Sturm and Liouville's Work on Ordinary Linear Differential Equations. The Emergence of Sturm-Liouville Theory

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Communicated by C. J. SCRIBA

Contents

Introduction

In a series of articles dating from 1836–37 STURM and LIOUVILLE created a whole new subject in mathematical analysis. The theory, later known as Sturm-Liouville theory, deals with the general linear second-order differential equation

$$(k(x)\,V'(x))' + (g(x)\,r - l(x))\,V(x) = 0 \text{ for } x \in \,]\alpha, \beta[\tag{1}$$

with the imposed boundary conditions:

$$k(x)\,V'(x) - hV(x) = 0 \text{ for } x = \alpha, \tag{2}$$

$$k(x)\,V'(x) + HV(x) = 0 \text{ for } x = \beta. \tag{3}$$

Here k, g, and l are given positive functions, h and H are given positive constants and r is a parameter. The boundary-value problem only allows non-trivial solutions (eigenfunctions) for certain values (eigenvalues) of r, which can be considered as roots of a certain transcendental equation

$$\Pi(r) = 0, \tag{4}$$

namely the equation obtained by inserting the general solution of (1) and (2) into (3)[1]. The questions studied by STURM and LIOUVILLE can roughly be divided into three groups:

1°. properties of the eigenvalues,
2°. qualitative behaviour of the eigenfunctions,
3°. expansion of arbitrary functions in an infinite series of eigenfunctions.

Of these, STURM investigated 1° and 2°, and LIOUVILLE examined 3°, finding further results related to 1° and 2° in the process.

Before 1820 the only question taken up in the theory of differential equations had been: given a differential equation, find its solution as an analytic expression. For the general equation (1) STURM could not find such an expression, and the expression found by LIOUVILLE by successive approximation was unsuited for

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[1] On notation and terminology. I have as far as possible unified, simplified and clarified the notation. For example letters have freely been replaced by others, the Lagrangean notation V' has been used instead of STURM and LIOUVILLE's Leibnizian notation $\frac{dV}{dx}$, parentheses have been inserted, and the variables have been introduced in the functions if clarity is gained (*i.e.* $V(x)$ instead of V). These changes do not alter the meaning at all.

I have freely used modern terms such as: eigenfunctions, eigenvalues, spectrum, spectral theory, orthogonality of two functions, *etc.*

Use of such modern terms abbreviates the discussion but also presents the danger of overinterpretation. For example, the anachronistic term orthogonal suggests a geometric interpretation of functions as points in a Hilbert space. However, such a way of thinking was not introduced before the work of E. SCHMIDT [1908].

The word "Fourier series" has throughout been used in its modern sense to describe the development in terms of any set of eigenfunctions of a Sturm-Liouville problem, whereas the Fourier series in the sense of the 19th century will be called trigonometric or ordinary Fourier series.

the investigation of the properties 1°–3° above. Instead they obtained the information about the properties of the solutions from the equation itself. This shows evidence of a new conception of the theory of differential equations characterized by a broader kind of question: given a differential equation, investigate some property of the solution.

Most conspicuous among the properties to be investigated in the early 19th century was existence. The existence theorem formulated and proved by CAUCHY [1824/1981, 1835/40] was the first to indicate the broader concept of differential equations.

The conceptual development in the field of differential equations ran parallel to the development in the field of algebraic equations. Here the works of ABEL and GALOIS shifted interest from the problem of finding solutions by radicals to a question of existence of such solutions and an investigation of their properties.

Since no workable explicit solutions to the general Sturm-Liouville problem could be found, the properties determined from the equation itself were necessarily qualitative in nature. Seen in this light, Sturm-Liouville theory was the first qualitative theory of differential equations, anticipating POINCARÉ'S approach to non-linear differential equations developed at the end of the century. In addition the Sturm-Liouville theory gave the first theorems on eigenvalue problems and as such it occupies a central place in the prehistory of functional analysis. But the Sturm-Liouville theory was important not only as a herald of coming ideas. It was, and has remained till this day, of importance in the technical treatment of many concrete problems in pure and applied mathematics and was as such of more than "just" conceptual importance.

Of the two conceptual novelties in the theory of differential equations in early 19th century France, existence theorems have generally received more attention in the secondary literature than the Sturm-Liouville theory despite the fact that the latter presented a wider range of innovations. It even included the former in the sense that the first widely circulated existence proof was published in three of LIOUVILLE'S papers on the Sturm-Liouville theory.

In two of the better surveys of the history of mathematics ([KLINE 1972, pp. 715–717] and [DIEUDONNÉ 1978, pp. 140–142]) STURM and LIOUVILLE'S theory has received brief treatment. More information, particularly on the role of the Sturm-Liouville theory in the history of functional analysis can be found in [DIEUDONNÉ 1981, pp. 16–21][2]. Richest on details are the two articles in the "Encyklopädie der mathematischen Wissenschaften" [HILB & SZÁSZ 1922] and [BÔCHER 1899/1916] and several of BÔCHER'S other works (*e.g.* [1911/12, 1912, 1917]). However it is rather difficult to extract a connected history from these older works since their primary goal is exposition of the mathematics, not of its history. [BÔCHER 1911/12] is an exception.

This paper is an attempt to supply a comprehensive and coherent treatment of the emergence of this beautiful theory, taking all published as well as unpublished sources into account. All unpublished material from STURM'S hand seems to be lost, but some of LIOUVILLE'S early mémoires presented to the Académie des

[2] When DIEUDONNÉ found the original theorems and arguments "long-winded and not very clear", he replaced them with more elegant formulations.

Sciences and his notes have been preserved. The handwritten mémoires [LIOUVILLE 1828, 1830/31] are kept in the Archive de l'Académie des Sciences. The Bibliothèque de l'Institut de France preserves LIOUVILLE's notebooks [LIOUVILLE Ms.]. I am indebted to the staff at these two institutions and to Professor TATON, Paris, who has helped me to get access to the unpublished material. I also wish to thank Professor UFFE HAAGERUP, Odense, for having cleared up mathematical problems (*cf.* Appendix), Lektor KIRSTI ANDERSEN, Aarhus, for encouragement and criticism, and LISBET LARSEN, Odense, who has painstakingly typed and retyped the manuscript.

I. The Friendship of Sturm and Liouville

1. The devoted friendship between STURM and LIOUVILLE began in the early 1830's [LIOUVILLE 1855]. At that time the Swiss born CHARLES-FRANÇOIS STURM (1803–1855) had already gained fame for his and D. COLLADON's prize-winning essay on the compression of fluids [1827/34] and for the celebrated theorem, called after him, on the number of real roots of a polynomial [1829a, 1835][3]. As a foreigner and a protestant, however, he had only in 1830 obtained a modest academic post as Professor at the Collège Rollin. The six years younger JOSEPH LIOUVILLE (1809–1882) had presented at least five mémoires on analysis and mathematical physics to the Paris Academy before he graduated from the École des Ponts et Chaussées in 1830. Though his results were not up to the triumphs of STURM, they gained him such a reputation that the following year he got a respected position as répétiteur (assistant) at the École Polytechnique. When in 1838 he advanced to Professor at the same school, his senior STURM was named his répétiteur. Later that year STURM got his own chair at the École Polytechnique.

2. In contrast to this reversed and unjust assignment of jobs STURM was the first of the two to be elected to the Académie des Sciences. The remarkable circumstances surrounding his election give a strong impression of the friendship between the two mathematicians. In 1833 both STURM and LIOUVILLE and their common friend J. M. C. DUHAMEL applied for the seat vacated by the death of

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[3] **Sturm's theorem.** Let $f(x) = 0$ be a real algebraic equation of arbitrary degree. Define $f_1(x) = f'(x)$ and f_n $(n \geq 2)$ successively as the negative of the remainder obtained by dividing f_{n-2} by f_{n-1}:

$$f_{n-2} = q_{n-1}(x) f_{n-1}(x) - f_n(x).$$

Let p be the number of variations of sign in the sequence

$$f(\alpha), f_1(\alpha), f_2(\alpha), \ldots, f_k(\alpha)$$

and let q be the number of variations of sign in the sequence

$$f(\beta), f_1(\beta), f_2(\beta), \ldots, f_k(\beta).$$

Then the number of real roots of $f(x)$ in $]\alpha, \beta[$ $(\alpha < \beta)$ is precisely equal to $p - q$.

A. M. LEGENDRE. A fourth applicant was G. LIBRI-CARUCCI, who was later charged with having stolen valuable books and manuscripts from the Academy. On March 18th, LIBRI was elected with 37 votes against DUHAMEL'S 16 and LIOUVILLE'S 1. Nobody voted for STURM [P. V. 1833, p. 227]. The next opportunity was offered after the death of AMPÈRE in the summer of 1836. Again the three friends applied for the vacant seat, together with a couple of others. Three weeks before the election of AMPÈRE'S successor, LIOUVILLE presented a paper to the Academy [1837a] in which he praised STURM'S two mémoires on the STURM-LIOUVILLE theory as ranking with the best works of LAGRANGE. Supporting a rival in this way was rather unusual in the competitive Parisian academic circles, and it must have been shocking when on the day of the election, December 5th, LIOUVILLE and DUHAMEL withdrew their candidacies to secure the seat for their friend. STURM was elected with an overwhelming majority.

3. During the dramatic events preceding LIOUVILLE'S election to the Academy three years later STURM repaid LIOUVILLE'S support by pleading for him in his disagreements with LIBRI, the mathematician whom the academy had preferred to STURM in 1833. The controversy started in February 1838 when LIOUVILLE had discovered grave mistakes in a paper by LIBRI. He presented the observation as a note to the Academy, which appointed an examining committee consisting of J. B. BIOT, S. D. POISSON, L. POINSOT and STURM. The fact that the committee never made a report was used against LIOUVILLE when LIBRI fought back in the Academy more than a year later, at the time when LIOUVILLE was seeking election to the seat vacated by the death of J. F. LALANDE. STURM defended LIOUVILLE for the following reason [C. R. May 20th, 1839]:

> "Malheureusement, M. Libri a voulu renouveler une discussion qui semblait terminée, dans le Mémoire qu'il a lu à la dernière séance de l'Académie, il a affirmé qu'il ne trouve aucun fondement dans les observations de M. Liouville, et il a attaqué à son tour une partie des travaux de ce géometre. M. Libri a sur M. Liouville l'avantage d'être membre de l'Académie, et il a choisi pour l'accuser d'erreur le moment où M. Liouville se présente comme candidat pour la section d'Astronomie. Il serait fâcheux que le silence de la Commission qui avait été chargée de décider la question controversée, reçût une interprétation défavorable à M. Liouville."

STURM also explained the silence of the commission:

> "M. Liouville, en publiant sa Note quelque temps après, dans son Journal [LIOUVILLE 1838b], nous dégagea de l'obligation de faire un rapport *qui pouvait n'être pas favorable à M. Libri*" (my italics).

After STURM had delivered his note LIBRI rose and even the dry factual report of the Compte Rendu hints to the dramatic scenes taking place:

> "Pendant que M. Libri continuait de se livrer à l'examen de la Note de M. Sturm, celui-ci a brusquement abandonné la discussion, malgré les instances répétées et les efforts inutiles de M. Libri pour le retenir."

Possibly as a result of STURM's intervention, LIOUVILLE was nominated to the academy on June 3rd, 1839. In this learned assembly he continued his quarrels with LIBRI. For example he defended DIRICHLET against LIBRI's unjust criticism [C. R. Feb. 17th–March 9th, 1840] which we also find described in LIOUVILLE's letters to DIRICHLET [TANNERY 1910]. In their correspondence they both expressed their indignation that STURM was underestimated relative to LIBRI:

> "Parmi ces injustices, il n'y en a pas de plus grandes et qui exigent une réparation plus prompte que celle qui a été commise envers notre ingénieux ami Sturm qui a été laissé dans une position subalterne et qui s'est vu préférer des charlatans adroits pour qui la Science n'est qu'un moyen de parvenir." (DIRICHLET to LIOUVILLE, May 6th, 1840 [TANNERY 1910, p. 8])

On July 7th, LIOUVILLE told DIRICHLET about the vacant position as Professor of Mechanics at the Sorbonne:

> "Croiriez-vous que les chances se balancent entre Sturm et Libri. Pauvres mathématiques!" [TANNERY 1910, p. 14].

At that moment, however, LIBRI was losing his popularity

> "... M. Libri ... un homme qui, dans l'Académie du moins, commence à être méprisé presque autant qu'il le mérite." (LIOUVILLE to DIRICHLET May?, 1840 [TANNERY 1910, p. 9–10])

Finally in 1850 STURM and LIOUVILLE must have been content to see LIBRI expelled from the Academy.

4. Even two years before STURM helped LIOUVILLE against LIBRI, LIOUVILLE had supported STURM in his competition with CAUCHY. When CAUCHY claimed the superiority of his method for finding the number of real roots of polynomials and published a method to calculate the number of imaginary roots inside a given contour, the two friends answered by publishing a joint paper [LIOUVILLE & STURM 1836], presenting an alternative solution to this last problem.

Thus STURM and LIOUVILLE had common friends and common competitors. With the exception of the few quarrels described above both STURM and LIOUVILLE seem to have been easy to get on with. At a time when animosity among the French scientists was the order of the day, the two friends were generally popular among their colleagues and particularly among their students. They were both eminent teachers who helped many younger mathematicians on their way [LIOUVILLE 1855], [FAYE 1882].

