Topology Vol. 8, pp. 357—359 Pergamon Press, 1969. Printed in Great Britain

SEMI-CHARACTERISTICS AND COBORDISM

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(*Received* 10 *March* 1969)

§1. STATEMENT OF RESULTS

LET $M = M^{2r+1}$ be a closed $(2r + 1)$-dimensional orientable manifold. The *semi-characteristic* of M with respect to a coefficient field F is defined to be

$$\sigma(M; F) = \sum_{i=0}^{r} \text{rank } H_i(M; F),$$

taken as an integer modulo 2. (Compare Kervaire [3]. For most applications one takes $F = Z_2$.)

In general the semi-characteristic of M depends on the choice of the field F. For example if M is a 3-dimensional lens space, then certainly $\sigma(M; F)$ depends on the choice of F. Our main result shows that under certain conditions $\sigma(M; F)$ is independent of F.

More generally, consider an orientable Poincaré space (that is a space M which satisfies Poincaré duality over the integers) of dimension $2r + 1$. The ith Wu class $v_i \in H^i(M; Z_2)$ is defined by the condition

$$v_i x = \text{Sq}^i x$$

for all $x \in H^{2r+1-i}(M; Z_2)$.

THEOREM. *If M is an orientable Poincaré space of dimension $2r + 1$ with r even, then the difference*

$$\sigma(M; Z_2) - \sigma(M; Q)$$

is equal to the Stiefel–Whitney number

$$(v_r \text{Sq}^1 v_r)[M] = (w_2 w_{2r-1})[M].$$

As an immediate corollary, we see that the equality $\sigma(M; Z_2) = \sigma(M; Q)$ holds whenever M is either a spin manifold ($w_2 M = 0$) or a boundary, providing that the dimension of M is a number of the form $4q + 1$. In these cases it follows that the semi-characteristic $\sigma(M; F)$ is completely independent of the field F. For if p is an odd prime, then the equality

$$\sigma(M^{4q+1}; Z_p) = \sigma(M^{4q+1}; Q)$$

follows easily from Browder [2] or de Rham [4]. (See Remark 1 below.)

Our Theorem can be considered as a clarification and completion of Browder's results in [2]. It replaces the last sentences in Theorems 1 and 2 of [2], which are false as stated.

It can also be considered as a generalization of a theorem of C.T.C. Wall. (Compare Barden [1, p. 372].)

Remarks. 1. The difference $\sigma(M; Z_p) - \sigma(M; Q)$ can be described more directly as the number, modulo 2, of p-primary cyclic summands in a direct sum decomposition of the group $H_r(M^{2r+1}; Z)$. But if r is even then according to [2, Theorem 1] or [4, Th. II, p. 165] the torsion subgroup of $H_r(M^{2r+1}; Z)$ is isomorphic either to a direct sum $A \oplus A$ of two mutually isomorphic groups or to a direct sum $A \oplus A \oplus Z_2$. Evidently the second alternative occurs if and only if $\sigma(M; Z_2) \neq \sigma(M; Q)$.

2. Note that our Theorem would be false in dimensions of the form $4q - 1$. For example the real projective space P^3 is parallelizable; but $\sigma(P^3; Z_2) \neq \sigma(P^3; Q)$.

3. Similarly the Theorem would be false for non-orientable manifolds. For example $\sigma(P^4 \times S^1; Z_2) \neq \sigma(P^4 \times S^1; Q)$.

4. For each $q > 0$ there does exist an orientable manifold M of dimension $4q + 1$ with $w_2 w_{4q-1}[M] \neq 0$. In fact for $q = 1$ the Wu 5-manifold has this property ([6, p. 80]); and for $q > 1$, using the identity

$$\sigma(M^{2r+1} \times M^{2s}; F) = \sigma(M^{2r+1}; F)\chi(M^{2s}),$$

we see that the product of the Wu 5-manifold and the $4(q - 1)$-dimensional complex projective space has the required property.

We want to thank M. F. Atiyah and I. M. Singer for asking us about the relationship between $\sigma(M; Z_2)$ *and* $\sigma(M; Q)$.

