F}^{\tau_0, C}$. To prove this, let $\mathcal{F}$ be an arbitrary compactly supported T-constructible sheaf on X; we will show that $\mathcal{F}$ belongs to the smallest stable subcategory of $\mathrm{Shv}^c_T(X; \mathcal{C})$ containing the sheaves $\mathcal{F}^{\tau_0, C}$. Since $\mathcal{F}$ is compactly supported, there exists a finite upward-closed subset $T_0 \subseteq T$ such that $\mathcal{F}(\tau) = 0$ for $\tau \notin T_0$. We proceed by induction on the size of T_0. If T_0 is empty, then $\mathcal{F} \simeq 0$ and there is nothing to prove. Otherwise, choose a minimal element $\tau_0 \in T_0$, let $C = \mathcal{F}(\tau_0)$, and form a cofiber sequence

$$\mathcal{F}^{\tau_0, C} \to \mathcal{F} \to \mathcal{F}'.$$

The desired result then follows by applying the inductive hypothesis to $\mathcal{F}'$.

Definition 6. Let T be a trianguation of X. We define the *global sections functor* $\Gamma : \mathrm{Shv}^{\mathcal{C}}_T(X;\mathcal{C})$ by the formula

$$\Gamma(\mathcal{F}) = \varprojlim_{\tau \in T} \mathcal{F}(\tau).$$

Example 7. Suppose that the polyhedron X is finite (or that $\mathcal{C}$ admits infinite limits). Let $C \in \mathcal{C}$ be an object and let $X_0 \subseteq X$ be a closed subset which is of simplices belonging to T. Define $\mathcal{F}_{X_0}$ by the formula

$$\mathcal{F}_{X_0}(\tau) = \begin{cases} C & \text{if } \tau \subseteq X_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Gamma(\mathcal{F}_{X_0})$ can be identified with the mapping object C^{X_0} (characterized by the universal property $\mathrm{Map}_{\mathcal{C}}(C', C^{X_0}) = \mathrm{Map}_{\mathcal{C}}(C', C)^{X_0}$).

Suppose now that τ is a simplex of T. The *open star* of τ is the union of the interiors of those simplices which contain τ (this is an open subset of X). Let X_0 be the complement of the open star of τ. We have a fiber sequence of functors

$$\mathcal{F}^{\tau, C} \to \mathcal{F}_X \to \mathcal{F}_{X_0}.$$

It follows that $\Gamma(\mathcal{F}^{\tau,C})$ can be identified with the mapping object $C^{(X,X_0)}$ (which is characterized by the universal property that $\mathrm{Map}_{\mathcal{C}}(C', C^{(X,X_0)})$ is the homotopy fiber of the map $\mathrm{Map}_{\mathcal{C}}(C', C)^X \to \mathrm{Map}_{\mathcal{C}}(C', C)^{X_0}$). It is not difficult to see that X_0 is a deformation retract of $X - \{x\}$, where x is any point belonging to the interior of τ. We can therefore write $\Gamma(\mathcal{F}^{\tau,C}) = C^{(X, X - \{x\})}$.

We now study the dependence of the ∞-category $\mathrm{Shv}_T(X;\mathcal{C})$ on the choice of triangulation T. Suppose that S is a triangulation of K which refines T. Then every simplex $\sigma \in S$ is contained in a simplex $\tau \in T$. We will denote the smallest such simplex by $i(\sigma)$. We regard i as a map of partially ordered sets $S \to T$, which induces (by composition) a functor $i^* : \mathrm{Shv}_T(X;\mathcal{C}) \to \mathrm{Shv}_S(X;\mathcal{C})$.

Proposition 8. *In the situation above, the functor i^* is fully faithful.*

Let us sketch the proof of Proposition 8. The pullback functor i^* has a left adjoint $i_+ : \mathrm{Shv}_S(X;\mathcal{C}) \to \mathrm{Shv}_T(X;\mathcal{C})$, given by left Kan extension along i. Concretely, this functor can be described by the formula

$$(i_+ \mathcal{F})(\tau) = \varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{F}(\sigma),$$

where $\mathcal{F} \in \mathrm{Shv}_S(X;\mathcal{C})$. To prove Proposition 8, we must show that for every $\mathcal{G} \in \mathrm{Shv}_T(X;\mathcal{C})$, the counit map $i_+ i^* \mathcal{G} \to \mathcal{G}$ is an equivalence. Evaluating at a simplex $\tau \in T$, we are required to prove that $\mathcal{G}(\tau)$ is given by the colimit $\varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(i(\sigma))$. In other words, we wish to show that the canonical map

$$\theta : \varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(i(\sigma)) \to \varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(\tau)$$

is an equivalence (the right hand side is given by $\mathcal{G}(\tau)$, since the diagram is indexed by a contractible partially ordered set: in fact, the geometric realization of this partially ordered set is homeomorphic to τ). Let $S_0 = \{\sigma \in S : \sigma \subseteq \tau\}$ and let $S_1 = \{\sigma \in S : i(\sigma) = \tau\}$. The map θ is determined by a natural transformation between diagrams $S_0 \to \mathcal{C}$, and this natural transformation is invertible when restricted to S_1. To prove that θ is invertible, it suffices to show that S_1 is cofinal in S_0. This is a special case of the following more general assertion (applied in the case $M = \tau$):

Lemma 9. *Let M be a piecewise linear n-manifold with boundary, equipped with a triangulation S. Let S_1 be the collection of simplices of S which are not contained in ∂M. Then the inclusion $S_1 \hookrightarrow S$ is cofinal.*

Remark 10. Lemma 9 can be regarded as an analogue of the assertion that a manifold with boundary is always homotopy equivalent to its interior.

To prove Lemma 9, we work by induction on n. Fix a simplex $\sigma \in S$; we wish to show that the set $V_1 = \{\sigma' \in S_1 : \sigma \subseteq \sigma'\}$ has weakly contractible nerve. If $\sigma \in S_1$ this is obvious (since the subset above contains σ as a smallest element). Let us therefore assume that σ is a simplex of the boundary ∂M. Let $V = \{\sigma \in S : \sigma \subsetneq \sigma'\}$. Then V can be identified with the partially ordered set of simplices of $\mathrm{lk}(\sigma)$, which (since M is a PL manifold with boundary) is PL isomorphic to a disk D^m for $m < n$. We can identify V_0 with the subset of V consisting of simplices which are not contained in ∂D^m. Using the inductive hypothesis, we deduce that the inclusion $V_0 \to V$ is cofinal. Since V has weakly contractible nerve, so does V_0.

Proposition 8 implies that, for every $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$, the canonical map

$$\Gamma(\mathcal{F}) \to \Gamma(i^* \mathcal{F})$$

is an equivalence. In particular, taking $\mathcal{F} = i_+ \mathcal{G}$, we obtain a canonical map

$$\Gamma(\mathcal{G}) \to \Gamma(i^* i_+ \mathcal{G}) \simeq \Gamma(i_+ \mathcal{G})$$

.

Proposition 11. *In the above situation, the map*

$$\Gamma(\mathcal{G}) \to \Gamma(i_+ \mathcal{G})$$

is an equivalence.

Proof. Let us assume for simplicity that $\mathcal{G}$ is compactly generated. Using Remark 5, we can assume that $\mathcal{F} = \mathcal{F}^{\sigma, C}$ for some simplex $\sigma \in S$. Then $i_+ \mathcal{F} \simeq \mathcal{F}^{\tau, C}$, where $\tau = i(\sigma)$. Choose a point $x \in X$ belonging to the interior of σ, so that x also belongs to the interior of τ. The calculation of Example 7 gives

$$\Gamma(\mathcal{G}) \simeq C^{(X, X - \{x\})} \simeq \Gamma(i_+ \mathcal{G}).$$

$\square$

Definition 12. Let K be a polyhedron and $\mathcal{C}$ an ∞-category. We let $\mathrm{Shv}_{\mathrm{const}}(K; \mathcal{C})$ denote the direct limit $\varprojlim_T \mathrm{Shv}_T(X; \mathcal{C})$, where T ranges over all triangulations of X. We will refer to $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ as the *∞-category of constructible $\mathcal{C}$-valued sheaves on K*. If $\mathcal{C}$ is stable, we let $\mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C}) = \varinjlim_T \mathrm{Shv}^c_T(X; \mathcal{C})$; we will refer to the objects of $\mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C})$ as *compactly supported* constructible $\mathcal{C}$-valued sheaves on X.

Construction 13. Let $\mathcal{C}$ be a stable ∞-category and let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a quadratic functor. Let K be a polyhedron with a triangulation T. We define $Q_T : \mathrm{Shv}^c_T(\mathcal{C})^{op} \to \mathrm{Sp}$ by the formula

$$Q_T(\mathcal{F}) = \varinjlim_{\tau \in T} Q(\mathcal{F}(\tau)).$$

It is not difficult to see that Q_T is a quadratic functor on $\mathrm{Shv}^c_T(K, \mathcal{C})^{op}$, whose associative bilinear functor is given by

$$B_T(\mathcal{F}, \mathcal{F}') = \varinjlim_{\tau \in T} B(\mathcal{F}(\tau), \mathcal{F}'(\tau)),$$

where B is the bilinear functor associated to Q.

In the situation of Construction 13, suppose we are given another triangulation S of X which refines T. Let $i^* : \mathrm{Shv}^c_T(X; \mathcal{C}) \to \mathrm{Shv}^c_S(X; \mathcal{C})$ be as defined earlier. The composition

$$\mathrm{Shv}^c_T(X; \mathcal{C})^{op} \xrightarrow{i^*} \mathrm{Shv}^c_S(X; \mathcal{C}) \xrightarrow{Q_S}$$

is given by the formula $\mathcal{F} \mapsto \varinjlim_{\sigma \in S} Q(\mathcal{F}(i(\sigma)))$. We claim that this composition is canonically equivalent to Q_T. To prove this, it suffices to show that i induces a cofinal map $S^{op} \to T^{op}$. Unwinding the definitions, we must show that for every simplex $\tau \in T$, the partially ordered set $\{\sigma \in S : i(\sigma) \subseteq \tau\}$ has weakly contractible nerve. This is clear, since the geometric realization of this nerve is homeomorphic to the simplex τ. It follows that the functors Q_T are compatible as T ranges over all triangulations of X, and amalgamate to a quadratic functor

$$Q_{\text{const}} : \text{Shv}^c_{\text{const}}(X; \mathcal{C})^{op} \to \text{Sp}\,.$$

Let us now assume that the quadratic functor Q on $\mathcal{C}$ is representable, and let $\mathbb{D}$ be the corresponding duality functor. We claim that for every triangulation T of K, the quadratic functor Q_T is also representable, and the corresponding duality functor $\mathbb{D}_T$ is given by the formula

$$\mathbb{D}_T(\mathcal{F})(\tau) = \varinjlim_{\sigma} \begin{cases} \mathbb{D}(\mathcal{F}(\sigma)) & \text{if } \tau \subseteq \sigma \\ 0 & \text{otherwise.} \end{cases}$$

(Note that this functor carries $\text{Shv}^c_T(K; \mathcal{C})$ to itself).

For simplicity, let us assume that $\mathbb{D}_T$ exists and show that it is given by the formula above. (A slightly more complication version of the same argument will show that it is given by the above formula.) Fix an object $C \in \mathcal{C}$ and a simplex $\tau \in T$. For every object $\mathcal{F} \in \text{Shv}_T(X; \mathcal{C})$, we have homotopy equivalences

$$\begin{aligned}
\text{Mor}_{\mathcal{C}}(C, (\mathbb{D}_T\, \mathcal{F})(\tau)) &\simeq \text{Mor}_{\text{Shv}_T(X;\mathcal{C})}(\mathcal{F}^{\tau,C}, \mathbb{D}_T\, \mathcal{F}) \\
&\simeq B_T(\mathcal{F}^{\tau,C}, \mathcal{F}) \\
&\simeq \varinjlim_{\tau' \in T} B(\mathcal{F}^{\tau,C}(\tau'), \mathcal{F}(\tau')) \\
&\simeq \varinjlim_{\tau'} \text{Mor}_{\mathcal{C}}(C, \begin{cases} \mathbb{D}\, \mathcal{F}(\tau') & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases}) \\
&\simeq \text{Mor}_{\mathcal{C}}(C, \varinjlim \begin{cases} \mathbb{D}\, \mathcal{F}(\tau') & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases})
\end{aligned}$$

depending functorially on C, so by Yoneda's lemma we get a canonical equivalence

$$(\mathbb{D}_T\, \mathcal{F})(\tau) = \varinjlim_{\tau'} \begin{cases} \mathbb{D}(\mathcal{F}(\tau)) & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases}$$

Example 14. Let $\mathcal{F}$ be a constant functor taking the value $C \in \mathcal{C}$.The above formula shows that

$$(\mathbb{D}_T\, \mathcal{F})(\tau) \simeq \mathbb{D}\Gamma(\mathcal{F}^{\tau,C}) = \mathbb{D}(C^{(X, X-\{x\})}) = \mathbb{D}(C) \wedge (X/X - \{x\})$$

(here $x \in X$ denotes a point belonging to the interior of σ, and we regard the stable ∞-category $\mathcal{C}$ as tensored over the ∞-category of finite pointed spaces).

Verdier Duality (Lecture 19)

March 9, 2011

Fix a polyhedron X and a stable ∞-category $\mathcal{C}$.

Construction 1. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a quadratic functor. Let T be a triangulation of X. We define $Q_T : \mathrm{Shv}^c_T(\mathcal{C})^{op} \to \mathrm{Sp}$ by the formula

$$Q_T(\mathcal{F}) = \varinjlim_{\tau \in T} Q(\mathcal{F}(\tau)).$$

It is not difficult to see that Q_T is a quadratic functor on $\mathrm{Shv}^c_T(K, \mathcal{C})^{op}$, whose associative bilinear functor is given by

$$B_T(\mathcal{F}, \mathcal{F}') = \varinjlim_{\tau \in T} B(\mathcal{F}(\tau), \mathcal{F}'(\tau)),$$

where B is the bilinear functor associated to Q.

In the situation of Construction 1, suppose we are given another triangulation S of X which refines T. Let $i^* : \mathrm{Shv}^c_T(X; \mathcal{C}) \to \mathrm{Shv}^c_S(X; \mathcal{C})$ be as defined earlier. The composition

$$\mathrm{Shv}^c_T(X; \mathcal{C})^{op} \xrightarrow{i^*} \mathrm{Shv}^c_S(X; \mathcal{C}) \xrightarrow{Q_S}$$

is given by the formula $\mathcal{F} \mapsto \varinjlim_{\sigma \in S} Q(\mathcal{F}(i(\sigma)))$. We claim that this composition is canonically equivalent to Q_T. To prove this, it suffices to show that i induces a cofinal map $S^{op} \to T^{op}$. Unwinding the definitions, we must show that for every simplex $\tau \in T$, the partially ordered set $\{\sigma \in S : i(\sigma) \subseteq \tau\}$ has weakly contractible nerve. This is clear, since the geometric realization of this nerve is homeomorphic to the simplex τ. It follows that the functors Q_T are compatible as T ranges over all triangulations of X, and amalgamate to a quadratic functor

$$Q_{\mathrm{const}} : \mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Sp}.$$

Let us now assume that the quadratic functor Q on $\mathcal{C}$ is representable, and let $\mathbb{D}$ be the corresponding duality functor. We claim that for every triangulation T of K, the quadratic functor Q_T is also representable, and the corresponding duality functor $\mathbb{D}_T$ is given by the formula

$$\mathbb{D}_T(\mathcal{F})(\tau) = \varinjlim_{\sigma} \begin{cases} \mathbb{D}(\mathcal{F}(\sigma)) & \text{if } \tau \subseteq \sigma \\ 0 & \text{otherwise.} \end{cases}$$

(Note that this functor carries $\mathrm{Shv}^c_T(K; \mathcal{C})$ to itself).

For simplicity, let us assume that $\mathbb{D}_T$ exists and show that it is given by the formula above. (A slightly more complication version of the same argument will show that it is given by the above formula.) Fix an object $C \in \mathcal{C}$ and a simplex $\tau \in T$. For every object $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$, we have homotopy equivalences

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{C}}(C, (\mathbb{D}_T \mathcal{F})(\tau)) &\simeq \mathrm{Mor}_{\mathrm{Shv}_T(X;\mathcal{C})}(\mathcal{F}^{\tau,C}, \mathbb{D}_T \mathcal{F}) \\
&\simeq B_T(\mathcal{F}^{\tau,C}, \mathcal{F}) \\
&\simeq \varinjlim_{\tau' \in T} B(\mathcal{F}^{\tau,C}(\tau'), \mathcal{F}(\tau')) \\
&\simeq \varinjlim_{\tau'} \mathrm{Mor}_{\mathcal{C}}(C, \begin{cases} \mathbb{D}\,\mathcal{F}(\tau') & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases}) \\
&\simeq \mathrm{Mor}_{\mathcal{C}}(C, \varinjlim \begin{cases} \mathbb{D}\,\mathcal{F}(\tau') & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases})
\end{aligned}$$

depending functorially on C, so by Yoneda's lemma we get a canonical equivalence

$$(\mathbb{D}_T \mathcal{F})(\tau) = \varinjlim_{\tau'} \begin{cases} \mathbb{D}(\mathcal{F}(\tau)) & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases}$$

Example 2. Let $\mathcal{F}$ be a constant functor taking the value $C \in \mathcal{C}$.The above formula shows that

$$(\mathbb{D}_T \mathcal{F})(\tau) \simeq \mathbb{D}\Gamma(\mathcal{F}^{\tau,C}) = \mathbb{D}(C^{(X, X-\{x\})}) = \mathbb{D}(C) \wedge (X/X - \{x\})$$

(here $x \in X$ denotes a point belonging to the interior of σ, and we regard the stable ∞-category $\mathcal{C}$ as tensored over the ∞-category of finite pointed spaces).

Now suppose that S is a triangulation refining T. We claim that the functor

$$i^* : \mathrm{Shv}^c_T(X; \mathcal{C}) \to \mathrm{Shv}^c_{T'}(X; \mathcal{C})$$

intertwines the duality functors $\mathbb{D}_T$ and $\mathbb{D}_{T'}$. Since $Q_T \simeq Q_{T'} \circ i^*$, for every object $\mathcal{F} \in \mathrm{Shv}^c_T(X; \mathcal{C})$ we have a canonical map $i^* \mathbb{D}_T(\mathcal{F}) \to \mathbb{D}_{T'}(i^* \mathcal{F})$. We claim that this map is invertible. To prove this, consider an arbitrary $\mathcal{G} \in \mathrm{Shv}^c_S(X; \mathcal{C})$. We wish to prove that the canonical map

$$B_T(i_+ \mathcal{G}, \mathcal{F}) \simeq \mathrm{Mor}_{\mathrm{Shv}^c_T}(i_+ \mathcal{G}, \mathbb{D}_T \mathcal{F})\, \mathrm{Mor}_{\mathrm{Shv}^c_S(X;\mathcal{C})}(\mathcal{G}, i^* \mathbb{D}_T \mathcal{F}) \to \mathrm{Mor}_{\mathrm{Shv}^c_S(X;\mathcal{C})}(\mathcal{G}, \mathbb{D}_S i^* \mathcal{F}) \simeq B_S(\mathcal{G}, i^* \mathcal{F})$$

is an equivalence. The left hand side is given by

$$\varinjlim_{\tau \in T} B(\varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(\sigma), \mathcal{F}(\tau)) \simeq \varinjlim_{\tau \in T} \varprojlim_{\sigma \in S, \sigma \subseteq \tau} B(\mathcal{G}(\sigma), \mathcal{F}(\tau)).$$

In the last lecture, we saw that the collection of $\sigma \subseteq \tau$ such that $i(\sigma) = \tau$ is cofinal in the collection of all $\sigma \subseteq \tau$. We may therefore rewrite the above limit as

$$\varinjlim_{\tau \in T} \varprojlim_{\sigma \in S, \sigma \subseteq \tau} B(\mathcal{G}(\sigma), \mathcal{F}(i(\sigma)).$$

Define $\mathcal{H} : S \to \mathrm{Sp}^{op}$ by the formula $\mathcal{H}(\sigma) = B(\mathcal{G}(\sigma), \mathcal{F}(i\sigma))$. Then the above construction is given by $\Gamma(i_+ \mathcal{H})$ (computed in the ∞-category Sp^{op}). In the last lecture, we saw that this is equivalent to

$$\Gamma(\mathcal{H}) = \varinjlim_{\sigma \in S} B(\mathcal{G}(\sigma), \mathcal{F}(i(\sigma))) = B_S(\mathcal{G}, i^* \mathcal{F}),$$

as desired.

Amalgamating the duality functors $\mathbb{D}_T$ as T runs over all triangulations of X, we obtain a duality functor

$$\mathbb{VD} : \mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C}).$$

We will refer to this functor as *Verdier duality*.

Suppose now that we have two finite polyhedra X and Y, and a PL map $f : X \to Y$. To f we can associate a pullback functor $f^* : \mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$. This pullback functor admits right adjoint, which we will denote by f_*. For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$, can explicitly describe $f_*(\mathcal{F})$ as follows. Choose triangulations S and T of X and Y, respectively, such that $\mathcal{F}$ is S-constructible and f is *simplicial*: that is, it induces a linear map from each simplex of S to each simplex of T (carrying vertices to vertices). Then $f_* \mathcal{F}$ can be identified with the T-constructible sheaf on Y given by the formula

$$(f_* \mathcal{F})(\tau) = \varprojlim_{\sigma \in S, \tau \subseteq f(\sigma)} \mathcal{F}(\sigma).$$

Using a cofinality argument, we can also write

$$(f_* \mathcal{F})(\tau) = \varprojlim_{f(\sigma) = \tau} \mathcal{F}(\sigma).$$

Let $\mathcal{G} \in \mathrm{Shv}_T(Y, \mathcal{C})$. Then we have a canonical equivalence

$$B_S(\mathcal{F}, f^* \mathcal{G}) = \varinjlim_{\sigma} B(\mathcal{F}(\sigma), \mathcal{G}(f(\sigma))) = \varinjlim_{\tau} \varinjlim_{f(\sigma)=\tau} B(\mathcal{F}(\sigma), \mathcal{G}(\tau)) \simeq \varinjlim_{\tau}(\varprojlim_{f(\sigma)=\tau} \mathcal{F}(\sigma), \mathcal{G}(\tau)) = B_T(f_* \mathcal{F}, \mathcal{G}).$$

We can rewrite the left side as

$$\mathrm{Mor}_{\mathrm{Shv}_S(X;\mathcal{C})}(f^* \mathcal{G}, \mathbb{D}_S \mathcal{F}) \simeq \mathrm{Mor}_{\mathrm{Shv}_T(Y;\mathcal{C})}(\mathcal{G}, f_* \mathbb{D}_S \mathcal{F})$$

and the right side as

$$\mathrm{Mor}_{\mathrm{Shv}_T(Y;\mathcal{C})}(\mathcal{G}, \mathbb{D}_T f_* \mathcal{F}).$$

Yoneda's lemma gives us a canonical isomorphism

$$\mathbb{D}_T f_* \mathcal{F} \simeq f_* \mathbb{D}_S \mathcal{F}.$$

Passing to the direct limit over all triangulations, we deduce that Verdier duality commutes with pushforwards.

Proposition 3. *Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a nondegenerate quadratic functor. Then for any finite polyhedron X equipped with a triangulation T, the quadratic functor $Q_T : \mathrm{Shv}_T(X; \mathcal{C})^{op} \to \mathrm{Sp}$ is nondegenerate. Passing to the direct limit over T, obtain a nondegenerate quadratic functor $Q_{\mathrm{const}} : \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Sp}$.*

Proof. Let $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$; we wish to show that the canonical map

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathbb{D}_T \mathbb{D}_T \mathcal{F}$$

is invertible. For every simplex $\tau \in T$ and every object $C \in \mathcal{C}$, define $\mathcal{F}_{\tau,C}$ by the formula

$$\mathcal{F}_{\tau,C}(\tau') = \begin{cases} C & \text{if } \tau' \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as in the previous lecture, we see that the objects $\mathcal{F}_{\tau,C}$ generate the stable ∞-category $\mathrm{Shv}_T(X; \mathcal{C})$. It will therefore suffice to prove that $\theta_{\mathcal{F}}$ is an equivalence when $\mathcal{F} = \mathcal{F}_{\tau,C}$. Note that $\mathcal{F}_{\tau,C} = f_* \mathcal{G}$, where f denotes the inclusion $\tau \to X$ and $\mathcal{G}$ is a constant functor taking the value C. Since Verdier duality commutes with pushforwards, we can replace X by τ and thereby reduce to the case where X is a simplex, T is a the collection of faces of X, and $\mathcal{F}$ is the constant functor taking the value $C \in \mathcal{C}$. Let n be the dimension of X.

We now compute

$$\mathbb{D}_T(\mathcal{F})(\sigma) = \mathbb{D}(C^{(X, X - \{x\})})$$

where x is a point belonging to the interior of σ. If σ is a proper face of X, then X and $X - \{x\}$ are both contractible so that $\mathbb{D}_T(\mathcal{F})(\sigma) = 0$. If $\sigma = X$, then the cofiber $X/(X - \{x\})$ is homotopy equivalent to S^n, so that $\mathbb{D}_T(\mathcal{F})(\sigma) \simeq \Sigma^n \mathbb{D}(C)$. Thus $\mathbb{D}_T(\mathcal{F})$ is the functor $\mathcal{F}'$ given by the formula

$$\mathcal{F}'(\sigma) = \begin{cases} \Sigma^n \mathbb{D}(C) & \text{if } \sigma = X \\ 0 & \text{otherwise.} \end{cases}$$

We now compute $\mathbb{D}_T \mathcal{F}'$ using the formula

$$(\mathbb{D}_T \mathcal{F}')(\tau) = \varinjlim_{\sigma} \begin{cases} \mathbb{D}\, \mathcal{F}'(\sigma) & \text{if } \tau \subseteq \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Using our formula for $\mathcal{F}'$, we can rewrite this as

$$\varinjlim_{\sigma} \begin{cases} \mathbb{D}(\Sigma^n \mathbb{D}(C)) & \text{if } \sigma = X \\ 0 & \text{otherwise.} \end{cases}$$

The nondegeneracy of Q gives an equivalence $\mathbb{D}\Sigma^n \mathbb{D}(C) \simeq \Sigma^{-n} C$. We therefore obtain $(\mathbb{D}_T \mathcal{F}') \simeq (X, \partial X) \wedge \Sigma^{-n} C \simeq C$ for every simplex $\tau \in T$, so that we have an isomorphism $\mathbb{D}_T \mathcal{F}' \simeq \mathcal{F}$. With a bit more effort, one can show that this isomorphism is given by the functor $\theta_{\mathcal{F}}$ defined above. $\square$

L-Groups of Polyhedra (Lecture 20)

March 11, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. Let X be a finite polyhedron. In the last lecture, we proved that Q determines a nondegenerate quadratic functor $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Sp}$. Let us denote this functor by Q_X, to emphasize its dependence on X. We let $L(X; \mathcal{C}, Q)$ denote the L-theory space of the pair $(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_X)$.

Example 1. When X consists of a single point, we have $L(X; \mathcal{C}, Q) \simeq L(\mathcal{C}, Q)$.

Remark 2. Let $f : X \to Y$ be a map of finite polyhedra, and choose triangulations S and T of X and Y such that f is linear on each simplex. Let $\mathcal{F} \in \mathrm{Shv}_S(X; \mathcal{C})$. Then we have a canonical map

$$Q_S(\mathcal{F}) \simeq \varinjlim_{\sigma \in S} Q(\mathcal{F}(\sigma)) \simeq \varinjlim_{\tau \in T} \varinjlim_{f(\sigma)=\tau} Q(\mathcal{F}(\sigma)) \to \varinjlim_{\tau \in T} Q(\varprojlim_{f(\sigma)=\tau} \mathcal{F}(\sigma)) = Q_T(f_* \mathcal{F}).$$

Taking the direct limit over triangulations, we obtain a natural transformation $Q_X \to Q_Y \circ f_*$. This natural transformation induces a natural transformation

$$f_* \circ \mathbb{V}\mathbb{D} \to \mathbb{V}\mathbb{D} \circ f_*$$

which we showed to be an equivalence in the previous lecture.

Consequently, the pushforward functor f_* carries quadratic objects of $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ to quadratic objects of $\mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C})$ and carries Poincare objects to Poincare objects. We obtain a map of classifying spaces $\mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_X) \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}), Q_Y)$. The same reasoning gives a map of simplicial spaces

$$\mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_X)_\bullet \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}), Q_Y)_\bullet$$

hence a map of L-theory spaces

$$L(X; \mathcal{C}, Q) \to L(Y; \mathcal{C}, Q).$$

In other words, $L(X; \mathcal{C}, Q)$ depends functorially on X.

We now study the functor $X \mapsto L(X; \mathcal{C}, Q)$.