5. The friendship of the two great mathematicians lasted till the premature death of STURM in December 1855. At the tomb LIOUVILLE gave a moving speech, the end of which bears witness to the intimate friendship which had bound them together:

"Ah! cher ami, ce n'est pas toi qu'il faut plaindre. Echappée aux angoisses de cette vie terrestre, ton âme immortelle et pure habite en paix dans le sein de Dieu, et ton nom vivra autant que la science.

"Adieu, Sturm, adieu." [LIOUVILLE 1855]

Two months later, when writing to DIRICHLET, LIOUVILLE was still mourning:

"Venez et vous serez le bien venu. Nous pleurons ensemble notre pauvre Sturm" [Ms 3640, dossier 1846–51, unpublished letter dated February 19th, 1856].

Though STURM and LIOUVILLE wrote only one joint paper on the theory called after them, several remarks in their works bear witness to their collaboration. They always praise each other's achievements and even cover up each other's mistakes (see note 35). They discussed each other's papers before their publication with the result that in some cases an elaboration of a certain discovery was published before the discovery itself.

6. The works of STURM and LIOUVILLE on linear differential equations fall into four periods. During the first period, 1829–1830, they formed and presented their initial ideas independently. In the middle of the period 1831–1835 STURM wrote his two large mémoires which were eventually published simultaneously with LIOUVILLE's first famous mémoires during the third period 1836–1837. LIOUVILLE had begun his generalisation of the theory to higher-order equations in 1835, but his main work in this area falls in the last period from 1838 to approximately 1840.

In the text below I break this chronology by analysing STURM's work before LIOUVILLE's. This is justifiable since LIOUVILLE's definitive work drew heavily on STURM's results, whereas STURM only accidentally commented on LIOUVILLE's. To throw the work of the two friends into relief, the prehistory of Sturm-Liouville theory is recorded in Chapter II. STURM's two impressive mémoires and their emergence are treated in the two following chapters, III and IV. In the last chapters V–VI I discuss LIOUVILLE's work on second-order and, in Chapter VII, that on higher-order equations. The first six chapters are based mainly on published sources whereas the LIOUVILLE Nachlass at the Institut de France has supplied valuable information on the subject treated in the last chapter.

At the end of the paper a chronological table is appended.

II. The Roots of Sturm-Liouville Theory

7. The following motivating considerations were presented in the opening phrases of STURM's first large paper on Sturm-Liouville theory.

"La résolution de la plupart des problèmes relatifs à la distribution de la chaleur dans des corps de formes diverses et aux petits mouvements oscillatoires des corps solides élastiques, des corps flexibles, des liquides et des fluides élastiques, conduit à des équations différentielles linéaires du second ordre ..." [STURM 1836a, p. 106].

In his second paper [1836b] he explained in more detail how the partial differential equations arising from the problems above can be solved by separating the variables, leading in general to a second-order ordinary differential equation with a parameter. The parameter must be chosen so that certain boundary conditions are satisfied.

As an example he discussed heat conduction in an inhomogeneous thin bar. In this case the temperature is governed by the equation

$$g\frac{\partial u}{\partial t} = \frac{\partial\left(k\dfrac{\partial u}{\partial x}\right)}{\partial x} - lu, \tag{5}$$

where $u(x, t)$ denotes the temperature at the point x and at the time t, and g, k and l are positive functions of x. If the surroundings of the bar are maintained at zero degrees the temperature u must satisfy boundary conditions at the end points α and β:

$$k\frac{\partial u}{\partial x} - hu = 0 \quad \text{for} \quad x = \alpha, \tag{6}$$

$$k\frac{\partial u}{\partial x} + Hu = 0 \quad \text{for} \quad x = \beta, \tag{7}$$

where h and H are positive constants which may become infinite (implying $u = 0$). Sometimes the temperature is known when $t = 0$. That gives rise to the initial condition:

$$u(x, 0) = f(x). \tag{8}$$

Ignoring (8), STURM first looked for solutions to (5)–(7) of the form[4]

$$u = V(x)\, e^{-rt}. \tag{9}$$

When substituted into (5)–(7) the factors e^{-rt} cancel, leaving the boundary-value problem (1)–(3) for V. If $V_1, V_2, \ldots, V_n, \ldots$ are the eigenfunctions to (1)–(3) corresponding to the eigenvalues $r_1, r_2, \ldots, r_n, \ldots$ the linear combination:

$$u = \sum_n A_n V_n(x)\, e^{-r_n t}$$

is also a solution of (5)–(7). The initial condition (8) thus poses the problem of determining the A_n's so that

$$\sum_n A_n V_n(x) = f(x). \tag{10}$$

This problem was taken up by LIOUVILLE.

[4] The technique of separating the variables by searching for solutions of the general form $F(x)f(t)$ had been introduced by FOURIER in [1822, § 167]. However, for simple equations like (5) both FOURIER and his successors knew the equation for $f(t)$ and its solution so well that they immediately wrote down the expression (9).

8. Eigenvalue problems of the form (1)–(3) had turned up in the early 18th century in the study of vibratory motions. In the papers of BROOK TAYLOR on the vibrating string [1713] and of JOHANN BERNOULLI [1728] on the hanging chain the first eigenvalue was found, corresponding to the fundamental mode. The higher modes were discovered by DANIEL BERNOULLI (1700–1782) in his continuation of his father's research on the vibrating hanging homogeneous chain [1733]. He derived the equation

$$\alpha \frac{d}{dx}\left(x \frac{dy}{dx}\right) + y = 0$$

for the shape $y(x)$ of the chain and found its solution as an infinite series which we would denote by

$$y = A J_0(2\sqrt{x/\alpha}),$$

J_0 being the zero$^{\text{th}}$ order Bessel function[5]. DANIEL BERNOULLI argued that there is an infinity of eigenvalues α satisfying $J_0(2\sqrt{l/\alpha}) = 0$, where l is the length of the chain, and investigated the distribution of the $n-1$ zeroes of the n^{th} eigenfunction in the interval $]0, l[$. Later he discovered also the possibility of superposing the eigenfunctions and in connection with the vibrating string he claimed that the general shape of the system could be obtained in this way.

TAYLOR and the BERNOULLIS had derived the ordinary differential equation directly from physical principles. When D'ALEMBERT and EULER from 1747 onwards derived the partial differential equations describing vibrating strings, chains and membranes, they obtained the eigenvalue problem by separating variables. Though they did not believe that they could get the complete solution by superposition of eigenfunctions they investigated many specific cases of (5)–(7) using this technique (*cf.* [TRUESDELL 1960]). From 1807 separation of variables was widely used in the theory of heat, first by FOURIER, and soon thereafter by almost all the younger French mathematicians. This vast complex of research presented ample motivation for STURM and LIOUVILLE.

Before 1830, mathematicians almost exclusively studied such particular cases of (1)–(3) for which they could find an explicit solution either in finite form or in infinite series. STURM and LIOUVILLE, however, could not find any manageable expression for the solution in the general case and therefore they had to draw their conclusions directly from the equations (1)–(3). This is the characteristic feature of the Sturm-Liouville theory. Because such a study of the equations had earlier been rendered unnecessary by the explicit knowledge of the solution, one can hardly find any anticipation of STURM'S and LIOUVILLE'S methods and results. Nevertheless, some exceptional investigations pointing toward the Sturm-Liouville theory were made by D'ALEMBERT, FOURIER, and POISSON. I shall discuss their researches in chronological order below.

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[5] DANIEL BERNOULLI also studied certain inhomogeneous chains and there he found a solution which we recognize as a first-order Bessel function.

9. JEAN LE ROND D'ALEMBERT (1717–83), who had solved the problem of the vibrations of a homogeneous string in his famous paper of [1747], turned to the nonhomogeneous string shortly after EULER had published his first investigations on this more difficult problem. In a letter of June 11th, 1769 (later published as [1763 (1770)]) to LAGRANGE, D'ALEMBERT set up the differential equation[6] governing the transversal amplitude $y(x, t)$:

$$\frac{\partial^2 y}{\partial x^2} = X \frac{\partial^2 y}{\partial t^2}, \tag{11}$$

where $X(x)$ is the distribution of mass along the string. He sought solutions of the form

$$y = \zeta(x) \cos \lambda t$$

for which the equation reduces to the form

$$\frac{d^2\zeta}{dx^2} = -X\lambda^2\zeta. \tag{12}$$

After this separation of the variables he let $\zeta = e^{\int p dx}$ and obtained for p the Riccati equation:

$$dx = -\frac{dp}{p^2 + X\lambda^2}. \tag{13}$$

Since he required the string to be fixed at the two endpoints,

$$y = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad y = 0 \quad \text{for} \quad x = a, \tag{14}$$

he was faced with the question

> "s'il est toujours possible de satisfaire à cette double condition, la valeur de X étant donnée; c'est un point que personne, ce me semble, n'a encore examiné en général." [D'ALEMBERT 1763 (1770), p. 242]

In order to show it possible to determine such a value of λ he considered the vibrations of a string of the uniform load $m = \min_{x \in [0,a]} X(x)$. It satisfies the equation

$$dx_1 = -\frac{dp_1}{p_1^2 + m\lambda^2} \tag{15}$$

corresponding to (13). He argued that if $\zeta_1 = e^{\int p_1 dx_1}$ is 0 at $x_1 = 0$, then p_1 must have vertical asymptotes at 0 and another point b_1, for which $\zeta_1(b_1) = 0$ (Fig. 1). A comparison of (13) and (15) shows that if $\zeta(0) = 0$, we must have $x < x_1$ at points where $p = p_1$. Therefore p must also have a vertical asymptote at a point $b \leqq b_1$ corresponding to $\zeta(b) = 0$ (Fig. 2). Finally D'ALEMBERT claimed it possible to choose λ in such a way that $b = a$; the two boundary

[6] The physical constants contained in D'ALEMBERT's equations have all been set equal to 1 in the following.

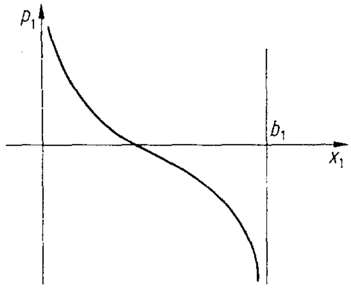

Fig. 1

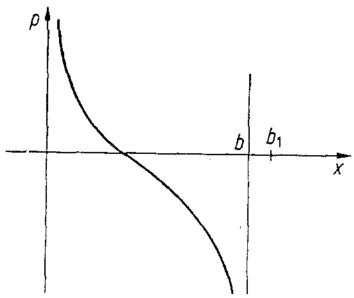

Fig. 2

conditions will then be fulfilled. The argument he had in mind, but did not write down, must have been the following:

One can prove that $b_1 \to 0(\infty)$ for $\lambda \to \infty(0)$, from which it follows that $b \to 0(\infty)$ for $\lambda \to \infty(0)$ and therefore λ can be chosen so as to make $b = a$. But in this argument only the implication $(b_1 \to 0$ for $\lambda \to \infty) \Rightarrow (b \to 0$ for $\lambda \to \infty)$ follows from D'ALEMBERT'S inequality $b < b_1$. In order to get the other inequality he could have compared equation (13) with the equation for a string of uniform load $M = \max X(x)$.

10. D'ALEMBERT established only the existence of one eigenvalue λ of the boundary-value problem (11), (14)[7]. In spite of these shortcomings D'ALEMBERT'S investigation was a remarkable anticipation of the Sturm-Liouville theory. Not only has the problem of existence of eigenvalues a central position in this theory, but also the method of basing the existence proof on a comparison with differential equations with constant coefficients points directly to STURM'S comparison theorem (*cf.* § 24). However, STURM did not refer to D'ALEMBERT'S paper.

[7] To a modern reader D'ALEMBERT'S assumption $\zeta = e^{\int p dx}$ for $p(x) \in \mathbb{R}$ limits the discussion to positive values of ζ in which case only the first value of λ can be found. However, this argument does not apply to D'ALEMBERT who believed that $\log x = \log(-x)$. In the letter considered here [D'ALEMBERT 1763 (1770), p. 250] gave a new argument for this standpoint based on the equation (11) for X constant.

11. The work of JOSEPH FOURIER (1768–1830), on the other hand, was well known to the young STURM, who was his protegé. In his main work "Théorie analytique de la chaleur" [1822], FOURIER treated only heat conduction in homogeneous media but was nevertheless led to differential equations with variable coefficients when he used spherical and cylindrical coordinates. In both cases he succeeded in finding explicit formulas for the solutions of the separated equations. When using spherical coordinates he found the solutions to be ordinary trigonometric functions so that the problem of finding the eigenvalues became a simple trigonometric problem. Heat conduction in an infinitely long homogeneous cylinder, on the other hand, posed problems similar to those DANIEL BERNOULLI had faced in his investigation of the hanging chain.