§2. PROOF OF THE THEOREM

For $x, y \in H^r(M; Z_2)$ define a bilinear inner product

$$\langle x, y \rangle = (x\mathrm{Sq}^1 y)[M] \in Z_2 .$$

The computation

$$(x\mathrm{Sq}^1 y + y\mathrm{Sq}^1 x)[M] = \mathrm{Sq}^1(xy)[M] = 0,$$

using the fact that $v_1 = 0$ since M is orientable, shows that $\langle x, y \rangle = \langle y, x \rangle$. Let $x_1, \ldots, x_n$ be a basis for $H^r(M; Z_2)$ and note that the rank of the matrix $\langle x_i, x_j \rangle$ is equal to the rank of $\mathrm{Sq}^1 \colon H^r(M; Z_2) \to H^{r+1}(M; Z_2)$ and hence is equal to the number of copies of Z_2 in the primary cyclic decomposition of $H_r(M; Z)$. Using Remark 1 above, this rank is congruent to $\sigma(M; Z_2) - \sigma(M; Q)$ modulo 2. (The hypothesis that r is even is needed here.)

Note also the identity $\langle x, x \rangle = \langle v_r, x \rangle$, which is proved in [2, Lemma 5]. We now consider three cases.

Case 1. If $\langle v_r, x \rangle = 0$ for all x then $\langle x, x \rangle = 0$ for all x and the matrix $\langle x_i, x_j \rangle$ can be put into a normal form with zeros or copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ along the diagonal. Hence it has even rank.

Case 2. If $\langle v_r, v_r \rangle \neq 0$, then the form $\langle x, y \rangle$ is non-singular on the 1-dimensional subspace $V \subset H^r(M; Z_2)$ which is spanned by v_r. Hence $H^r(M; Z_2) = V \oplus V^{\perp}$, and as in

Case 1 we see that the form $\langle x, y\rangle$ restricted to $V^{\perp}$ has even rank. Hence $\langle x_i, x_j\rangle$ has odd rank.

Case 3. If $\langle v_r, v_r\rangle = 0$ *but* $\langle v_r, x\rangle \neq 0$ for some x, then the subspace $X \subset H^r(M; Z_2)$ spanned by v_r and x has as matrix $\left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)$. So $H^r(M; Z_2) = X \oplus X^{\perp}$, and it follows that $\langle x_i, x_j\rangle$ has even rank.

So in all three cases the rank of the matrix $\langle x_i, x_j\rangle$ is congruent to

$$\langle v_r, v_r\rangle = (v_r \mathrm{Sq}^1 v_r)[M] \text{ modulo } 2.$$

To finish the proof we show that

$$(v_r \mathrm{Sq}^1 v_r)[M] = (w_2 w_{2r-1})[M]$$

for any orientable $(2r+1)$-dimensional Poincaré space. Wu's theorem ([5, p. 350]) states that

$$w_k = \sum \mathrm{Sq}^i v_{k-i}.$$

In particular $w_2 = v_2$ and $w_{2r-1} = \mathrm{Sq}^{r-1} v_r$ (making use of the fact that $v_j = 0$ for $j > \frac{1}{2}\dim M$). Therefore

$$\begin{aligned} w_2 w_{2r-1} &= v_2(\mathrm{Sq}^{r-1} v_r) = \mathrm{Sq}^2(\mathrm{Sq}^{r-1} v_r) \\ &= \left(\mathrm{Sq}^r \mathrm{Sq}^1 + \binom{r-2}{2}\mathrm{Sq}^{r+1}\right) v_r \\ &= \mathrm{Sq}^r(\mathrm{Sq}^1 v_r) = v_r(\mathrm{Sq}^1 v_r); \end{aligned}$$

using the Adem relations to simplify $\mathrm{Sq}^2\mathrm{Sq}^{r-1}$. This completes the proof. (If r is odd, note that $v_r = 0$, so that this Stiefel–Whitney number is necessarily zero.)

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