Lemma 3. *Let $n \geq 0$ be an integer, and suppose that $f, g : X \to Y$ are homotopic PL maps of finite polyhedra. Then f and g induce the same map $L_n(X; \mathcal{C}, Q) \to L_n(Y; \mathcal{C}, Q)$.*

Proof. Replacing Q by $\Sigma^{-n} Q$, we can reduce to the case $n = 0$. Let $(\mathcal{F}, q)$ be a Poincare object of $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$. We wish to show that the Poincare objects $f_* \mathcal{F}$ and $g_* \mathcal{F}$ are cobordant. Choose a PL map $h : X \times [0, 1] \to Y$ which is a homotopy from f to g. Let $i_0 : X \simeq X \times \{0\} \hookrightarrow X \times [0, 1]$ be the canonical map, and define i_1 similarly. Since the pushforward functor h_* carries cobordisms to cobordisms, it will suffice to show that $i_{0,*} \mathcal{F}$ and $i_{1,*} \mathcal{F}$ are cobordant as Poincare objects of $\mathrm{Shv}_{\mathrm{const}}(X \times [0, 1]; \mathcal{C})$. It now suffices to observe that a bordism between these objects is given by $p^* \mathcal{F}$, where $p : X \times [0, 1] \to X$ denotes the projection. $\square$

From Lemma 3 we immediately deduce the following consequence:

Proposition 4. *Let $f : X \to Y$ be a PL homotopy equivalence between finite polyhedra. Then f induces a homotopy equivalence $L(X; \mathcal{C}, Q) \to L(Y; \mathcal{C}, Q)$.*

Let Poly denote the category whose objects are finite polyhedra and whose morphisms are PL maps. The construction $X \mapsto L(X; \mathcal{C}, Q)$ determines a functor from the category Poly to the ∞-category $\mathcal{S}$ of spaces. It follows from Proposition 4 that this functor factors through $\mathrm{Poly}[W^{-1}]$, where $\mathrm{Poly}[W^{-1}]$ denotes the ∞-category obtained from Poly by formally inverting all homotopy equivalences between finite polyhedra. The ∞-category $\mathrm{Poly}[W^{-1}]$ is equivalent to the full subcategory $\mathcal{S}^{\mathrm{fin}} \subseteq \mathcal{S}$ spanned by those spaces which are homotopy equivalent to a finite polyhedron (or equivalently, to a finite CW complex). We may therefore regard the functor $X \mapsto L(X; \mathcal{C}, Q)$ as defined on the ∞-category $\mathcal{S}^{\mathrm{fin}}$ of finite spaces.

To continue our analysis, it will be convenient to introduce a slight variation on the above construction. Let X be a finite polyhedron, and let $Y \subseteq X$ be a closed subpolyhedron. We then have a fully faithful embedding $i_* : \mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ which commutes with Verdier duality. It follows that the quotient ∞-category $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}) / \mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C})$ inherits a nondegenerate quadratic functor. This quotient can be identified with a full subcategory of $\mathrm{Shv}_{\mathrm{const}}(X, Y; \mathcal{C}) \subseteq \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$: namely, the subcategory spanned by those sheaves $\mathcal{F}$ such that $i^* \mathcal{F} \simeq 0$. (Note that, for any $\mathcal{F} \in \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$, the ∞-category of sheaves $\mathcal{F}' \in \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ equipped with a map $\mathcal{F}' \to \mathcal{F}$ whose cofiber is supported on Y has a final object, given by the extension by zero of $\mathcal{F}|(X - Y)$.) We let $L(X, Y; \mathcal{C}, Q)$ denote the L-theory space of $(\mathrm{Shv}_{\mathrm{const}}(X, Y; \mathcal{C}), Q_X)$. We have seen that there is a fiber sequence of spaces

$$L(Y; \mathcal{C}, Q) \to L(X; \mathcal{C}, Q) \to L(X, Y; \mathcal{C}, Q).$$

More generally, for $Z \subseteq Y \subseteq Z$, we have a fiber sequence

$$L(Y, Z; \mathcal{C}, Q) \to L(X, Z; \mathcal{C}, Q) \to L(X, Y; \mathcal{C}, Q).$$

Note that the ∞-category $\mathrm{Shv}_{\mathrm{const}}(X, Y; \mathcal{C})$ can be identified with the full subcategory of $\mathrm{Shv}_{\mathrm{const}}(X/Y; \mathcal{C})$ spanned by those sheaves which vanish at the base point of X/Y. For every pointed polyhedron Z, let $L^{\mathrm{red}}(Z; \mathcal{C}, Q)$ denote the relative L-theory space $L(Z, *; \mathcal{C}, Q)$. The construction $Z \mapsto L^{\mathrm{red}}(Z; \mathcal{C}, Q)$ is functorial with respect to pointed PL maps between pointed finite polyhedra. Moreover, Proposition 4 implies that it carries homotopy equivalences to homotopy equivalences, and therefore extends (in an essentially unique way) to a map

$$L^{\mathrm{red}}(\bullet; \mathcal{C}, Q) : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{S},$$

where $\mathcal{S}^{\mathrm{fin}}_*$ denotes the ∞-category of pointed finite spaces.

Proposition 5. *The functor $L^{red}(\bullet; \mathcal{C}, Q) : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{S}$ is excisive: that is, it carries homotopy pushout squares to homotopy pullback squares.*

Proof. Consider a homotopy pushout square of finite pointed spaces

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y'. \end{tikzcd}$$

Without loss of generality, we may assume that each of these spaces is a finite polyhedron, each of the maps are PL, the horizontal maps are inclusions. Consider the diagram

$$\begin{tikzcd} L^{\mathrm{red}}(X; \mathcal{C}, Q) \arrow[r] \arrow[d] & L^{\mathrm{red}}(X'; \mathcal{C}, Q) \arrow[r] \arrow[d] & L^{\mathrm{red}}(X'/X; \mathcal{C}, Q) \arrow[d, "\theta"] \\ L^{\mathrm{red}}(Y; \mathcal{C}, Q) \arrow[r] & L^{\mathrm{red}}(Y'; \mathcal{C}, Q) \arrow[r] & L^{\mathrm{red}}(Y'/Y; \mathcal{C}, Q). \end{tikzcd}$$

Since the rows are fiber sequences, to show that the left square is a homotopy pullback, it will suffice to show that θ is a homotopy equivalence. This is clear, since the map $X'/X \to Y'/Y$ is a homotopy equivalence, by virtue of our assumption that σ is a homotopy pushout square. $\square$

It follows from Proposition 5 that we can write

$$L^{\mathrm{red}}(X; \mathcal{C}, Q) \simeq \Omega^{\infty}(X \wedge \mathbf{L}(\mathcal{C}, Q))$$

for some spectrum $\mathbb{L}(\mathcal{C}, Q)$, which we will call the *L-theory spectrum* of the pair $(\mathcal{C}, Q)$. In particular, $L(X; \mathcal{C}, Q) \simeq L^{\mathrm{red}}(X_+; \mathcal{C}, Q)$ can be identified with the zeroth space of $X_+ \wedge \mathbb{L}(\mathcal{C}, Q)$. Taking X to be a point, we get $\Omega^{\infty}\mathbb{L}(\mathcal{C}, Q) = L(\mathcal{C}, Q)$, so that the homotopy groups of the spectrum $\mathbb{L}(\mathcal{C}, Q)$ are the L-groups of the pair $(\mathcal{C}, Q)$. More generally,

$$L_n(X; \mathcal{C}, Q) \simeq \pi_n(X_+ \wedge \mathbb{L}(\mathcal{C}, Q))$$

is the nth homology group of X with coefficients in the spectrum $\mathbb{L}(\mathcal{C}, Q)$.

Locally Constant Sheaves (Lecture 21)

March 21, 2011

Let X be a finite polyhedron with a triangulation T and let $\mathcal{C}$ be an ∞-category. We will say that T-constructible sheaf $\mathcal{F} : T \to \mathcal{C}$ is *locally constant* if $\mathcal{F}(\tau) \to \mathcal{F}(\tau')$ is invertible whenever $\tau \leq \tau'$. We let $\mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$ denote the full subcategory of $\mathrm{Shv}_T(X;\mathcal{C})$ spanned by the locally constant sheaves. This ∞-category does not depend on the choice of triangulation: if S is a refinement of T, then the pullback functor $\mathrm{Shv}_T(X;\mathcal{C}) \to \mathrm{Shv}_S(X;\mathcal{C})$ induces an equivalence on the full subcategories spanned by the locally constant sheaves.

Let R be an A_∞-ring, fixed for the remainder of this lecture. Our goal is to study *local systems* of R-modules on $\mathcal{C}$: that is, locally constant sheaves on $\mathcal{C}$ with values in the ∞-category of R-modules.

Fix a finite polyhedron X and a triangulation T of X. Let $\mathrm{Shv}_T(X : R)$ denote the ∞-category of T-constructible sheaves on X with values in $\mathrm{LMod}_R^{\mathrm{fp}}$: that is, contravariant functors $T^{op} \to \mathrm{LMod}_R^{\mathrm{fp}}$. The formation of R-linear duals gives a contravariant equivalence of $\mathrm{LMod}_R^{\mathrm{fp}}$ with $\mathrm{RMod}_R^{\mathrm{fp}}$; we may therefore identify $\mathrm{Shv}_T(X;R)$ with the opposite of the ∞-category $\mathrm{coShv}_T(X;\mathrm{RMod}_R^{\mathrm{fp}})$ of T-constructible *cosheaves* with values in $\mathrm{RMod}_R^{\mathrm{fp}}$ (that is, functors $T^{op} \to \mathrm{RMod}_R^{\mathrm{fp}}$). This is contained in the larger ∞-category $\mathrm{coShv}_T(X;\mathrm{RMod}_R)$ of cosheaves with values in RMod_R. In fact, we can identify $\mathrm{coShv}_T(X;\mathrm{RMod}_R)$ with the ∞-category of Ind-objects $\mathrm{Ind}(\mathrm{coShv}_T(X;\mathrm{RMod}_R^{\mathrm{fp}})$.

Let $\mathrm{coShv}_{\mathrm{lc}}(X : \mathrm{RMod}_R)$ denote the full subcategory of $\mathrm{coShv}_T(X;\mathrm{RMod}_R)$ spanned by the locally constant cosheaves (that is, those cosheaves for which $\mathcal{F}(\tau) \to \mathcal{F}(\tau')$ is an equivalence for every $\tau' \subseteq \tau \in T$). Note that $\mathrm{coShv}_{\mathrm{lc}}(X;\mathrm{RMod}_R)$ is closed under all limits and colimits in $\mathrm{coShv}_T(X;\mathrm{RMod}_R)$. It follows that the inclusion

$$\mathrm{coShv}_{\mathrm{lc}}(X;\mathrm{RMod}_R) \hookrightarrow \mathrm{coShv}_T(X;\mathrm{RMod}_R)$$

admits both left and right adjoints. We will denote a left adjoint to this inclusion by L. Let $\mathrm{coShv}_T^0(X;\mathrm{RMod}_R)$ denote the full subcategory of $\mathrm{coShv}_T(X;\mathrm{RMod}_R)$ spanned by those objects $\mathcal{F}$ such that $T(\mathcal{F}) \simeq 0$: that is, those objects $\mathcal{F}$ such that $\mathrm{Mor}_{\mathrm{coShv}_T(X;\mathrm{RMod}_R)}(\mathcal{F},\mathcal{G}) \simeq 0$ whenever $\mathcal{G}$ is locally constant. We let $\mathrm{coShv}_T^0(X;\mathrm{RMod}_R^{\mathrm{fp}})$ denote the intersection $\mathrm{coShv}_T^0(X;\mathrm{RMod}_R) \cap \mathrm{coShv}_T(X;\mathrm{RMod}_R^{\mathrm{fp}})$.

Lemma 1. *The full subcategory* $\mathrm{coShv}_T^0(X;\mathrm{RMod}_R)$ *is generated (under filtered colimits) by* $\mathrm{coShv}_T^0(X;\mathrm{RMod}_R^{\mathrm{fp}})$. *Consequently, we have a canonical equivalence*

$$\mathrm{coShv}_T^0(X;\mathrm{RMod}_R) \simeq \mathrm{Ind}\,\mathrm{coShv}_T^0(X;\mathrm{RMod}_R^{\mathrm{fp}}).$$

Proof. Since every object of $\mathrm{coShv}_T^0(X;\mathrm{RMod}_R^{\mathrm{fp}})$ is a compact object of $\mathrm{coShv}_T(X;\mathrm{RMod}_R)$, we get a fully faithful embedding

$$\mathrm{Ind}(\mathrm{coShv}_T^0(X;\mathrm{RMod}_R^{\mathrm{fp}}) \to \mathrm{coShv}_T(X;\mathrm{RMod}_R).$$

Let $\mathcal{C}$ denote the essential image of this embedding; it is a full subcategory of $\mathrm{Shv}_T(X;\mathrm{RMod}_R)$. We clearly have $\mathcal{C} \subseteq \mathrm{coShv}_T^0(X;\mathrm{RMod}_R)$. Let us prove the reverse inclusion. For any object $\mathcal{F} \in \mathrm{coShv}_T^0(X;\mathrm{RMod}_R)$, we can choose a fiber sequence

$$\mathcal{F}' \to \mathcal{F} \xrightarrow{\alpha} \mathcal{F}''$$

where $\mathcal{F}' \in \mathcal{C}$ and $\mathrm{Mor}(\mathcal{G}, \mathcal{F}'') \simeq 0$ for every $\mathcal{G} \in \mathcal{C}$ (here $\mathcal{F}'$ is given by the colimit of the filtered diagram of all objects of $\mathrm{coShv}^0_T(X; \mathrm{RMod}^{\mathrm{fp}}_R)$ equipped with a map to $\mathcal{F}$). For every simplex $\tau \in T$, let $\mathcal{F}_\tau \in \mathrm{coShv}_T(X; \mathrm{RMod}_R)$ be given by the formula

$$\mathcal{F}_\tau(\sigma) = \begin{cases} R & \text{if } \sigma \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

For any cosheaf $\mathcal{G}$, we have $\mathrm{Mor}(\mathcal{F}_\tau, \mathcal{G}) \simeq \mathcal{G}(\tau)$. Let $\tau' \subseteq \tau$, and form a cofiber sequence

$$\mathcal{F}_{\tau'} \to \mathcal{F}_\tau \to \mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'}\,.$$

If $\mathcal{G}$ is locally constant, we have $\mathrm{Mor}(\mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'}, \mathcal{G}) \simeq \mathrm{fib}(\mathcal{G}(\tau) \to \mathcal{G}(\tau')) \simeq 0$, so that $\mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'} \in \mathrm{coShv}^0_T(X; \mathrm{RMod}^{\mathrm{fp}}_R)$. It follows that $\mathrm{Mor}(\mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'}, \mathcal{F}'') \simeq 0$, so that $\mathcal{F}''(\tau) \simeq \mathcal{F}''(\tau')$. Since τ and τ' are arbitrary, we deduce that $\mathcal{F}''$ is locally constant. Since $\mathcal{F} \in \mathrm{coShv}^0_T(X; \mathrm{RMod}_R)$, the map α is nullhomotopic. Then $\mathcal{F}$ is a direct summand of $\mathcal{F}'$, and therefore belongs to $\mathcal{C}$ as desired. $\square$

Under the contravariant equivalence of ∞-categories $\mathrm{coShv}_T(X; \mathrm{RMod}^{\mathrm{fp}}_R) \simeq \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, the subcategory $\mathrm{coShv}^0_T(X; \mathrm{RMod}^{\mathrm{fp}}_R)$ corresponds to a full subcategory $\mathrm{Shv}^0_T(X : \mathrm{LMod}^{\mathrm{fp}}_R) \subseteq \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, which is evident closed under the formation of direct summands. Several lectures ago, we constructed a quotient ∞-category

$$\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R) / \mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$$

as a full subcategory of $\mathrm{Pro}(\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)) \simeq \mathrm{Ind}(\mathrm{coShv}_T(X; \mathrm{RMod}^{\mathrm{fp}}_R))^{op} \simeq \mathrm{coShv}_T(X; \mathrm{RMod}_R)^{op}$. Unwinding the definitions, we see that this subcategory consists precisely of those objects of the form $L\,\mathcal{F}$, where $\mathcal{F} \in \mathrm{coShv}_T(X; \mathrm{RMod}_R)^{\mathrm{fp}})$. Let us denote this subcategory by $\mathrm{coShv}^{\mathrm{fp}}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$.

We now study the ∞-category $\mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \simeq \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ in more detail. For simplicity, let us restrict our attention to the case where X is connected. For every point $x \in X$, let $i_x : \{x\} \to X$ denote the inclusion map. Pullback along i_x determines a functor $x^* : \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \to \mathrm{RMod}_R$ (given by evaluation at the unique simplex $\tau \in T$ containing x in its interior). This functor commutes with all limits and colimits. In particular, it admits a left adjoint, which we will denote by $x_+ : \mathrm{RMod}_R \to \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. Since x^* commutes with filtered colimits, $x_+(R)$ is a compact object of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. Moreover, it is a compact *generator* of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$: if $\mathcal{F} \in \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$, then $\mathrm{Mor}(x_+(R), \mathcal{F}) \simeq 0$ if and only if $\mathrm{Mor}_{\mathrm{RMod}_R}(R, x^*\,\mathcal{F}) = x^*\,\mathcal{F}$ vanishes. Since X is connected, this is equivalent to the vanishing of *all* stalks of $\mathcal{F}$: that is, to the condition that $\mathcal{F} \simeq 0$. It follows that $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ is equivalent to the ∞-category $\mathrm{RMod}_{R'}$, where R' is the A_∞-ring whose underlying spectrum is $\mathrm{Mor}_{\mathrm{Shv}_{\mathrm{lc}}(X;\mathrm{RMod}_R)}(x_+(R), x_+(R)) \simeq \mathrm{Mor}_{\mathrm{RMod}_R}(R, x^* x_+ R) \simeq x^* x_+ R$.

We can describe R' more explicitly. More generally, suppose we are given any pair of points $x, y \in X$. We can form a homotopy pullback diagram of topological spaces (commutative up to canonical homotopy)

$$\begin{tikzcd} P_{x,y} \arrow[r, "\phi"] \arrow[d, "\psi"] & \{x\} \arrow[d, "x"] \\ \{y\} \arrow[r, "y"] & X, \end{tikzcd}$$

where $P_{x,y}$ is the path space $\{p : [0,1] \to X : p(0) = x, p(1) = y\}$. Let ϕ^* and ψ^* denote the pullback functors on locally constant sheaves of right R-modules, and let ϕ_+ and ψ_+ be their left adjoints. There is a natural "base-change" isomorphism $y^* x_+ \simeq \psi_+ \phi^*$ of functors from RMod_R to itself. Consequently, $y^* x_+(R)$ is given by ψ_+ of the constant sheaf on $P_{x,y}$ with values in R. This is given by the smash product spectrum $P_{x,y} \wedge R$ (here we regard $P_{x,y}$ as an *unpointed* space) whose homotopy groups are given by the R-homology groups $R_*(P_{x,y})$ of the space $P_{x,y}$. In particular, we have $R' = P_{x,x} \wedge R$. If R is connective, we obtain

$$\pi_0 R' = R_0(P_{x,x}) = \bigoplus_{\eta \in \pi_0 P_{x,x}} \pi_0 R \simeq (\pi_0 R)[\pi_1 X].$$

Let us now describe the full subcategory $\mathrm{coShv}^{\mathrm{fp}}_{\mathrm{lc}}(X;\mathrm{RMod}_R) \subseteq \mathrm{coShv}_{\mathrm{lc}}(X;\mathrm{RMod}_R) \overset{\theta}{\simeq} \mathrm{RMod}_{R'}$. This is a stable subcategory, consisting of those objects of the form $L(\mathcal{F})$, where $\mathcal{F} \in \mathrm{coShv}_T(X;\mathrm{RMod}^{\mathrm{fp}}_R)$. Note that $\mathrm{coShv}_T(X;\mathrm{RMod}^{\mathrm{fp}}_R)$ is generated, as a stable ∞-category, by objects of the form $\mathcal{F}_{\tau,M}$, where

$$\mathcal{F}_{\tau,M}(\sigma) = \begin{cases} M & \text{if } \sigma \subseteq \tau \\ 0 & \text{otherwise} \end{cases}$$

and M is a finitely presented right R-module. It is therefore generated as a stable ∞-category by objects of the form $\mathcal{F}_{\tau,R} = \mathcal{F}_\tau$. We observe that $L\,\mathcal{F}_\tau \simeq y_+(R)$, where y is any point in the interior of τ: indeed, for any object $\mathcal{G} \in \mathrm{coShv}_{\mathrm{lc}}(X;\mathrm{RMod}_R)$ we have

$$\mathrm{Mor}(L\,\mathcal{F}_\tau, \mathcal{G}) \simeq \mathrm{Mor}(\mathcal{F}_\tau, \mathcal{G}) \simeq \mathcal{G}(\tau) \simeq y^*\,\mathcal{G} \simeq \mathrm{Mor}(y_+(R), \mathcal{G}).$$

Let x be our fixed base point of X. Since X is connected, for any point $y \in X$ there is an isomorphism $x_+(R) \simeq y_+(R)$ in $\mathrm{coShv}_{\mathrm{lc}}(X;\mathrm{RMod}_R)$ (obtained by choosing a path joining x and y). Consequently, the full subcategory $\mathrm{coShv}^{\mathrm{fp}}_{\mathrm{lc}}(X;\mathrm{RMod}_R) \subseteq \mathrm{coShv}_{\mathrm{lc}}(X;\mathrm{RMod}_R)$ is generated, as a stable subcategory, by the object $x_+(R)$. In particular, it corresponds to the full subcategory

$$\mathrm{RMod}^{\mathrm{fp}}_{R'} \subseteq \mathrm{RMod}_R$$

under the equivalence θ. Passing to opposite ∞-categories, we obtain the following result:

Theorem 2. *Let X be a connected finite polyhedron with base point x, T a triangulation of X, R an A_∞-ring, and $R' = P_{x,x} \wedge R$ the A_∞-ring constructed above. Let $\mathrm{Shv}^0_T(X;\mathrm{LMod}^{\mathrm{fp}}_R) \subseteq \mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)$ be defined as above. Then there is a canonical equivalence of ∞-categories*

$$\mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)/\,\mathrm{Shv}^0_T(X;\mathrm{LMod}^{\mathrm{fp}}_R) \simeq \mathrm{LMod}^{\mathrm{fp}}_{R'}\,.$$

Let $\mathrm{Shv}^0_{\mathrm{const}}(X;\mathrm{LMod}^{\mathrm{fp}}_R) \simeq \varinjlim_T \mathrm{Shv}^0_T(X;\mathrm{LMod}^{\mathrm{fp}}_R$. Passing to the direct limit over T, we obtain an equivalence

$$\mathrm{Shv}_{\mathrm{const}}(X;\mathrm{LMod}^{\mathrm{fp}}_R)/\,\mathrm{Shv}^0_{\mathrm{const}}(X;\mathrm{LMod}^{\mathrm{fp}}_R) \simeq \mathrm{LMod}^{\mathrm{fp}}_{R'}\,.$$

Warning 3. Working with perfect module spectra in place of finitely presented module spectra, one can construct a fully faithful embedding

$$\mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{perf}}_R)/\,\mathrm{Shv}^0_T(X;\mathrm{LMod}^{\mathrm{fp}}_R) \simeq \mathrm{LMod}^{\mathrm{perf}}_{R'}\,.$$

This embedding is generally not an equivalence, which is why we have generally confined our attention to the study of finitely presented modules rather than perfect modules.

Definition 4. Let X be a spectrum. We say that X is *invertible* if there exists another spectrum Y and a homotopy equivalence $X \wedge Y \simeq S$, where S is the sphere spectrum. One can show that a spectrum X is invertible if and only if $X \simeq \Sigma^n S$ for some integer n. We let $\mathrm{Sp}^{\mathrm{inv}}$ denote the full subcategory of Sp spanned by the invertible spectra.

Let X be a space (for now, let's say a polyhedron). A *spherical fibration* over X is a locally constant sheaf on X with values in $\mathrm{Sp}^{\mathrm{inv}}$.

Example 5. Let M be a piecewise linear manifold of dimension n, and let $\mathcal{F}$ be the constant sheaf of spectra taking the value S. Then the Verdier dual $\mathbb{D}(\mathcal{F})$ is a spherical fibration over M. For every point $x \in M$, we have seen that the stalk $x^*\mathbb{D}(\mathcal{F})$ can be described as the suspension spectrum of the homotopy quotient $M/M - \{x\}$, which is (noncanonically) homotopy equivalent to an n-sphere.

Now suppose that R is an A_∞-ring equipped with an involution σ. Let $Q : (\mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Sp}$ denote either of the quadratic functors Q^q or Q^s. Let X be a polyhedron equipped with a triangulation T, and let $\zeta : T \to \mathrm{Sp}^{\mathrm{inv}}$ be a spherical fibration on X. We define a quadratic functor $Q_{T,\zeta} : \mathrm{Shv}_T(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Sp}$ by the formula

$$Q_{T,\zeta}(\mathcal{F}) = \varinjlim_{\tau \in T} \zeta(\tau) \wedge Q(\mathcal{F}).$$

Since ζ is a constant functor locally on X, the work of the previous lectures shows that $Q_{T,\zeta}$ is a nondegenerate quadratic functor on $\mathrm{Shv}_T(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op}$. Passing to the limit over T, we obtain a nondegenerate quadratic functor

$$Q_\zeta : \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Sp}\,.$$

In particular, we obtain a "twisted" Verdier duality functor $\mathbb{D}_\zeta : \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{fp})$, given by

$$\mathbb{D}_\zeta(\mathcal{F}) = \zeta \wedge \mathbb{D}(\mathcal{F})$$

(where $\mathbb{D}$ denotes the standard Verdier duality functor discussed earlier).

In the next lecture, we will apply the paradigm of Lecture 8 to the cofiber sequence of stable ∞-categories

$$\mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}})$$

and the quadratic functor Q_ζ.

Assembly (Lecture 22)

March 22, 2011

Let R be an A_∞-ring and let X be a connected finite polyhedron with a triangulation T. In the last lecture, we defined a subcategory $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R) \subseteq \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$. Moreover, we showed that the quotient

$$\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R) / \mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R$$

can be identified with the ∞-category $(\mathrm{RMod}^{\mathrm{fp}}_{R'})^{op} \simeq \mathrm{LMod}^{\mathrm{fp}}_{R'}$ of finitely presented R'-module spectra, where $R' \simeq R \wedge \Omega(X)$ is the A_∞-ring whose modules are local systems of R-modules on X.

Our goal in this lecture is to study quadratic functors on $\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ which descend to the quotient category.

Definition 1. Let X be a spectrum. We say that X is *invertible* if there exists another spectrum Y and a homotopy equivalence $X \wedge Y \simeq S$, where S is the sphere spectrum. One can show that a spectrum X is invertible if and only if $X \simeq \Sigma^n S$ for some integer n. We let $\mathrm{Sp}^{\mathrm{inv}}$ denote the full subcategory of Sp spanned by the invertible spectra.

Let X be a space (for now, let's say a polyhedron). A *spherical fibration* over X is a locally constant sheaf on X with values in $\mathrm{Sp}^{\mathrm{inv}}$.

Example 2. Let M be a piecewise linear manifold of dimension n, and let $\mathcal{F}$ be the constant sheaf of spectra taking the value S. Then the Verdier dual $\mathbb{D}(\mathcal{F})$ is a spherical fibration over M. For every point $x \in M$, we have seen that the stalk $x^* \mathbb{D}(\mathcal{F})$ can be described as the suspension spectrum of the homotopy quotient $M/M - \{x\}$, which is (noncanonically) homotopy equivalent to an n-sphere.

Now suppose that R is an A_∞-ring equipped with an involution σ. Let $Q : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ denote either of the quadratic functors Q^q or Q^s. Let $\zeta : T \to \mathrm{Sp}^{\mathrm{inv}}$ be a spherical fibration on X. We define a quadratic functor $Q_\zeta : \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ by the formula

$$Q_\zeta(\mathcal{F}) = \varinjlim_{\tau \in T} \zeta(\tau) \wedge Q(\mathcal{F}).$$

Since ζ is a constant functor locally on X, the work of the previous lectures shows that Q_ζ is a nondegenerate quadratic functor on $\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)^{op}$. In particular, we obtain a "twisted" Verdier duality functor $\mathbb{D}_\zeta : \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Shv}_T(X; \mathrm{LMod}^{fp}_R)$, given by

$$\mathbb{D}_\zeta(\mathcal{F}) = \zeta \wedge \mathbb{D}(\mathcal{F})$$

(where $\mathbb{D}$ denotes the standard Verdier duality functor discussed earlier).

Lemma 3. *The subcategory* $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ *is stable under the twisted Verdier duality functor* $\mathbb{D}_\zeta$.

Proof. For each simplex $\tau \in T$, define $\mathcal{F}_\tau \in \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ by the formula

$$\mathcal{F}_\tau(\sigma) = \begin{cases} R & \text{if } \sigma \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Whenever $\tau' \subseteq \tau$, we have a canonical map $\mathcal{F}_\tau \to \mathcal{F}_{\tau'}$. Let us denote the fiber of this map by $\mathcal{F}_{\tau,\tau'}$. We saw in the previous lecture that $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ is generated by the objects $\mathcal{F}_{\tau,\tau'}$. It will therefore suffice to show that each $\mathbb{D}_\zeta \mathcal{F}_{\tau,\tau'}$ belongs to $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$.