12. After having set up the heat equation in cylindrical coordinates [1822 § 118–120] FOURIER assumed as he usually did that the temperature is of the form $e^{-mt}\,u(x)$ and found [1822 § 306] for $u(x)$ the equation

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\frac{m}{k}u=0, \tag{16}$$

where k is a positive constant and x is the distance from the axis. FOURIER imagined the cylinder to be immersed in a medium of constant temperature. Then u must satisfy the boundary condition

$$hu+\frac{du}{dx}=0 \quad \text{for} \quad x=\beta, \tag{17}$$

where h is a constant and β is the radius of the cylinder.

FOURIER found the solution to (16) expressed as an infinite series[8]:

$$u=1-\frac{mx^2}{k2^2}+\frac{m^2x^4}{k^2 2^2\cdot 4^2}-\frac{m^3x^6}{k^3 2^2\cdot 4^2\cdot 6^2}+\dots. \tag{18}$$

This is the Bessel function $J_0\left(x\sqrt{\frac{m}{k}}\right)$. Now m must be chosen such that (17) is satisfied. Since FOURIER wanted to represent any initial temperature $f(x)$ as a sum of solutions

$$f(x)=a_1u_1(x)+a_2u_2(x)+\dots+a_nu_n(x)+\dots, \tag{19}$$

he needed to show that there is an infinite number of such values of m (eigenvalues). Setting $\theta=\frac{m\beta^2}{k\,2^2}$ and

$$f(\theta)=1-\theta+\frac{\theta^2}{2^2}-\frac{\theta^3}{2^2\,3^2}+\dots$$

[8] Here FOURIER has left out one multiplicative constant. Another constant has been determined from the implicit boundary value condition that $u(x)$ is regular at $x=0$.

makes it possible to write the boundary condition as follows:

$$\frac{h\beta}{2} + \theta \frac{f'(\theta)}{f(\theta)} = 0. \tag{20}$$

FOURIER noted that $f(\theta)$ is a solution of the differential equation

$$y + \frac{dy}{d\theta} + \theta \frac{d^2y}{d\theta^2} = 0, \tag{21}$$

from which he deduced the existence of infinitely many roots of (20) in the following way: Successive differentiation of (21) yields

$$\frac{d^iy}{d\theta^i} + (i+1)\frac{d^{i+1}y}{d\theta^{i+1}} + \theta \frac{d^{i+2}y}{d\theta^{i+2}} = 0, \tag{22}$$

which shows that when $f^{(i+1)}$ has a root $f^{(i)}$ and $f^{(i+2)}$ have opposite signs. FOURIER claimed that if such a relation holds between the real roots of a function and its successive derivatives then the function has no imaginary roots. For polynomials this theorem is valid and it is closely related to FOURIER'S earlier investigation [1820] of the number of real roots of algebraic equations between given limits[9]. However, in a debate with FOURIER, POISSON pointed out that the theorem is not always true for transcendental functions [POISSON 1823b, p. 383, and 1830]. Nevertheless, FOURIER applied it to the transcendental function $f(\theta)$ above and concluded that it had no imaginary roots and hence infinitely many real roots (he probably considered f as a polynomial of infinite degree, possessing therefore an infinity of roots)[10]. A simple argument, left out by FOURIER, then shows that (20) or (17) has infinitely many real roots. In spite of the inadequacy of the proof the result is correct.

13. Having thus obtained an infinity of eigenfunctions $u_1, u_2, \ldots$ to (16) and (17) FOURIER desired to prove [1822, § 310–319] that any initial state $f(x)$ can be developed in a Fourier series (19). As in all other cases of this kind FOURIER considered this to be proved if he could find a formula for the Fourier coefficients a_i. After long calculations involving only the equations (16) and (21) but not the

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[9] FOURIER'S theorem states: Let $f(x) = 0$ be a real algebraic equation of the k^{th} degree, let p denote the number of variations of signs in the sequence

$$f(\alpha), f'(\alpha), \ldots, f^{(k)}(\alpha),$$

and let q denote the number of variations of signs in the sequence:

$$f(\beta), f'(\beta), \ldots, f^{(k)}(\beta).$$

Then the number of real roots of $f(x)$ in $]\alpha, \beta[$ $(\alpha < \beta)$ is at most equal to $p - q$.

STURM'S theorem (note 3) was announced as an improvement of this theorem.

[10] FOURIER defended his use of the above-mentioned method against POISSON'S criticism in [FOURIER 1831].

formula (18) for the solution he found that

$$\int_0^\beta x u_j(x)\, u_i(x)\, dx = \begin{cases} 0 \text{ for } i \neq j \\ \left[1 + \left(\frac{h\beta}{2\sqrt{\theta_i}}\right)^2\right] \frac{\beta^2 u_i^2(\beta)}{2} \quad \text{for} \quad i = j. \end{cases} \tag{23}$$

From this statement of orthogonality and (19) it follows that

$$a_i = \frac{2 \int_0^\beta x f(x)\, u_i(x)}{\beta^2 u_i(\beta) \left[1 + \frac{k h^2}{m_i}\right]}. \tag{24}$$

FOURIER's treatment of heat conduction in a cylinder anticipates the Sturm-Liouville theory in the sense that the conclusions are drawn directly from the differential equations (21) and (16). However, since the deduction of (21) rested on the knowledge of the explicit expression (18) for the solution u, this equation is foreign to the Sturm-Liouville theory.

14. The only mathematician who proved general theorems in Sturm-Liouville theory before STURM and LIOUVILLE was SIMÉON-DENIS POISSON (1781–1840). He obtained his results in connection with his above-mentioned debate with FOURIER over the reality of the roots of the transcendental equations determining the eigenvalues of various problems in heat conduction. Even in the paper [1823b] in which he first criticized FOURIER's proof he had presented what he called an "a posteriori" proof of the reality of the eigenvalues for the boundary value problem describing heat conduction in a sphere consisting of two concentric homogeneous materials. He simply noted [1823b, p. 381] that by using two different methods [1823a, § VII] and [1823b, § V] he could express the temperature in two ways as a sum $\sum_m u_m$ which were identical except that one sum ranged over all eigenvalues whereas in the other only real eigenvalues were taken into account. Since the two expressions had to be equal, POISSON concluded that no complex eigenvalues existed.

15. In [1823b, p. 382] he acertained that no "a priori" method of proving the reality of the eigenvalues was known, but three years later he presented such a method in a note read in the Société Philomatique [1826]. The proof was based on the orthogonality of the eigenfunctions. He had proved this relation in the case of the double layer sphere in [1823b, p. 380] but he had not noticed that it implied the reality of the eigenfunctions as a simple consequence. In [1826] POISSON treated the particular case of (4):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + X(x)\, u \tag{25}$$

with the boundary conditions

$$\frac{du}{dx} - hu = 0 \quad \text{for} \quad x = \alpha, \tag{26}$$

$$\frac{du}{dx} + Hu = 0 \quad \text{for} \quad x = \beta. \tag{27}$$

As usual he noted that the function $y_\varrho(x)\, e^{\varrho t}$ solves this boundary value problem if y_ϱ is a solution to the ordinary differential equation:

$$\varrho y_\varrho = \frac{d^2 y_\varrho}{dx^2} + X y_\varrho \tag{28}$$

and satisfies boundary conditions similar to (26) and (27). In order to arrive at the orthogonality relations he considered a particular eigenfunction $y_{\varrho'}$ and the solution of the original problem (25)–(27):

$$u = \Sigma y_\varrho e^{\varrho t}, \tag{29}$$

where the summation ranges over all the eigenvalues. Multiplying (25) with $y_{\varrho'}$ and integrating over $[\alpha, \beta]$ he obtained

$$\frac{d \int\limits_\alpha^\beta u y_{\varrho'}\, dx}{dt} = \int\limits_\alpha^\beta \frac{d^2 u}{dx^2} y_{\varrho'}\, dx + \int\limits_\alpha^\beta X u y_{\varrho'}\, dx. \tag{30}$$

By partial integration the first term on the right-hand side is transformed into

$$\int\limits_\alpha^\beta \frac{d^2 u}{dx^2} y_{\varrho'}\, dx = \int\limits_\alpha^\beta u \frac{d^2 y_{\varrho'}}{dx^2}\, dx, \tag{31}$$

because the boundary terms cancel when both u and $y_{\varrho'}$ satisfy (26) and (27). Similarly multiplication of (28) by u and integration over $[\alpha, \beta]$ yields

$$\varrho' \int\limits_\alpha^\beta u y_{\varrho'}\, dx = \int\limits_\alpha^\beta u \frac{d^2 y_{\varrho'}}{dx^2}\, dx + \int\limits_\alpha^\beta X u y_{\varrho'}\, dx. \tag{32}$$

From (30)–(32) Poisson deduced the equation

$$\frac{d \int\limits_\alpha^\beta u y_{\varrho'}\, dx}{dt} = \varrho' \int\limits_\alpha^\beta u y_{\varrho'}\, dx,$$

which can readily be integrated to give

$$\int\limits_\beta^\alpha u y_{\varrho'}\, dx = A e^{\varrho' t}, \tag{33}$$

where A is an arbitrary constant. By reintroducing the particular form (29) of u into (33), and equating the coefficients of similar exponentials POISSON demonstrated orthogonality:

$$\int_\alpha^\beta y_\varrho y_{\varrho'}\,dx = 0 \quad \text{for} \quad \varrho \neq \varrho'. \tag{34}$$

16. POISSON then assumed ϱ to be a complex eigenvalue with the eigenfunction y_ϱ. Since he implicitly assumed $X(x)$, h and H to be real, he concluded that $\bar{\varrho}$ must also be an eigenvalue with the eigenfunction $\overline{v_\varrho} = v_{\bar{\varrho}}$, and since $\varrho \neq \bar{\varrho}$, equation (34) implies

$$\int_\alpha^\beta |y_\varrho|^2\,dx = \int_\alpha^\beta y_\varrho \overline{y_\varrho}\,dx = 0. \tag{35}$$

Hence $y_\varrho \equiv 0$ in $[\alpha, \beta]$. This is a contradiction and therefore POISSON concluded that all eigenvalues must be real.

In his "Théorie mathématique de la chaleur" [1835] POISSON carried over this proof to the more general boundary-value problem

$$c(\bar{x})\frac{\partial u}{\partial t} = \sum_{i=1}^{3} \frac{\partial k(\bar{x})\left(\frac{\partial u_i}{\partial x_i}\right)}{\partial x_i} \quad \text{in} \quad A, \tag{36}$$

$$k\,(\overline{\text{grad}\,u} \cdot \bar{n}) + pu = 0 \text{ on } \partial A, \tag{37}$$

where A is a domain in $\mathbb{R}^3$ and $\bar{n}$ is the outer normal of its boundary ∂A[11]. This is the three-dimensional analogue of STURM'S problem (5)–(7). In this case POISSON found the following orthogonality:

$$\int_A c(\bar{x})\,P_i(\bar{x})\,P_j(\bar{x})\,d\bar{x} = 0 \text{ for } i \neq j. \tag{38}$$

17. In the two works mentioned above POISSON provided both theorems and methods of lasting value for the Sturm-Liouville theory. The orthogonalities (34) and (38) and the theorem on the reality of the eigenvalues were adopted with due credit in STURM'S papers and the method used to obtain (33), particularly the application of partial integration, is still used, though in a slightly simplified form found by STURM. Nonetheless, POISSON'S researches are of a limited scope compared with the gigantic advances in the field made by STURM and LIOUVILLE within two years of the publication of POISSON'S last results.

III. Sturm's First Mémoire

18. STURM'S mathematical masterpieces grew out of the blend of theorems on differential equations and roots of equations found by FOURIER and POISSON. His famous algebraic theorem, improving FOURIER'S theorem on the determination

[11] POISSON did not use vector notation but wrote his equations in components.

of real roots (note 3), was presented to the Academy on May 25th, 1829 [1829a], and during the following half year he presented a series of papers on transcendental equations and differential equations [1829b–f]. The papers are all lost[12] but, from the short summaries of some of them [c and d] in the Bulletin de Férussac and from CAUCHY's report on [f], one sees that STURM got most of his ideas on the Sturm-Liouville theory during this period. He proved that certain systems of differential equations [c] and algebraic equations [d] have real eigenvalues, he determined Fourier coefficients [c], he found a version of his oscillation theorem [f], and he applied the theorems to determine the temperature distribution in different bodies for large values of the time.

In the concluding remarks of the first of his large printed papers on the Sturm-Liouville theory STURM shed more light on his approach to both the algebraic and the analytical theorems:

> "La théorie exposée dans ce mémoire sur les équations différentielles linéaires de la forme
>
> $$L\frac{d^2V}{dx^2} + M\frac{dV}{dx} + NU = 0 \tag{39}$$
>
> correspond à une théorie tout-à-fait analogue que je me suis faite antérieurement sur les équations linéaires du second ordre à différences finies de cette forme
>
> $$LU_{i+1} + MU_i + NU_{i-1} = 0 \tag{40}$$
>
> i est un indice variable qui remplace la variable continue x; L, M, N, sont des fonctions de cet indice i et d'une indéterminée m, qu'on assujettit à certaines conditions. C'est en étudiant les propriétés d'une suite de fonctions U_0, U_1, U_2, U_3, ... liées entre elles par un système d'équations semblables à la précédente que j'ai rencontré mon théorème sur la détermination du nombre des racines réelles d'une équation numérique comprises entre deux limites quelconques, lequel est renfermé comme cas particulier dans la théorie que je ne fais que indiquer ici. Elle devient celle qui fait le sujet de ce mémoire, par le passage des différences finies aux différences infiniment petites. [STURM 1836a, p. 186]

However, in the extant summaries of STURM's early papers there is no explicit mention of this theory of difference equations. To be sure, one of the papers [1829d] deals with systems of equations resembling (40) but the aim of the paper is so different from the one described in the quotations that it cannot be the work STURM had in mind. Where, how and why did STURM then study the difference equation (40)? A convincing answer to these questions has been given by BÔCHER in his interesting paper [1911/12]. BÔCHER argues that the problem of STURM's paper "Sur la distribution de la chaleur dans un assemblage de vases" [1829e], of which only the title is known, lead STURM to the difference equation (40), and

[12] Many letters in STURM's personal file bear witness to several fruitless attempts to find the mémoires.

he shows how an analysis of this equation can lead to STURM's theorem as well as to a discrete version of the theorems found in STURM's first published paper on the Sturm-Liouville theory. The reader is referred to BÔCHER's paper for a further discussion of STURM's unpublished papers on algebraic and differential equations.