If $\tau' \subset \tau'' \subset \tau$, then we have a fiber sequence

$$\mathcal{F}_{\tau,\tau''} \to \mathcal{F}_{\tau,\tau'} \to \mathcal{F}_{\tau'',\tau'}\,.$$

We may therefore reduce to proving the lemma for the pairs (τ, τ'') and (τ'', τ'). Suppose that $\tau \simeq \Delta^n$ and that τ' is a face of τ; let K denote the closure of $\partial\,\tau - \tau'$. Since τ is contractible, ζ is constant on τ; it will therefore suffice to show that $\mathbb{D}\,\mathcal{F}_{\tau,\tau'}$ belongs to $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$. A simple calculation shows that $\Sigma^{-n}\mathbb{D}\,\mathcal{F}_{\tau,\tau'}$ is given by the formula

$$\sigma \mapsto \begin{cases} R & \text{if } \sigma \subseteq \tau, \sigma \nsubseteq K \\ 0 & \text{otherwise.} \end{cases}$$

We can choose a different triangulation of X for which τ and K are simplices of the triangulation, in which case the sheaf abvoe is given by $\mathcal{F}_{\tau,K}$, and therefore belongs to $\mathrm{Shv}^0_{\mathrm{const}}(T; \mathrm{LMod}^{\mathrm{fp}}_R)$. □

Using the formalism of Lecture 8, we see that Q_ζ descends to give a quadratic functor Q_{lc} on $\mathrm{LMod}^{\mathrm{fp}}_{R'}$. When $Q = Q^s$, we will denote this functor by Q^s_{lc}; when $Q = Q^q$, we will denote this functor by Q^q_{lc}. Using the fact that Q^q and Q^s have the same associated bilinear functor B, we deduce that Q^s_{lc} and Q^q_{lc} have the same associated bilinear functor B_{lc}. Let us try to describe this bilinear functor more explicitly. Set $Q = Q^s$, so that R is a Poincare object of $\mathrm{LMod}^{\mathrm{fp}}_R$. Let $x \in X$ be our chosen base point. We will assume that x is a vertex of the triangulation T, and suppose that the invertible spectrum ζ_x is homotopy equivalent to the sphere spectrum (something that can always be achieved by an appropriate shift). Let $x_*(R) \in \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ denote the skyscraper sheaf with stalk R at the point x (and vanishing elsewhere). Then $x_*(R)$ has the structure of a Poincare object of $\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$. It follows that the image of $x_*(R)$ in the ∞-category $\mathrm{LMod}^{\mathrm{fp}}_{R'}$ has the structure of a Poincare object of $\mathrm{LMod}^{\mathrm{fp}}_{R'}$. By construction, this image can be identified with R' itself. Arguing as in Lecture 10, we deduce that R' is equipped with an involution σ, and that the bilinear functor $B_{\mathrm{lc}} : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \times (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ is given by $(M, N) \mapsto \mathrm{Mor}_{R'-R'}(M \wedge N, R')$.

Remark 4. To be even more explicit, one would like to describe the involution on the A_∞-ring R'. This involves a mixture of three ingredients:

(*i*) The given involution on R.

(*ii*) The involution of the loop space $\Omega(X) \simeq P_{x,x}$, given by reading each path in the opposite direction.

(*iii*) The nontriviality of the spherical fibration ζ

Suppose for example that R is connective, so that R' is connective and we have a canonical isomorphism of associative rings $\pi_0 R' \simeq (\pi_0 R)[\pi_1 X]$. Then the involution on $\pi_0 R'$ is given by

$$\sum_{g \in \pi_1 X} \lambda_g g \mapsto \sum_{g \in \pi_1 X} \epsilon(g)\sigma(\lambda_g) g^{-1}$$

where σ denotes the underlying involution on $\pi_0 R$ and $\epsilon : \pi_1 X \to \pm 1$ is the obstruction to choosing an orientation of the spherical fibration ζ.

Now armed with our description of B_{lc}, let us try to to describe Q_{lc} in the special case where $Q = Q^q$. Let us regard Q as a covariant functor $\mathrm{RMod}^{\mathrm{fp}}_R \to \mathrm{Sp}$, given by $Q(M) = B(M, M)_{h\Sigma_2}$. Then Q_ζ corresponds to the covariant functor $\mathrm{coShv}_T(X; \mathrm{RMod}^{\mathrm{fp}}_R) \to \mathrm{Sp}$ given by

$$Q_\zeta(\mathcal{F}) = \varinjlim_{\tau \in T} \zeta(\tau) \wedge B(\mathcal{F}(\tau), \mathcal{F}(\tau))_{h\Sigma_2}.$$

We can extend this to a functor

$$\widehat{Q}_\zeta(\mathcal{F}) : \mathrm{Ind}(\mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}})) \simeq \mathrm{coShv}_T(X; \mathrm{RMod}_R) \to \mathrm{Sp}$$

which commutes with filtered colimits; this is again given by the formula

$$\widehat{Q}_\zeta(\mathcal{F}) = \varinjlim_{\tau} \zeta(\tau) \wedge (\mathcal{F}(\tau) \wedge_R \mathcal{F}(\tau))_{h\Sigma_2}$$

The quadratic functor

$$Q_{\mathrm{lc}} : (\mathrm{LMod}_{R'}^{\mathrm{fp}})^{op} \simeq \mathrm{RMod}_{R'}^{\mathrm{fp}} \to \mathrm{Sp}$$

is given by composing $\widehat{Q}_\zeta$ with the fully faithful embedding

$$\theta : \mathrm{RMod}_{R'}^{\mathrm{fp}} \subseteq \mathrm{RMod}_{R'} \simeq \mathrm{coShv}_{\mathrm{lc}}(X : \mathrm{RMod}_R) \subseteq \mathrm{coShv}_T(X; \mathrm{RMod}_R).$$

It follows that the polarization B_{lc} is given by composing θ with the map $\widehat{B}_\zeta$, given by

$$(\mathcal{F}, \mathcal{G}) \mapsto \varinjlim_{\tau} \zeta(\tau) \wedge (\mathcal{F}(\tau) \wedge_R \mathcal{G}(\tau)).$$

We deduce that the natural map $B_{\mathrm{lc}}(M, M)_{h\Sigma_2} \to Q_{\mathrm{lc}}^q$ is an equivalence. This proves the following:

Proposition 5. *Let R be an A_∞-ring with involution, let X be a connected finite polyhedron with base point x, let ζ be a spherical fibration over X equipped with a trivialization at x, and let $R' \simeq R \wedge \Omega(X)$ be defined as above. Then the L-theory spectrum $\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q_{\mathrm{lc}}^q)$ can be identified with the quadratic L-theory spectrum $\mathbb{L}^q(R')$ of the A_∞-ring R', equipped with the involution described above. In particular, if R is connective, there is a canonical equivalence $\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q_{\mathrm{lc}}^q) \simeq \mathbb{L}^q((\pi_0 R)[\pi_1 X])$, where the involution on the group ring $(\pi_0 R)[\pi_1 X]$ is described in Remark 4.*

The analogous statement is generally not true for symmetric L-theory, because the construction $M \mapsto (M \wedge_R M)^{h\Sigma_2}$ generally does not commute with filtered colimits. This motivates the following definition:

Definition 6. Let R be an A_∞-ring with involution, let X be a finite polyhedron, and let ζ be a spherical fibration on X. We let $\mathbf{L}^v_{(X,\zeta)}(R)$ denote the L-theory spectrum associated to $(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q_{\mathrm{lc}}^s)$. We will refer to $\mathbf{L}_{(X,\zeta)}{}^v(R)$ as the *visible L-theory spectrum* of X.

Assume that X is connected with base point x. We then have a commutative diagram

$$\begin{tikzcd}
\mathbf{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q_\zeta^q) \arrow[r] \arrow[d] & \mathbf{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q_\zeta^s) \arrow[d] \\
\mathbf{L}^q(R') \arrow[r] & \mathbf{L}^v_{(X,\zeta)}(R).
\end{tikzcd}$$

If we assume that ζ is trivial and that R is connective, this simplifies to a commutative diagram of spectra

$$\begin{tikzcd}
X \wedge \mathbf{L}^q(R) \arrow[r] \arrow[d] & X \wedge \mathbf{L}^s(R) \arrow[d] \\
\mathbf{L}^q((\pi_0 R)[\pi_1 X]) \arrow[r] & \mathbf{L}^v_{(X,\zeta)}(R).
\end{tikzcd}$$

The vertical maps here are referred to as *assembly maps.*

Proposition 7. *The diagrams above are homotopy pullback squares.*

Proof. It suffices to show that we get a homotopy equivalence between the homotopy fibers of the vertical maps. Using localization theorem of Lecture 8, we can identify these homotopy fibers with the L-theory spectra of $\mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ with respect to Q^q_ζ and Q^s_ζ, respectively. It will therefore suffice to show that the canonical map $Q^q_\zeta \to Q^s_\zeta$ is an equivalence when evaluated on any $\mathcal{F} \in \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)$.

Let $U : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ be the homotopy cofiber of the map $Q^q \to Q^s$, so that U is given by the formula $U(M) = \mathrm{Mor}_{R-R}(M \wedge M, R)^{t\Sigma_2}$. Then U is an exact functor. It follows that $U(R)$ has the structure of an R-module spectrum, and that U is given by the formula $U(M) = \mathrm{Mor}_R(M, U(R))$. We deduce that for $\mathcal{F} \in \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, the cofiber of the map $Q^q_\zeta(\mathcal{F}) \to Q^s_\zeta(\mathcal{F})$ is given by

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge U(\mathcal{F}(\tau)) = \mathrm{Mor}_R(\varprojlim_{\tau} \zeta(\tau)^{-1} \wedge \mathcal{F}(\tau), U(R)).$$

If $\mathcal{F} \in \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, then the limit $\varprojlim_\tau \zeta(\tau)^{-1} \wedge \mathcal{F}(\tau)$ vanishes (since it is the spectrum of maps from the locally constant sheaf $\zeta \wedge R$ into $\mathcal{F}$). $\square$

Orientations of L-Theory (Lecture 23)

March 24, 2011

In the last lecture, we introduced the L-theory spectra $\mathbb{L}^q(X, \zeta, R)$ and $\mathbb{L}^s(X, \zeta, R)$, where R is an A_∞-ring with involution, X is a finite polyhedron, and ζ is a spherical fibration on X. When ζ is trivial, these spectra are given simply by $X \wedge \mathbb{L}^q(R)$ and $X \wedge \mathbb{L}^s(R)$, respectively. In general, they depend on the spherical fibration ζ. However, our excision argument generalizes to show that $\mathbb{L}^q(X, \zeta, R)$ is given by the homotopy colimit

$$\varinjlim_{\tau \in T} \mathbb{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta(\tau) \wedge Q^q)$$

where T denotes any triangulation of X. In other words, the homotopy groups of $\mathbb{L}^q(X, \zeta, R)$ are given by the homology of X with coefficients in a local system of spectra, given by $(x \in X) \mapsto \mathbb{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta(x) \wedge Q^q)$. This raises the following general question:

Question 1. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate functor Q, and let E be an invertible spectrum. What is the relationship between the L-theory spectra $\mathbb{L}(\mathcal{C}, Q)$ and $\mathbb{L}(\mathcal{C}, E \wedge Q)$?

In the situation of Question 1, we can write $E \simeq S^{-n}$ for some integer n. We have seen that there is a canonical isomorphism $L_k(\mathcal{C}, \Omega^n Q) = L_{k+n}(\mathcal{C}, Q)$, suggesting that we should have an equivalence of L-theory spectra $\mathbb{L}(\mathcal{C}, \Omega^n Q) \simeq \Omega^n \mathbb{L}(\mathcal{C}, Q)$. In other words, we have a homotopy equivalence

$$\theta_E : \mathbb{L}(\mathcal{C}, E \wedge Q) \simeq E \wedge \mathbb{L}(\mathcal{C}, Q).$$

For our purposes, we need to know this not just for an individual invertible spectrum E, but in the case where E ranges over the fibers of some spherical fibration. It is therefore important that our analysis be functorial with respect to automorphisms of E. In fact, it is not possible to choose θ_E to be functorial with respect to all automorphisms of E. However, we will show that it can be chosen to depend naturally on automorphisms which are of geometric origin.

Definition 2. Let M be PL manifold, and let $\underline{S}$ denote the local system of spectra on M taking the constant value S (where S is the sphere spectrum). The Verdier dual $\mathbb{D}(\underline{S})$ is a spherical fibration over M. We will denote the *inverse* of this spherical fibration by ζ_M. We refer to ζ_M as the *normal spherical fibration of* M. Unwinding the definitions, it can be described by the formula

$$\zeta_M(x) = (\Sigma^\infty(M/M - \{x\}))^{-1}.$$

There is a canonical map of spectra $S \to \Gamma(M; \underline{S})$. If M is compact, this dualizes to give a map

$$\Gamma(M; \mathbb{D}\underline{S}) \simeq \mathbb{D}\Gamma(M; \underline{S}) \to S.$$

This map gives a point in the zeroth space of the spectrum

$$\mathrm{Mor}_{\mathrm{Sp}}(\varprojlim_{\tau \in T} (\mathbb{D}\underline{S})(\tau), S) \simeq \varinjlim_{\tau \in T} \zeta_M(\tau)$$

where T denotes some triangulation of M. We will denote this point by $[M]$ and refer to it as the *fundamental class* of M.

More generally, if M is a PL manifold with boundary, we let ζ_M denote the local system of spectra on M obtained by extending the normal spherical fibration from the interior of M (note that the interior of M is homotopy equivalent to M, so there exists an essentially unique extension). In this case, we have a fundamental class

$$[M] \in \Omega^\infty(\varinjlim_{\tau \in T} \begin{cases} \zeta_M(\tau) & \text{if } \tau \not\subseteq \partial M \\ 0 & \text{otherwise.} \end{cases}$$

Let us now fix a PL manifold with boundary M. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. For each triangulation T of M, let

$$Q_{\zeta_M, T} : \mathrm{Shv}_T(M, \partial M; \mathcal{C})^{op} \to \mathrm{Sp}$$

be given by the formula

$$\varinjlim_{\tau} \begin{cases} Q_{\zeta_M, T}(\mathcal{F}(\tau)) \wedge \zeta_M(\tau) & \text{if } \tau \not\subseteq \partial M \\ 0 & \text{otherwise.} \end{cases}$$

Let $C \in \mathcal{C}$ be an object, and let $\underline{C}$ denote the constant sheaf on M with taking the value C (which we will identify with its image in $\mathrm{Shv}_T(M, \partial M; \mathcal{C})$). We then obtain a homotopy equivalence

$$Q_{\zeta_M, T}(\underline{C}) \simeq Q(C) \wedge \varinjlim_{\tau \in T} \begin{cases} \zeta_M(\tau) & \text{if } \tau \not\subseteq \partial M \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the fundamental class $[M]$ determines a map

$$Q(C) \to Q_{\zeta_M, T}(\underline{C}),$$

which we will denote by $q \mapsto q_{[M]}$. This constuction carries Poincare objects to Poincare objects, and induces a map of L-theory spectra

$$\Phi : \mathbb{L}(\mathcal{C}, Q) \to \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \partial M; \mathcal{C}); Q_{\zeta_M})$$

(where Q_{ζ_M} denotes the amalgamation of the quadratic functors $Q_{\zeta_M, T}$ where T ranges over all triangulations of M).

Example 3. Let M be a piecewise linear disk. For every point x in the interior of M, we have a canonical homotopy equivalence of pairs $(M, \partial M) \to (M, M - \{x\})$. Consequently, ζ_M is canonically equivalent to the constant sheaf taking the value E, where $E^{-1} = \Sigma^\infty(M/\partial M)$. It follows that $\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \partial M; \mathcal{C}), Q_\zeta)$ is given by $(M, \partial M) \wedge \mathbb{L}(\mathcal{C}, E \wedge Q) \simeq E^{-1} \wedge \mathbb{L}(\mathcal{C}, E \wedge Q)$. We may therefore identify Φ with a map of spectra $E \wedge \mathbb{L}(\mathcal{C}, Q) \to \mathbb{L}(\mathcal{C}, E \wedge Q)$.

Suppose $M \simeq \Delta^n$. Then Φ determines maps $L_{k+n}(\mathcal{C}, Q) \to L_k(\mathcal{C}, \Omega^n Q)$, which can be identified with the shift isomorphisms defined earlier. It follows that Φ is a homotopy equivalence whenever M is a piecewise linear disk.

The construction of Φ is functorial with respect to piecewise linear homeomorphisms of the PL manifold M.

Let us now introduce some terminology to describe the situation more systematically.

Let X be a polyhedron. A *closed n-disk bundle* over X is a map of polyhedra $q : D \to X$ such that every point $x \in X$ has an open neighborhood U for which there is a PL homeomorphism $q^{-1}U \simeq U \times \Delta^n$ (which commutes with the projection to U).

There is a canonical bijection between isomorphism classes of closed n-disk bundles over X and homotopy classes of maps $X \to B\,\mathrm{Disk}(n)$, where $B\,\mathrm{Disk}(n)$ denotes the classifying space of the (simplicial) group $\mathrm{Disk}(n)$ of PL homeomorphisms of Δ^n.

The disjoint union $\coprod_n B\,\mathrm{Disk}(n)$ is equipped with a multiplication which is associative up to coherent homotopy, classifying the formation of products of closed n-disk bundles. We can describe the group completion of $\coprod_n B\,\mathrm{Disk}(n)$ as a product $\mathbf{Z} \times \mathrm{BPL}$, where BPL is the direct limit $\varinjlim_n B\,\mathrm{Disk}(n)$.

Let $\mathrm{Pic}(S)$ denote the classifying space for invertible spectra (so that homotopy classes of maps $X \to \mathrm{Pic}(S)$ correspond to equivalence classes of spherical fibrations over X). Every closed disk bundle $q : D \to X$ has an associated spherical fibration, given by $x \mapsto \Sigma^\infty(D_x / \partial D_x)$. This construction determines a map $\coprod_n B\,\mathrm{Disk}(n) \to \mathrm{Pic}(S)$, which is multiplicative up to coherent homotopy and therefore extends to a map $\mathbf{Z} \times \mathrm{BPL} \to \mathrm{Pic}(S)$.

Fix $(\mathcal{C}, Q)$ as above. Over the space $\mathrm{Pic}(S)$, we have two local systems of spectra: one given by the formula $E \mapsto \mathbb{L}(\mathcal{C}, E \wedge Q)$ and one given by the formula $E \mapsto E \wedge \mathbb{L}(\mathcal{C}, Q)$. The above analysis implies that these two local systems are canonically equivalent when restricted to $\mathbf{Z} \times \mathrm{BPL}$. This proves the following:

Proposition 4. *Let X be a finite polyhedron with triangulation T, $\mathcal{C}$ a stable ∞-category equipped with a nondegenerate quadratic functor Q, and ζ a spherical fibration on X, classified by a map $X \to \mathrm{Pic}(S)$. Suppose that this classifying map factors through $\mathbf{Z} \times \mathrm{BPL}$ (that is, that the spherical fibration ζ arises from a closed disk bundle, at least stably). Then there is a homotopy equivalence (depending canonically on the factorization)*

$$\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q).$$

We can also make the analysis of the preceding discussion read in a different way. Let us suppose that $\mathcal{C} = \mathcal{D}^{\mathrm{fp}}(\mathbf{Z})$ is the ∞-category of perfect complexes of $\mathbf{Z}$-modules, and let Q be either Q^q or Q^s. Then Q is a spectrum valued functor which factors through the ∞-category of $\mathbf{Z}$-module spectra. It follows that for every spectrum E, we can write $E \wedge Q \simeq (E \wedge \mathbf{Z}) \wedge_{\mathbf{Z}} Q$, so that $E \wedge Q$ depends only on the generalized Eilenberg-MacLane spectrum $E \wedge \mathbf{Z}$. Let ζ be a spherical fibration on a polyhedron X, and suppose that ζ assigns to each point $x \in X$ a spectrum $\zeta(x)$ which is homotopy equivalent to $\Sigma^n S$. Suppose further that ζ is orientable. A choice of orientation determines a canonical homotopy equivalence of each $\zeta(x) \wedge \mathbf{Z}$ with $\Sigma^n \mathbf{Z}$, and therefore a natural isomorphism $Q_\zeta \simeq \Sigma^n Q$. It follows that we obtain a canonical homotopy equivalence

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), \Sigma^n Q) \simeq \Sigma^n \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q) \simeq \Sigma^n (X \wedge \mathbb{L}(\mathcal{C}, Q)).$$

This proves:

Proposition 5. *If ζ is an oriented spherical fibration (of dimension n) on X classified by a map $X \to \mathrm{Pic}(S)$ which factors through $\mathbf{Z} \times \mathrm{BPL}$, then we have homotopy equivalences (depending canonically on the choice of factorization)*

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^q(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^q(\mathbf{Z}))$$

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^s(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^s(\mathbf{Z}))$$

Remark 6. Proposition 5 can be interpreted as saying that every orientable PL bundle is *oriented* with respect to the ring spectrum $\mathbb{L}^s$. We will return to this point in the next lecture.

L-Theory Orientations of Manifolds (Lecture 24)

March 30, 2011

Let us begin by introducing some terminology to systematize the ideas of the previous lecture. Let X be a polyhedron. A *closed n-disk bundle* over X is a map of polyhedra $q : D \rightarrow X$ such that every point $x \in X$ has an open neighborhood U for which there is a PL homeomorphism $q^{-1}U \simeq U \times \Delta^n$ (which commutes with the projection to U).

There is a canonical bijection between isomorphism classes of closed n-disk bundles over X and homotopy classes of maps $X \rightarrow B\,\mathrm{Disk}(n)$, where $B\,\mathrm{Disk}(n)$ denotes the classifying space of the (simplicial) group $\mathrm{Disk}(n)$ of PL homeomorphisms of Δ^n.

The disjoint union $\coprod_n B\,\mathrm{Disk}(n)$ is equipped with a multiplication which is associative up to coherent homotopy, classifying the formation of products of closed n-disk bundles. We can describe the group completion of $\coprod_n B\,\mathrm{Disk}(n)$ as a product $\mathbf{Z} \times \mathrm{BPL}$, where BPL is the direct limit $\varinjlim_n B\,\mathrm{Disk}(n)$.

Let $\mathrm{Pic}(S)$ denote the classifying space for invertible spectra (so that homotopy classes of maps $X \rightarrow \mathrm{Pic}(S)$ correspond to equivalence classes of spherical fibrations over X). Every closed disk bundle $q : D \rightarrow X$ has an associated spherical fibration, given by $x \mapsto \Sigma^\infty(D_x / \partial\, D_x)$. This construction determines a map $\coprod_n B\,\mathrm{Disk}(n) \rightarrow \mathrm{Pic}(S)$, which is multiplicative up to coherent homotopy and therefore extends to a map $\mathbf{Z} \times \mathrm{BPL} \rightarrow \mathrm{Pic}(S)$.

Fix $(\mathcal{C}, Q)$ as above. Over the space $\mathrm{Pic}(S)$, we have two local systems of spectra: one given by the formula $E \mapsto \mathbb{L}(\mathcal{C}, E \wedge Q)$ and one given by the formula $E \mapsto E \wedge \mathbb{L}(\mathcal{C}, Q)$. The constructions of the previous lecture show that these spherical fibrations are (canonically) equivalent when restricted to $\mathbf{Z} \times \mathrm{BPL}$.:

Proposition 1. *Let X be a finite polyhedron with triangulation T, $\mathcal{C}$ a stable ∞-category equipped with a nondegenerate quadratic functor Q, and ζ a spherical fibration on X, classified by a map $X \rightarrow \mathrm{Pic}(S)$. Suppose that this classifying map factors through $\mathbf{Z} \times \mathrm{BPL}$ (that is, that the spherical fibration ζ arises from a closed disk bundle, at least stably). Then there is a homotopy equivalence (depending canonically on the factorization)*

$$\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q).$$

Now suppose that M is a compact PL manifold with boundary and let ζ_M be its normal fibration. Then ζ_M factors canonically through $\mathbf{Z} \times \mathrm{BPL}$. Let T be a triangulation of M, so that we have a canonical orientation

$$[M] : S \rightarrow \varinjlim_{\tau \in T} \begin{cases} 0 & \text{if } \tau \subseteq \partial\, M \\ \zeta_M(\tau) & \text{otherwise.} \end{cases}$$

Smashing with $\mathbb{L}(\mathcal{C}, Q)$ and using Proposition 1, we obtain a map

$$\begin{aligned} \mathbb{L}(\mathcal{C}, Q) &\rightarrow \varinjlim_{\tau \in T} \begin{cases} 0 & \text{if } \tau \subseteq \partial\, M \\ \zeta_M(\tau) \wedge \mathbb{L}(\mathcal{C}, Q) \end{cases} \\ &\simeq \varinjlim_{\tau \in T} \begin{cases} 0 & \text{if } \tau \subseteq \partial\, M \\ \mathbb{L}(\mathcal{C}, \zeta_M(\tau) \wedge Q) \end{cases} \\ &\simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \partial\, M; \mathcal{C}), Q_{\zeta_M}). \end{aligned}$$

This can be identified with the map constructed in the previous lecture, which carries each Poincare object (C, q) to a Poincare object $(\underline{C}, q_{[M]})$, where $\underline{C}$ denotes the constant sheaf taking the value C.

We can also make the analysis of the preceding discussion read in a different way. Let us suppose that $\mathcal{C} = \mathcal{D}^{\mathrm{fp}}(\mathbf{Z})$ is the ∞-category of perfect complexes of $\mathbf{Z}$-modules, and let Q be either Q^q or Q^s. Then Q is a spectrum valued functor which factors through the ∞-category of $\mathbf{Z}$-module spectra. It follows that for every spectrum E, we can write $E \wedge Q \simeq (E \wedge \mathbf{Z}) \wedge_{\mathbf{Z}} Q$, so that $E \wedge Q$ depends only on the generalized Eilenberg-MacLane spectrum $E \wedge \mathbf{Z}$. Let ζ be a spherical fibration on a polyhedron X, and suppose that ζ assigns to each point $x \in X$ a spectrum $\zeta(x)$ which is homotopy equivalent to $\Sigma^n S$. Suppose further that ζ is orientable. A choice of orientation determines a canonical homotopy equivalence of each $\zeta(x) \wedge \mathbf{Z}$ with $\Sigma^n \mathbf{Z}$, and therefore a natural isomorphism $Q_\zeta \simeq \Sigma^n Q$. It follows that we obtain a canonical homotopy equivalence

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), \Sigma^n Q) \simeq \Sigma^n \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q) \simeq \Sigma^n (X \wedge \mathbb{L}(\mathcal{C}, Q)).$$

This proves:

Proposition 2. *If ζ is an oriented spherical fibration (of dimension n) on X classified by a map $X \to \mathrm{Pic}(S)$ which factors through $\mathbf{Z} \times \mathrm{BPL}$, then we have homotopy equivalences (depending canonically on the choice of factorization)*

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^q(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^q(\mathbf{Z}))$$
$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^s(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^s(\mathbf{Z}))$$

Let $\mathbb{L}$ denote the symmetric L-theory spectrum $\mathbb{L}^s(\mathbf{Z})$. Tensor product of chain complexes endows $\mathbb{L}$ with the structure of a ring spectrum (in fact, an E_∞-ring spectrum). Let X be a space and let ζ be an oriented spherical fibration (of dimension n) over X whose classifying map factors through $\mathbf{Z} \times \mathrm{BPL}$. The above argument shows that the local system

$$(x \in X) \mapsto \Sigma^{-n}(\zeta(x) \wedge \mathbb{L})$$

is equivalent to the constant local system taking the value $\mathbb{L}$. In particular, this local system has a canonical global section; let us denote it by α.

Let us identify the field $\mathbf{Q}$ of rational numbers with the corresponding Eilenberg-MacLane spectrum. Since ζ is oriented, the local system

$$(x \in X) \mapsto \Sigma^{-n}(\zeta(x) \wedge \mathbf{Q})$$

is (canonically) equivalent to the constant local system taking the value $\mathbf{Q}$. Let $\mathbb{L}_{\mathbf{Q}} = \mathbb{L}\mathbf{Q}$ denote the rationalization of $\mathbb{L}$. Smashing with $\mathbb{L}$, we obtain a trivialization of the local system

$$(x \in X) \mapsto \Sigma^{-n}(\zeta(x) \wedge \mathbb{L}_{\mathbf{Q}}).$$

Under this equivalence, α determines a section of the constant local system taking the value $\mathbb{L}_{\mathbf{Q}}$. That is, we can identify α with an (invertible) element $l(\zeta)$ in the cohomology ring $(\mathbb{L}_{\mathbf{Q}})_0(X)$. This element depends naturally on the pair (X, ζ) (together with the orientation and PL-structure on ζ). We may therefore regard $l(\zeta)$ as a *characteristic class* of oriented PL bundles ζ. It measures the discrepancy between two $\mathbb{L}_{\mathbf{Q}}$-orientations of ζ: one coming from the $\mathbb{L}$-orientation defined above, and the other from the $\mathbf{Q}$-orientaiton on ζ.

Using our calculation of the homotopy groups of $\mathbb{L}^q(\mathbf{Z})$, we can be more explicit. We have seen that there is an isomorphism of commutative rings $\pi_* \mathbb{L} \simeq \mathbf{Q}[t^{\pm 1}]$, where t lies in degree -4. We therefore have a homotopy equivalence of spectra $\mathbb{L}_{\mathbf{Q}} \simeq \prod_{k \in \mathbf{Z}} \Sigma^{-4k} \mathbf{Q}$, and therefore an isomorphism of graded rings

$$(\mathbb{L}_{\mathbf{Q}})_*(X) \simeq \prod_{k \in \mathbf{Z}} \mathrm{H}^{*+4k}(X; \mathbf{Q}) \simeq \mathrm{H}^*(X; \mathbf{Q})[[t]]$$

In particular, we can identify $(\mathbb{L}_{\mathbf{Q}})_0(X)$ with the ring $\prod_{k\geq 0} \mathrm{H}^{4k}(X;\mathbf{Q})$, so that $l(\zeta)$ can be written as a formal sum

$$l_0(\zeta) + l_1(\zeta) + \cdots$$

where $l_k(\zeta) \in \mathrm{H}^{4k}(X;\zeta)$.

Now suppose that M is a compact closed oriented PL manifold of dimension n. Then M has orientations with respect to both $\mathbf{Q}$ and the L-theory spectrum $\mathbb{L}$, giving us fundamental classes

$$[M]_{\mathbf{Q}} \in \mathrm{H}_n(X;\mathbf{Q}) \qquad [M]_{\mathbb{L}} \in \mathbb{L}_n(M).$$

Let us identify $[M]_{\mathbf{Q}}$ and $[M]_{\mathbb{L}}$ with their images in $(\mathbb{L}_{\mathbf{Q}})_n(M)$. We then have

$$[M]_{\mathbf{Q}} = l(\zeta)[M]_{\mathbb{L}}$$

where ζ denotes the normal fibration of M. The projection map $p : M \to *$ induces a pushforward

$$p_* : (\mathbb{L}_{\mathbf{Q}})_n(M) \to (\mathbb{L}_{\mathbf{Q}})_n(*) = \begin{cases} \mathbf{Q} & \text{if } n = 4k \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $[M]_{\mathbb{L}}$ is represented by the constant sheaf on M taking the value $\mathbf{Z}$, regarded as a Poincare object of $\mathrm{Shv}_{\mathrm{const}}(M)$, so that $p_*([M]_{\mathbb{L}})$ can be represented by the cochain complex $C^*(M;\mathbf{Z})$, regarded as a Poincare object of $\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}$. If $n = 4k$, then the isomorphism $(\mathbb{L}_{\mathbf{Q}})_n \simeq \mathbf{Q}$ carries $p_*(M)$ to the signature σ_M of the manifold M. This proves an abstract version of the Hirzebruch signature formula: we have

$$\sigma_M = p_*([M]_{\mathbb{L}}) \simeq p_*(l(\zeta)[M]_{\mathbf{Q}}) = l(\zeta)_k[M],$$

where $l(\zeta)_k$ denotes the kth term in the characteristic class $l(\zeta)$ described above.

The Hirzebruch Signature Formula (Lecture 25)

March 30, 2011

In the last lecture, we defined a characteristic class $l(\zeta) \in \prod_k \mathrm{H}^{4k}(X; \mathbf{Q})$ associated to an oriented PL bundle on a space X, given by

$$l(\zeta) = l_0(\zeta) + l_4(\zeta) + \cdots$$

If M is a compact oriented PL manifold of dimension $4k$ and ζ is its (stable) normal bundle, then the signature of M is given by

$$\sigma_M = l_{4k}(\zeta)[M].$$

To obtain the Hirzebruch signature formula, we need to describe the characteristic class $l(\zeta)$ explicitly. For simplicity, let us restrict our attention to the case where ζ comes from a (virtual) vector bundle of rank zero, so that ζ is classified by a map $X \xrightarrow{\chi} \mathrm{BSO} = \varinjlim \mathrm{BSO}(n)$. Then ζ is the pullback of a universal bundle ζ_0 on BSO, so that $l(\zeta) = \chi^* l(\zeta_0)$. It will therefore suffice to describe $l(\zeta_0)$ as an element of the cohomology ring $\prod_k \mathrm{H}^{4k}(\mathrm{BSO}; \mathbf{Q}) \simeq \mathbf{Q}[[p_1, p_2, \ldots]]$; here p_i denote the universal Pontryagin classes.

Remark 1. Our restriction to virtual vector bundles of rank zero involves no essential loss of generality. First, the characteristic class $l(\zeta)$ does not change if we add a trivial bundle to ζ, so we may as well assume that ζ is of degree zero and classified by a map $X \to \mathrm{BSPL}$ (where SPL denotes the index two subgroup of PL corresponding to homeomorphisms which preserve orientation). At the very end of this course, we will see that the map $\mathrm{BSO} \to \mathrm{BSPL}$ is a rational homotopy equivalence.

Every complex vector bundle can be regarded as an oriented real vector bundle. This observation determines a map of classifying spaces $\mathrm{BU} \to \mathrm{BSO}$. The induced map

$$\mathrm{H}^*(\mathrm{BSO}; \mathbf{Q}) \to \mathrm{H}^*(\mathrm{BU}; \mathbf{Q})$$

is injective on cohomology. Consequently, to describe $l(\zeta_0)$, it will suffice to describe its image in $\mathrm{H}^*(\mathrm{BU}(n); \mathbf{Q})$ for every integer n. According to the splitting principle for complex line vector bundles, the canonical map

$$\mathrm{H}^*(\mathrm{BU}(n); \mathbf{Q}) \to \mathrm{H}^*(\mathrm{BU}(1)^n; \mathbf{Q}) \simeq \mathrm{H}^*(\mathrm{BU}(1); \mathbf{Q})^{\otimes n}$$

is injective on cohomology. In other words, it suffices to describe $l(\zeta)$ in the special case where ζ comes from a complex vector bundle of rank n that is written as a direct sum $\zeta_1 \oplus \cdots \oplus \zeta_n$ of complex line bundles. Our construction of L-theory orientations is compatible with the formation of direct sums of PL disk bundles, so we have

$$l(\zeta) = \prod_{1 \leq i \leq n} l(\zeta_i).$$

We are therefore reduced to describing $l(\zeta)$ when ζ arises from a complex line bundle. Here it suffices to treat the case $X = \mathrm{BU}(1) = \mathbf{C}\mathrm{P}^\infty$, so that $\mathrm{H}^*(X; \mathbf{Q}) \simeq \mathbf{Q}[[H]]$ where $H \in \mathrm{H}^2(X; \mathbf{Q})$ denotes the first Chern class of the universal line bundle. We can therefore write $l(\zeta) = f(H)$ for some power series f with rational coefficients, which are yet to be determined. Note that since $f(H)$ lies in $\prod_k \mathrm{H}^{4k}(X; \mathbf{Q})$, the power series f is even: that is, we have $f(H) = f(-H)$, so that f can be written as a function of H^2.

We can determine the power series f by applying the signature formula to the manifolds $\mathbf{C}\mathrm{P}^n$, where n ranges over the integers. Note that the tangent bundle T to $\mathbf{C}\mathrm{P}^n$ fits into an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T \to 0.$$

If we let $\zeta_{\mathbf{C}\mathrm{P}^n}$ denote the normal bundle to $\mathbf{C}\mathrm{P}^n$, we have $l(\zeta_{\mathbf{C}\mathrm{P}^n}) = l(T)^{-1} = l(\mathcal{O}(1))^{-n-1} = f(H)^{-n-1}$. The fundamental class of $\mathbf{C}\mathrm{P}^n$ is dual to H^n. It follows that the signature $\sigma(\mathbf{C}\mathrm{P}^n)$ of $\mathbf{C}\mathrm{P}^n$ is given by the coefficient of H^n in the power series $f(H)^{-n-1}$. Equivalently, the signature of $\mathbf{C}\mathrm{P}^n$ is given by the residue of (formal) differential $(\frac{1}{Hf(H)})^{n+1}dH$.

We can describe this residue more explicitly using the Lagrange inversion formula. Since the constant term of f is invertible, $Hf(H)$ is an invertible power series. That is, if we set $U = Hf(H)$, then there is a power series g such $H = g(U) = \sum c_i U^i$. Then $(\frac{1}{Hf(H)})^{n+1}dH = \frac{1}{U^{n+1}}dg(U) = \frac{1}{U^{n+1}}g'(U)dU$. The residue of this formal series is given by $(n+1)c_{n+1}$. It follows that

$$g(U) == \sum \frac{\sigma(\mathbf{C}\mathrm{P}^n)}{n+1} U^{n+1} = U + \frac{U^3}{3} + \frac{U^5}{5} + \cdots.$$

This power series describes the transcendental function $\tanh^{-1}$, so that $Hf(H) = \tanh(H)$ and $f(H) = \frac{\tanh(H)}{H}$.

Let us now make the above formula more explicit. When ζ comes from a complex line bundle, it has a first Chern class $c_1(\zeta)$, and $l(\zeta)$ is given by $f(c_1(\zeta))$. Since f is really a function of H^2, we can write $f(c_1(\zeta))$ as a power series in the Pontryagin class $p_1(\zeta) = c_1(\zeta)^2$. Informally, we have

$$l(\zeta) = \frac{\tanh(\sqrt{p_1(\zeta)})}{\sqrt{p_1(\zeta)}}$$

If ζ is a sum of complex line bundles $\zeta_1, \zeta_2, \ldots, \zeta_n$, then we have

$$l(\zeta) = \prod_i l(\zeta_i) = \prod_i \frac{\tanh(\sqrt{p_1(\zeta_i)})}{\sqrt{(}p_1(\zeta_i)}.$$

This can be written as an (infinite) sum of symmetric polynomials in the variables $p_1(\zeta_1), \ldots, p_1(\zeta_n)$. It can therefore be *rewritten* as an (infinite) sum of polynomials in the elementary symmetric polynomials in the variables $p_1(\zeta_1), \ldots, p_1(\zeta_n)$, which are the Pontryagin classes $p_1(\zeta), p_2(\zeta), \ldots, p_n(\zeta)$. Taking the limit as $n \to \infty$, we obtain a formula for $l(\zeta)$ in terms of the Pontryagin classes of ζ (which is valid for any orientable real vector bundle, by virtue of the above remarks concerning injectivity on cohomology).

We can recast the above discussion in the language of complex oriented cohomology theories. The symmetric L-theory spectrum $\mathbb{L}^s(\mathbf{Z})$ is orientable with respect to all oriented real vector bundles, and in particular has a canonical complex orientation. Following Quillen, we can associate to $\mathbb{L}^s(\mathbf{Z})$ a formal group law. The above calculation can be interpreted as saying that (modulo torsion) this formal group law is given by

$$(X, Y) \mapsto \tanh(\tanh^{-1}(X) + \tanh^{-1}(Y)).$$

That is, tanh is the inverse of the logarithm for the formal group law determined by our complex orientation on $\mathbb{L}^s$. Let

$$p(X) = X + \frac{X^2}{2} + \frac{X^3}{6} + \cdots = e^X - 1$$

be the inverse of the logarithm for the multiplicative formal group law. Then

$$\tanh(X) = \frac{e^X - e^{-X}}{e^X + e^{-X}} = \frac{e^{2X} - 1}{e^{2X} + 1} = \frac{p(2X)}{p(2X) + 2}.$$

After inverting 2, we see that $\tanh(X)$ and $p(X)$ differ by an invertible change of coordinate. It follows that the formal group law

$$(X, Y) \mapsto \tanh(\tanh^{-1}(X) + \tanh^{-1}(Y))$$

is isomorphic to the multiplicative formal group after inverting 2. Using Landweber exactness, we see that the spectrum $\mathbb{L}^s(\mathbf{Z})[\frac{1}{2}] \simeq \mathbb{L}^q(\mathbf{Z})[\frac{1}{2}]$ is determined up to homotopy equivalence by the graded ring $\pi_* \mathbb{L}^s(\mathbf{Z})[\frac{1}{2}] \simeq \mathbf{Z}[\frac{1}{2}][t^{\pm 1}]$ together with the associated formal group, which is the multiplicative group. This yields a proof of the following result:

Proposition 2. *There is a homotopy equivalence of spectra* $\mathbb{L}^s(\mathbf{Z})[\frac{1}{2}] \simeq \mathrm{KO}[\frac{1}{2}]$, *where* KO *denotes the real* K*-theory spectrum.*

Poincare Spaces and Spivak Fibrations (Lecture 26)

April 1, 2011

Let X be a topological space and $\mathcal{C}$ an ∞-category. We let $\mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$ denote the ∞-category of maps from the Kan complex $\mathrm{Sing}_{\bullet}(X)$ into $\mathcal{C}$. We will refer to $\mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$ as the ∞-category of *locally constant* $\mathcal{C}$-valued sheaves on X, or sometimes as the ∞-category of *local systems* of $\mathcal{C}$-valued sheaves on X. If X is a polyhedron with triangulation T, then we can identify $\mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$ with the full subcategory of $\mathrm{Shv}_T(X;\mathcal{C})$ spanned by those functors which carry each inclusion $\tau \subseteq \tau'$ of simplices to an invertible morphism of $\mathcal{C}$.

Let $f : X \to Y$ be a map of topological spaces. Then f induces a pullback functor $f^* : \mathrm{Shv}_{\mathrm{lc}}(Y;\mathcal{C}) \to \mathrm{Shv}_{\mathrm{lc}}(X;\mathcal{C})$. Suppose that $\mathcal{C}$ is the ∞-category of spectra. Then f^* preserves all limits and colimits, and therefore admits both a left adjoint $f_!$ and a right adjoint f_*.

In the special case where Y is a point, we will denote the functors $f_!$ and f_* by $C_*(X;\bullet)$ and $C^*(X;\bullet)$, respectively. If X si a polyhedron with triangulation T, these are described by the formulas

$$C_*(X;\mathcal{F}) = \varinjlim_{\tau} \mathcal{F}(\tau) \qquad C^*(X;\mathcal{F}) = \varprojlim_{\tau} \mathcal{F}(\tau).$$

If X is a *finite* polyhedron, we conclude that the construction $C^* : \mathrm{Shv}_{\mathrm{lc}}(X;\mathrm{Sp}) \to \mathrm{Sp}$ commutes with homotopy colimits.

Suppose now that X is connected with base point x. Then $\mathrm{Shv}_{\mathrm{lc}}(X;\mathrm{Sp})$ can be identified with the ∞-category of modules over the A_∞-ring $R = \Sigma^\infty_+ \Omega(X)$. Any functor $F : \mathrm{LMod}_R \to \mathrm{Sp}$ is determined by its value $F(R) \in \mathrm{Sp}$, together with its right R-module structure. Indeed, the fact that F commutes with homotopy colimits implies that F is given by $F(M) \simeq F(R) \wedge_R M$. We can identify $F(R)$ with a local system ζ on X, so that F is given by the formula $F(M) = C_*(X; M \wedge \zeta)$. This description generalizes immediately to the case where X is not assumed to be connected:

Proposition 1. *Let $F : \mathrm{Shv}_{\mathrm{lc}}(X;\mathrm{Sp}) \to \mathrm{Sp}$ be a functor which commutes with homotopy colimits. Then F is given by $F(\mathcal{F}) = C_*(\mathcal{F} \wedge \zeta)$, where ζ is a local system of spectra on X. Moreover, the local system ζ is determined uniquely up to equivalence.*

In particular, if X is a finite polyhedron (or any space equivalent to a finite polyhedron) and $f : X \to *$ denotes the projection map, we have an equivalence of functors

$$f_*(\bullet) \simeq C_*(\bullet \wedge \zeta_X)$$

for some local system ζ_X on X.

Definition 2. We say that a finite polyhedron X is a *Poincare space* if ζ_X is a spherical fibration (that is, if each of the fibers $\zeta_X(x)$ is an invertible spectrum). In this case, we say that ζ_X is the *Spivak normal fibration* of X.

Remark 3. Let X be a finite polyhedron containing a point x. Let $i : \{x\} \to X$ denote the inclusion and $p : X \to *$ the projection map, so that $p \circ i$ is a homeomorphism. Then we have a homotopy equivalence of spectra

$$i^*\zeta_X \simeq (p \circ i)_! i^* \zeta_X \simeq p_!(i_! i^* \zeta_X) \simeq p_!((i_! S) \wedge \zeta_X) \simeq C^*(X; i_! S).$$

In other words, the stalk $\zeta_X(x)$ is given by taking global sections of the local system of spectra on X that assigns to each point $y \in X$ the suspension spectra $\Sigma^\infty_+ P_{x,y}$, where $P_{x,y}$ denotes the path space $\{p : [0,1] \to X : p(0) = x, p(1) = y\}$.

Remark 4. Let $\underline{S}$ denote the constant local system on X with value the sphere spectrum, so we have a canonical map $S \to C^*(X; \underline{S}) \simeq C_*(X; \underline{S} \wedge \zeta_X) \simeq C_*(X; \zeta_X)$. We can identify this map with a point of $\Omega^\infty C_*(X; \zeta_X)$, which we will refer to as the *fundamental class* of X and denote by $[X]$.

The fundamental class determines the equivalence of functors $C^*(X; \bullet) \simeq C_*(X; \bullet \wedge \zeta_X)$: it is given by

$$C^*(X; \mathcal{F}) \simeq \mathrm{Mor}(\underline{S}, \mathcal{F}) \to \mathrm{Mor}(\zeta_X, \mathcal{F} \wedge \zeta_X) \to \mathrm{Mor}(C_*(X; \zeta_X), C_*(X; \mathcal{F} \wedge \zeta_X)) \stackrel{[X]}{\to} C_*(X; \mathcal{F} \wedge \zeta_X).$$

Example 5. Let X be a simply connected finite polyhedron. Then X is a Poincare space if and only if there exists a fundamental class $\eta_X \in \mathrm{H}_n(X; \mathbf{Z})$ which induces cap product isomorphisms $\phi_i : \mathrm{H}^i(X; \mathbf{Z}) \to \mathrm{H}_{n-i}(X; \mathbf{Z})$. The "only if" direction is obvious: if X is a Poincare space, then $\zeta_X \wedge \mathbf{Z}$ is necessarily equivalent to $\Sigma^{-n}\underline{\mathbf{Z}}$ (orientability is obvious, since X is simply connected) so we can take η_X to be the image of the fundamental class $[X]$; the desired result then follows from the equivalence

$$C^*(X; \underline{\mathbf{Z}}) \simeq C_*(X; \underline{\mathbf{Z}} \wedge \zeta_X) \simeq C_*(X; \Sigma^{-n}\underline{\mathbf{Z}}).$$

The converse requires the simple connectivity of X. Note that η_X induces a map of spectra $C^*(X; \underline{\mathbf{Z}}) \to \Sigma^{-n}\mathbf{Z}$, hence a map $C_*(X; \underline{Z} \wedge \zeta_X) \to \Sigma^{-n}\mathbf{Z}$ which is adjoint to a map $\theta : \underline{Z} \wedge \zeta_X \to \Sigma^{-n}\underline{\mathbf{Z}}$. We claim that θ is invertible (from which it will follow that each fiber of ζ_X is equivalent to the invertible spectrum $\Sigma^{-n}S$). Since X is simply connected (and the fibers of ζ_X are k-connective for $k \ll 0$), it will suffice to show that θ induces an equivalence after applying the functor C_*. That is, we must show that the canonical map

$$C^*(X; \underline{\mathbf{Z}}) \simeq C_*(X; \underline{Z} \wedge \zeta_X) \to \Sigma^{-n} C_*(\underline{\mathbf{Z}})$$

is a homotopy equivalence. On the level of homotopy groups, this is precisely the condition that the maps ϕ_i are isomorphisms.

Let us now depart from our previous convention and regard quadratic functors as *covariant* functors from a stable ∞-category $\mathcal{C}$ to spectra. If R is an A_∞-ring with involution, we have a quadratic functor $Q^s : \mathrm{RMod}_R \to \mathrm{Sp}$ given by

$$Q^s(M) = (M \wedge_R M)^{h\Sigma_2},$$

which restricts to a nondegenerate quadratic functor on $\mathrm{RMod}_R^{\mathrm{fp}}$. If X is a space equipped with a spherical fibration ζ and $f : X \to *$ denotes the projection map, then we obtain a quadratic functor $Q_\zeta : \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \to \mathrm{Sp}$ given by the formula $Q_\zeta(\mathcal{F}) = C_*(X; \zeta \wedge Q^s(\mathcal{F}))$, which is nondegenerate when restricted to the ∞-category of compact objects of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$.

Let $\underline{R}$ denote the constant sheaf on X having the value R. Given a map of spectra $\eta : S \to C_*(X; \zeta)$, we obtain a map $S \to C_*(X; \zeta \wedge R^{\mathrm{h}\Sigma_2}) \simeq Q_\zeta(\underline{R})$, which we will denote by q. Then the pair $(\underline{R}, q)$ is a quadratic object of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. Let B_ζ denote the polarization of Q_ζ, given by the formula

$$B_\zeta(\mathcal{F}, \mathcal{F}') = C_*(X; \mathcal{F} \wedge_R \mathcal{F}' \wedge \zeta).$$

If X is a Poincare space and $\zeta = \zeta_X$ is its Spivak normal fibration, then we have a homotopy equivalence

$$B_\zeta(\underline{R}, \mathcal{F}) = C_*(X; \underline{R} \wedge_R \mathcal{F} \wedge \zeta) \simeq C_*(X; \mathcal{F} \wedge \zeta) \simeq C^*(X; \mathcal{F}) \simeq \mathrm{Mor}(\underline{R}, \mathcal{F}).$$

This tells us that $\underline{R}$ is self-dual: that is, $(\underline{R}, q)$ is a Poincare object of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. We therefore obtain an element $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$, called the *visible symmetric signature* of the Poincare complex X.

For later use, we will need a slight generalization of the notion of a Poincare complex. Suppose we are given a map of finite spaces $\partial\, X \to X$ (which, up to homotopy equivalence, we may as well suppose is given by an inclusion between finite polyhedra). Given a local system of spectra $\mathcal{F}$ on X, we let $\mathcal{F}\,|\,\partial X$ denote the pullback of $\mathcal{F}$ to ∂X, and form fiber sequences

$$C_*(\partial X; \mathcal{F}\,|\,\partial X) \to C_*(X; \mathcal{F}) \to C_*(X, \partial X; \mathcal{F})$$

$$C^*(X, \partial X; \mathcal{F}) \to C^*(X; \mathcal{F}) \to C^*(\partial X; \mathcal{F}\,|\,\partial X).$$

Arguing as above, we see that $C^*(X, \partial X; \bullet)$ commutes with homotopy colimits and is therefore given by $\mathcal{F} \mapsto C_*(X; \zeta_{(X,\partial X)} \wedge \mathcal{F})$ for some local system $\zeta_{(X,\partial X)}$. The equivalence between $C^*(X, \partial X; \bullet)$ is determined by a fundamental class $[X] : S \to C_*(X, \partial X; \zeta_{(X,\partial X)})$. Note that $[X]$ determines a composite map

$$[\partial X] : S \xrightarrow{[X]} C_*(X, \partial X; \zeta_{(X,\partial X)}) \to \Sigma C_*(\partial X; \zeta_{(X,\partial X)}\,|\,\partial X) \simeq C_*(\partial X; \Sigma\zeta_{(X,\partial X)}\,|\,\partial X).$$

Definition 6. A pair of finite spaces $(X, \partial X)$ is a *Poincare pair* if the following conditions are satisfied:

(1) The local system $\zeta_{(X,\partial X)}$ defined above is a spherical fibration.

(2) The map $[\partial X]$ is a fundamental class for ∂X: that is, it induces a homotopy equivalence

$$C^*(\partial X; \mathcal{F}) \to C_*(\partial X; (\Sigma\zeta_{X,\partial X}\,|\,\partial X) \wedge \mathcal{F})$$

for every local system $\mathcal{F}$ on ∂X. (So that the Spivak normal fibration of ∂X is given by $\Sigma(\zeta_{X,\partial X}\,|\,\partial X)$.)

Remark 7. Let $(X, \partial X)$ be a pair of finite spaces $\mathcal{F}$ be a local system of spectra on X. We have a commutative diagram of fiber sequences

$$\begin{tikzcd}
C^*(X, \partial X; \mathcal{F}) \arrow[r] \arrow[d] & C^*(X; \mathcal{F}) \arrow[r] \arrow[d] & C^*(\partial X; \mathcal{F}\,|\,\partial X) \arrow[d] \\
C_*(X; \zeta_{(X,\partial X)} \wedge \mathcal{F}) \arrow[r] & C_*(X, \partial X; \zeta_{(X,\partial X)} \wedge \mathcal{F}) \arrow[r] & C_*(\partial X; (\Sigma\zeta_{(X,\partial X)} \wedge \mathcal{F})\,|\,\partial X)
\end{tikzcd}$$

where the vertical maps are given by cap product with $[X]$ and $[\partial X]$. The left vertical map is a homotopy equivalence by construction, and the right vertical map is a homotopy equivalence when $(X, \partial X)$ is a Poincare pair. It follows that if $(X, \partial X)$ is a Poincare pair, then the middle map is also a homotopy equivalence: that is, the cap product map

$$C^*(X; \mathcal{F}) \to C_*(X, \partial X; \zeta_{(X,\partial X)} \wedge \mathcal{F})$$

is a homotopy equivalence.

Suppose that $i : \partial X \to X$, and let R be an A_∞-ring with involution. We have a visible symmetric signature $\sigma^{vs}_{\partial X} \in \mathbb{L}^{vs}(\partial X, \zeta_{\partial X}, R)$, given by $(\underline{R}, q)$. Then q determines a symmetric bilinear form q_∂ on the object $i_! \underline{R} \in \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ with respect to the quadratic functor $Q_{\Sigma\zeta_{(X,\partial X)}}$. We have a canonical map $i_! \underline{R} \to \underline{R}$, and a fiber sequence

$$C_*(X, \partial X; \zeta_{(X,\partial X)} \wedge \underline{R}^{h\Sigma_2}) \to C_*(X; \Sigma\zeta_{(X,\partial X)} \wedge i_! \underline{R}^{\mathrm{h}\Sigma_2}) \to C_*(X; \Sigma\zeta_{(X,\partial X)} \wedge \underline{R}^{\mathrm{h}\Sigma_2}).$$

Consequently, the fundamental class $[X]$ provides a nullhomotopy of the image of q_∂ in $Q_{\Sigma\zeta_{(X,\partial X)}}(\underline{R})$. This nullhomotopy exhibits $\underline{R}$ as a Lagrangian for the Poincare object $(i_! \underline{R}; q_\partial)$. In other words, it gives a canonical lifting of $\sigma^{vs}_{\partial X}$ to the homotopy fiber of the map

$$\mathbb{L}^{vs}(\partial X, \zeta_{\partial X}, R) \to \mathbb{L}^{vs}(X, \Sigma\zeta_{(X,\partial X)}, R).$$

Let us denote this lifting by σ^{vs}_X. We will refer to it as the *visible symmetric signature of* X (or the *visible symmetric signature* of the Poincare pair $(X, \partial X)$).

Notation 8. Let $f : Y \to X$ be a map of spaces and let ζ be a spherical fibration on X. We let $\mathbb{L}^{vs}(X, Y, \zeta, R)$ denote the homotopy cofiber of the map

$$\mathbb{L}^{vs}(Y, f^*\zeta, R) \to \mathbb{L}^{vs}(X, \zeta, R).$$

Equivalently $\mathbb{L}^{vs}(X, Y, \zeta, R)$ is the homotopy fiber of the map

$$\mathbb{L}^{vs}(Y, f^*\Sigma\zeta, R) \to \mathbb{L}^{vs}(X, \Sigma\zeta, R).$$

The upshot of the above discussion is that if $(X, \partial X)$ is a Poincare pair, we can identify σ_X^{vs} with a point in the 0th space of $\mathbb{L}^{vs}(X, \partial X, \zeta_{(X,\partial X)}, R)$. When $\partial X = \emptyset$, this specializes to the definition of the visible symmetric signature of a Poincare space described earlier.

The Structure Space (Lecture 27)

April 4, 2011

Let X be a Poincare space. The goal of this course is to answer the following:

Question 1. When is X homotopy equivalent to a manifold?

In this lecture, we will formulate this question more precisely by introducing the (piecewise-linear) *structure space* of X, which we will denote by $\mathbb{S}(X)$. We first need to introduce a bit of terminology.

Let $f : M \to \Delta^k$ be a map of finite polyhedra. We will say that f is *neat* if, for every face $\tau \subseteq \Delta^k$ of codimension p, the following condition is satisfied:

$(*)$ There exists a neighborhood U of τ which is PL homeomorphic to $\tau \times \mathbb{R}^p_{\geq 0}$, such that the induced map $f^{-1}(U) \to \tau \times \mathbb{R}^p_{\geq 0} \to \mathbb{R}^p_{\geq 0}$ is a (necessarily trivial) PL fiber bundle.

Example 2. If f is a fiber bundle, then it is neat.

We will be particularly interested in the case where M is a PL manifold with boundary of dimension n. In this case, if $f : M \to \Delta^k$ is neat, then for every face $\tau \subseteq \Delta^k$ of codimension p, the inverse image $f^{-1}\tau$ is a PL manifold with boundary of dimension $n - p$. Moreover, the boundary of $f^{-1}\tau$ is the intersection $f^{-1}\tau \cap \partial M$.

Example 3. In the special case $k = 1$, a neat map $f : M \to \Delta^k$ determines a bordism from the PL manifold $f^{-1}\{0\}$ to the PL manifold $f^{-1}\{1\}$. Conversely, any bordism M between PL manifolds M_0 and M_1 admits neat map $M \to \Delta^1$, choosing any PL function $M \to [0, 1]$ which is sufficiently well-behaved near the two ends.

Now suppose that $(X, \partial X)$ is a Poincare pair. We may assume without loss of generality that X is a finite polyhedron and that ∂X is given a closed subpolyhedron of X. Let us suppose furthermore that ∂X is a PL manifold. For every integer m, we define a simplicial set $\mathbb{S}(X, \mathbb{R}^m)$ as follows: a k-simplex of $\mathbb{S}(X, \mathbb{R}^m)$ is a finite subpolyhedron $M \subseteq X \times \mathbb{R}^m \times \Delta^k$ satisfying the following conditions:

(a) M is a compact PL manifold with boundary.