19. The publication of a comprehensive version of STURM's ideas was delayed until 1836 when LIOUVILLE urged STURM [*cf.* LIOUVILLE 1855] to publish in his newly founded journal a mémoire presented to the Academy three years earlier[13] [STURM 1836a]. A summary of STURM's paper had appeared in the journal L'Institut of [1833a][14] but it contained only the main results, not the methods. Thanks to its conciseness, however, the summary displayed a clarity which the eighty pages of the mémoire lacked. In the latter every detail was proved, sometimes with several proofs, and this combined with the lack of emphasis on the important theorems created a rather unreadable paper. STURM's reason for being so elaborate was probably the novelty of his methods.

> "Le principe sur lequel reposent les théorèmes que je développe, n'a jamais, si je ne me trompe, été employé dans l'analyse". [STURM 1836a, p. 107]

The paper dealt with the general second-order linear differential equation (39) in which the coefficient functions depends on a real parameter r. For convenience STURM transcribed the equation into its self-adjoint form

$$(K(r, x)\, V_r'(x))' + G(r, x)\, V_r(x) = 0, \quad x \in \,]\alpha, \beta[\tag{41}$$

which generalizes the equation (1)[15,16]. However, in the first paper STURM did not discuss a boundary-value problem with two boundary conditions of the kind (2) and (3). He postponed the treatment of spectral theory to the second paper [STURM 1836b] and imposed in the first paper only one boundary condition of this kind:

$$K(r, x)\, V_r'(x) - h(r)\, V_r(x) = 0 \text{ for } x = \alpha, \tag{42*}$$

or equivalently

$$\frac{K(r,x)\, V_r'(x)}{V_r(x)} = h(r) \text{ for } x = \alpha, \tag{42}$$

[13] STURM's mémoire was presented to the Academy on September 28th, 1833 and not on September 30th, as is stated in [STURM 1833a] and [PROUHET 1856].

[14] The analysis of STURM's mémoire in [1833a] is written in the third person; thus it is not certain but is most probable that STURM is the author.

[15] For $L(x) \neq 0$ in (39) STURM found the functions of (41) to be

$$K(x) = e^{\int \frac{M(x)}{L(x)} dx} \quad \text{and} \quad G(x) = \frac{N(x)}{L(x)} e^{\int \frac{M(x)}{L(x)} dx}.$$

[16] In this paper $'$ as in f' always means differentiation with respect to x; δ will denote differentiation with respect to the parameter r.

in which he even allowed the "constant" h to vary with r. STURM remarked that the following proposition secured the existence of a solution and its uniqueness, to within a multiplicative constant, to (41) and (42)[17]:

Proposition A. *Suppose* V_r *is a solution to* (41) *and suppose* $V_r(\alpha)$ *and* $V_r'(\alpha)$ *are given. Then* V_r *"a une valeur déterminée et uinque pour chaque valeur de* x*".*

By the 1830's this basic theorem of existence and uniqueness had been generally believed for a century, but the first proof had only recently been provided by LIOUVILLE (see § 34).

Thus the problem (41), (42) leads to a continuous family of solutions V_r, one for each r. The aim of STURM's first paper was to study the qualitative behaviour of these solutions $V_r(x)$ and particularly how it varied with r. STURM was primarily interested in the oscillatory properties of the $V_r(x)$'s, for example their zeroes, their changes of sign, and their maxima and minima in the interval $]\alpha, \beta[$. He obtained this information "par la seule considération des équations différentielles en elles-mêmes, sans qu'on ait besoin de leur intégration" [STURM 1836a, p. 107]. In this way he made explicit the method which had been indicated by FOURIER and used rather unconsciously by POISSON.

20. Central in STURM's paper is his investigation of the number of roots of V_r in $]\alpha, \beta[$, from which all the other results follow as easy corollaries. This investigation has two components:

1°. Proof that under variation of the parameter r a root $x(r)$ of V_r can appear or disappear from the interval $]\alpha, \beta[$ only if it crosses one of the boundaries ($x(r) = \alpha$ or $x(r) = \beta$).

2°. Determination of how the roots $x(r)$ of V_r move for varying r (also outside $]\alpha, \beta[$), particularly how they enter the interval $]\alpha, \beta[$.

The first property is easily established for if a root $x(r)$ appears or disappears without passing α or β it gives rise to a double root (Fig. 3). However, this cannot happen according to STURM's proposition:

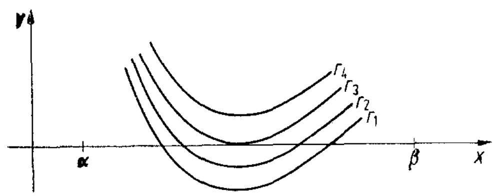

Fig. 3

[17] STURM formulated only one theorem (here called theorem E) in the whole paper. In order to make this discussion easier to grasp than STURM's paper, I have formulated as propositions many of the properties proved by STURM.

Proposition B. *When V_r is a non-trivial solution to* (41), *then V_r and V_r' have no common roots.*

STURM offered two proofs of Proposition B. First he pointed out that if $V_r(c) = V_r'(c) = 0$ it follows from (41) that all the higher derivatives $V_r^{(n)}(c)$ will vanish in c "et par suite [by TAYLOR's theorem] V serait nulle pour toutes les valeurs de x". This simple proof is followed by a second "démonstration plus rigoureuse" which rests on the constancy of the Wronskian [STURM 1836a, p. 109–110]:

$$K(V_1V_2' - V_2V_1') = C,$$

where V_1 and V_2 are two arbitrary solutions to (41) and C is a constant[18].

The two proofs reflect the radical changes taking place in the foundation of analysis during the early 19th century. The first proof reveals an author attached to the tradition of LAGRANGE, whereas the need to give an alternative proof would occur only to a mathematician influenced by the new standards of rigour. In particular, CAUCHY's example [CAUCHY 1829, 10. leçon] of a function whose Taylor series converges to a sum different from the function expanded, shows that STURM's first proof is invalid.

21. Proposition B shows that it makes sense to follow a particular root x_r of V_r, when r varies, as is required in 2°) above. In order to describe this variation STURM studied the behaviour of the two fractions $\frac{V}{KV'}$ and $\frac{KV'}{V}$. For brevity I shall discuss only the thorems concerning $\frac{KV'}{V'}$ of which the first and most central stated [STURM 1836a, p. 116]:

Proposition C. *If V is a solution to* (41) *and* (42) *and if*

$$\left\{\begin{array}{ll} K>0 \quad \forall r,\ \forall x \in [\alpha,\beta], & \\ G \text{ is an increasing function of } r & \forall x \in [\alpha,\beta], \\ K \text{ is a decreasing function of } r & \forall x \in [\alpha,\beta], \\ \left[\frac{KV'}{V}\right]_{x=\alpha} = h(r) \text{ is a decreasing function of } r, & \end{array}\right\} \tag{43}$$

then $\frac{KV'}{V}$ is a decreasing function of r for all values of $x \in [\alpha,\beta]$[19].

---

[18] STURM's second proof: If $V_1 \not\equiv 0$, there is a point a such that $V_1(a) \neq 0$. STURM then takes a solution V_2 with such values of $V_2(a)$ and $V_2'(a)$ that the Wronskian evaluated at the point a is different from zero. Then the Wronskian is everywhere different from zero and hence V_1 and V_1' can not both be equal to zero.

Without comment STURM has here used the existence part of Proposition A when he chose V_2. On the other hand, Proposition B is a simple consequence of the uniqueness part of Proposition A. STURM clearly did not see these relations between Propositions A and B for he would probably consider the assertion of existence of A as a stronger statement than the assertion of uniqueness.

[19] When $V_r(x)$ for fixed x and variable r becomes zero the quotient $\frac{KV'}{V}$ thus jumps from $-\infty$ to ∞.

Since both the proposition and its proof are fundamental in STURM's theory, I shall consider the proof in detail.

Without bothering about differentiability[20], STURM differentiated equation (41) with respect to r and found:

$$\frac{d\delta(KV')}{dx} + G\,\delta V + V\,\delta G = 0, \tag{44}$$

where he has interchanged differentiation with respect to x (denoted d) and differentiation with respect to r (denoted δ). Multiplying (41) by $\delta V\,dx$, and subtracting (44) multiplied by $V\,dx$ yields

$$\delta V\,d(KV') - V\,d\delta(KV') = V^2\,\delta G\,dx, \tag{45}$$

which, integrated by parts between α and x, gives

$$\delta V(KV') - V\,\delta(KV') = C + \int_\alpha^x V^2\,\delta G\,dx - \int_\alpha^x (V')^2\,\delta K dx, \tag{46}$$

where C is the value of the left-hand side for $x = \alpha$. Since this left-hand side is equal to

$$-V^2\,\delta\left(\frac{KV'}{V}\right),$$

STURM finally obtained

$$-V^2\,\delta\left(\frac{KV'}{V}\right) = \left[-V^2\,\delta\left(\frac{KV'}{V}\right)\right]_{x=\alpha} + \int_\alpha^x V^2\,\delta G\,dx - \int_\alpha^x (V')^2\,\delta K\,dx. \tag{47}$$

If the assumptions (43) of Proposition C are fulfilled, we have

$$\left[\delta\left(\frac{KV'}{V}\right)\right]_{x=\alpha} < 0$$

and

$$\delta G > 0 \text{ and } \delta K < 0 \text{ for } x \in [\alpha, \beta]. \tag{48}$$

Hence the left-hand side of (47) is positive, *i.e.*

$$\delta\left(\frac{KV'}{V}\right) < 0 \quad \text{for} \quad x \in [\alpha, \beta], \tag{49}$$

which proves Proposition C.

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[20] In the 1830's the concept of differentiability had not been introduced. Following CAUCHY, the rigorists differentiated only continuous functions, but even continuity of V with respect to r is never questioned by STURM. LIOUVILLE raised this question once, when he had obtained completely inadmissible results (*cf.* § 56). In the following presentation I take for granted sufficient smoothness in V_r as a function of r.

22. From Proposition C STURM easily obtained the desired description of the variation of a root $x(r)$, satisfying $V_r(x(r)) = 0$, when the conditions (43) are fulfilled. When r increases to $r + dr$, x will change to $x + dx$ and V to the value $V_{r+dr}(x(r + dr)) = V_r(x(r)) + \frac{\partial V_r(x)}{\partial x} dx + \frac{\partial V_r(x)}{\partial r} dr$. Since $V_{r+dr}(x(r + dr)) = V_r(x(r)) = 0$ we have:

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial r} dr = 0$$

or

$$\frac{dr}{dx} = -\frac{\dfrac{\partial V}{\partial x}}{\dfrac{\partial V}{\partial r}}. \tag{50}$$

STURM then used (46) to see that $\frac{\partial V}{\partial r}$ (or δV) and $\frac{\partial V}{\partial x}$ (or V') have the same signs when $V = 0$. Therefore

$$\frac{dr}{dx} < 0,$$

which implies the proposition:

Proposition D. *If the assumptions* (43) *are satisfied, the roots $x(r)$ of the solution $V_r(x)$ to* (41) *and* (42) *are decreasing with r.*

Thus new roots of V_r may enter the interval $]\alpha, \beta[$ through its right-hand end point β, but no roots can leave the interval through β. This means, according to property 1° (§ 20) that, if $V_r(\alpha) \neq 0$ for all r (*i.e.* $h(r) \neq \infty$ in (42) or (43)), the number of roots of V_r in $]\alpha, \beta[$ increases with r.

23. STURM could then easily deduce what he considered his main theorem:

Theorem E. *Let V_1 and V_2 denote solutions to the equations*

$$(K_i V_i')' + G_i V_i = 0 \quad \forall x \in]\alpha, \beta[, \tag{51}$$

$$\frac{K_i V_i'}{V_i} = h_i \quad \text{for} \quad x = \alpha \tag{52}$$

for $i = 1$ and $i = 2$ respectively. Suppose further that

$$G_2(x) \geqq G_1(x), \quad K_2(x) \leqq K_1(x) \quad \forall x \in [\alpha, \beta] \tag{53}$$

and

$$h_2 < h_1. \tag{54}$$

Then V_2 vanishes and changes sign at least as many times as V_1 in $]\alpha, \beta[$; and if one lists the roots of V_1 and V_2 in increasing order from α the roots of V_1 are larger than the roots of V_2 of the same order.