(b) Let $N \subseteq M$ be the closure of boundary of $M \cap X \times \mathbb{R}^m \times (\Delta^k - \partial \Delta^k)$. Then the induced map $F : M \to X \times \Delta^k$ induces a PL homeomorphism $N \to \partial X \times \Delta^k$.

(c) Let $f : M \to \Delta^k$ be the composition of F with projection onto the second factor. Then f is neat.

(d) For every face $\tau \subseteq \Delta^k$, the induced map $f^{-1}(\tau) \to X$ is a homotopy equivalence.

We let $\mathbb{S}(X)$ denote the direct limit $\varinjlim_m \mathbb{S}(X, \mathbb{R}^m)$. We will refer to $\mathbb{S}(X)$ as the *structure space* of the pair $(X, \partial X)$.

Remark 4. Assume for simplicity that the boundary ∂M is empty. Informally, giving a k-dimensional simplex of $\mathbb{S}(X)$ is equivalent to giving a neat map $f : M \to \Delta^k$ such that, for each simplex $\tau \subseteq \Delta^k$, the inverse image $f^{-1}\tau$ is equipped with a homotopy equivalence to X (which is compatible with enlargement of τ). There is also an auxiliary datum of a map from M into $\mathbb{R}^m$ for $m \gg 0$, but this data should be ignored (it is there to rigidify the problem, so that the k-simplices of $\mathbb{S}(X)$ form a set rather than a topological groupoid).

Remark 5. The structure space $\mathbb{S}(X)$ defined above is a Kan complex.

Following Remark 4 (and still assuming $\partial X = \emptyset$), we can describe the low-dimensional simplices of $\mathbb{S}(X)$ as follows:

- Giving a 0-simplex of $\mathbb{S}(X)$ is equivalent to giving a compact PL manifold M equipped with a homotopy equivalence $M \to X$.
- A 1-simplex of $\mathbb{S}(X)$ joining 0-simplex corresponding to $f_0 : M \to X$ and $f_1 : N \to X$ is a bordism B from M to N, together with a map $f : B \to X$ extending f_0 and f_1, such that f is a homotopy equivalence. It follows that the inclusions $M \hookrightarrow B \hookleftarrow N$ are homotopy equivalences: that is, B is an *h-cobordism* from M to N.
- ...

In particular, $\pi_0\mathbb{S}(X)$ is the collection of equivalence classes of PL manifolds equipped with a homotopy equivalence to X, where the equivalence relation is given by h-cobordism.

Variant 6. Suppose that we modify the definition of $\mathbb{S}(X)$ by adding the following additional condition on a k-simplex of $\mathbb{S}(X; \mathbb{R}^m)$:

(e) The map $f : M \to \Delta^k$ is a PL fiber bundle.

We then obtain a different notion of structure space $\mathbb{S}^+(X)$, where $\pi_0\mathbb{S}^+(X)$ is the collection of PL manifolds equipped with a homotopy equivalence to X, up to the equivalence relation given by PL homeomorphism. We have a map of structure spaces $\mathbb{S}^+(X) \to \mathbb{S}(X)$. The structure space $\mathbb{S}^+(X)$ is much more difficult to describe than $\mathbb{S}(X)$: to obtain information about it, one must go far beyond the tools we have introduced in this class.

We can now restate our main goal as follows:

Question 7. Given a Poincare space X (or a Poincare pair $(X, \partial X)$ where ∂X is a PL manifold), what is the homotopy type of the structure space $\mathbb{S}(X)$?

To answer this question, let us fix an A_∞-ring R with involution. Assume for simplicity that $\partial X = \emptyset$. Let M be a compact closed PL manifold, and let ζ_X and ζ_M be the Spivak normal bundles to X and M, respectively. Since M is a PL manifold, the constant sheaf $\underline{R}$ is a Poincare object of $(\mathrm{Shv}_{\mathrm{const}}(M; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_M})$. Suppose that we are given a homotopy equivalence $f : M \to X$, so that $\zeta_M \simeq f^*\zeta_X$. Then $f_*\underline{R}$ determines a Poincare object of $(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X})$, and therefore a point of the space

$$\mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X}).$$

Elaborating on this construction, we obtain a map of simplicial spaces

$$\mathbb{S}(X)_\bullet \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X})_\bullet.$$

Passing to geometric realizations, we get a map

$$\mathbb{S}(X) \to L(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X}) \simeq \Omega^\infty \mathbb{L}^s(X, \zeta_X, R).$$

Let us denote this map by σ^s (for "symmetric signature"). It is closely related to the visible symmetric signature of X defined in the previous lecture. More precisely, the composite map

$$\mathbb{S}(X) \xrightarrow{\sigma} \Omega^\infty \mathbb{L}^s(X, \zeta_X, R) \to \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$$

is homotopic to a constant map, taking the value σ_X^{vs}. (Unwinding the definitions, this amounts to the observation that when $f : M \to X$ is a homotopy equivalence, then the functor $f_!$ carries the locally constant sheaf $\underline{R}$ on M to the locally constant sheaf $\underline{R}$ on X.)

We can summarize the situation informally as follows. If X is a Poincare space, we can define a visible symmetric signature $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$. This invariant reflects the fact that X satisfies a global form of Poincare duality (that is, we have a Poincare duality isomorphism for locally constant sheaves on X). However, if we are given a homotopy equivalence $f : M \to X$ where M is a PL manifold, then we can say more. The manifold M satisfies Poincare duality not only globally but also locally: that is, the constant sheaf on M is self-dual in the category of all (constructible) sheaves on M, rather than merely the locally constant sheaves. We therefore obtain an invariant $\sigma^s(M) \in \Omega^\infty(\mathbb{L}^s(X, \zeta_X, R)$, which refines the visible symmetric signature (and depends on the homotopy equivalence $f : M \to X$).

To describe the situation more systematically, let F_R denote the homotopy fiber of the map $\Omega^\infty \mathbf{L}^s(X, \zeta_X, R) \to \Omega^\infty \mathbf{L}^{vs}(X, \zeta_X, R)$ (taken over the point σ_X^{vs}). The above constructions determine a map of spaces $\mathbb{S}(X) \to F$.

Non-Theorem 1. *Let X be a Poincare space and let $R = \mathbf{Z}$. Then the map $\mathbb{S}(X) \to F_R$ is a homotopy equivalence.*

Assertion 1 is not quite right. To correct it, we need to make three modifications:

(1) We must work in the setting of topological manifolds, rather than piecewise linear manifolds. (In dimensions ≥ 5, there is not too much difference between the two. As a consequence, Assertion 1 is not too far from being correct in the PL setting, as we will see later.)

(2) We must restrict our attention to the case where the Poincare space X has dimension at least 5 (the domain of high-dimensional topology). If we work in the topological setting, this can be slightly relaxed: one can allow 4-dimensional Poincare spaces provided that their fundamental groups are not too big.

(3) We need to replace F by an appropriate subset $F_R^0 \subseteq F_R$ which we now describe. Recall that $\mathbf{L}^s(X, \zeta_X, R)$ can be identified with $C_*(X; \mathbf{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s))$. Since X is a Poincare space, we can rewrite this spectrum as

$$C^*(X; \zeta_X^{-1} \mathbf{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s).$$

That is, we can identify points of $\Omega^\infty \mathbb{L}^s(X, \zeta_X, R)$ with maps of local systems $\zeta_X \to \mathcal{F}$, where $\mathcal{F}$ is the local system of spectra which assigns to each point $x \in X$ the spectrum $\mathbf{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s)$. We have $R = \mathbf{Z}$, so that $\mathbb{L}^s(R)$ is a ring spectrum. Consequently, any point of $\Omega^\infty \mathbb{L}^s(X, \zeta_X, R)$ determines, for each $x \in X$, a map of spectra

$$\theta_x : \zeta_X(x) \wedge \mathbb{L}^s(R) \to \mathbb{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s).$$

Let $(\Omega^\infty \mathbb{L}^s(X, \zeta_X, R))^\times$ denote the subspace of of $\Omega^\infty \mathbb{L}^s(X, \zeta_X, R)$ corresponding to those points for which each θ_x is a homotopy equivalence, and let F_R^0 denote the homotopy fiber of the induced map

$$\Omega^\infty \mathbb{L}^s(X, \zeta_X, R))^\times \to \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$$

(taken over the visible symmetric signature of R). Note that the map $\mathbb{S}(X) \to F_R$ factors through F_R^0: when X is a PL manifold, the map θ_x defined above is determined by the orientation of $\mathbb{L}^s(\mathbf{Z})$ with respect to the PL tangent bundle of X.

With these amendments, assertion 1 becomes a theorem. This is Ranicki's theory of the *total surgery obstruction*. We will not prove the theorem in this course (though we will get close), primarily because we do not want to talk about topological manifolds.

We close this lecture by observing that the above analysis can be generalized to the case where $(X, \partial X)$ is a Poincare pair with ∂X a piecewise linear manifold. In this case, we have defined a visible symmetric signature

$$\sigma_{(X,\partial X)}^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \partial X, \zeta_X, R) = \Omega^\infty \operatorname{cofib}(\mathbb{L}^{vs}(\partial X, \zeta_X | \partial X, R) \to \mathbb{L}^{vs}(X, \zeta_X, R)).$$

Let $\mathbb{L}^s(X, \partial X, \zeta_X, R)$ denote the cofiber of the map

$$\mathbb{L}^s(\partial X, \zeta_X | \partial X, R) \to \mathbb{L}^s(X, \zeta_X, R),$$

and let F_R denote the homotopy fiber of the map

$$\Omega^\infty \mathbb{L}^s(X, \partial X, \zeta_X, R) \to \Omega^\infty \mathbb{L}^{vs}(X, \partial X, \zeta_X, R)$$

over the point $\sigma^{vs}_{(X, \partial X)}$. The constructions above generalize to give a map $\mathbb{S}(X) \to F_R$. Roughly speaking, our goal over the next few lectures is to show that this map is close to being a homotopy equivalence.

Transversality (Lecture 28)

April 6, 2011

Let X be a Poincare space. Our goal is to determine whether or not there exists a homotopy equivalence $f : M \to X$, where M is a compact PL manifold. Suppose that this is the case. In this lecture, we will consider the first obstruction: X must admit a PL tangent bundle. More precisely, if we let $\chi : X \to \mathrm{Pic}(S)$ be the map classifying the Spivak normal bundle of X, then χ must factor (up to homotopy) through the classifying space $\mathbf{Z} \times \mathrm{BPL}$.

Let us be more precise. Let X be a finite complex, and choose a map $X \to \mathbf{Z} \times \mathrm{BPL}$, classifying a (stable) PL bundle on X which we will denote by ζ. Let ζ_0 denote the underlying spherical fibration of ζ (so that ζ_0 is a local system of invertible spectra on X). By a *normal structure* on ζ, we mean a map of spectra $S \to C_*(X; \zeta_0)$ which induces a homotopy equivalence $C^*(X; \mathcal{F}) \to C_*(X; \zeta_0 \wedge \mathcal{F})$ for every local system of spectra $\mathcal{F}$ on X. If ζ admits a normal structure, then X is a Poincare space, and a choice of normal structure amounts to a choice of homotopy equivalence $\zeta_0 \simeq \zeta_X$, where ζ_X denotes the Spivak normal fibration of X.

Let $\mathbb{S}^{tn}(X)$ denote a classifying space for pairs $(\zeta, [X])$, where ζ is a stable PL bundle on X and $[X] : S \to C_*(X; \zeta_0)$ is a normal structure on ζ. We will refer to $\mathbb{S}^{tn}(X)$ as the space of *tangential normal structures* on X. If X is a Poincare space, then $\mathbb{S}^{tn}(X)$ can be described as the homotopy fiber of the map

$$(\mathbf{Z} \times \mathrm{BPL})^X \to \mathrm{Pic}(S)^X,$$

taken over the map $X \to \mathrm{Pic}(S)$ classifying the Spivak normal fibration of X. (If X is not a Poincare space, then $\mathbb{S}^{tn}(X)$ is empty.)

Our goal in this lecture is to obtain a more geometric description of the space $\mathbb{S}^{tn}(X)$.

Definition 1. Let X be a finite complex and let ζ be a stable PL bundle over X. A *normal map* to X consists of the following data:

(*a*) A map $f : M \to X$, where M is a compact PL manifold.

(*b*) An equivalence $\alpha : f^*\zeta \simeq -T_M$ of stable PL bundles (where T_M denotes the PL tangent bundle of M).

In this case, we obtain a map of spectra

$$S \overset{[M]}{\to} C_*(M; \zeta_M) \overset{\alpha}{\simeq} C_*(M; f^*\zeta_0) \to C_*(X; \zeta_0).$$

where ζ_M denotes the Spivak normal bundle of M (which underlies the stable PL bundle $-T_M$) and $[M]$ denotes the fundamental class of M. We will say that f is a *degree one normal map* if this composite map determines a normal structure on ζ.

We will generally abuse notation by identifying a normal map $(f, \alpha) : (M, -T_M) \to (X, \zeta)$ with the underlying map $f : M \to X$. Note that if there exists a degree one normal map $M \to X$, then X must be a Poincare space. Conversely, suppose that X is a Poincare space and that ζ is a stable PL bundle refining the Spivak normal bundle. Then we have $C_*(X; \zeta_0) \simeq C^*(X; \underline{S})$, so that any normal map $f : M \to X$ determines a section of the trivial bundle $\underline{S}$. The condition that f be degree one is the condition that this section determines an equivalence from $\underline{S}$ to itself. More concretely, it means that the fundamental class $[M]$ pushes forward to a fundamental class of X.

Suppose we are given a finite polyhedron X and a stable PL bundle ζ on X. If there exists a degree one normal map $f : M \to X$, then ζ admits a normal structure. The converse is true as well. Suppose we are given a normal structure $u : S \to C_*(X; \zeta_0)$. We can identify $C_*(X; \zeta_0)$ with the Thom spectrum X^{ζ_0}. This Thom spectrum can be described more concretely as follows: choose a PL disk bundle $D \to X$ and an equivalence of stable PL bundles $D \simeq \zeta \oplus \underline{\mathbb{R}}^n$, for $n \gg 0$. Then $\Sigma^n X^{\zeta_0}$ is the suspension spectrum of the Thom space $D/\partial D$ of the disk bundle D. In particular, after enlarging n, we can assume that u is induced by a map of pointed spaces $e : S^n \to D/\partial D$. Choose a PL section of the map $D \to X$ whose image Z is disjoint from ∂D (which we will refer to as the *zero section*). Modifying e by a homotopy, we may assume that it is piecewise linear and in general position with respect to the zero section (if we work in the setting of vector bundles rather than PL bundles, we should assume here that e is transverse to the zero section). Then $M = e^{-1}Z$ is a compact PL submanifold of S^n. Moreover, we have an equivalence of stable PL bundles $T_M \oplus e^* D \simeq \underline{R}^n$, so that e induces a normal map $M \to X$. By construction, this normal map carries the Thom-Pontryagin collapse map $S \to M^{-T_M}$ to the normal structure u, and is therefore of degree one.

The discussion above involved some arbitrary choices, so the normal map $f : M \to X$ is not uniquely determined by the normal structure on ζ. Let us now explain how to account for this ambiguity.

Definition 2. Let $k \geq 0$ be an integer, let X be a polyhedron, and let ζ be a stable PL bundle on X. A *Δ^k-family of normal maps to X* consists of the following data:

- (a) A map $M \to X \times \Delta^k$, inducing a neat map $e : M \to \Delta^k$ and a map $f : M \to X$, where M is a PL manifold with boundary $\partial M = e^{-1} \partial \Delta^k$.
- (b) An equivalence of stable PL bundles $\alpha : T_M \simeq e^* T_{\Delta^k} - f^*\zeta$.

In this case, for every vertex v of Δ^k, f induces a normal map from the PL manifold $M_v = g^{-1}\{v\}$ to X. We will say that $(M \to \Delta^k \times X, \alpha)$ has *degree one* if the induced normal map $M_v \to X$ has degree one. Note that this condition is independent of the choice of v (because the underlying map $S \to C_*(X; \zeta_0)$ is independent of v, up to homotopy).

Definition 3. Let $k \geq 0$ be an integer and let X be a finite complex. We let $\mathbb{S}^n(X)_k$ denote a classifying space for the following data:

- (i) A stable PL bundle ζ on X.
- (ii) A Δ^k-family of normal maps $M \to \Delta^k \times X$ having degree one.

The definition of $\mathbb{S}^n(X)_k$ is functorial in k; we therefore obtain a simplicial space $\mathbb{S}^n(X)_\bullet$. We will denote the realization of $\mathbb{S}^n(X)_\bullet$ by $\mathbb{S}^n(X)$, and refer to it as the *normal structure space of X*.

Every Δ^k-family of normal maps to a pair (X, ζ) determines a normal structure on the stable PL bundle ζ (given by the pushforward of the fundamental class of M along the map $M \to X \times \Delta^k$). We therefore obtain a map

$$\mathbb{S}^n(X) \to \mathbb{S}^{tn}(X).$$

Proposition 4. *The map $\mathbb{S}^n(X) \to \mathbb{S}^{tn}(X)$ is a homotopy equivalence.*

In other words, any stable PL structure on the Spivak normal bundle of a Poincare complex X determines a normal map $f : M \to X$, where M is well-defined up to (normal) bordism.

Proof. Let $Z_\bullet$ denote the constant simplicial space taking the value $\mathbb{S}^{tn}(X)$. It will suffice to show that the map of simplicial spaces $\mathbb{S}^n(X)_\bullet \to Z_\bullet$ is a trivial Kan fibration. In other words, for every integer k, we must show that the map

$$\mathbb{S}^n(X)_k \to M_k(\mathbb{S}^n(X)) \times_{\mathbb{S}^{tn}(X)^{\partial \Delta^k}} \mathbb{S}^{tn}(X)$$

is surjective on connected components. When $k = 0$, this amounts to the argument given above. The proof in general is similar (it involves general position/ transversality arguments). □

Now let X be an arbitrary Poincare space, and suppose we are given a homotopy equivalence $f : M \to X$, where M is a compact PL manifold. Since f is a homotopy equivalence, we can choose a stable PL bundle ζ on X and an equivalence $-T_M \simeq f^*\zeta$, so that f has the structure of a degree one normal map. More generally, if we are given a datum

$$(f, e, j) : M \hookrightarrow X \times \Delta^k \times \mathbb{R}^m$$

defining a k-simplex of the structure space $\mathbb{S}(X)$, then there is an essentially unique PL bundle ζ on X such that $f^*\zeta \simeq e^* T_{\Delta^k} - T_M$, for which the pair (f, e) becomes a Δ^k-family of normal maps determining a point of $\mathbb{S}^n(X)_k$. We therefore obtain a map of structure spaces

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \simeq \mathbb{S}^{tn}(X) \simeq \mathrm{fib}((\mathbf{Z} \times \mathrm{BPL})^X \to \mathrm{Pic}(S)^X).$$

Our goal for the rest of this course will be to analyze the homotopy fiber of *this* map. The work of this lecture gives us a good starting point: a point of $\mathbb{S}^n(X)$ determines a degree one normal map $f : M \to X$, where M is well-defined up to (normal) bordism. We now wish to determine if it is possible to modify M by a (normal) bordism so as to replace f by a homotopy equivalence.

Remark 5. Let $(X, \partial X)$ be a Poincare pair, where ∂X is a PL manifold. We can then define relative versions of the structure spaces $\mathbb{S}^n(X)$ and $\mathbb{S}^{tn}(X)$, and the above discussion carries over essentially without change.

Quadratic Refinements (Lecture 29)

April 11, 2011

Let X be a Poincare space. Our goal is to understand the homotopy type of the piecewise linear structure space $\mathbb{S}(X)$. In the last lecture, we introduced the structure space $\mathbb{S}^n(X)$ of normal maps to X. Roughly speaking, the points of $\mathbb{S}(X)$ are compact PL manifolds M equipped with a homotopy equivalence $f : M \to X$, while the points of $\mathbb{S}^n(X)$ are degree one normal maps $f : M \to X$. Every homotopy equivalence $M \to X$ can be viewed as a degree one normal map (for an essentially unique PL reduction of the Spivak bundle of X), and this observation underlies a map of structure spaces

$$\theta : \mathbb{S}(X) \to \mathbb{S}^n(X).$$

Since the homotopy type of $\mathbb{S}^n(X)$ can be understood by means of obstruction theory, we are reduced to studying the map θ and its homotopy fibers.

We therefore ask the following question: given a PL bundle ζ on X and a normal map $f : M \to X$, how far is f from a homotopy equivalence? We would like to give an answer in terms of the formalism of L-theory. Fix an A_∞-ring R with involution, and let ζ_0 denote the underlying spherical fibration of ζ (which can be identified with the Spivak fibration of X), so that X has a visible symmetric signature $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \zeta_0, R)$. Similarly, M has a visible symmetric signature $\sigma_M^{vs} \in \Omega^\infty \mathbb{L}^{vs}(M, f^*\zeta_0, R)$. Since M is a PL manifold, this lifts canonically to a point $\sigma_M^s \in \Omega^\infty \mathbb{L}^s(M, f^*\zeta_0, R)$. The map f induces pushforward maps in L-theory, which we will denote by $f_!$. If f is a homotopy equivalence, then $f_!(\sigma_M^{vs}) \simeq \sigma_X^{vs}$ so that $f_!(\sigma_M^s)$ is a preimage of σ_X^{vs} under the assembly map. In general, this need not be the case: a degree one normal map $f : M \to X$ generally does not carry the visible symmetric signature of M to the visible symmetric signature of X.

Example 1. Suppose that we are given a degree one normal map $f : M \to S^{4k}$. The signature of M need not be zero (despite the fact that the signature of S^{4k} is zero). However, the signature of M is constrained. Since the stable normal bundle of S^{4k} is trivial, we conclude that the stable normal bundle to M is trivial. In particular, all Stiefel-Whitney classes of M are trivial, so that the Wu class of M vanishes. It follows that the intersection form on the middle dimensional homology of M is even: that is, it refines to a quadratic form, so that the signature of M must be divisible by 8.

Our goal in this lecture is to demonstrate that the phenomenon of Example 1 is quite general: if $f : M \to X$ is a degree one normal map, then the difference $f_!(\sigma_M^{vs}) - \sigma_X^{vs}$ can be canonically lifted to the *visible quadratic* L-theory space $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_0, R)$.

The assertion above has nothing to do with the fact that M is actually a manifold. Let us therefore work a little bit more generally.

Definition 2. Let $f : Y \to X$ be a map of spaces, both of which have the homotopy type of a finite complex. A *degree one structure* on f consists of the following data:

(1) A spherical fibration ζ_X on X.

(2) A map of spectra $[Y] : S \to C_*(Y; f^*\zeta_X)$ satisfying the following conditions:

   (a) For every local system of spectra $\mathcal{F}$ on Y, cap product with $[Y]$ induces a homotopy equivalence $C^*(Y; \mathcal{F}) \to C_*(Y; \mathcal{F} \wedge f^*\zeta_X)$.

(b) Let $[X]$ denote the composite map $S \xrightarrow{[Y]} C_*(Y; f^*\zeta_X) \to C_*(X; \zeta_X)$. For every local system of spectra $\mathcal{F}$ on X, cap product with $[X]$ induces a homotopy equivalence $C^*(X; \mathcal{F}) \to C_*(X; \mathcal{F} \wedge \zeta_X)$.

Remark 3. Condition (a) of Definition 2 guarantees that Y is a Poincare space with Spivak bundle $f^*\zeta_X$, and condition (b) guarantees that X is a Poincare space with Spivak bundle X.

A degree one structure on f can be regarded as a compatibility between the Poincare duality isomorphisms of X and Y. It can be thought of as consisting of two pieces of data:

(i) An identification of the Spivak bundle of Y with the pullback of the Spivak bundle of X.

(ii) An identification of the fundamental class of X with the pushforward of the fundamental class of Y (which makes sense by virtue of the datum (i)).

Remark 4. Let $f : Y \to X$ be a homotopy equivalence of Poincare spaces. Then f admits an essentially unique degree one structure. In other words, if X is a Poincare space, then the data of the pair $(\zeta_X, [X] : S \to C_*(X; \zeta_X))$ is determined uniquely up to a contractible space of choices.

Let $f : Y \to X$ be a map with a degree one structure $(\zeta_X, [Y])$. Then X and Y have visible symmetric signatures $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$ and $\sigma_Y^{vs} \in \Omega^\infty \mathbb{L}^{vs}(Y, f^*\zeta_X, R)$. Our goal in this lecture is to prove the following result:

Theorem 5. *In the above situation, we can canonically construct a point $\sigma_f^{vq} \in \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X, R)$ and a path from $\sigma_X^{vs} + U(\sigma_f^{vq})$ to $f_! \sigma_Y^{vs}$ in the space $\Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$. Here U denotes the canonical map of spectra $\mathbb{L}^{vq}(X, \zeta_X, R) \to \mathbb{L}^{vs}(X, \zeta_X, R)$.*

In fact, we can be even more precise. The visible symmetric signatures σ_X^{vs} and σ_Y^{vs} have canonical representatives $(\underline{R}, q_X)$ and $(\underline{R}, q_Y)$ by quadratic objects of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ and $\mathrm{Shv}_{\mathrm{lc}}(Y; \mathrm{RMod}_R)$, respectively. We will construct a canonical representative of σ_f^{vq}, so that the identity

$$\sigma_X^{vs} + U(\sigma_f^{vq}) \simeq f_! \sigma_Y^{vs}$$

is visible at the level of Poincare objects (that is, it comes from an equivalence of Poincare objects, rather than a bordism of Poincare objects).

In what follows, there is no loss of generality in treating the universal case where R is the sphere spectrum (equipped with the trivial involution). We will henceforth assume that we are in this case, and we will therefore omit mention of R in our notation.

Recall that $\mathbb{L}^{vs}(X, \zeta_X)$ can be identified with the L-theory of the finitely presented part of the ∞-category $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{Sp})$, equipped with the quadratic functor $Q^s_{\zeta_X}$ given by the formula

$$Q^s_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2}).$$

Similarly, $\mathbb{L}^{vq}(X, \zeta_X)$ is given by the L-theory of the same ∞-category, equipped with the quadratic functor

$$Q^q_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}).$$

Using our assumption that X is a Poincare space with Spivak bundle ζ_X, we can rewrite

$$Q^s_{\zeta_X}(\mathcal{F}) = C^*(X; (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2}) \qquad Q^q_{\zeta_X}(\mathcal{F}) = C^*(X; (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}).$$

In these terms, the visible symmetric signature σ_X^{vs} is easy to describe: it is represented by the Poincare object $(\underline{S}, q_X)$ where q_X classifies the evident global section of $(\underline{S} \wedge \underline{S})^{h\Sigma_2}$.

Let us now describe the pushforward of the visible symmetric signature of Y under the map f. This is given by a nondegenerate symmetric form q' on the local system $f_! \underline{S}$ on X. Let us describe this form more explicitly. We may assume without loss of generality that f is a fibration. For each point $x \in X$, we let Y_x

denote the fiber $f^{-1}\{x\}$. Unwinding the definitions, we see that the local system $f_!\underline{S}$ is given by the formula $x \mapsto \Sigma^\infty_+ Y_x$.

Let us now invoke our assumption that f is a degree one map. The fundamental class of $[Y]$ gives a map of spectra

$$S \to C_*(Y; f^*\zeta_X) \simeq C_*(X; f_! f^*\zeta_X) \simeq C_*(X; \zeta_X \wedge f_!\underline{S}) \simeq C^*(X; f_!\underline{S}).$$

We may view this as a map of local systems $u : \underline{S} \to f_!\underline{S}$. That is, for every point $x \in X$, we have a canonical map of spectra $u_x : S \to \Sigma^\infty_+ Y_x$. The condition that f be of degree one guarantees that the composite map

$$\underline{S} \to f_!\underline{S} \to \underline{S}$$

is homotopic to the identity. That is, each u_x can be regarded as a *section* of the canonical map $\Sigma^\infty_+ Y_x \to \Sigma^\infty_+\{x\} \simeq S$.

Unwinding the definitions, we see that $q' \in \Omega^\infty Q^s_{\zeta_X}(f_!\underline{S})$ can be identified with the global section of $(f_!\underline{S} \wedge f_!\underline{S})^{h\Sigma_2}$ given by the composition

$$\underline{S} \xrightarrow{u} f_!\underline{S} \xrightarrow{\delta} (f_!\underline{S} \wedge f_!\underline{S})^{h\Sigma_2},$$

where δ is induced by the diagonal map $Y \to Y \times_X Y$. Fiberwise, this is given by the map of spectra

$$S \xrightarrow{u_x} \Sigma^\infty_+ Y_x \to (\Sigma^\infty_+ Y_x \wedge \Sigma^\infty_+ Y_x)^{h\Sigma_2},$$

where the second map is induced by the diagonal $Y_x \to Y_x \times Y_x$.

Theorem 5 can be obtained by fiberwise application of the following claim:

Proposition 6. *Let Z be a space (in our case of interest, the homotopy fiber of a map of Poincare spaces $f : X \to Y$), and suppose we are given a map of spectra $e : S \to \Sigma^\infty_+ Z$ which splits the canonical map $\Sigma^\infty_+ Z \to \Sigma^\infty_+\{*\} \simeq S$. Let q denote the composition*

$$S \to \Sigma^\infty_+ Z \to (\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2}$$

where the second map is induced by the diagonal embedding $Z \to Z \times Z$. Then the quadratic object $(\Sigma^\infty_+ Z, q)$ splits (canonically!) as a direct sum $(S, q_+) \oplus (E, q_-)$, where $q_+ : S \to (S \wedge S)^{h\Sigma_2}$ is the evident map and $q_- : S \to (E \wedge E)^{h\Sigma_2}$ factors (canonically!) through the transfer map $(E \wedge E)_{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2}$

The splitting $\Sigma^\infty_+ Z \simeq S \oplus E$ at the level of spectra is easy: it is determined by our choice of e. We therefore have an identification

$$(\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2} \simeq (S \wedge S)^{h\Sigma_2} \oplus (E \wedge S \oplus S \wedge E)^{h\Sigma_2} \oplus (E \wedge E)^{h\Sigma_2}.$$

We may therefore identify q with a trio of maps

$$q_+ : S \to (S \wedge S)^{h\Sigma_2}$$

$$q_0 : S \to (E \wedge S \oplus S \wedge E)^{h\Sigma_2} \simeq E$$

$$q_- : S \to (E \wedge E)^{h\Sigma_2}.$$

Unwinding the definitions, we see that q_0 is given by the composition

$$S \xrightarrow{e} \Sigma^\infty_+ Z \to E,$$

where the second map is projection onto the summand E. It follows that q_0 is canonically nullhomotopic, so that we obtain a direct sum decomposition of $(\Sigma^\infty_+ Z, q)$ as a quadratic object. Moreover, it is obvious that $q_+ : S \to (S \wedge S)^{h\Sigma_2}$ is as described. The only nontrivial point is to show that $q_- : S \to (E \wedge E)^{h\Sigma_2}$ admits a quadratic refinement (that is, it factors through the transfer map). We will take this up in the next lecture.

Statement of the Main Theorem (Lecture 30)

April 11, 2011

Our first goal in this lecture is to complete the analysis of the previous lecture, constructing a quadratic refinement of difference $f_!(\sigma_Y^{vs}) - \sigma_X^{vs}$ when $f : Y \to X$ is a map of Poincare spaces with a degree one structure. Recall that we need to show the following:

Proposition 1. *Let Z be a space, and suppose we are given a splitting of the canonical map of spectra $\Sigma^\infty_+ Z \to \Sigma^\infty_+ * \simeq S$, giving a map of spectra $e : S \to \Sigma^\infty_+ Z$ and a decomposition of spectra $\Sigma^\infty_+ Z \simeq E \oplus S$. Then the composite map*

$$q_- : S \xrightarrow{e} \Sigma^\infty_+ Z \to (\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2}$$

factors canonically through the transfer map $\mathrm{tr} : (E \wedge E)_{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2}$.

In other words, we claim that there is a canonical nullhomotopy of the composite map

$$S \xrightarrow{e} \Sigma^\infty_+ Z \to (\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2} \to (E \wedge E)^{t\Sigma_2}.$$

For every spectrum M, let $T(M) = (M \wedge M)^{t\Sigma_2}$. Then $T(M)$ is an exact functor. If M is a finite spectrum, we can write $T(M) = M \wedge T(S) = M \wedge S^{t\Sigma_2}$. More generally, we obtain a map $\alpha_M : M \wedge S^{t\Sigma_2} \to T(M)$, which need not be a homotopy equivalence when M is not finite.

Let Z be a space and $M = \Sigma^\infty_+ Z$. The diagonal $Z \to Z \times Z$ induces a map of spectra $M \to (M \wedge M)^{h\Sigma_2}$. Composing with the map $(M \wedge M)^{h\Sigma_2} \to (M \wedge M)^{t\Sigma_2}$, we obtain a map $\beta_Z : M \to T(M)$, depending functorially on Z. Taking Z to be a point, we obtain a map $\beta_0 : S \to T(S)$. Note that β_0 induces a map

$$\gamma : M = C_*(Z; \underline{S}) \to C_*(Z; \underline{T(S)}) = M \wedge T(S).$$

By naturality, we deduce that β_Z factors as a composition

$$M \xrightarrow{\gamma} M \wedge T(S) \xrightarrow{\alpha_M} T(M).$$

We wish to show that the composite map

$$S \xrightarrow{e} M \xrightarrow{\beta_Z} T(M) \to T(E)$$

is nullhomotopic. Using the above analysis, we can write this map as a composition

$$S \xrightarrow{e} M \xrightarrow{M \wedge \beta_0} M \wedge T(S) \xrightarrow{\alpha_M} T(M) \to T(E),$$

which is also the composition

$$S \xrightarrow{e} M \to E \xrightarrow{E \wedge \beta_0} E \wedge T(S) \xrightarrow{\alpha_E} T(E).$$

Since E is defined as the cofiber of e, the composition of the first two maps is canonically nullhomotopic. This completes the proof of Proposition 1.

Let us now recall our application of Proposition 1. Let $f : Y \to X$ be a map of Poincare spaces with a degree one structure. Let $\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}$ denote the smallest stable subcategory of locally constant sheaves of spectra on X which contains $i_! S$, for every point $i : \{x\} \hookrightarrow X$ (if X is connected with base point x, so that $\mathrm{Shv}_{\mathrm{lc}}(X)$ can be identified with the ∞-category of right modules over $R : \Sigma^\infty_+ \Omega(X)$, then $\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}$ is the full subcategory spanned by the finitely presented R-modules). Let ζ_X denote the Spivak bundle of X, so that we have quadratic functors

$$Q^s_{\zeta_X}, Q^q_{\zeta_X} : \mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}} \to \mathrm{Sp}$$

given by

$$Q^s_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2}) \simeq C^*(X; (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2})$$
$$Q^q_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}) \simeq C^*(X; (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}).$$

Applying Proposition 1 to the homotopy fibers of the map f, we obtain a quadratic object $(\mathcal{E}, q_f)$ of $(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})$, where $\mathcal{E}$ denotes the homotopy fiber of the map of local systems $f_! \underline{S} \to \underline{S}$. Let $\mathrm{tr}(q_f)$ denote the image of q_f under the transfer map $Q^q_{\zeta_X}(\mathcal{E}) \to Q^s_{\zeta_X}(\mathcal{E})$. We then have an equivalence of Poincare objects

$$(\mathcal{E}, \mathrm{tr}(q_f)) \oplus (\underline{S}, q_X) \simeq f_!(\underline{S}, q_Y)$$

where $(\underline{S}, q_X)$ represents the visible symmetric signature of X and $(\underline{S}, q_Y)$ represents the visible symmetric signature of Y. It follows that $(\mathcal{E}, \mathrm{tr}(q_f))$ is a Poincare object of $(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^s_{\zeta_X})$, so that $(\mathcal{E}, q_f)$ is a Poincare object of $(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})$. It therefore determines a point in the zeroth space of the visible quadratic L-theory spectrum $\mathbb{L}^{vq}(X, \zeta_X) = \mathbb{L}^{vq}(X; \zeta_X, S)$. We will denote this point by σ^{vq}_f.

Let us now fix the Poincare space X, and let $\mathbb{S}^n(X)_0$ denote the classifying space for stable PL bundles ζ on X and degree one normal maps $f : M \to X$, where M is a compact PL manifold. The above construction determines a map of classifying spaces

$$\mathbb{S}^n(X)_0 \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})_0.$$

More generally, given a Δ^k-family of normal maps $f : M \to X \times \Delta^k$, we can apply the above construction to each of the induced maps $f^{-1}(X \times \tau) \to X$, to obtain a Poincare object of the ∞-category $\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}_{[k]}$ (see Lecture 6). This construction is classified by a map of spaces

$$\mathbb{S}^n(X)_k \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})_k.$$

Passing to geometric realizations, we obtain a map

$$\mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X).$$

Note that if a degree one normal map $f : M \to X$ is actually a homotopy equivalence, then $f_! \underline{S} \to \underline{S}$ is invertible, so that the quadratic object $(\mathcal{E}, q_f)$ constructed above is trivial. It follows that the composite map

$$\mathbb{S}(X)_\bullet \to \mathbb{S}^n(X)_\bullet \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})_\bullet$$

is canonically nullhomotopic (as a map of simplicial spaces), so that the composite map

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$$

is likewise nullhomotopic.

We are now ready to state the main theorem of this course:

Theorem 2. *Let X be a Poincare space of dimension ≥ 5. Then*

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$$

is a fiber sequence. In other words, the structure space $\mathbb{S}(X)$ can be regarded as the homotopy fiber of a map $\mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$, which carries every degree one normal map $f : M \to X$ to the signature σ^{vq}_f.

Using the π-π Theorem, we can be more precise. Let us assume that X is connected. Choose a base point $x \in X$ and a trivialization $\zeta_X(x) \simeq S^{-n}$ of the Spivak fibration at the point x (so that n is the dimension of the Poincare space X). Then $\Sigma^n \zeta_X$ has a trivialization at x, so that we have

$$\mathbb{L}^{vq}(X, \zeta_X) \simeq \Sigma^{-n} \mathbb{L}^{vq}(X, \Sigma^n \zeta_X) \simeq \Sigma^{-n} \mathbb{L}^q(R) \simeq \Sigma^{-n} \mathbb{L}^q(\mathbf{Z}[\pi_1 X]),$$

where R is the A_∞-ring $\Sigma^\infty_+(\Omega X)$ and the last equivalence follows from the π-π theorem. Here $\mathbf{Z}[\pi_1 X]$ is equipped with the involution given by $\gamma \mapsto \pm\gamma^{-1}$ for $\gamma \in \pi_1 X$, where the signs are given by the obstruction to orienting the Spivak bundle ζ_X.

Given a stable PL reduction ζ of the Spivak bundle of X, we obtain a point of $\mathbb{S}^{tn}(X) \simeq \mathbb{S}^n(X)$, which determines an element γ of the abelian group $L^q_n(\mathbf{Z}[\pi_1 X])$ (represented by σ^{vq}_f, where $f : M \to X$ is a degree one normal map to (X, ζ)). This element vanishes if we can choose f to be a homotopy equivalence. Theorem 2 asserts the converse: if the dimension n is greater than 5 and γ vanishes, then X is homotopy equivalent to a PL manifold M (having stable normal bundle given by the pullback of ζ).

The Main Theorem: First Reduction (Lecture 31)

April 12, 2011

Let X be a Poincare space of dimension ≥ 5. Our goal for the next few lectures is to sketch a proof of the main theorem of surgery theory: there is a fiber sequence of spaces

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X).$$

In other words, we have a diagram

$$\begin{tikzcd} \mathbb{S}(X) \arrow[r, "\phi"] \arrow[d] & \mathbb{S}^n(X) \arrow[d] \\ * \arrow[r, "\phi'"] & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X) \end{tikzcd}$$

and we wish to show that it is a homotopy pullback square.

Fix a point of $\mathbb{S}^n(X)$, corresponding to a PL reduction ζ of the Spivak bundle of X and a degree one normal map $f_0 : M_0 \to X$. We will abuse notation by simply referring to f as a point of $\mathbb{S}^n(X)$. Let $\sigma^{vq}_{f_0}$ denote the image of f_0 in $\mathbb{L}^{vq}(X, \zeta_X)$. We wish to prove that the above diagram induces a homotopy equivalence from the homotopy fiber of ϕ over f to the homotopy fiber of ϕ' over $\sigma^{vq}_{f_0}$.

We now wish to describe the homotopy fiber of ϕ in more explicit geometric terms. Recall that ϕ is given as the geometric realization of a map of simplicial spaces $\phi_\bullet : \mathbb{S}(X)_\bullet \to \mathbb{S}^n(X)_\bullet$. Unfortunately, this map is not a Kan fibration, so the homotopy fiber cannot be computed "pointwise". We will therefore need to make an auxiliary construction.

We define a simplicial space $P_\bullet$ so that P_k is the homotopy fiber product $\{f_0\} \times_{\mathbb{S}^n(X)_0} \mathbb{S}^n(X)_{k+1}$. More informally, P_k is a classifying space for families of degree one normal maps $f : M \to X \times \Delta^{k+1}$ such that the fiber of f over the vertex $\{0\} \in \Delta^{k+1}$ is the normal map $f_0 : M_0 \to X$.

Restricting to the faces opposite the 0th vertex, we obtain a map of simplicial spaces $P_\bullet \to \mathbb{S}^n(X)_\bullet$, which is easily seen to be a Kan fibration. One can also see that the map $P_\bullet \to \{f_0\}$ is a trivial Kan fibration, so that the geometric realization $|P_\bullet|$ is contractible. This realization is a model for the space of paths in $\mathbb{S}^n(X)$ beginning at the point f_0.

The homotopy fiber of ϕ over f_0 is given by the fiber product

$$\{f_0\} \times_{\mathbb{S}^n(X)} \mathbb{S}(X) \simeq |P_\bullet| \times_{|\mathbb{S}^n(X)_\bullet|} |\mathbb{S}(X)_\bullet|.$$

Since the map $P_\bullet \to \mathbb{S}^n(X)_\bullet$ is a Kan fibration, we can compute this as $|U_\bullet|$, where $U_k \simeq P_k \times_{\mathbb{S}^n(X)_k} \mathbb{S}(X)_k$. More explicitly, U_k is a classifying space for degree one normal maps $f : M \to X \times \Delta^{k+1}$ for which the fiber over $\{0\} \subseteq \Delta^{k+1}$ is $f_0 : M_0 \to X$, and f induces a homotopy equivalence $f^{-1}(X \times \tau) \to X \times \tau$ for every simplex τ which does not contain the vertex 0.

Let P' be the space of paths in $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ from $\sigma^{vq}_{f_0}$ to the base point, and let $P'_\bullet$ be the constant simplicial space taking the value P'. We have a map of simplicial spaces

$$\psi : U_\bullet \to P'_\bullet.$$

To prove the main theorem, it will suffice to show that this map induces a homotopy equivalence of geometric realizations. In fact, we claim that ψ is a trivial Kan fibration of simplicial spaces. In other words, we claim that for each $k \geq 0$, the canonical map

$$\theta_k : U_k \to U_\bullet(\partial \Delta^k) \times_{\mathrm{Map}(\partial \Delta^k, P)} \mathrm{Map}(\Delta^k, P)$$

is surjective on connected components.

Let us spell out this out more explicitly in the special case $k = 0$. The space U_0 classifies degree one maps $f : M \to X \times \Delta^1$ such that $f^{-1}(X \times \{0\}) \simeq M_0$ and $M_1 = f^{-1}(X \times \{1\})$ is homotopy equivalent to X. In other words, U_0 is a classifying space for *normal bordisms* from M_0 to a manifold homotopy equivalent to X. Every such bordism determines a Lagrangian in the Poincare object representing $\sigma^{vq}_{f_0}$, which we can regard as a path from $\sigma^{vq}_{f_0}$ to the base point in $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$. This path is the image of M under the map θ_0. Consequently, when $k = 0$, we wish to prove the following:

Theorem 1. *Let X be a Poincare space of dimension $n \geq 5$ equipped with a PL reduction of its tangent bundle, let $f_0 : M_0 \to X$ be a degree one normal map determining a Poincare object $\sigma^{vq}_{f_0}$ of $(\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{Sp})^{\mathrm{fp}}, Q^q_{\zeta_X})$, and let L be a Lagrangian in $\sigma^{vq}_{f_0}$. Then L is cobordant to a Lagrangian arising from a normal bordism from $f_0 : M_0 \to X$ to a homotopy equivalence $f_1 : M_1 \to X$.*

Let us now extend our analysis to the case where k is arbitrary. Let $D \subseteq \Delta^{k+1}$ be the subset obtained by deleting the interior and the face opposite the 0th vertex. Then D is a PL disk equipped with preferred triangulation. Every point of $U_\bullet(\partial \Delta^k)$ determines D-family of normal maps $f : M \to X \times D$ such that the fiber over the vertex $\{0\} \subseteq \Delta^{k+1}$ (which we can think of as the "center" of the disk D) is given by M_0, and $f^{-1}(X \times \tau)$ is homotopy equivalent to $X \times \tau$ for every simplex $\tau \subseteq \partial D$. Such data determines a map $u : D \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ which is trivial on ∂D (and carries the center of D to the point $\sigma^{vq}_{f_0}$, but this is not important in what follows). We may think of u as a point of $\Omega^{\infty+k} \mathbb{L}^{vq}(X, \zeta_X)$, which we can represent by Poincare object $(\mathcal{F}, q)$ of $(\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{Sp}^{\mathrm{fp}}, \Omega^k \zeta_X)$.

Lifting f to a point of U_k amounts to extending f to a Δ^{k+1}-family of normal maps $\overline{f} : \overline{M} \to X \times \Delta^{k+1}$, such that $\overline{f}^{-1}(X \times \tau)$ is homotopy equivalent to $X \times \tau$ for every face $\tau \subseteq \Delta^{k+1}$ not containing the 0th vertex. Such an extension can be described by first specifying the inverse image of the face of Δ^{k+1} opposite the 0th vertex: this is a PL manifold N equipped with a homotopy equivalence $g : N \to X \times \Delta^k$ (neat over Δ^k) such that $g^{-1}(X \times \partial \Delta^k)$ is PL homeomorphic to $f^{-1}(X \times \partial D)$, together with a normal bordism from M to N which is compatible with the PL homeomorphism on the boundaries.

Such a lifting determines an extension of σ^{vq}_f to a map $\sigma^{vq}_{\overline{f}} : \Delta^{k+1} \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ which vanishes on the face of Δ^{k+1} opposite the 0th vertex. In particular, we obtain a Lagrangian in the Poincare object $(\mathcal{F}, q)$, which is well-defined up to bordism. Unwinding the definitions, the surjectivity of θ_k amounts to the following assertion: every Lagrangian in $(\mathcal{F}, q)$ arises (up to cobordism) from a normal bordism from M to N as in the above paragraph.

It is convenient to rephrase the above construction by replacing the Poincare space X by the Poincare pair $(X \times D, X \times \partial D)$. Note that the Spivak bundle $\zeta_{X \times D}$ is given by $p^*(\Omega^k \zeta_X)$, where p denotes the projection $X \times D \to X$.

Let now embark on a brief digression about how some of the constructions of the last few lectures generalize to the setting of Poincare pairs. Our definition of a degree one structure on a map of Poincare spaces $f : Y \to X$ generalizes to the setting of a map of Poincare pairs $f : (Y, \partial Y) \to (X, \partial X)$. To such a map, one can assign a signature

$$\sigma^{vq}_f : S \to \mathbb{L}^{vq}(X, \partial X, \zeta_X) = \mathrm{cofib}(\mathbb{L}^{vq}(\partial X, \zeta_X | \partial X) \to \mathbb{L}^{vq}(X, \zeta_X).$$

We have a fiber sequence of spectra

$$\mathbb{L}^{vq}(X, \zeta_X) \to \mathbb{L}^{vq}(X, \partial X, \zeta_X) \to \mathbb{L}^{vq}(\partial X, \zeta_{\partial X}),$$

where the second map carries σ_f^{vq} to $\sigma^{vq}\partial f$, where $\partial f : \partial Y \to \partial X$ denotes the induced degree one map of Poincare spaces. In the special case where ∂f is a homotopy equivalence, we obtain a canonical lifting of σ_f^{vq} to a point of $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$.

Let us now return to the problem of showing that the map

$$\theta_k : U_k \to U_\bullet(\partial \Delta^k) \times_{\mathrm{Map}(\partial \Delta^k, P)} \mathrm{Map}(\Delta^k, P)$$

is surjective. A point of the codomain determines a degree one normal map of Poincare pairs $f : (M, \partial M) \to (X \times D, X \times \partial D)$ which is a homotopy equivalence on the boundary, The relative signature σ_f^{vq} can be identified with the map u under the homotopy equivalence $\mathbb{L}^{vq}(X \times D, \zeta_{X \times D}) \simeq \mathbb{L}^{vq}(X, \zeta_X)$. Consequently, verifying the surjectivity of θ_k amounts to verifying that Theorem 2 is satisfied, where we replace the Poincare space X by the Poincare pair $(X \times D, X \times \partial D)$. This is in turn a special case of the following more general assertion:

Theorem 2. *Let $(X, \partial X)$ be a Poincare pair of dimension $n \geq 5$ equipped with a PL reduction of its Spivak bundle, let $f : (M, \partial M) \to (X, \partial X)$ be a degree one normal map inducing a homotopy equivalence $\partial M \to \partial X$, and let L be a Lagrangian in Poincare object representing σ_f^{vq}. Then L is cobordant to a Lagrangian which arises from a normal bordism (constant along ∂M) from M to a homotopy equivalence $g : (N, \partial M) \to (X, \partial X)$.*

Remark 3. Using Theorem 2, we can extend the statement of the main theorem of the previous lecture to the case of Poincare pairs. That is, if $(X, \partial X)$ is a Poincare pair of dimension $n \geq 5$ where ∂X is a PL manifold of dimension $n-1$, then we have a fiber sequence

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \mathbb{L}^{vq}(X, \zeta_X).$$

Surgery (Lecture 32)

April 14, 2011

Our goal today is to begin the proof of the following:

Theorem 1. *Let X be a Poincare pair of dimension $n \geq 5$, ζ a stable PL bundle on X, and $f : M \to X$ a degree one normal map, where M is a PL manifold. Let $\sigma_f^{vq} \in \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ be the relative signature of f, and suppose we are given a path p from σ_f^{vq} to the base point of $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$. (We can identify such a path with a Lagrangian in the Poincare object representing σ_f^{vq}, which is well-defined up to bordism). Then there exists a Δ^1-family of degree one normal maps $F : B \to X \times \Delta^1$, where B is a bordism from $M = F^{-1}(X \times \{0\})$ to a PL manifold $N = F^{-1}(X \times \{1\})$ such that F induces a homotopy equivalence $f' : N \to X$. Moreover, we can arrange that F determines a path from σ_f^{vq} to $\sigma_{f'}^{vq} = 0$ which is homotopic to p.*

Remark 2. In the last lecture, we sketched the formulation of a more general version of Theorem 1, where we replace X by a Poincare pair $(X, \partial X)$ where ∂X is already a PL manifold. To simplify the discussion, we will restrict our attention to the case where $\partial X = \emptyset$, but the ideas introduced in this lecture generalize to the relative case.

To prove Theorem 1, we need a method for producing bordisms between PL manifolds. For this, we will use the method of *surgery*. Fix a PL manifold M of dimension n. Write $n = p + q + 1$. Let D^{p+1} and D^{q+1} denote PL disks of dimension $p+1$ and $q+1$, respectively. Let S^p and S^q denote their boundaries (spheres of dimension p and q, respectively.

Definition 3. A *p-surgery datum* on M is a PL embedding $\alpha : S^p \times D^{q+1} \to M$.

To a first approximation, a p-surgery datum α on M is given by an embedding of PL manifolds $\alpha_0 : S^p \hookrightarrow M$ (given by restricting α to the product of S^p with the center of D^{q+1}). To obtain a surgery datum from α_0, we must additionally specify that α_0 extends to a PL homeomorphism between $S^p \times D^{q+1}$ and a neighborhood of the image of α_0. Such a homeomorphism determines a smooth structure on M along the image of α_0, with respect to which α_0 is a smooth embedding with trivialized normal bundle. Conversely, suppose we are given an embedding $\alpha_0 : S^p \to M$ and a smoothing of M along the image of α_0, such that α_0 is a smooth map. Then α_0 has a normal bundle N_{α_0}, and there is a neighborhood of $\alpha_0(S^p)$ in M which is diffeomorphic to the unit sphere bundle of N_{α_0}. In particular, if N_{α_0} is trivial, we obtain a diffeomorphism (and therefore a PL homeomorphism) of a neighborhood of $\alpha_0(S^p)$ with $S^p \times D^{q+1}$. This argument shows that we can identify a p-surgery datum on M with three pieces of data:

(i) A PL embedding $\alpha_0 : S^p \to M$.

(ii) A smoothing of M along the image of α_0 (with respect to which α_0 is a smooth map).

(iii) A trivialization of the normal bundle to α_0 (as a vector bundle).

Construction 4. Let M be a PL manifold of dimension $n = p + q + 1$ and let $\alpha : S^p \times D^{q+1} \hookrightarrow M$ be a p-surgery datum. We let $B(\alpha)$ denote the polyhedron given by

$$(M \times [0,1]) \coprod_{\{1\} \times S^p \times D^{q+1}} (D^{p+1} \times D^{q+1}).$$

Then $B(\alpha)$ is a PL manifold with boundary, given by the disjoint union of $M \times \{0\}$ and

$$N = M - (S^p \times (D^{q+1})^\circ) \coprod_{S^p \times S^q} (D^{p+1} \times S^q).$$

We refer to N as the *PL manifold obtained from M via surgery along α*, and to $B(\alpha)$ as the *trace of the surgery*.

More informally: N is the manifold obtained from M by removing the interior of $S^p \times D^{q+1}$ (thereby creating a manifold with boundary $S^p \times S^q$) and gluing on a copy of $D^{p+1} \times S^q$.

Remark 5. Let N be a PL manifold obtained from surgery on a PL manifold M along a map $\alpha : S^p \times D^{q+1} \hookrightarrow M$. Then there is an evident embedding $\beta : D^{p+1} \times S^q \to N$, which is a q-surgery datum in N. Performing surgery on N along β recovers the manifold M.

We will be interested in using surgery to construct *normal bordisms* between normal maps to a Poincare complex. For this, we need a slight variation on Definition 3. Let M be a PL manifold, so that the stable normal bundle of M is classifies by a map $\chi : M \to \mathbf{Z} \times \mathrm{BPL}$. If we are given a p-surgery datum $\alpha : S^p \times D^{q+1} \to M$, then $\chi \circ \alpha$ extends canonically to a map $\gamma : D^{p+1} \times D^{q+1} \to \mathbf{Z} \times \mathrm{BPL}$.

Suppose now that X is a space equipped with a stable PL bundle ζ, and that we are given a normal map $f : M \to X$. Then ζ is classified by a map $\chi_X : X \to \mathbf{Z} \times \mathrm{BPL}$, and the normal structure on f gives a homotopy $h_0 : \chi \simeq \chi_X \circ f$.

Definition 6. In the situation above, a *normal p-surgery datum* on M consists of the following data:

(i) A p-surgery datum $\alpha : S^p \times D^{q+1} \to M$.

(ii) A map $\beta : D^{p+1} \times D^{q+1} \to X$ extending $f \circ \alpha$.

(iii) A homotopy h from $\chi_X \circ \beta$ to γ, extending the homotopy determined by h.

Given a normal p-surgery datum, we can use α to construct a bordism $B(\alpha)$ from M to a PL manifold N, β to construct a map $F : B(\alpha) \to X$ extending $f : M \to X$, and h to endow F with the structure of a Δ^1-family of normal maps.

Remark 7. Let us think of a p-surgery datum on a PL manifold M as an embedding $\alpha_0 : S^p \to M$, together with a choice of trivial normal bundle to α_0. If $f : M \to X$ is a degree one normal map, then to obtain a normal p-surgery datum we need to choose a nullhomotopy of the composite map $(f \circ \alpha_0) : S^p \to X$, which is *compatible* with the nullhomotopy of the map

$$S^p \xrightarrow{\alpha_0} M \xrightarrow{f} X \to \mathbf{Z} \times \mathrm{BPL}$$

determined by the choice of trivial normal bundle.

Let us now see what surgery can do for us in low degrees. Assume that X is a Poincare space of dimension $n \geq 5$, ζ a stable PL bundle on X, and $f : M \to X$ is a degree one normal map.

Let us begin by doing surgery in the case $p = -1$. In this case, S^p is empty and therefore a surgery datum $\alpha : S^p \times D^{q+1} \to M$ is unique. To promote α to a normal surgery datum, we need to choose a map $\beta : D^{n+1} \to X$ (up to homotopy, this a point $x \in X$), together with a trivialization of $\beta^*\zeta$. Unwinding the definitions, we see that $B(\alpha)$ is the disjoint union $(M \times [0,1]) \coprod D^{n+1}$, regarded as a bordism from M to $M \coprod S^n$. If we have chosen β and the trivialization of $\beta^*\zeta$, then we can regard this as a normal bordism from f to a map $M \coprod S^n \to X$, whose restriction to S^n is determined by β. By performing surgeries of this type, we can always arrange that the map $M \to X$ is surjective on connected components.

Now suppose that $f : M \to X$ fails to be *injective* on connected components. Then we can choose two points $x, y \in M$ belonging to different components of M and a path joining $f(x)$ to $f(y)$. Choosing small disks around the points x and y, we obtain a 0-surgery datum $\alpha : S^0 \times D^n \hookrightarrow M$. A choice of path p from $f(x)$ to $f(y)$ determines the datum (ii) required by Definition 6. We cannot always extend α to a normal surgery datum: our choice of disks determines trivializations of the fibers $\zeta_{f(x)}$ and $\zeta_{f(y)}$, which may or may not extend to a trivalization of ζ over the path p. However, the obstruction is slight by virtue of the following (non-obvious!) fact:

Claim 8. The fundamental group $\pi_1(\mathbf{Z} \times \mathrm{BPL})$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. In other words, every orientation-preserving PL automorphism of $\mathbb{R}^n$ is isotopic to the identity, for $n \gg 0$.