In order to deduce Theorem E from the remarks following Proposition D, STURM "connected" the two situations ($i = 1$ and $i = 2$) by a continuous family of equations (41), (42) satisfying:

$$G(x, r_i) = G_i(x), \quad K(x, r_i) = K_i(x) \quad i = 1, 2 \tag{55}$$

and

$$h(r_i) = h_i \tag{56}$$

in such a way as to satisfy the conditions (43). The possibility of selecting such a continuous family is secured by (53) and (54).

The introduction of the continuous family of equations also allowed STURM to evaluate the difference Δ between the number of roots of V_1 and V_2 in $]\alpha, \beta[$. He showed that

Proposition F. $\Delta =$ *the number of roots of* $V_r(\beta)$ *for* $r \in]r_1, r_2[$;

that is, the number of zeroes which the solution V of the continuous family (41) and (42) assumes at the right-hand end point of the interval, when r varies from r_1 to r_2. This is the theorem which most clearly shows the connection between STURM's theorem in algebra [*cf.* note 3] and his analytical investigations.

24. The rest of the mémoire consists of consequences, refinements and variations of Theorem E and Proposition F. Of these I shall in the following only discuss the most important, particularly those that were used in the subsequent development of the Sturm-Liouville theory. The first interesting consequence of Theorem E can be found in Section 16 of [STURM 1836a]. It is:

Sturm's Comparison Theorem. *If V_1 and V_2 satisfy* (51) *for $i = 1$ and $i = 2$ respectively, and G_i, K_i ($i = 1, 2$) satisfy* (55), *the interval between two consecutive roots of V_1 will contain at least one root of V_2.*

Sections 19 to 35 investigate how

$$\left.\frac{KV' + HV}{V}\right|_{x=\beta} \tag{57}$$

behaves as a function of r (H is a constant) and how

$$KV' + p(x)\, V \tag{58}$$

behaves as a function of x and r when $V_r(x)$ is a solution to (41) and (42). As a result of this analysis STURM proved

Proposition G. $[KV' + HV]_{x=\beta}$ *considered as a function of r vanishes Δ or $\Delta + 1$ times for $r \in [r_1, r_2]$*

where Δ has the meaning explained above by Proposition F. A related consequence [STURM 1836a, p. 141] of the analysis is

Sturm's Oscillation Theorem. *Let ϱ and ϱ' be two consecutive values of r which satisfy $KV_r' + HV_r = 0$ for $x = \beta$. Then $V_{\varrho'}$ has one more root in $]\alpha, \beta[$ than V_ϱ.*

25. In the last seven sections STURM gave methods of approximating the solutions V to (41) and their roots. During this part of the investigation he transformed the general equation (41) into the simpler form

$$V_r''(x) + \Gamma(x, r)\, V_r(x) = 0 \tag{59}$$

by altering either the independent variable x or the dependent variable V. He obtained the approximations by comparing equation (59) with the corresponding two differential equations which emerge when Γ is replaced by its maximum and minimum in $[\alpha, \beta]$. The solution of these two differential equations with constant coefficients are trigonometric functions and from their familiar properties STURM could get approximations to V using the main Theorem E. This important method had, as we saw in § 9, been perceived by D'ALEMBERT.

By way of this method STURM also proved in passing [STURM 1836a, section 40]

Proposition H. *If $\Gamma(x, r) \to \infty$ for $r \to \infty$, $x \in [\alpha, \beta]$ and if* (43) *is satisfied and if $V_r(\alpha)$ has the same sign for all r, then*

$$[KV' + HV]_{x=\beta}$$

vanishes infinitely many times when $r \to \infty$.

An immediate consequence of Proposition H is that the boundary value problem (1)–(3) has an infinity of eigenvalues, but STURM did not make this observation explicit here.

26. I have treated STURM's paper in such detail to give an impression of the wealth of new ideas and new results it presents. Of the Propositions A–H only the first two had been known earlier, and even the idea of proving theorems like C–H was new with STURM. The results were to constitute the basis of all the work of STURM and LIOUVILLE in the theory called after them. The individual elements in STURM's proofs, such as differentiation with respect to a parameter and partial integration, were all well known but STURM combined them in an original way to obtain qualitative statements about the solutions.

As in all early 19$^{\text{th}}$ century analysis STURM's methods do not meet the modern standards of rigour, but only minor alterations are needed to make them acceptable to a modern reader. Nevertheless STURM's exposition differs considerably from the presentation in modern textbooks. Today the solutions V_r are thought of as vectors in a HILBERT space and therefore their oscillation becomes uninteresting. If it is discussed at all, both boundary conditions (*e.g.* (2) and (3)) are usually introduced from the beginning, and so only the behaviour of the eigenfunctions is studied, whereas the continuous family of equations and solutions V_r do not occur. In addition the modern presentation generally concentrates on specific types of coefficients (*e.g.* as in (1)). Such limitations were applied by STURM in his second mémoire on Sturm-Liouville theory, devoted to spectral theory.

IV. Sturm's Second Mémoire

27. STURM'S mémoire on spectral theory was published later in the year of 1836 in LIOUVILLE'S Journal [STURM 1836b]. In its final form it must have been composed during that year for it includes some comments on results obtained by LIOUVILLE towards the end of 1835 [LIOUVILLE 1836c & d]. However, it is clear both from LIOUVILLE'S reference to the mémoire in the papers mentioned above and from a one-page summary in "L'Institut" of 1833 [STURM 1833b] that STURM had written a preliminary version in 1833 in connection with the composition of the first paper [1836a]. As discussed in § 18, the scanty evidence of STURM'S early works even shows that he must have had some of the ideas even in 1829, but probably not in the polished form he gave them in the mémoire of 1836.

28. I have already (§ 7) summarized how STURM in the second mémoire deduced the equation (1) with the boundary conditions (2)–(3) from a problem of heat conduction and how the eigenvalues can be considered as roots of a transcendental equation $\Pi(r) = 0$ (4). This is the situation about which STURM proved a number of results, the most important being propositions J–N.

Proposition J. *There are infinitely many roots (eigenvalues) of the equation $\Pi(r) = 0$; they are all real and positive and there are no multiple roots.*

Let $r_1 < r_2 < \ldots < r_n < \ldots$ be the eigenvalues[21] with the corresponding eigenfunctions $V_1, V_2, \ldots, V_n, \ldots$. In this notation STURM formulated the following results:

Proposition K (orthogonality).

$$\int_\alpha^\beta g(x)\, V_m(x)\, V_n(x)\, dx = 0 \quad \text{for} \quad m \neq n. \tag{60}$$

Proposition L. *V_n never becomes infinite in $[\alpha, \beta]$ and has in $]\alpha, \beta[$ $n - 1$ roots, at each of which it always changes sign.*

Proposition M. *Between two consecutive roots of V_m there is precisely one root of V_{m-1}.*

[21] STURM did not argue in favor of such an ordering of the eigenvalues. However, it is a consequence of LIOUVILLE'S expression for the asymptotic behaviour of the eigenfunctions (96).

Proposition N. *Let $m < n \in \mathbb{N}$ and $A_m, A_{m+1}, \ldots, A_n$ be constants which are not all zero. Define*

$$\psi(x) = A_m V_m(x) + A_{m+1} V_{m+1}(x) + \ldots + A_n V_n(x). \tag{61}$$

Then $\psi(x)$ has at least $m - 1$ roots and at most $n - 1$ roots counted with multiplicity.

If h or H are infinite, the boundary conditions (2) and (3) must be read $V(\alpha) = 0$ and $V(\beta) = 0$. Then these roots must be given particular treatment in L, M, and N. STURM painstakingly took care of these particular cases[22]. In the following brief discussion of the proofs such details will be ignored, and so the principal ideas will become clearer.

29. *ad J and K* STURM offered three proofs that the eigenvalues are real. The third was inspired by a method used in perturbation theory by LAPLACE in his Mécanique Celeste [1799, book II, chapter 6] and the second was a revision of POISSON's proof, to which STURM referred. Instead of POISSON's use of one solution to the ordinary differential equation (1) and one solution to the partial differential equation (5), STURM started out at once with two different solutions V_n, V_m of the ordinary differential equation. This idea made the proof easier and independent of the partial differential equation. Following POISSON, he demonstrated orthogonality as an intermediate step.

ad J STURM's proof that there are infinitely many eigenvalues has now become standard. It rests upon comparing equation (1) with the equation

$$U_1'' + n(r)^2 U_1 = 0, \tag{62}$$

where $n(r)$ is a constant independent of x, so selected that

$$g(x)\, r - l(x) > n(r)^2 \sup_{x \in [\alpha,\beta]} k(x) \text{ for } x \in [\alpha, \beta]$$

and that $n(r) \to \infty$ for $r \to \infty$.

The well known solution $C \sin(nx + c)$ of (62) has $\left[\dfrac{n(r)(\beta - \alpha)}{\pi}\right]$ roots in $]\alpha, \beta[$ and according to the comparison theorem of STURM's first paper $V_r(x)$ has at least as many roots. STURM had shown that $V_1(x)$ has no roots in $]\alpha, \beta[$; hence it follows from Proposition G of the first paper that $KV_r'(\beta) + HV_r(\beta)$ has at least $\left[\dfrac{n(R)(\beta - \alpha)}{\pi}\right]$ roots when r runs through $[0, R]$. Therefore there are infinitely many eigenvalues of (1)–(3). The proof is only a slight modification of the proof of Proposition H in the first mémoire.

ad L and M They are simple consequences of the oscillation theorem and the observation that $V_1(x)$ has no roots in $]\alpha, \beta[$.

[22] In the last paragraph of [STURM 1836b, section 9], STURM claimed that the root $x = \beta$ must be counted among the $n - 1$ roots of V_n. This mistake was corrected in the Errata.

ad N STURM obtained this result through careful investigation of the solution of the original boundary-value problem for the partial differential equation (5)–(7):

$$C_m V_m(x)\, e^{-r_m t} + C_{m+1} V_{m+1}(x)\, e^{-r_{m+1} t} + \ldots + C_n V_n(x)\, e^{-r_n t}, \tag{63}$$

where $r_m < r_{m+1} < \ldots < r_n$. For large positive values of t the solution (63) will be dominated by the first term, which has $m - 1$ roots, and for large negative values of t it will be dominated by the last term which has $n - 1$ roots. STURM proved that for other values of t the number of roots lay between these extremes.

In connection with the argument above STURM concluded that the temperature distribution in the bar

$$C_1 V_1 e^{-r_1 t} + C_2 V_2 e^{-r_2 t} + \ldots + C_n V_n e^{-r_n t} + \ldots$$

will eventually have m nodes where m is the smallest value of i for which C_i is different from zero. This value of m can be found from the initial temperature distribution $f(x)$ since, in virtue of orthogonality C_i is determined by[23]

$$C_i = \frac{\int_\alpha^\beta g(x)\, V_i(x) f(x)\, dx}{\int_\alpha^\beta g(x)\, V_i^2(x)\, dx}. \tag{64}$$

This result, which generalizes some of FOURIER's and POISSON's theorems, had already been indicated by STURM in [1829b].

30. Few other papers in the history of mathematics can rival STURM's two papers [1836a, b] for novelty of problem, methods, techniques and results. Above I have tried to summarize the theory as it appeared in 1833[24]. When preparing the last paper for publication STURM added a few remarks caused by LIOUVILLE's entrance on the scene (*cf.* § 50). He returned to the subject only once, namely in the following year with a paper written in collaboration with LIOUVILLE [LIOUVILLE & STURM 1837b] (*cf.* § 49). Otherwise he left it to LIOUVILLE to extend his researches on differential equations.

V. Loiuville's Youthful Work on Heat Conduction

31. LIOUVILLE's work on the Sturm-Liouville theory concentrated on two major problems: expansion of functions in Fourier series of eigenfunctions and generalisation of the theory to other types of differential equations. In his cele-

[23] STURM did not question the interchange of integration and summation involved in determination of the Fourier coefficients C_i.

[24] In the above account of STURM's second mémoire I have referred only to those ideas which do not result from LIOUVILLE's improvements. I think that thereby I have referred to STURM's original (1833) approach but it can not be excluded that in rewriting his mêmoire of 1833 STURM made alterations other than those due to LIOUVILLE's intervention.

brated papers on these two problems published during the three years 1836–1838 he built directly on STURM's research. His repeated reference to STURM creates the impression that the work of his friend had been the starting point of his interest in these matters. However, an inspection of LIOUVILLE's youthful works reveals that in fact his interest goes as far back as STURM's earliest investigations, and has the same inspiration, namely the study of FOURIER's and especially of POISSON's work on the theory of heat.

LIOUVILLE presented his first papers on the theory of heat to the Academy on June 29th, 1829, February 15th, 1830 and August 16th, 1830. In the case of LIOUVILLE we are not reduced to guessing what these early papers dealt with, as we were in STURM's case, because the last of them entitled "Recherches sur la théorie physico-mathématique de la Chaleur" is still preserved *in extenso* in the Archive de l'Académie des Sciences and is partially published [1830/31][25]. According to LIOUVILLE's introduction its main merit is the determination of the law of radiation of heat between the molecules of a medium as a function of distance and temperature. This law, which was a matter of much discussion among the physicists and mathematicians of the day, was derived in the fifth and final section of LIOUVILLE's paper. However this part of the paper was rejected by J. D. GERGONNE[26] to whom LIOUVILLE submitted his mémoire for publication in "Annales de Mathématiques". The major part of the rest of the mémoire was accepted for publication [1830/31] only because its author bore the double title of engineer and former student at the École Polytechnique. In fact GERGONNE had not read the paper properly before accepting it because he had misplaced it during a removal (*cf.* [LIOUVILLE 1830/31, p. 181, footnote]). When he finally came to read LIOUVILLE's paper he found the style so awful that he felt obliged to apologize in a footnote to the reader for its publication:

> "Je crois devoir m'excuser, vis-à-vis du lecteur, de lui livrer un mémoire aussi maussadement, je puis même dire, aussi inintelligiblement rédigé. ... Je ne prétends contester aucunement la capacité mathématique de M. Liouville; mais à quoi sert cette capacité, si elle n'est accompagnée de l'art de disposer, de l'art de se faire lire, entendre et goûter. [LIOUVILLE 1830/31, p. 181, footnote]

In fact LIOUVILLE's paper is rather confused but not much worse than STURM's first published paper, so perhaps STURM ranks among the "too many young people" whom GERGONNE accused of this indifference to style. LIOUVILLE's style improved in his later works.

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[25] The paper presented on February 15th bears the same title as the paper presented on August 16th. The two seem to have been identical, or almost so. In [1830/31] LIOUVILLE stated that the paper, which is almost word for word the same as that of August 16th, was an extract from the mémoire of February 15th.

[26] As a protégé of AMPÈRE, LIOUVILLE seems to have valued mathematical physics more than pure mathematics in his youth. Early in the 1830's his interests shifted toward pure mathematics. GERGONNE possibly saw LIOUVILLE's greater talent for mathematics earlier than did LIOUVILLE himself and therefore excluded the most physical part of LIOUVILLE's paper.

32. It may have been the harsh criticism in GERGONNE'S footnote which restrained LIOUVILLE from referring to this paper in his later works. Otherwise such references would have been appropriate since many of his most important ideas in the Sturm-Liouville theory can be found here in a preliminary or fully developed form. In order to give an account of these ideas I shall discuss in some detail those sections of the printed part of the mémoire [LIOUVILLE 1830/31] where they were set forth. The bulk of the paper was devoted to a generalization of POISSON'S theory of heat conduction in a thin metallic bar to the case where the surface is unequally polished and the material is inhomogeneous[27].

[27] According to LIOUVILLE [1830/31, § 1] the problem offers "des difficultés presque insurmontables" if the bar is not very thin. In the original mémoire from August 16th, 1830, LIOUVILLE indicates what kind of problems were involved in determining the stationary distribution of temperature in a metal plate (Fig. 7) kept at zero degrees along its two horizontal edges and radiating heat into a medium of constant temperature along its left, unequally polished, and vertical edge.

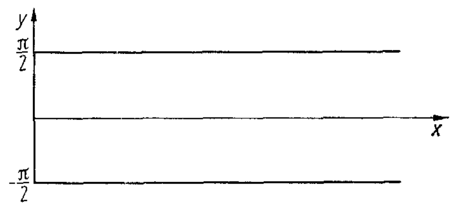

Fig. 7

The mathematical formulation of this problem is the boundary-value problem

$$\text{a)} \qquad \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0,$$

$$\text{b)} \qquad u = 0 \quad \text{for} \quad y = \pm\frac{\pi}{2},$$

$$\text{c)} \qquad \frac{\partial u}{\partial x} = f(y)\,(1-u) \quad \text{for} \quad x = 0,$$

Under the condition that $f(y) = f(-y)$ LIOUVILLE solved a) and b) by separating the variables and inserted the result into c). Thus he got the equation

$$\text{d)} \qquad \sum_{m=0}^{\infty} mA_m \cos my = f(y) - f(y) \sum_{m=0}^{\infty} A_m \cos my$$

which he could not solve for A_m.

During the next few years he often wrote notes on this problem in his notebooks [Ms. 3615 (3, 4)] and on March 17th, 1834, he presented a partial solution to the Academy. He converted equation d) into an integrodifferential equation, which he could solve for a particular class of functions f. His mémoire was printed in full in [1838d], but an extract of it had appeared in [1836b].

In a note from December 1835 [Ms. 3615 (4), pp. 87v–90v], LIOUVILLE returned to the solution of d) but this time he substituted eigenfunctions V_m to a general Sturm-Liouville problem for the trigonometric functions cos my. The note ends with the words "cela ne se peut".

First LIOUVILLE considered the homogeneous bar for which the temperature $u(x, t)$ must satisfy the following special case of (5)[28]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f(x)\, u, \qquad x \in\,]0, \beta[, \tag{65}$$

where $f(x) > 0$ in $[0, \beta]$.

He assumed the temperature at the end points 0 and β of the bar to be fixed at θ and θ' degrees:

$$u(x, t) = \theta \text{ for } x = 0, \tag{66}$$

$$u(x, t) = \theta' \text{ for } x = \beta. \tag{67}$$

Since these boundary conditions, unlike STURM's conditions (6) and (7) are not homogeneous, the temperature distribution has a nontrivial stationary state. In the first part of [1830/31] LIOUVILLE described two different determinations of this stationary state u from the equation

$$u'' = f(x)\, u \tag{68}$$

and the boundary conditions (66) and (67).

The first method [§ IV–§ XI] consisted in approximating $f(x)$ by a polygon, solving the equation in this case and letting the number of sides of the polygon tend to infinity. LIOUVILLE claimed, without proof, that the process would converge to a solution of the original equation. This polygon method, already developed in the mémoire of June 1829 [LIOUVILLE 1830/31, p. 157], is of little interest to us whereas the second method, invented in 1830, is very important.

33. LIOUVILLE began the second method by observing that if $u_0, u_1, u_2, \ldots$ are solutions of the infinite system of differential equations:

$$\frac{d^2 u_0}{dx^2} = 0, \quad \frac{d^2 u_1}{dx^2} = f(x)\, u_0, \quad \frac{d^2 u_2}{dx^2} = f(x)\, u_1, \ldots, \tag{69}$$

then their sum

$$u = u_0 + u_1 + u_2 + \ldots \tag{70}$$

is a solution of (68). The solutions of (69) can be written

$$\begin{aligned} u_0(x) &= A + Bx, \\ u_1(x) &= \int_0^x dx \int_0^x (A + Bx) f(x)\, dx, \\ u_2(x) &= \int_0^x dx \int_0^x f(x)\, dx \int_0^x dx \int_0^x (A + Bx) f(x)\, dx, \end{aligned} \tag{71}$$

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[28] I have set all the physical constants appearing in LIOUVILLE's paper equal to 1, and have from the start assumed the interval to be of the form $[0, \beta]$. In LIOUVILLE's paper this simplification is made only in the last part.

and so the general solution of (68) is of the form

$$A\left\{1+\int_0^x dx\int_0^x f(x)\,dx+\int_0^x dx\int_0^x f(x)\int_0^x dx\int_0^x f(x)\,dx+\dots\right\}+ \quad (72)$$
$$B\left\{x+\int_0^x dx\int_0^x xf(x)\,dx+\int_0^x dx\int_0^x f(x)\int_0^x dx\int_0^x xf(x)\,dx+\dots\right\}$$

Substituting $M=\max_{x\in[0,\beta]} f(x)$ for $f(x)$ in the first series in (72), LIOUVILLE obtained the series

$$\left\{1+\frac{Mx^2}{2!}+\frac{M^2x^4}{4!}+\dots\right\}. \quad (73)$$

Since this series converges, LIOUVILLE concluded that the first series of (72) converges as well. Similarly he proved that the second series in (72) converges. The two arbitrary constants A and B can finally be determined from the boundary conditions (66) and (67).

34. In the above argument LIOUVILLE has used the method of successive approximation, later ascribed to E. PICARD [1890], to show that a solution of the differential equation (68) such as to fulfil conditions (66) and (67) exists. Thus LIOUVILLE's anticipation of PICARD's proof of existence did not originate in [1836c and 1837c] as is generally believed but was developed six years earlier.

It is now a well established fact that CAUCHY gave another type of existence proof for a general first-order differential equation in his lectures at the École Polytechnique during the 1820's [CAUCHY 1824/1981] (publ. [1835/40]) and it is often conjectured (*e.g.* by KLINE [1972, p. 719] and by BIRKHOFF [1973, p. 243]) that CAUCHY at that time knew also the proof of existence using successive approximation. The only argument in favour of that conjecture is MOIGNO's inclusion of the method in his "Leçons de calcul différentiel et de calcul intégral rédigées principalement d'après les méthodes de M. A.-L. Cauchy" [1844, pp. 702–707]. Recently C. GILAIN (*cf.* [CAUCHY 1824/1981]) has argued convincingly that CAUCHY did not apply successive approximation to prove existence but only used this method to find approximate solutions. Further GILAIN is of the opinion that MOIGNO got his proof there. However it is striking that MOIGNO used the method of successive approximation to a second-order linear differential equation, just as LIOUVILLE did, whereas CAUCHY proved existence for first-order equations and partial differential equations. This fact more than indicates that MOIGNO borrowed this part of his "Leçons" directly from LIOUVILLE (probably from [1837c]). Therefore I think LIOUVILLE's proof in [1830/31] is his own original contribution to the theory of differential equations. If that is true, LIOUVILLE [1830/31] both presented the first published proof of the existence of a solution of a differential equation and the first application—published or unpublished—of the method of successive approximation for that purpose[29].

[29] Admittedly LIOUVILLE never formulated as explicitly as CAUCHY the central question: Given a differential equation and certain boundary conditions or initial conditions; does a solution exist? In [1830/31] LIOUVILLE differed from CAUCHY also by focusing on boundary-value problems. In [1836d, 1837c] LIOUVILLE imposed Cauchy data at only one point.

35. After having thus found the stationary solution, hereafter called $V_0(x)$, of (65) with the boundary values (66) and (67) LIOUVILLE supposed the general solution to (65)–(67) to be of the form

$$V_0(x) + C_1 V_1(x)\, e^{-r_1 t} + C_2 V_2(x)\, e^{-r_2 t} + C_3 V_3(x)\, e^{-r_3 t} \ldots, \tag{74}$$

where each of the eigenfunctions must satisfy the special case of (1),

$$-r V_r = V_r'' - f(x)\, V_r, \tag{75}$$

and the homogeneous boundary conditions,

$$V_r(x) = 0 \text{ for } x = 0, \tag{76}$$

$$V_r(x) = 0 \text{ for } x = \beta. \tag{77}$$

Again the method of successive approximation produces an expression for the solution as a sum of two series of which one vanishes if we take the boundary condition (76) into account (corresponding to $A = 0$ in (72)). In the remaining expression for V_r

$$V_r = x + \int_0^x dx \int_0^x x(f(x) - r)\, dx \tag{78}$$

$$+ \int_0^x dx \int_0^x (f(x) - r)\, dx \int_0^x dx \int_0^x x(f(x) - r)\, dx + \ldots$$

the eigenvalue r must be chosen so that the boundary condition (77) is satisfied. As usual this gives rise to a transcendental equation (4) for r.

36. LIOUVILLE went on to prove the orthogonality (60) (for $g \equiv 1$) in essentially the same way as STURM later did, that is without using the partial differential equation as POISSON had done in [1826]. Apparently LIOUVILLE did not even know of POISSON's paper, for he used orthogonality only to determine the arbitrary constants C_i in (74) from a given initial temperature distribution $f(x) + V_0(x)$ (as in (64) with $g \equiv 1$), but used another method to prove the eigenvalues real:

> "Nous ferons voir 1° que l'équation d'où résultent les valeurs de m [r] a toutes ses racines réelles et positives; 2° nous prouverons que la série que forme la valeur de u [(74)] est une série convergente, ce qui est nécessaire pour compléter la solution." [LIOUVILLE 1830/31, p. 164]

37. LIOUVILLE's proof of 1° is both clumsy and unrigorous compared with POISSON's proof. It rests on the inspection of the transcendental equation $\Pi(r) = 0$ found by substituting (78) into (77) (in the following arguments he takes $\beta = 1$):

$$1 + \int_0^1 dx \int_0^x x(f(x) - r)\, dx + \int_0^1 dx \int_0^x (f(x) - r)\, dx \int_0^x dx \int_0^x x(f(x) - r)\, dx + \ldots = 0. \tag{79}$$

Since $f(x) > 0$, it is clear that no negative value of r solves this equation. LIOUVILLE believed that he would exclude all non-real eigenvalues if only he could prove the existence of an infinity of positive eigenvalues. In the most far-reaching of the two proofs he supplied for the latter, he substituted for $f(x)$ a loosely described "mean value" P_r independent of x, reducing (79) to

$$1 - \frac{(r - P_r)}{3!} + \frac{(r - P_r)^3}{5!} - \frac{(r - P_r)^5}{7!} + \ldots = 0 \tag{80}$$

or

$$\frac{\sin\sqrt{r - P_r}}{\sqrt{r - P_r}} = 0. \tag{81}$$

In fact this argument is not valid, since one can not use the same "mean value" in the different terms of (79). But taking this for granted and accepting LIOUVILLE's loose argument that P_r remains bounded as a function of r because $f(x)$ is bounded as a function of x, we can conclude with LIOUVILLE that there are infinitely many positive solutions of (79) or (81) of the form:

$$r = P_r + n^2\pi^2, \qquad n \in \mathbb{N}. \tag{82}$$

The argument even shows that for large values of n the eigenvalues are "très-approchée $r = n^2\pi^2$" in the sense that $|r_n - n^2\pi^2| \leq \max P_r < \infty$.