In fact, more is true: the map $\pi_i(\mathbf{Z} \times \mathrm{BO}) \to \pi_i(\mathbf{Z} \times \mathrm{BPL})$ induces an isomorphism for $i \leq 6$ and a surjection when $i = 7$ (using smoothing theory, this is equivalent to the assertion that there are no exotic smooth structures on piecewise linear spheres of dimensions ≤ 6). In this lecture, we will need something much weaker: namely, that the above map is bijective for $i \leq 1$ and surjective for $i \leq 2$. Using smoothing theory, this is equivalent to the (reasonably obvious) claim that there are no exotic smooth structures on spheres of dimension ≤ 1.

In our situation, we cannot necessarily extend an *arbitrary* $\alpha : S^0 \times D^n \hookrightarrow M$ to a normal surgery datum. However, we always do so after modifying α by applying an orientation-reversing automorphism to one of the disks D^n. After making this modification, we obtain a normal bordism from M to a PL manifold with fewer connected components. Applying this procedure finitely many tiimes, we may replace $f : M \to X$ by a degree one normal map which induces an isomorphism $\pi_0 M \to \pi_0 X$.

Let us now assume that X and M are connected, and choose a base point $x \in M$. Suppose that the map $\pi_1 M \to \pi_1 X$ is not surjective. Choose another point y in M and a path q from y to x. Choose any class γ in $\pi_1 X$, and a path p from $f(x)$ to $f(y)$ such that the loop composing p with $f(q)$ represents γ. Choosing small disks around x and y, we obtain a surgery datum $\alpha : S^0 \times D^n \hookrightarrow M$ as before. The path p supplies the datum (ii) required by Definition 6, and we can argue as before (modifying α if necessary) to obtain the datum (iii). Let N be obtained from M by normal surgery along α. Since $n \geq 3$, deleting small disks around x and y does not change the fundamental group of M. Using van Kampen's theorem, we compute that $\pi_1 N$ is obtained from $\pi_1 M$ by freely adjoining an additional generator, and the map $\pi_1 N \to \pi_1 X$ carries this generator to γ (here we are being sloppy about base points here). Since X is a finite complex, its fundamental group is finitely generated. We may therefore perform this procedure finitely many times to reduce to the situation where the degree one normal map $f : M \to X$ induces a surjection $\pi_1 M \to \pi_1 X$.

Now suppose that $\pi_1 M \to \pi_1 X$ fails to be injective. Choose an element of $\pi_1 M$ whose image in $\pi_1 X$ is trivial. We can represent this element by a map $\alpha_0 : S^1 \to M$. Since the dimension of M is ≥ 3, a general position argument allows us to assume that α_0 is an embedding. The composite map $S^1 \to M \to X$ is nullhomotopic, so that the stable normal bundle of M is trivial in a neighborhood of α_0 and we may therefore assume that M is smooth in a neighborhood of α_0. The normal bundle to α_0 is stable trivial, hence orientable and therefore trivial. We may therefore extend α_0 to an embedding $\alpha : S^1 \times D^{n-1} \hookrightarrow M$. Choose a nullhomotopy of $f \circ \alpha$. As before, it is not clear that we can choose datum (iii) required by Definition 6: we encounter an obstruction in $\pi_2(\mathbf{Z} \times \mathrm{BPL})$. However, since the map $\pi_2(\mathbf{Z} \times \mathrm{BO}) \to \pi_2(\mathbf{Z} \times \mathrm{BPL})$ is surjective, we can adjust our original embedding α (choosing a different trivialization of the normal bundle to α_0) to make this obstruction vanish. This allows us to perform a normal surgery on the manifold M, thereby obtaining a cobordant degree one normal map $f' : N \to X$. Since the dimension of M is ≥ 4, removing a neighborhood of $\alpha_0(S^1)$ does not change the fundamental group of M. Consequently, we can use van Kampen's theorem to compute the fundamental group of N: it is obtained from the fundamental group of M by killing the normal subgroup generated by γ.

Since X is a finite complex, the fundamental group $\pi_1 X$ is finitely presented. Since $\pi_1 M$ is finitely generated, the surjective map $\pi_1 M \to \pi_1 X$ exhibits $\pi_1 X$ as the quotient of $\pi_1 M$ by the normal subgroup generated by finitely many elements of $\pi_1 M$. It follows that, after a finite number of applications of the above procedure, we may replace $f : M \to X$ by a degree one normal map which induces an isomorphism of fundamental groups.

(Geometric) Surgery Below the Middle Dimension (Lecture 34)

April 20, 2011

In the last lecture, we reduced the main theorem of this course to the following assertion:

Theorem 1. *Let X be a Poincare space of dimension $n \geq 5$, let ζ be a stable PL bundle on X, and let $f : M \to X$ be a degree one normal map, where M is a compact PL manifold. Assume that M and X are connected and that f induces an isomorphism $\pi_1 M \simeq \pi_1 X \simeq G$, so that σ_f^{vq} can be represented by a Poincare object (V, q), where $V \in \mathrm{LMod}_{\mathbf{Z}[G]}$ satisfies $C_*(\widetilde{M}; \mathbf{Z}) \simeq C_*(\widetilde{X}; \mathbf{Z}) \oplus \Sigma^n V$ (here $\widetilde{M}$ and $\widetilde{X}$ denote universal covers of M and X, respectively).*

Assume that f is p-connected, that we are given a map $u : \Sigma^{p-n}\mathbf{Z}[G] \to V$ a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$, so that (algebraic) surgery along u determines a bordism bordism from (V, q) to another Poincare object (V', q'). Then this (algebraic) bordism can be obtained by performing (geometric) surgery with respect to a normal surgery datum $\alpha : S^p \times D^{q+1} \hookrightarrow M$.

In this lecture, we will treat the "easy" case of Theorem 1, where p is strictly smaller than $\frac{n}{2}$. Let $f : M \to X$ be as in Theorem 1. Assume that f is p-connected, and choose any map $e : \Sigma^p \mathbf{Z}[G] \to \Sigma^n V$, classified by an element of $\pi_{p-n} V \simeq \ker(\mathrm{H}_p(\widetilde{M}; \mathbf{Z}) \to \mathrm{H}_p(\widetilde{X}; \mathbf{Z})) = \mathrm{H}_{p+1}(\widetilde{X}, \widetilde{M}; \mathbf{Z})$. Since f is p-connected, the Hurewicz theorem allows us to identify this group with the relative homotopy group $\pi_{p+1}(\widetilde{X}, \widetilde{M})$: that is, with $\pi_p F$, where F denotes the homotopy fiber of the map $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ (note that F can also be identified with the homotopy fiber of the map $f : M \to X$. Consequently, e determines a map $\alpha_0 : S^p \to M$, together with a nullhomotopy of the composite map $S^p \to M \to X$.

Since $p < \frac{n}{2}$, we can assume (modifying the map $\overline{e}$ by a homotopy if necessary) that α_0 is an embedding. Note that we have an equivalence of stable PL bundles

$$\alpha_0^*(-T_M) \simeq \overline{e}^* f^* \zeta.$$

Since the composition $f \circ \alpha_0$ is nullhomotopic, we can trivialize the (stable) tangent bundle of M in a neighborhood of the image of α_0. It follows from smoothing theory that M admits a smooth structure in a neighborhood U of $\alpha_0(S^p)$, and this smooth structure admits a framing. Modifying α_0 by a homotopy if necessary, we may assume that it is a smooth embedding from S^p into U. This embedding has a normal bundle, which we will denote by $\mathcal{E}$. Since U and S^p are framed, the bundle $\mathcal{E}$ is framed: that is, it is stably trivial. In order to extend α_0 to a normal surgery datum $\alpha : S^p \times D^{n-p} \hookrightarrow M$, we need to promote this stable trivialization of $\mathcal{E}$ to an actual trivialization of $\mathcal{E}$.

The bundle $\mathcal{E}$ is classified by a map $\chi : S^p \to \mathrm{BO}(n-p)$. Our stable framing gives a nullhomotopy of the composite map $S^p \xrightarrow{\chi} \mathrm{BO}(n-p) \to \mathrm{BO}(N)$ for N large. We wish to lift this to a nullhomotopy of the map χ itself. We can carry out this lifting in stages. Suppose that we have a map $\psi : S^p \to \mathrm{BO}(k)$ and a trivialization of the composite map $S^p \to \mathrm{BO}(k) \to \mathrm{BO}(k+1)$. This nullhomotopy determines a factorization of ψ through the homotopy fiber of the map $\mathrm{BO}(k) \to \mathrm{BO}(k+1)$, which is homotopy equivalent to the quotient $O(k+1)/O(k) \simeq S^k$. If $p < k$, such a map is automatically nullhomotopic, and therefore the lifting is possible. This analysis applies in our case of interest: for any $k \geq n-p$, we have $p \leq k$, since we have assumed that $2p < n$.

The above argument shows that every map $e : \Sigma^p \mathbf{Z}[G] \to \Sigma^n V$ can be lifted to a map $\alpha_0 : S^p \to M$ which extends to a normal surgery datum $\alpha : S^p \times D^{n-p} \hookrightarrow M$. However, this is not quite sufficient to prove Theorem 1. A normal surgery datum determines both a map $e : \Sigma^p \mathbf{Z}[G] \to \Sigma^n V$ *and* a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$. In the situation of Theorem 1, there is generally no guarantee that this coincides with the nullhomotopy we are interested in.

Note that $q|\Sigma^{p-n}\mathbf{Z}[G]$ can be identified with a point in the 0th space of the spectrum

$$\Sigma^{-n} Q^q(\Sigma^{p-n}\mathbf{Z}[G]) = \Sigma^{-n}(\Sigma^{2n-2p}\mathbf{Z}[G])_{h\Sigma_2} = \Sigma^{n-2p}(\mathbf{Z}[G])_{h\Sigma_2},$$

where on the right hand side the permutation group acts on $\mathbf{Z}[G]$ by means of the involution of the previous lecture together with the sign $(-1)^{n-p}$. Since $n > 2p$, this spectrum is always connected. If $n > 2p+1$, this spectrum is simply connected: it follows that a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$ is uniquely determined, up to homotopy. This completes the proof of Theorem 1 in this case.

If $n = 2p+1$, we need to work a little bit harder. Let T be the set of homotopy classes of trivializations of $q|\Sigma^{p-n}\mathbf{Z}[G]$. Then T is a torsor for the group $\pi_1 \Sigma^{-n} Q^q(\Sigma^{p-n}\mathbf{Z}[G]) \simeq \mathbf{Z}[G]_{\Sigma_2}$, where Σ_2 acts on the group $\mathbf{Z}[G]$ as indicated above. In particular, there is a transitive action of the group ring $\mathbf{Z}[G]$ (regarded as an abelian group under addition) on the set T. If $\alpha : S^p \times D^{n-p} \hookrightarrow M$ is a normal surgery datum whose restriction α_0 to $S^p \times \{0\}$ represents the homology class $e \in \pi_{p-n} V$, then α determines a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$. Let us denote the corresponding element of T by $t(\alpha)$. To prove Theorem 1, it will suffice to show that for any element $x \in \mathbf{Z}[G]$, we can find another normal surgery datum α' (still representing the homology class e) such that $t(\alpha') = x + t(\alpha)$. Since $\mathbf{Z}[G]$ is generated as abelian group by the elements of G, we may assume that $x = \pm g$, for some element $g \in G$. In this case, we assert without proof that there is a specific geometric construction for obtaining the surgery datum α' (it is determined by writing an isotopy through immersions from α_0 to another embedding α_0'). (Some pictures are provided in class; we will not reproduce them here.)

Surgery in the Middle Dimension (Lecture 35)

April 22, 2011

Recall that our goal is to prove the following:

Theorem 1. *Let X be a Poincare space of dimension $n \geq 5$, let ζ be a stable PL bundle on X, and let $f : M \to X$ be a degree one normal map, where M is a compact PL manifold. Assume that M and X are connected and that f induces an isomorphism $\pi_1 M \simeq \pi_1 X \simeq G$, so that σ_f^{vq} can be represented by a Poincare object (V, q), where $V \in \mathrm{LMod}_{\mathbf{Z}[G]}$ satisfies $C_*(\widetilde{M}; \mathbf{Z}) \simeq C_*(\widetilde{X}; \mathbf{Z}) \oplus \Sigma^n V$ (here $\widetilde{M}$ and $\widetilde{X}$ denote universal covers of M and X, respectively).*

Assume that f is p-connected, that we are given a map $u : \Sigma^{p-n}\mathbf{Z}[G] \to V$ a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$, so that (algebraic) surgery along u determines a bordism bordism from (V, q) to another Poincare object (V', q'). Then this (algebraic) bordism can be obtained by performing (geometric) surgery with respect to a normal surgery datum $\alpha : S^p \times D^{q+1} \hookrightarrow M$.

In the last lecture, we explained why this was true in the case where $p < \frac{n}{2}$. Our goal in this lecture is sketch the proof in the case $p = \frac{n}{2}$. Let us therefore assume that $n = 2k$, that f is k-connected, and that we are given a class $u : \Sigma^{-k}\mathbf{Z}[G] \to V$ and a nullhomotopy of $q|\Sigma^{-k}\mathbf{Z}[G]$. We would like to lift this data to a normal surgery datum in M.

Recall that, in order to identify $\mathbb{L}^{vq}(X, \zeta_X)$ with $\Sigma^{-n}\mathbb{L}^q(\mathbf{Z}[G])$, we chose a base point of X and a trivialization of the Spivak bundle of X at the base point. Without loss of generality, we may suppose that this base point is the image a point $x_0 \in M$, that G is given canonically as $\pi_1(M, x_0)$, and that x_0 is the base point used in constructing the universal cover $\widetilde{M}$ (so that x_0 lifts canonically to a point $\widetilde{x}_0 \in \widetilde{M}$). The trivialization of ζ_X at $f(x_0)$ gives an orientation of the manifold M at the point x_0, hence an orientation of $\widetilde{M}$ at the point $\widetilde{x_0}$ (which extends to an orientation of the whole of $\widetilde{M}$, since $\widetilde{M}$ is simply connected). We have a group homomorphism $\epsilon : G \to \{\pm 1\}$, where $\epsilon(g) = 1$ if the action of G on $\widetilde{M}$ preserves the orientation, and $\epsilon(g) = -1$ otherwise.

Let us now try to lift the class u to a normal surgery datum in X. The first step in the argument proceeds just as in the previous lecture. Since f is k-connected, we can apply the relative Hurewicz theorem to the pair (X, M) to represent u by a diagram

$$\begin{array}{ccc} S^k & \longrightarrow & \widetilde{M} \\ \downarrow{\scriptstyle \widetilde{\alpha}_0} & & \downarrow \\ D^{k+1} & \longrightarrow & \widetilde{X}. \end{array}$$

Let $\alpha_0 : S^k \to M$ be the composition of $\widetilde{\alpha}_0$ with the covering map $\widetilde{M} \to M$.

We now encounter our first difficulty: since $n = 2k$, we cannot use general position arguments to ensure that the map $\alpha_0 : S^k \to M$ is an embedding. However, we get *almost* this much for free: we can assume that α_0 is an immersion with a finite number of points of simple self-intersection. Then α_0 is factors as a composition

$$S^k \xrightarrow{\beta} U \to M,$$

where the map $U \to M$ is a local homeomorphism and β is both an embedding and a homotopy equivalence. Since $f \circ \alpha_0$ is canonically nullhomotopic, the pullback $\beta^* - T_M$ is trivialized, so that $-T_M$ is trivial along U. In particular, we can assume that U is equipped with a smooth structure (and a framing). Modifying β by a homotopy if necessary, we may assume that β is a smooth embedding with normal bundle $\mathcal{E}$, classified by a map $S^k \to \mathrm{BO}(k)$. Using the framings of S^k and U, we obtain a nullhomotopy of the composite map $S^k \to \mathrm{BO}(k) \to \mathrm{BO}$. Arguing as in the previous lecture, we can lift this to a nullhomotopy of the composite map $S^k \to \mathrm{BO}(k) \to \mathrm{BO}(k+1)$. Such a nullhomotopy cannot always be lifted to a trivialization of $\mathcal{E}$: we encounter an obstruction given by a map $S^k \to O(k+1)/O(k) \simeq S^k$, which has some degree $d \in \mathbf{Z}$. When k is even, this integer is half of the Euler class of $\mathcal{E}$; when k is odd, the integer d is really only well-defined modulo 2 (that is, the exact integer d depends on a choice of trivialization of $\mathcal{E} \oplus \underline{R}$). In either case, we can get the obstruction to vanish by introducing some "kinks" in the map β (that is, by replacing β by a map which is homotopic but not isotopic to β, possibly introducing some additional double points). By means of this procedure, we can ensure that the stable framing of $\mathcal{E}$ lifts to a trivialization of $\mathcal{E}$.

If the map α_0 were an embedding, then (possibly after shrinking U) we can identify U with an open subset of M, and the above argument allows us to extend α_0 to a normal surgery datum $\alpha : S^k \times D^k \hookrightarrow M$. However, α_0 is generally not an embedding: we certainly cannot arrange this for an arbitrary map $u : \Sigma^{-k} \to V$. We must take advantage of the algebraic information encoded in the nullhomotopy of $q|\Sigma^{-k}\mathbf{Z}[G]$. Note that $q|\Sigma^{-k}\mathbf{Z}[G]$ can be regarded as a point of the space $\Omega^{\infty+2k}Q^q(\mathbf{Z}[G]) = \Omega^{\infty}(\mathbf{Z}[G])_{h\Sigma_2}$. Here the action of the permuation group Σ_2 on $\mathbf{Z}[G]$ is given by $g \mapsto (-1)^k \epsilon(g) g^{-1}$, and the group of connected components of $\Omega^{\infty}(\mathbf{Z}[G])$ is given by the abelian group of coinvariants $\mathbf{Z}[G]_{\Sigma_2}$. The homotopy class of $q|\Sigma^{-k}\mathbf{Z}[G]$ is an element of this abelian group depending on u; let us denote it by $q(u)$.

Let us describe the image of $q(u)$ under the norm map

$$\mathrm{tr} : \mathbf{Z}[G]_{\Sigma_2} \to \mathbf{Z}[G]^{\Sigma_2} \subseteq \mathbf{Z}[G].$$

Unwinding the definition, we see that this is given by restricting the $\mathbf{Z}[G]$-valued intersection form on $C_*(\widetilde{M}; \mathbf{Z})$ along u. In particular, we have $\mathrm{tr}(q(u)) = \sum_{g \in G} j(g) g$, where $j(g)$ is the number of points of intersection (counted with multiplicity) of $\widetilde{\alpha}_0(S^k)$ with its translate under g. Let us describe this number more explicitly. Let $Y \subseteq S^k$ be the (finite) set of points over which $\alpha_0 : S^k \to M$ fails to be an immersion. For each $y \in Y$, there exactly one other element $\widehat{y} \in Y$ such that $\alpha_0(y) = \alpha_0(\widehat{y})$. We then have $\widetilde{\alpha}_0(\widehat{y}) = g_y \widetilde{\alpha}_0(y)$ for some element $g_y \in G$. Moreover, the orientations of S^k at y and $\widehat{y}$ determine an orientation of $\widetilde{M}$ at $\widetilde{\alpha}_0(\widehat{y})$. Define $\eta(y) \in \{\pm 1\}$ so that $\eta(y) = 1$ if this orientation agrees with our given orientation on $\widetilde{M}$, and $\eta(y) = -1$ otherwise. Note that $g(\widehat{y}) = g(y)^{-1}$, and that $\eta(\widehat{y}) = (-1)^k \epsilon(g(y))$ (the first sign comes from the fact that permuting the factors of $\mathbb{R}^k \times \mathbb{R}^k$ is orientation-reversing when k is odd, and the second from the fact that translation by $g(y)$ is orientation-reversing when $\epsilon(g(y)) = -1$). Unwinding the definitions (and using the triviality of the normal bundle $\mathcal{E}$), we see that the self-intersection of the homology class represented by $\widetilde{\alpha}_0$ is given by the expression

$$E = \sum_{y \in Y} \eta(y) g(y).$$

Note that $\sum_{y \in Y} \eta(y) g(y) = \sum_{y \in Y} \eta(\widehat{y}) (-1)^k \epsilon(g(y)) g(\widehat{y})^{-1}$. In particular, this sum is invariant under the involution $g \mapsto (-1)^k \epsilon(g) g^{-1}$, on $\mathbf{Z}[G]$, as we know it must be.

We now observe that E lies in the image of the transfer map $\mathrm{tr} : \mathbf{Z}[G]_{\Sigma_2} \to \mathbf{Z}[G]^{\Sigma_2}$. Namely, choose a decomposition $Y = Y_- \coprod Y_+$, where for every element $y \in Y$ exactly one of the points $\{y, \widehat{y}\}$ belongs to Y_-. Then write $E_0 = \sum_{y \in Y_-} \eta(y) g(y)$. Then $E_0 \in \mathbf{Z}[G]$ determines an element of $\mathbf{Z}[G]_{\Sigma_2}$, whose transfer is given by E. The specific element $E_0 \in \mathbf{Z}[G]$ depends on the choice of decomposition $Y = Y_- \cup Y_+$, but its image in $\mathbf{Z}[G]_{\Sigma_2}$ does not.

We assert the following without proof:

Claim 2. The element E_0 defined above is a representative for $q(u) \in \mathbf{Z}[G]_{\Sigma_2}$.

Remark 3. When X is simply connected and n is divisible by 4, the transfer map $\mathrm{tr} : \mathbf{Z}[G]_{\Sigma_2} \simeq \mathbf{Z} \to \mathbf{Z} \simeq \mathbf{Z}[G]^{\Sigma_2}$ is injective (it is given by multiplication by 2), so that Claim 2 follows from our analysis of the intersection pairing).

Let us now study the meaning of the condition that $q(u) = 0$. Let $G_{(2)} \subseteq G$ be the subset consisting of 2-torsion elements. Write $G_{(2)} = G^-_{(2)} \cup G^+_{(2)}$, where $g \in G^+_{(2)}$ if $(-1)^k \epsilon(g) = 1$, and $g \in G^-_{(2)}$ if $(-1)^k \epsilon(g) = -1$. Choose a subset $H \subseteq G - G_{(2)}$ such that for every element $g \in G$ which is no 2-torsion, exactly one of the elements $\{g, g^{-1}\}$ belongs to H. Then $\mathbf{Z}[G]_{\Sigma_2}$ is given by the direct sum

$$(\bigoplus_{g \in H} \mathbf{Z}g) \oplus (\bigoplus_{g \in G^+_{(2)}} \mathbf{Z}g) \oplus (\bigoplus_{g \in G^-_{(2)}} \mathbf{Z}/2\mathbf{Z}g).$$

We may assume that Y_- is chosen such that for $y \in Y_+$, if $g(y)$ is not 2-torsion, then $g(y) \in H$. Then $q(u) = 0$ implies the following:

(*i*) If $g \in H$, then the sum $\sum_{y \in Y_-, g(y) = g} \eta(y)$ vanishes.

(*ii*) If $g \in G^+_{(2)}$, then the sum $\sum_{y \in Y_-, g(y) = g} \eta(y)$ vanishes.

(*iii*) If $g \in G^-_{(2)}$, then the sum $\sum_{y \in Y_-, g(y) = g} \eta(y)$ is even.

Note that when $g(y) \in G^-_{(2)}$, then replacing y by $\widehat{y}$ changes the sign $\eta(y)$. We therefore see that $q(u) = 0$ if and only if we can choose Y_- such that the sum E_0 vanishes. In this case, we can partition Y into subsets of the form $\{y, \widehat{y}, y', \widehat{y}'\}$ where $g(y) = g(y')$ and $\eta(y) = -\eta(y')$.

The argument now proceeds by invoking Whitney's trick for cancelling double points. Suppose we are given a quadruple $\{y, \widehat{y}, y', \widehat{y}'\}$ as above. Choose a path γ from y to y' in S^p, and another path $\widehat{\gamma}$ from $\widehat{y}'$ to $\widehat{y}$. We may assume that the paths γ and $\widehat{\gamma}$ are embedded, do not intersect one another, and do not meet any points of Y except at their endpoints.

After applying the map α_0, we obtain paths in M which can be concatenated to yield a closed loop. The condition that $g(y) = g(y')$ implies that this loop is nullhomotopic (it lifts a loop in $\widetilde{M}$, given by concatenating $\widetilde{\alpha}_0(\gamma)$ with the translate by $g(y)^{-1}$ of $\widetilde{\alpha}_0(\widehat{\gamma})$). We may therefore represent this loop by the restriction to the boundary of a continuous map $D^2 \to M$. Since the dimension of M is at least 5 (in fact, at least 6, since $n = 2k$ is even), we may assume that the map $D^2 \to M$ is an embedding which does not intersect the image of α_0 except at the boundary. By analyzing a small neighborhood of the image of D^2, we can "push" the immersion α_0 to obtain a new (isotopic) immersion, in which the double points $\{y, y', \widehat{y}, \widehat{y}'\}$ have been eliminated. Iterating this procedure, we can reduce to the case where $Y = \emptyset$, so that α_0 is an embedding as desired.

The above analysis allows us to construct a normal surgery datum $\alpha : S^k \times D^k \hookrightarrow M$ representing any map $u : \Sigma^{-k}\mathbf{Z}[G] \to V$ such that $q(u) = 0$. However, this is not quite enough to prove Theorem **??**. Let T denote the collection of homotopy classes of trivializations of $q|\Sigma^{-k}\mathbf{Z}[G]$. Then T is a torsor for the fundamental group $\pi_1 \Omega^{\infty+n} Q^q(\Sigma^{-k}\mathbf{Z}[G]) \simeq \mathrm{H}_1(\Sigma_2; \mathbf{Z}[G])$, where the permutation group Σ_2 acts on $\mathbf{Z}[G]$ as indicated above. A simple calculation shows that this group is given by the direct sum $A = \bigoplus_{g \in G^+_{(2)}} \mathbf{Z}/2\mathbf{Z}$. Every normal surgery datum α representing u determines a nullhomotopy of $q|\Sigma^{-k}\mathbf{Z}[G]$, giving an element $t(\alpha) \in T$. In order to prove Theorem **??**, we need to show that every element of T has the form $t(\alpha)$, for some normal surgery datum α. For this, we need a procedure to modify a given normal surgery datum α to change the invariant $t(\alpha)$ in a controlled way.

The above analysis suggests such a procedure. Let $g \in G^+_{(2)}$. Suppose we begin with a fixed embedding $\alpha_0 : S^k \to M$. By "reversing" the Whitney trick described above, we can modify α_0 by an isotopy to obtain an immersion $\alpha'_0 : S^k \to M$ which fails to be an embedding at a set of points $\{y, y', \widehat{y}, \widehat{y}'\}$, satisfying $g(y) = g(y') = g$ and $\eta(y) = -\eta(y')$. Since $g = g^{-1}$ and $(-1)^k \epsilon(g) = 1$, we have $g(y) = g(\widehat{y})$ and $\eta(y) = \eta(\widehat{y})$. We may therefore switch the roles of y and $\widehat{y}$ in Whitney's construction, to obtain a *different* isotopy from

α_0' to an embedding α_0''. Since α_0 and α_0'' are isotopic (though immersions), every normal surgery datum α extending α_0 determines a normal surgery datum α'' extending α_0''. We assert that $t(\alpha) = t(\alpha'') + x$, where x is a generator for the summand of A corresponding to the element $g \in G$. Since the collection of such elements generate the abelian group A and A acts transitively on T, this claim completes the proof of Theorem 1.

The Homotopy Groups of G/PL (Lecture 36)

April 25, 2011

Let X be a Poincare space of dimension n. Recall that the normal structure space $\mathbb{S}^n(X)$ is homotopy equivalent to $\mathbb{S}^{tn}(X)$, which is given by the homotopy fiber of the map $\mathrm{BPL}^X \to \mathrm{Pic}(S)^X$ (which classifies stable PL reductions of the Spivak normal bundle of X). The map $\mathrm{BPL} \to \mathrm{Pic}(S)$ is a map of infinite loop spaces, whose fiber we denote by G/PL (here one should think of $G = \mathrm{GL}_1(S)$ as the automorphism group of the sphere spectrum). Consequently, if nonempty, the normal structure space $\mathbb{S}^{tn}(X)$ is homotopy equivalent to a torsor for the infinite loop space $(G/PL)^X$. Our goal in this lecture is to describe the homotopy type of G/PL.

More generally, if $(X, \partial X)$ is a Poincare pair such that ∂X is a compact PL manifold, we can describe $\mathbb{S}^{tn}(X)$ as the homotopy fiber of the canonical map

$$\mathrm{BPL}^X \to \mathrm{Pic}(S)^X \times_{\mathrm{Pic}(S)^{\partial X}} \mathrm{BPL}^{\partial X}$$

(over the point classifying the Spivak bundle of X together with its PL reduction on ∂X). Assume that the PL tangent bundle to ∂X is stably trivial, and that this trivialization extends to a trivialization of the Spivak bundle of X. Then $\mathbb{S}^n(X)$ can be identified with the space of maps of pairs from $(X, \partial X)$ to $(G/PL, *)$. Taking $X = D^n$ and $\partial X = S^{n-1}$, we obtain a canonical homotopy equivalence $\mathbb{S}^{tn}(X) = \Omega^n G/PL$.

Recall that in the above situation, we have a homotopy commutative diagram

$$\begin{tikzcd} \mathbb{S}(X) \arrow[r] \arrow[d] & \mathbb{S}^{tn}(X) \arrow[d, "\theta_n"] \\ * \arrow[r] & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X). \end{tikzcd}$$

In the special case $(X, \partial X) = (D^n, S^{n-1})$, the Spivak bundle ζ_X is the constant sheaf with value $\Sigma^{-n} S$. Since X is simply connected, it follows that $\mathbb{L}^{vq}(X, \zeta_X)$ is given by $\Sigma^{-n}\mathbb{L}^q(\mathbf{Z})$. We therefore have $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X) = \Omega^n L^q(\mathbf{Z})$, where $L^q(\mathbf{Z})$ denotes the zeroth space of the spectrum $\mathbb{L}^q(\mathbf{Z})$.

Taking $n = 0$, we obtain a map $\theta : G/PL \to L^q(\mathbf{Z})$. With a little bit of effort, one can show that *all* of the maps θ_n appearing above are induced by θ by passing to n-fold loop spaces. We would like to use the map θ to obtain information about the homotopy type of G/PL. We will obtain this information by combining two facts:

(a) If $n \geq 5$, then we have a homotopy pullback diagram

$$\begin{tikzcd} \mathbb{S}(D^n) \arrow[r] \arrow[d] & \Omega^n(G/PL) \arrow[d, "\Omega^n \theta"] \\ * \arrow[r] & \Omega^n L^q(\mathbf{Z}). \end{tikzcd}$$

This is a special case of our main theorem.

(b) If $n \geq 5$, the structure space $\mathbb{S}(D^n)$ is contractible.

Let us provide an argument for (b). Recall that the structure space $\mathbb{S}(D^n)$ is given as the geometric realization of a simplicial space $\mathbb{S}(D^n)_\bullet$. Let $Y_\bullet$ be the constant simplicial space which consists of a single point in each degree. We claim that the map $\mathbb{S}(D^n)_\bullet \to Y_\bullet$ is a trivial Kan fibration. In other words, we claim that for every integer k, the map $\mathbb{S}(D^n)_k \to \mathbb{S}(D^n)_\bullet(\partial\,\Delta^k)$ is surjective on connected components.

When $k = 0$, this says that $\mathbb{S}(D^n)$ is nonempty: this is obvious, since D^n is already a PL manifold. We now give the proof when $k = 1$; the proof of the general case is the same. Suppose we are given an element of $\mathbb{S}(D^n)_\bullet(\partial\,\Delta^1)$, consisting of two contractible PL manifolds M and M' having boundary S^{n-1}. To lift to a point of $\mathbb{S}(D^n)_1$ we must write an h-cobordism from M to M', trivial along S^{n-1}. This is equivalent showing that the manifold $M \coprod_{S^{n-1}} M'$ bounds a contractible PL manifold of dimension $n+1$. Since M and M' are contractible, $M \coprod_{S^{n-1}} M'$ is homotopy equivalent to a sphere S^n. Since $n \geq 5$, the generalized Poincare conjecture implies that $M \coprod_{S^{n-1}} M'$ is PL homeomorphic to S^n, which bounds the disk D^{n+1}.

Remark 1. In fact, something much stronger is true: each of the spaces $\mathbb{S}(D^n)_k$ is contractible. One can prove this by combining the generalized Poincare conjecture with the Alexander trick.

Warning 2. In proving (b), it is important that we work in the PL category rather than the smooth category. The smooth structure space of D^n is generally not contractible because of the existence of exotic spheres. We can appreciate the importance of the PL condition by examining Smale's proof of the generalized Poincare conjecture. Let M be a manifold of dimension n which is homotopy equivalent to a sphere, and assume that $n \geq 6$ (the case $n = 5$ requires additional effort). Choose two distinct points $x, x' \in M$, and let D and D' be disjoint small disks around x and x', respectively. Let M° be the manifold obtained by removing the interiors of D and D'. The condition that M is homotopy equivalent to S^n guarantees that M° is an h-cobordism from $\partial\,D$ to $\partial\,D'$. The h-cobordism theorem then gives $M^\circ \simeq S^{n-1} \times [0,1]$. If we work in the PL category, we can recover M from M° by "collapsing" the two ends, thereby obtaining $M \simeq S^n$. In the smooth category, this argument does not apply: given a manifold with a boundary sphere, there is no canonical way to assign a smooth structure to the manifold obtained by the collapsing the sphere.

Combining observations (a) and (b), we conclude the following:

(c) If $n \geq 5$, the homotopy fiber of the map $\Omega^n(\theta) : \Omega^n G/PL \to \Omega^n L^q(\mathbf{Z})$ is contractible. In other words, the map

$$\pi_i G/PL \to \pi_i L^q(\mathbf{Z})$$

is injective when $i = 5$, and bijective for $i > 5$.

In fact, we can do a little bit better. Since $\pi_5 L^q(\mathbf{Z})$ is trivial, the fact that $\pi_5 G/PL \to \pi_5 L^q(\mathbf{Z})$ is injective implies that $\pi_5 G/PL$ is trivial. It follows that $\Omega^5(\theta)$ is a homotopy equivalence.

Combining (c) with our calculation of the homotopy groups $L^q_n(\mathbf{Z})$, we obtain the following:

Corollary 3. *Let $n \geq 5$. Then we have a canonical isomorphism*

$$\pi_n G/PL \simeq \begin{cases} 8\mathbf{Z} & \text{if } n = 4k \\ 0 & \text{if } n = 4k+1 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 4k+2 \\ 0 & \text{if } n = 4k+3. \end{cases}$$

Let us now turn our attention to calculating the homotopy groups of G/PL in *low* degrees. We have maps

$$\mathbf{Z} \times \mathrm{BO} \to \mathbf{Z} \times \mathrm{BPL} \to \mathrm{Pic}(S),$$

giving rise to a fiber sequence of spaces

$$PL/O \to G/O \to G/PL.$$

Smoothing theory gives an identification of $\pi_n PL/O$ with the collection of smooth structures on S^n (compatible with the standard PL structure on S^n. It follows that the map

$$\pi_n G/O \to \pi_n G/PL$$

is bijective provided that the PL spheres S^n and S^{n-1} admit unique smoothings. This is true for $n \leq 6$.

We can use this observation to compute $\pi_n G/PL$ for small values of n, since the homotopy groups of $\mathrm{Pic}(S)$ and $\mathbf{Z} \times \mathrm{BO}$ are known. Let us summarize them in the following table:

	$\mathbf{Z} \times \mathrm{BO}$	$\mathrm{Pic}(S)$
π_5	0	0
π_4	$\mathbf{Z}$	$\mathbf{Z}/24\mathbf{Z}$
π_3	0	$\mathbf{Z}/2\mathbf{Z}$
π_2	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
π_1	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
π_0	$\mathbf{Z}$	$\mathbf{Z}$.

The map from the groups on the left to the groups on the right is given by the J-homomorphism. This map is an isomorphism on π_1 and π_2 and a surjection on π_4. We therefore obtain

$$\pi_n G/PL = \begin{cases} 24\mathbf{Z} & \text{if } n = 4 \\ 0 & \text{if } n = 3 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2 \\ 0 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

The map $\pi_n(G/PL) \to \pi_n L^q(\mathbf{Z})$ is obviously an isomorphism if $n = 1$ or $n = 3$, since both sides vanish. Let us show that it is an isomorphism when $n = 2$. Let M be an oriented surface equipped with a spin structure (or "theta characteristic"). Using the spin structure, we can trivialize the tangent bundle of M outside of a small disk $D \subseteq M$, thereby obtaining a degree one normal map $f : M \to D/\partial D \simeq S^2$, which in turn represents an element in $\pi_0 \mathbb{S}^n(S^2) \simeq \pi_0 (G/PL)^{S^2} \simeq \pi_2(G/PL)$. The choice of spin structure determines a quadratic refinement q of the intersection form on $\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})$, making $\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})$ into a nondegenerate quadratic space over the finite field $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$. To prove that the map

$$\mathbf{Z}/2\mathbf{Z} \simeq \pi_2(G/PL) \to \pi_2 L^q(\mathbf{Z}) \simeq \pi_2 L^q(\mathbf{F}_2) \simeq W(\mathbf{F}_2) \simeq \mathbf{Z}/2\mathbf{Z}$$

is nontrivial, it suffices to show that we can choose M (and its spin structure) so that the quadratic space $(\mathrm{H}^1(M; \mathbf{F}_2), q)$ has Arf invariant 1. This is always possible. The collection of spin structures on M is a torsor for $\mathrm{H}^1(M;\mathbf{F}_2)$. If q has Arf invariant 0, then we can modify its Arf invariant by modifying the spin structure by an element $v \in \mathrm{H}^1(M;\mathbf{F}_2)$ satisfying $q(v) = 1$ (which always exists provided that the genus of M is positive).

Let us now compute the map

$$\pi_4 G/PL \to \pi_4 L^q(\mathbf{Z}) = 8\mathbf{Z}$$

determined by θ. Choose a map $u : S^4 \to G/O$, representing a generator of $[u] \in \pi_4 G/O \simeq \pi_4 G/PL$. Then u determines a point of the structure space $\mathbb{S}^{tn}(S^4)$, corresponding to a PL (in fact smooth) 4-manifold M equipped with a degree one normal map $f : M \to S^4$. By construction, the image of $[u]$ in $L^q_4(\mathbf{Z}) \simeq 8\mathbf{Z}$ is the difference of signatures $\sigma_M - \sigma_{S^4}$. Since the signature of S^4 vanishes, this is just the signature of M. The Hirzebruch signature formula shows that this is given by $\frac{p_1(T_M)}{3}[M]$.

Let $v : S^4 \to \mathrm{BO}$ represent a generator $[v] \in \pi_4 \mathrm{BO} \simeq \mathbf{Z}$. Then the composite map $S^4 \xrightarrow{u} G/O \to \mathrm{BO}$ represents $24[v]$. Let us regard p_1 as an element in the integral cohomology ring $\mathrm{H}^4(\mathrm{BO};\mathbf{Z})$. Since f has degree 1, we can identify $\frac{p_1(T_M)}{[}M]$ with $-24v^*(p_1) \in \mathrm{H}^4(S^4;\mathbf{Z}) \simeq \mathbf{Z}$ (the sign comes from our convention that the map u classifies the normal bundle, rather than the tangent bundle). It follows that the image of $[u]$ is given by $-8v^*(p_1) \in 8\mathbf{Z}$.

To compute $v^*(p_1)$, let us identify S^4 with the quaternionic projective space $\mathbf{H}\mathrm{P}^1$, Then we can take $v : \mathbf{H}\mathrm{P}^1 \to \mathrm{BO}$ to classify the real vector bundle $\mathcal{E}$ underlying quaternionic line bundle $\mathcal{O}(1)$. In particular, $\mathcal{E}$ admits the structure of a complex vector bundle , so that

$$p_1(\mathcal{E}) = -c_2(\mathcal{E} \otimes \mathbf{C}) = -c_2(\mathcal{E} \oplus \overline{\mathcal{E}}) = -c_2(\mathcal{E}) - c_1(\mathcal{E})c_1(\overline{\mathcal{E}}) - c_2(\overline{\mathcal{E}}) = -2c_2(\mathcal{E}) = -2e(\mathcal{E}),$$

where e denotes the Euler class of $\mathcal{E}$. Since $\mathcal{E}$ admits a section having exactly one simple zero, we obtain $v^*(p_1) = -2$.

Corollary 4. *The map $\theta : (G/PL) \to L^q(\mathbf{Z})$ induces an isomorphism $\pi_n(G/PL) \to \pi_n L^q(\mathbf{Z})$ for all $n \neq 0, 4$. The map $\pi_4(G/PL) \to \pi_4 L^q(\mathbf{Z})$ is injective, and its image is the subgroup $16\mathbf{Z} \subseteq 8\mathbf{Z} \simeq \pi_4 L^q(\mathbf{Z})$.*

Remark 5. The failure of the map $\pi_4(G/PL) \to \pi_4 L^q(\mathbf{Z})$ to be surjective is a consequence of Rohlin's theorem: a smooth, compact, spin 4-manifold M has signature divisible by 16.

We can regard Corollary **??** as a calculation of the homotopy groups of the homotopy fiber of the map θ. Since these homotopy groups are concentrated in a single degree, the structure of the homotopy fiber can be explicitly determined:

Corollary 6. *We have a homotopy fiber sequence $K(\mathbf{Z}/2\mathbf{Z}, 3) \to G/PL \to L^q(\mathbf{Z})$*

It is possibly to carry out many of the constructions of this course in the setting of topological manifolds (rather than PL manifolds). Stable topological bundles are classified by a space $\mathbf{Z} \times \mathrm{BTOP}$ fitting into a diagram

$$\mathbf{Z} \times \mathrm{BPL} \to \mathbf{Z} \times \mathrm{BTOP} \to \mathrm{Pic}(S).$$

There is an associated fiber sequence $\mathrm{Top}/PL \to G/PL \to G/\mathrm{Top}$. The map $G/PL \to L^q(\mathbf{Z})$ factors through G/Top, and G/Top is homotopy equivalent to the identity component of $L^q(\mathbf{Z})$. Assuming this, Corollary 6 gives a homotopy equivalence $\mathrm{Top}/PL \simeq K(\mathbf{Z}/2\mathbf{Z}, 3)$. We therefore have a fiber sequence of infinite loop spaces

$$\mathbf{Z} \times \mathrm{BPL} \to \mathbf{Z} \times \mathrm{BTOP} \xrightarrow{\phi} K(\mathbf{Z}/2\mathbf{Z}, 4).$$

The map ϕ classifies the *Kirby-Siebenmann obstruction.* If M is a topological manifold, its stable (topological) tangent bundle is classified by a map $M \to \mathbf{Z} \times \mathrm{BTOP}$. Composing this map with ϕ, we get a map $M \to K(\mathbf{Z}/2\mathbf{Z}, 4)$, classified by a cohomology class $\nu \in \mathrm{H}^4(M;\mathbf{Z}/2\mathbf{Z})$. This class vanishes if and only if the stable tangent bundle of M admits a PL reduction. In particular, ν vanishes whenever M admits a PL structure. One can show that the converse holds if the dimension of M is different from 4.

The Total Surgery Obstruction (Lecture 37)

April 27, 2011

Let X be a finite polyhedron, ζ_X a spherical fibration on X, and R and A_∞-ring with involution. Recall that we have a homotopy pullback diagram of spectra

$$\begin{tikzcd} \mathbb{L}^q(X, \zeta_X, R) \arrow[r] \arrow[d] & \mathbb{L}^s(X, \zeta_X, R) \arrow[d] \\ \mathbb{L}^{vq}(X, \zeta_X, R) \arrow[r] & \mathbb{L}^{vs}(X, \zeta_X, R), \end{tikzcd}$$

where the vertical maps are given by assembly. From now on, we will fix R to be the ring $\mathbf{Z}$ of integers, and omit it from the notation (we could just as well take R to be the sphere spectrum). Let $\widehat{\mathbb{L}}(X, \zeta_X)$ denote the cofibers of the horizontal maps, so that we have a diagram

$$\begin{tikzcd} \mathbb{L}^q(X, \zeta_X) \arrow[r] \arrow[d] & \mathbb{L}^s(X, \zeta_X) \arrow[r] \arrow[d] & \widehat{\mathbb{L}}(X, \zeta_X) \arrow[d] \\ \mathbb{L}^{vq}(X, \zeta_X) \arrow[r] & \mathbb{L}^{vs}(X, \zeta_X) \arrow[r] & \widehat{\mathbb{L}}(X, \zeta_X). \end{tikzcd}$$

When the spherical fibration ζ_X is trivial; we will omit it from the notation. If, in addition, X is a point, then the vertical maps are homotopy equivalences and we have a single fiber sequence

$$\mathbb{L}^q(\mathbf{Z}) \to \mathbb{L}^s(\mathbf{Z}) \to \widehat{\mathbb{L}}.$$

The upper row of the diagram above consists of functors which are excisive in X. We may therefore write $\widehat{\mathbb{L}}(X, \zeta_X) = C_*(X, \widehat{\mathbb{L}}(\zeta_X))$, where $\widehat{\mathbb{L}}(\zeta_X)$ is the local system on X which assigns to each point $x \in X$ the spectrum $\widehat{\mathbb{L}}(\zeta_X(x))$.

Suppose we are given a map of spaces $i : \partial X \to X$. We let $\mathbb{L}^{vs}(X, \partial X, \zeta_X)$ denote the cofiber of the map $\mathbb{L}^{vs}(\partial X, \zeta_X | \partial X) \to \mathbb{L}^{vs}(X, \zeta_X)$, and define $\mathbb{L}^{vq}(X, \partial X, \zeta_X)$ and $\widehat{\mathbb{L}}(X, \partial X, \zeta_X)$ similarly. By functoriality, we obtain vertical maps fitting into a commutative diagram of spectra

$$\begin{tikzcd} \mathbb{L}^q(\mathbf{Z}) \wedge (X/\partial X) \arrow[r] \arrow[d] & \mathbb{L}^s(\mathbf{Z}) \wedge (X/\partial X) \arrow[r] \arrow[d] & \widehat{\mathbb{L}} \wedge (X/\partial X) \arrow[d] \\ \mathbb{L}^{vq}(X, \partial X) \arrow[r] & \mathbb{L}^{vs}(X, \partial X) \arrow[r] & \widehat{\mathbb{L}}(X, \partial X). \end{tikzcd}$$

The right vertical map is a homotopy equivalence (by excision). If X and ∂X have the same fundamental groupoid, then the π-π theorem implies that $\mathbb{L}^{vq}(X, \partial X)$ vanishes. The above diagram then gives a canonical homotopy equivalence

$$\widehat{\mathbb{L}} \wedge (X/\partial X) \simeq \mathbb{L}^{vs}(X, \partial X).$$

In particular, we can take $X = D^n$ and $\partial X = S^{n-1}$ for $n \geq 3$, to obtain a homotopy equivalence

$$\widehat{\mathbb{L}} \simeq \Omega^n \mathbb{L}^{vs}(D^n, S^{n-1}).$$

Recall that the symmetric L-theory spectrum $\mathbb{L}^s(\mathbf{Z})$ is an E_∞-ring spectrum, with multiplication induced by the tensor product of chain complexes. If we have pairs of spaces $(X, \partial X)$ and $(Y, \partial Y)$, there is a similar multiplicaiton

$$\mathbb{L}^{vs}(X, \partial X) \wedge \mathbb{L}^{vs}(Y, \partial Y) \to \mathbb{L}^{vs}(X \times Y, \partial(X \times Y)),$$

where $\partial(X \times Y)$ denotes the homotopy pushout $(\partial X \times Y) \coprod_{\partial X \times \partial Y} (X \times \partial Y)$. Taking X and Y to be disks of dimension ≥ 3, this gives a multiplication

$$\widehat{\mathbb{L}} \wedge \widehat{\mathbb{L}} \to \widehat{\mathbb{L}}.$$

Using more elaborate reasoning along the same lines, we see that $\widehat{\mathbb{L}}$ also has the structure of an E_∞-ring spectrum, and that the canonical map $\mathbb{L}^s(\mathbf{Z}) \to \widehat{\mathbb{L}}$ can be promoted to a map of E_∞-ring spectra.

Now suppose that $(X, \partial X)$ is a Poincare pair with Spivak fibration ζ_X, so that the visible symmetric signature $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^s(X, \partial_X, \zeta_X)$ is defined. Let $\widehat{\sigma}_X$ denote the image of σ_X^{vs} in $\Omega^\infty \widehat{\mathbb{L}}(X, \partial X, \zeta_X)$.

Let us now suppose that $\partial X = S^{n-1}$ is a sphere, and that $X = D^n$ is the cone on ∂X. Then the Spivak bundle of X is canonically equivalent to the constant sheaf whose value is an invertible specturm E, given by the *inverse* of $\Sigma^\infty(X/\partial X)$. Then $\widehat{\mathbb{L}}(X, \partial X, \zeta_X)$ can be identified with the spectrum $E^{-1} \wedge \widehat{\mathbb{L}}(*, E)$. We can then identify $\widehat{\sigma}_X$ with a map $\widehat{\mathbb{L}}$-modules $\widehat{\phi} : E \wedge \widehat{\mathbb{L}} \to \widehat{\mathbb{L}}(E)$, which is a homotopy equivalence. In Lecture 23, we discussed an analogous homotopy equivalence

$$\phi : E \wedge \mathbb{L}^s(\mathbf{Z}) \simeq \mathbb{L}^s(*, E).$$

These maps fit into a commutative diagram

$$\begin{tikzcd}
E \wedge \mathbb{L}^s(\mathbf{Z}) \arrow[r, "\phi"] \arrow[d] & \mathbb{L}^s(*, E) \arrow[d] \\
E \wedge \widehat{\mathbb{L}} \arrow[r, "\widehat{\phi}"] & \widehat{\mathbb{L}}(*, E).
\end{tikzcd}$$

However, there is an important difference: to write down the map ϕ, we needed to realize the spectrum $\mathbb{L}^s(X, \partial X, \underline{E})$ in terms of constructible sheaves on X; the map ϕ itself was given by choosing the constant sheaf on X, which is Verdier-self dual (up to a twist). Consequently, ϕ is functorial with respect to PL homeomorphisms of $X = D^n$. However, the construction of $\widehat{\phi}$ depends only on the realization of $(X, \partial X)$ as a Poincare pair, and is therefore functorial with respect to all homotopy equivalences of $\partial X = S^{n-1}$.

Elaborating on the above construction, we obtain the following:

Proposition 1. *Let E be any invertible spectrum. Then there is a canonical homotopy equivalence*

$$E \wedge \widehat{\mathbb{L}} \simeq \widehat{\mathbb{L}}(*, E).$$

Now suppose we are given a point $\eta \in G/PL = \mathrm{fib}(\mathbf{Z} \times \mathrm{BPL} \to \mathrm{Pic}(S))$. Then η classifies a stable PL bundle ζ over a point, together with a trivialization of the underlying invertible spectrum E. Using ζ, we can write down a homotopy equivalence of $\mathbb{L}^s$-modules

$$\phi_\zeta : E \wedge \mathbb{L}^s(\mathbf{Z}) \to \mathbb{L}^s(*, E).$$

Since E is trivial, we can think of ϕ_ζ as an invertible map from $\mathbb{L}^s(\mathbf{Z})$ to itself. This construction gives a map

$$G/PL \to \mathrm{GL}_1(\mathbb{L}^s(\mathbf{Z}))$$

Since the automorphism of $\widehat{\mathbb{L}}$ determined by ϕ_ζ depends only on the underlying spherical fibration of ζ, this automorphism is trivial. Consequently, the composite map

$$G/PL \to \mathrm{GL}_1(\mathbb{L}^s(\mathbf{Z})) \to \mathrm{GL}_1(\widehat{\mathbb{L}})$$

is canonically nullhomotopic. We therefore obtain a map $G/PL \to F$, where F denotes the homotopy fiber of the map $\mathrm{GL}_1(\mathbb{L}^s) \to \mathrm{GL}_1(\widehat{\mathbb{L}})$. Note that F is the identity component of the space

$$L^q(\mathbf{Z}) = \Omega^\infty \operatorname{fib}(\mathbb{L}^s \to \widehat{\mathbb{L}}).$$

The above construction recovers the map $\theta : G/PL \to L^q(\mathbf{Z})$ described in the previous lecture. Recall that this map is *almost* a homotopy equivalence: we have a fiber sequence $K(\mathbf{Z}/2\mathbf{Z}, 3) \to G/PL \to L^q(\mathbf{Z})$.

Now let X be any finite polyhedron and ζ_X any spherical fibration on X. Using excision and Proposition 1, we obtain homotopy equivalences

$$\widehat{\mathbb{L}}(X, \zeta_X) \simeq C_*(X; \widehat{\mathbb{L}}(\zeta_X)) \simeq C_*(X; \zeta_X \wedge \widehat{\mathbb{L}}).$$

Here $\widehat{\mathbb{L}}(\zeta_X)$ denotes the local system on X which assigns to each point $x \in X$ the spectrum $\widehat{\mathbb{L}}(\{x\}, \zeta_X(x))$. In the special case where X is a Poincare space and ζ_X is its Spivak bundle, we obtain

$$\widehat{\mathbb{L}}(X, \zeta_X) \simeq C^*(X; \underline{\widehat{\mathbb{L}}}).$$

Under this homotopy equivalence, the point $\widehat{\sigma}_X \in \Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)$ corresponds to the global section of $\underline{\widehat{\mathbb{L}}}$ given by the unit of $\widehat{\mathbb{L}}$.

A similar calculation gives $\mathbb{L}^s(X, \zeta_X) \simeq C^*(X; \zeta_X^{-1} \wedge \mathbb{L}^s(\zeta_X))$, where $\mathbb{L}^s(\zeta_X)$ denotes the local system on X which assigns to each point $x \in X$ the spectrum $\mathbb{L}^s(\{x\}, \zeta_X(x))$. In other words, we can identify $\mathbb{L}^s(X, \zeta_X)$ with the $\mathrm{Mor}(\zeta_X \wedge \mathbb{L}^s(\mathbf{Z}), \mathbb{L}^s(\zeta_X))$ in the ∞-category of local systems of $\mathbb{L}^s(\mathbf{Z})$-modules on X. Let $\mathbb{L}^s(X, \zeta_X)^\times$ denote the subspace of $\Omega^\infty \mathbb{L}^s(X, \zeta_X)$ corresponding to *isomorphisms* $\zeta_X \wedge \mathbb{L}^s(\mathbf{Z}) \to \mathbb{L}^s(\zeta_X)$. We have a map

$$\mathbb{L}^s(X, \zeta_X)^\times \to \Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X).$$

The homotopy fiber of this map over $\widehat{\sigma}_X$ can be identified with the space of sections of a fibration $X' \to X$, having fiber $F = \operatorname{fib}(\mathrm{GL}_1(\mathbb{L}^s(\mathbf{Z})) \to \mathrm{GL}_1(\widehat{\mathbb{L}}))$.

Any stable PL structure on the bundle ζ_X gives an isomorphism of local systems of $\mathbb{L}^s(\mathbf{Z})$-modules $\zeta_X \wedge \mathbb{L}^s(\mathbf{Z}) \to \mathbb{L}^s(\zeta_X)$, corresponding to a point of $\mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)} \{\widehat{\sigma_X}\}$. The collection of such PL structures is classified by the space of sections of a fibration $X \times_{\mathrm{Pic}(S)} (\mathbf{Z} \times \mathrm{BPL}) \to X$, having fiber G/PL. We have a map of spaces over X

$$X \times_{\mathrm{Pic}(S)} (\mathbf{Z} \times \mathrm{BPL}) \to X'$$

having homotopy fiber $K(\mathbf{Z}/2\mathbf{Z}, 3)$. Consequently, every section s of the map $X' \to X$ determines a fibration $Y \to X$ with fiber $K(\mathbf{Z}/2\mathbf{Z}, 3)$, where $Y = X \times_{X'} (X \times_{\mathrm{Pic}(S)} (\mathbf{Z} \times \mathrm{BPL}))$. This fibration is classified by a map $X \to K(\mathbf{Z}/2\mathbf{Z}, 4)$, which depends on the choice of section s. We therefore obtain a map

$$\mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)} \{\widehat{\sigma_X}\} \to K(\mathbf{Z}/2\mathbf{Z}, 4)^X,$$

whose fiber can be identified with the structure space $\mathbb{S}^{tn}(X)$.

Let $\mathbb{S}'(X)$ denote the homotopy fiber product $\mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \mathbb{L}^{vs}(X, \zeta_X)} \{\sigma_X^{vs}\}$. Recall that if M is a manifold equipped with a homotopy equivalence $f : M \to X$, then f determines a lifting of σ_X^{vs}, giving a point of $\mathbb{S}'(X)$. Elaborating on this construction, we get a map $\mathbb{S}(X) \to \mathbb{S}'(X)$, which fits into a commutative diagram

$$\begin{tikzcd}
\mathbb{S}(X) \arrow[r] \arrow[d] & \mathbb{S}^n(X) \arrow[r] \arrow[d] & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X) \arrow[d] \\
\mathbb{S}'(X) \arrow[r] & \mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)} \{\widehat{\sigma}_X\} \arrow[r] & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X).
\end{tikzcd}$$

Assume that the dimension of X is at least 5. The main theorem of this course asserts that the upper row is a fiber sequence, and the bottom row is obviously a fiber sequence. Since the right vertical map is a homotopy equivalence, we see that the square on the left is homotopy Cartesian. Combining this with the above analysis, we obtain:

Theorem 2. *Let X be a Poincare space of dimension ≥ 5. Then we have a fiber sequence of spaces*

$$\mathbb{S}(X) \to \mathbb{S}'(X) \to K(\mathbf{Z}/2\mathbf{Z}, 4)^X,$$

where $\mathbb{S}'(X) = \mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \mathbb{L}^{vs}(X,\zeta_X)} \{\sigma_X^{vs}\}$.

Remark 3. It is possible to prove the main results of this course in the setting of topological, rather than piecewise linear manifolds. However, things work slightly differently in low degrees: we actually get a homotopy equivalence from G/Top to the base point component of $L^q(\mathbf{Z})$. The analysis above shows that $\mathbb{S}'(X)$ can be identified with the *topological* structure space of X, parametrizing h-cobordism classes of compact topological manifolds in the homotopy type of X. The map $\psi : \mathbb{S}'(X) \to K(\mathbf{Z}/2\mathbf{Z}, 4)^X$ is a version of the *Kirby-Siebenmann obstruction*: it assigns to every topological manifold M of dimension ≥ 5 a cohomology class $\eta \in \mathrm{H}^4(M; \mathbf{Z}/2\mathbf{Z})$, which vanishes if and only if M admits a PL structure.