38. This approximation of the eigenvalues was applied by LIOUVILLE in his subsequent proof of the convergence of the Fourier series (74) for $t = 0$, which will imply the convergence for $t > 0$. If we consider only large values of r (which suffices in a proof of convergence) $f(x)$ can be ignored in the expression (78) of V_r leaving the approximate eigenfunction:

$$V_r = \frac{\sin\sqrt{r}\,x}{\sqrt{r}}. \tag{83}$$

Using this value of V_r in (74) and in formula (64) for the Fourier coefficients, LIOUVILLE was led to the approximate value of u for $t = 0$:

$$\sum_r \frac{\sin\sqrt{r}\,x}{\beta} \int_0^\beta f(x)\sin\sqrt{r}\,x\,dx. \tag{84}$$

If the "approximate" value $n^2\pi^2$ is substituted for r, (84) reduces to an ordinary Fourier series. LIOUVILLE believed that FOURIER and POISSON had provided proofs of convergence for the ordinary Fourier series and concluded that since its terms coincided with the terms of (74) for large values of r, the latter, more general Fourier series must also converge.

In the last sections [1830/31, § 24–27] LIOUVILLE generalized all these considerations to equation (5) for the heterogeneous bar[30].

[30] LIOUVILLE correctly deduced the orthogonality (60) [LIOUVILLE 1830/31, p. 179] but when he used it to derive the expression (64) for the Fourier coefficients he forgot the factor $g(x)$. [LIOUVILLE 1830/31, pp. 179–180]

39. In spite of the lack of rigour in the last arguments LIOUVILLE'S [1830/31] is of the greatest importance in the development of Sturm-Liouville theory because it constitutes the germ of LIOUVILLE's subsequent contributions to this theory. Together with STURM's now lost mémoires from 1829 and POISSON's [1826], it presents the earliest advances in this branch of analysis. I have argued that LIOUVILLE did not know POISSON's work and according to his own testimony in the introduction of [1830/31] he also did not know STURM's works:

> "Ces questions jusqu'ici n'avaient été traitées par aucun géomètre: elles semblent offrir des difficultés presque insurmontables, lorsqu'on suppose aux corps leur trois dimensions [*cf.* note 27]. Si l'on fait abstraction de deux d'entre elles, le problème est complètement résolu par mon travail."

There is also a striking difference between STURM'S and LIOUVILLE'S approaches to the theory. Though they share some theorems in common, as for example the statement of orthogonality, the reality of eigenvalues and the determination of the Fourier coefficients, the bulk of their papers have different goals, STURM tending toward the qualitative behaviour of the eigenfunctions, and LIOUVILLE toward expansion in Fourier series.

VI. Liouville's Mature Papers on Second-Order Differential Equations. Expansion in Fourier Series

40. Though LIOUVILLE's interests after 1830 turned to other fields, such as fractional differentiation and integration in finite terms, he continued to work on problems related to those treated in the paper of [1830/31] (*cf.* [LIOUVILLE 1836a, 1836b, 1836c]; see note 26). However, these problems were all concerned with trigonometric series and he did not publish anything on the more general type of Fourier series until the three large "Mémoires sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable" [1836d, 1837c, 1837e].

In these three important papers LIOUVILLE was chiefly concerned with the questions he had treated in the mémoire of [1830/31]. However he now succeeded in rigorizing the theory considerably, partly by building on STURM's much more detailed investigation of the eigenfunctions, partly by refining his own earlier arguments. He immediately turned to STURM's general equation (1) with the boundary conditions (2) and (3) and repeated the solution [1830/31] by successive approximation, in this case for the CAUCHY problem (1) and (2). In the first mémoire [1836d, p. 255] he merely wrote down the formulas similar to (72) and did not write one word on the question of convergence. However, in the second paper [1837c, pp. 19–22] he amply made up for this omission by supplying in addition to the proof of convergence [1830/31] another proof based on conversion of the differential equation into an integral equation [*cf.* LÜTZEN 1981].

41. Otherwise LIOUVILLE devoted his second large mémoire [1837c] to the other great question of convergence treated in the [1830/31] mémoire, that of the Fourier series

$$F(x) = \sum_{n=1}^{\infty} \frac{V_n(x) \int_\alpha^\beta g(x)\, V_n(x) f(x)\, dx}{\int_\alpha^\beta g(x)\, V_n^2(x)\, dx}. \tag{85}$$

The central idea of the improved proof of convergence is the following uniform inequality:

> "... si l'on désigne par n un indice très grand, par u_n la valeur absolue du $n^{\text{ième}}$ terme de la série (85) et par M un certain nombre indépendant de n, ... on a $u_n < \frac{M}{n^2}$." [LIOUVILLE 1837c, p. 18]

To obtain this conclusion LIOUVILLE introduced the new dependent and independent variables z and U defined by

$$z = \int_\alpha^x \sqrt{\frac{g(x)}{k(x)}}\, dx, \tag{86}$$

$$V(x) = \theta(x)\, U(x), \qquad \text{where} \quad \theta = \frac{1}{\sqrt[4]{g(x)\, k(x)}}, \tag{87}$$

and

$$r = \varrho^2.$$

Expressed in these new variables, the original problem (1)–(3) is reduced to the simpler problem

$$U''(z) + \varrho^2 U(z) = \lambda(z)\, U(z), \qquad z \in [0, \gamma], \tag{88}$$

$$U'(z) - h' U(z) = 0 \quad \text{for} \quad z = 0, \tag{89}$$

$$U'(z) + H' U(z) = 0 \quad \text{for} \quad z = \gamma \tag{90}$$

where

$$\gamma = \int_\alpha^\beta \sqrt{\frac{g(x)}{k(x)}}\, dx,$$

$$\lambda = \frac{1}{\theta \sqrt{gk}} \left(l \sqrt{\frac{k}{g}}\, \theta - \frac{d\sqrt{gk}}{dz} \frac{d\theta}{dz} - \sqrt{gk}\, \frac{d^2\theta}{dz^2} \right) \tag{91}$$

and h', H' are constants which are not necessarily positive. The elegant transformation (86) and (87) is now called the Liouville transformation after its inventor. It combines STURM's transformations of the dependent and independent variables (see § 25) in such a way that the function k disappears in the transformed equation (88) at the same time as the coefficient to the undifferentiated term U retains its simple form.

42. From the transformed equation (88)–(90) LIOUVILLE deduced the integral equation for U[31]:

$$U(z) = \cos \varrho z + \frac{h' \sin \varrho z}{\varrho} + \frac{1}{\varrho} \int_0^z \lambda(z')\, U(z') \sin \varrho(z - z')\, dz'. \tag{92}$$

Various estimates applied to this integral equation then lead to the desired bounds of the numerator and denominator of the Fourier series (85):

$$\left| V_n \int_\alpha^\beta g(x)\, V_n(x) f(x)\, dx \right| < \frac{1}{r_n} K_1 \left(1 + \left(\frac{h'}{\varrho_n}\right)^2\right), \tag{93}$$

$$\int_\alpha^\beta g(x)\, V_n^2(x)\, dx > K_2 \left(1 + \left(\frac{h'}{\varrho_n}\right)^2\right), \tag{94}$$

K_1, K_2 being positive constants. Hence the absolute value of the n^{th} term u_n of the Fourier series is bounded by

$$|u_n| < \frac{M}{r_n}. \tag{95}$$

As in the proof of [1830/31] LIOUVILLE then needed only show that r_n tends to infinity fast enough when n tends to infinity. The idea of the proof is the same as in [1830/31] but in [1837c] the trigonometric behaviour of the eigenfunctions for large values of r was established from the integral equation (92). Using this asymptotic behaviour of U_r and STURM's oscillation theorem,[32] LIOUVILLE rigorously established the asymptotic behaviour of the eigenvalues:

$$\varrho_n = \sqrt{r_n} \sim \frac{(n-1)\pi}{\gamma} + \frac{P_0}{(n-1)\pi} \tag{96}$$

where P_0 is a constant. Combining this statement with (95), he finally obtained the desired estimate:

$$|u_n| < \frac{M'}{n^2},$$

implying the convergence of the Fourier series (85).

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[31] In equation (92) LIOUVILLE has fixed the arbitrary multiplicative constant in such a way that $U(0) = 1$.

[32] In the first place LIOUVILLE's analysis shows only that U_n behaves asymptotically like $\cos \frac{m\pi}{\gamma} x$. This has $m - 1$ roots in $]0, \gamma[$, and since U_n, according to STURM's oscillation theorem, has $n - 1$ roots in $]0, \gamma[$, LIOUVILLE concludes that $m = n$.

43. LIOUVILLE'S deduction contained several innovations in addition to the convergence theorem itself. I have already mentioned the LIOUVILLE transformation (86) and (87). Just as important are the asymptotic expressions for the eigenvalues (96) and the corresponding approximate eigenfunctions to be found from (92). According to LIOUVILLE the latter complemented STURM'S methods of approximating the eigenfunctions (§ 25), which was manageable only for small values of n.

Finally the ingenious application of the equation (92) marks an important instance in the early theory of integral equations (see [LÜTZEN 1981]).

In the above proof of convergence LIOUVILLE had explicitly assumed that $g(x)$, $k(x)$, $f(x)$ and their "dérivées premières et secondes conservent toujours des valeurs finies" ($g, k, f \in \mathscr{C}^2[\alpha, \beta]$). Implicitly he also assumed that $f(x)$ satisfies the boundary conditions (2) and (3). Under these assumptions LIOUVILLE'S proof even proves rigorously that the Fourier series converges uniformly. However, LIOUVILLE could not appreciate this virtue of the proof since the difference between pointwise and uniform convergence had not been realized by then.

44. In November of the same year LIOUVILLE published a new proof of convergence [1837e] which did not make use of the too restrictive assumptions mentioned in the last section.

> "Je me propose ici de faire disparaître, autant qu'il me serà possible, ces restrictions diverses, et surtout celles relatives à la fonction $f(x)$." [LIOUVILLE 1837e, p. 419]

In particular he had discovered that it was unnecessary to impose the boundary conditions (2) and (3) on f:

> "ces conditions, que j'ai imposées mal à propos à la fonction $f(x)$ dans mes deux premiers mémoires, sont inutiles et doivent être absolument mises de coté". [LIOUVILLE 1837e, p. 421].

In place of them LIOUVILLE assumed f to be continuous, and instead of $g, k, l \in \mathscr{C}^2[\alpha, \beta]$ he assumed only that λ defind by (91) be absolutely integrable[33].

An investigation of LIOUVILLE'S proof reveals that he actually used more assumptions on g, k, l and λ such as differentiability of k and piecewise monotonicity of f. The last property is implicitly used in the proof [1837e, Sect. 4]

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[33] LIOUVILLE wrote: "que l'intégrale $\int_0^{\gamma} \sqrt{\lambda^2}\, dz$ ait une valeur finie et puisse être regardée comme équivalente à la somme de ses éléments". The last remark no doubt means that the integral is not be taken in its 18[th]-century sense, as the opposite of the differential, but must be defined as a sum in the way FOURIER and more precisely CAUCHY had done. LIOUVILLE probably thought of CAUCHY'S extension of the integral to functions with isolated discontinuities.

of a strong version of RIEMANN's lemma:

$$\left|\int_0^z f(z)\sin \varrho z\, dz\right| < \frac{K}{\varrho}; \tag{97}$$

further it is presupposed at the end of the proof because LIOUVILLE referred to DIRICHLET's proof of convergence [1829] for trigonometric Fourier series, which was explicitly carried through for piecewise monotone functions.

45. In spite of these insufficiencies LIOUVILLE's attitude towards the relation between assumption, proof, and theorem was very modern. He had realized that a theorem may be improved by relaxing its assumptions and he had seen that the best assumptions can be found by examining the proof. For example, the assumption on λ clearly stems from the proof. In this respect LIOUVILLE was more far-sighted than his leading compatriot CAUCHY, who did not understand this interplay[34]. By 1837 the mathematician who had most explicitly put forward such proof-generated assumptions was DIRICHLET in his paper on Fourier series [1829]. LIOUVILLE may have been influenced by his friend DIRICHLET but he took a step further by searching a new, and according to himself less elegant, proof of an already established theorem with the sole aim of weakening the assumptions. Such an understanding of the interplay between assumptions and proofs did not catch on until the end of the 19th century.

46. Clearly LIOUVILLE could not use under the new assumptions the proof of convergence [1837e] with its uniform estimates. Instead he produced a rigorized version of the proof [1830/31] by using instead of the loose argument concerning the asymptotic behaviour of the eigenfunctions an intricate application of the integral equation (92), leading again to the asymptotic eigenvalues (96). In this way he obtained the following expression for the terms of the Fourier series:

$$u_n(x) = \frac{2}{\gamma}\cos nz \int_0^\gamma F(z)\cos nz\, dz + \frac{\psi_1(x, n)}{n^2}, \tag{98}$$

where $\psi_1(x, n)$ is bounded. From this expression the convergence follows easily from the convergence of trigonometric Fourier series, for which LIOUVILLE this time referred to CAUCHY's "et surtout l'excellent Mémoire de M. LEJEUNE DIRICHLET", [1829].

47. In his two proofs of convergence [1837c, 1837e] LIOUVILLE did not have to concern himself with finding the limit of the Fourier series because he had already settled that problem in the first of the three large mémoires [1836d], presented to the Academy on November 30th, 1835. LIOUVILLE claimed to prove "par un procédé rigoureux" that the "valeur de la série" (85) is $f(x)$. The proof rests on two lemmas.

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[34] CAUCHY usually made the stereotype assumption that the functions be continuous no matter what properties he actually used in his proofs. Related remarks have recently been made by GILAIN (*cf.* [CAUCHY 1824/1981, pp. XLI–XLIX]).

Lemma I. *Define inductively the functions $P_i^j(x)$ by*

$$P_i^1(x) = V_1(a_1)\, V_i(x) - V_i(a_1)\, V_1(x), \qquad i = 2, 3, \ldots$$
$$P_i^2(x) = P_2^1(a_2)\, P_i^1(x) - P_i^1(a_2)\, P_2^1(x), \qquad i = 3, 4, \ldots$$
$$\vdots$$
$$P_i^j(x) = P_j^{j-1}(a_j)\, P_i^{j-1}(x) - P_i^{j-1}(a_j)\, P_j^{j-1}(x), \qquad i = j+1, j+2, \ldots$$

where $a_1, a_2, \ldots, a_j, \ldots$ are different points of $]\alpha, \beta[$. Then $P_{j+1}^j(x)$ vanishes and changes sign at $a_1, a_2, \ldots, a_j, \ldots$, and it has no other roots.

It is easily verified that $a_1, a_2, \ldots, a_j$ are roots of P_{j+1}^j, and since P_{j+1}^j is of the form $\sum_{i=1}^{j+1} A_i V_i(x)$, it follows from STURM's proposition N (§ 28) that it has at most j roots counted with multiplicity. Thus P_{j+1}^j has precisely the simple roots $a_1, a_2, \ldots, a_j$.

Lemma II. *Let $\varphi(x)$ be a function of $x \in [\alpha, \beta]$. If*

$$\int_\alpha^\beta \varphi(x)\, V_n(x)\, dx = 0 \tag{99}$$

for all eigenfunctions V_n of (1)–(3), *then* $\varphi \equiv 0$.

LIOUVILLE gave an indirect proof: Suppose φ changes sign j times, say in $a_1, a_2, \ldots, a_j$. As in the first lemma he constructs P_{j+1}^j corresponding to this series of roots. Since P_{j+1}^j is a linear combination of eigenfunctions, we have

$$\int_\alpha^\beta \varphi(x)\, P_{j+1}^j(x)\, dx = 0, \tag{100}$$

which contradicts the fact that $\varphi(x)\, P_{j+1}^j(x)$ is not identically zero in $[\alpha, \beta]$ where it conserves its sign. Therefore $\varphi(x)$ cannot change sign a finite number of times. Consequently, LIOUVILLE says, φ must be identically zero in $[\alpha, \beta]$.

It was an easy matter for LIOUVILLE to deduce the main theorem from this lemma. He multiplied both sides of (85) by $g(x)\, V_m(x)\, dx$ and integrated from α to β. Because of the orthogonality relations (60) only the m^{th} term on the right-hand side survives:

$$\int_\alpha^\beta g(x)\, V_m(x)\, F(x)\, dx = \int_\alpha^\beta g(x)\, V_m(x) f(x)\, dx \qquad \forall m = 1, 2, \ldots$$

or

$$\int_\alpha^\beta g(x)\,(F(x) - f(x))\, V_m(x)\, dx = 0 \qquad \forall m = 1, 2, \ldots . \tag{101}$$

According to Lemma 2 $F(x) - f(x) = 0$, so that the sum in question equals $f(x)$. This completes LIOUVILLE's proof.

48. In spite of its simplicity this proof presents the most profound mistake in STURM's and LIOUVILLE's theory of second-order linear differential equations. I do not refer to the curious neglect of the problem of convergence, solved the following year[35], nor to the term-by-term integration[36], but to the last step in Lemma 2, where LIOUVILLE concludes that a function with infinitely many roots in $[\alpha, \beta]$ must be identically zero there. This would be true for an analytic function φ, but it is not easy to prove that F is analytic even if f is analytic. In fact one has to use a totally different approach to prove the theorem.

LIOUVILLE later came to realize at least a part of the problem. On March 28th (or 29th) 1838 he noted in his notebooks [Ms. 3616 (2), p. 56v] that $\varphi(x)$ could be different from zero for isolated values of x, but if φ is continuous that can not happen[37]. Both here and in his repetition of this insufficient argument in [1838c, p. 603, 612] LIOUVILLE dismissed the difficulty as a minor detail. However his subsequent attempts to supply another proof reveals that he was disturbed by the problem.

49. Even in 1837 LIOUVILLE had devised a different proof of the expansion theorem in collaboration with STURM [LIOUVILLE & STURM] 1837b. They proved that $\varphi(x) \equiv F(x) - f(x)$ satisfies the equation

$$\int_{\alpha}^{\beta} \varphi(x)\, V_r(x)\, dx = 0 \qquad \forall r \in \mathbb{R} \tag{102}$$

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[35] It is impossible to tell whether LIOUVILLE in [1836d] had forgotten the convergence problems of the succesisve approximation and of the Fourier series, which he had discussed in [1830/31], or whether he consciously postponed treatment of these problems to the following papers. As LIOUVILLE expressed himself in [1836d], he must have created the impression that he was unable to reach CAUCHY's standards of rigour, which he had earlier explicitly stressed:

> "Les séries divergentes, amenant le plus souvent des résultats fautifs, doivent être tout-à-fait bannis de l'analyse". [LIOUVILLE 1832, p. 77]

When STURM in [1836b, p. 411] referred to LIOUVILLE's theorem he was kind enough to formulate it not as LIOUVILLE had done in [1836d] but as he ought have done:

> "... la somme de la série [85], si cette série est convergente, ne peut qu'être égale à $f(x)$, pour toutes les valeurs de x comprises entre α et β."

In [LIOUVILLE 1838c, § 35] LIOUVILLE formulated the theorem in this way as well.

[36] Under the assumptions made in LIOUVILLE's first convergence proof [1836d] term-by-term integration is allowed since the Fourier series converges uniformly.

[37] In a note from September 7th, 1840 [Ms. 3616 (5), pp. 45r–46r] LIOUVILLE showed a deeper understanding of the problem. There he mentioned the difficulties occurring if φ can "s'évanouir un nombre infini de fois dans chacun des intervalles infinimens petits compris entre α et $\alpha + \varepsilon$, $\beta - \varepsilon$ et β" (LIOUVILLE does not use the letters α and β).

for *all* solutions V_r of (1) satisfying only the boundary condition (2)[38]. However, in order to conclude that (102) implied $\varphi \equiv 0$ they referred to LIOUVILLE's original proof in [1836d], and so their argument was no more convincing than the original. STURM & LIOUVILLE's proof was a small extract of a large mémoire on Sturm-Liouville theory presented to the Academy on May 8th, 1837. Unfortunately the rest of the mémoire is lost.

LIOUVILLE returned to this approach to the expansion theorem three years later. However he left only an unfinished and insufficient sketch of a proof in a draft of a letter to an unnamed colleague [Ms. 3616 (5), pp. 45r–46r].

Indications of other approaches can be found in LIOUVILLE's notes from March 8th, 1838 [Ms. 3616 (2)], August 21st, 1839 [Ms. 3616 (1)] and September 1839 [Ms. 3616 (5)]. Of these notes only the second, written in Bruxelles, led anywhere. The central idea in the proof is to make the coefficients depend upon a new parameter m.

At first the new method did not work satisfactorily[39], but after some revision LIOUVILLE was so content with it that he sent the proof to DIRICHLET in a letter dated by TANNERY February 1841 [TANNERY 1910, pp. 17–19]. He considered the simplified form (88)–(90) of the problem and chose $\lambda(z, m)$, $h'(m)$, $H'(m)$ such as to converge to zero as $m \to 0$. Thus for $m = 0$ the eigenfunctions were simple trigonometric functions and in that case LIOUVILLE knew that $\varphi(x, m) = F(x, m) - f(x) \equiv 0$. He then argued that if $\varphi(x, m) \equiv 0$ for some value of m it vanishes necessarily in a whole neighbourhood of m. Hence he concluded that $\varphi(x, m) \equiv 0$ for all m[40].

Unfortunately there is a grave mistake in this proof as well, namely in the proof of convergence of a certain series which he claimed to be "très facile" in the letter to DIRICHLET. His faulty proof has been preserved in a note from the first half of 1840 [Ms. 3616 (5), pp. 14v–15r][41].

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38 STURM and LIOUVILLE built their conclusion on a development of

$$\frac{V_r}{\Pi(r)}$$

in simple fractions ($\Pi(r)$ being defined by formula (4)).

39 LIOUVILLE rejected the first deduction because it used the Taylor theorem. He explicitly referred to CAUCHY's objections to this theorem.

40 LIOUVILLE did not notice that the size ε of the neighbourhood $]m - \varepsilon, m + \varepsilon[$ was dependent on m. Therefore he thought he could exhaust an interval $[0, M]$ with a finite number of neighbourhoods. This part of LIOUVILLE's argument can be made rigorous if we note that in modern terms LIOUVILLE claims that

$$\{m > 0 \mid \varphi(m, x) \equiv 0\}$$

is an open subset of $\mathbb{R}_+$. Under sufficiently strong conditions of regularity it is clear that the set is also closed and hence it is equal to $\mathbb{R}_+$.

41 The crucial mistake in this note occurs at the end where LIOUVILLE states that a series of the form

$$\sum_{i=1}^{\infty} \frac{1}{i} f(z, i) \cos iz$$

50. LIOUVILLE'S work on second-order linear differential equations almost exclusively dealt with Fourier expansion of "arbitrary" functions. However, in his second publication on Sturm-Liouville theory from 1836 [1836e] he gave a time-independent analysis leading to STURM'S proposition N (*cf.* § 28). Using the techniques of STURM, he showed that

$$A_m(r_m - r_1)\,V_m + A_{m+1}(r_{m+1} - r_1)\,V_{m+1} + \ldots + A_n(r_n - r_1)\,V_n \tag{103}$$

has at least as many roots in $]\alpha, \beta[$ as

$$A_m V_m + A_{m+1} V_{m+1} + \ldots + A_n V_n. \tag{61}$$

Repeated use of this observation shows that

$$A_m \left(\frac{r_m - r_1}{r_n - r_1}\right)^k V_m + A_{m+1} \left(\frac{r_{m+1} - r_1}{r_n - r_1}\right)^k + \ldots + A_n V_n \tag{104}$$

has at least as many roots as (61) and since $\left(\dfrac{r_{m+1} - r_1}{r_n - r_1}\right) < 0$, LIOUVILLE concluded that in the limit $k = \infty$ the number of roots of V_n $(= n - 1)$ is greater or equal to the number of roots of (61)[42]. A similar argument gives the lower bound on the number of roots of (61). Inspired by LIOUVILLE, STURM added a similar proof to his [1836b]. It was built on the observation that

$$A_m r_m V_m + A_{m+1} r_{m+1} V_{m+1} + \ldots + A_n r_n V_n \tag{105}$$

has at least as many roots as (61). His proof had the advantage of taking into account possible roots at the end points, a problem LIOUVILLE had left aside (as have I in §§ 28 and 29).

LIOUVILLE'S new proof of proposition N can be viewed as the last step in a process of establishing Sturm-Liouville theory as a self-contained subject, independent of the physical problems and partial differential equations whence it had emerged. STURM had taken the first step by freeing POISSON'S proof that the eigenvalues were real from the unnecessary use of a solution of the partial differential equation. After [LIOUVILLE 1836e] all theorems in Sturm-Liouville theory rested only on the equations (1)–(3).

51. In addition to these papers on second-order linear differential equations LIOUVILLE published some results which were inspired by specific ideas in Sturm-

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is convergent when $f(z, i)$ is a bounded function. He probably thought that this could be inferred from the behaviour of the Fourier series

$$\sum_{i=1}^{n} \frac{\cos iz}{i}.$$

[42] The argument works only because V_n has no double root. LIOUVILLE'S careful argument in [1836e, p. 275] takes this fact into account.

Liouville theory[43]. For example he proved that Lemma II is valid when x^n is substituted for V_n [1837b], and concluded that the Fourier expansion of $f(x)$

[43] In secondary sources it is often claimed that LIOUVILLE proved BESSEL's inequality for the general Sturm-Liouville problem (1)–(3) (see *e.g.* [KLINE 1972, p. 716–717], [BIRKHOFF 1973, p. 276] and (DIEUDONNÉ 1981, p. 21]). BESSEL's inequality states that

$$\sum_{n=1}^{\infty} C_n^2 \leqq \int_{\alpha}^{\beta} g(x) f^2(x)\, dx,$$

where C_n are the Fourier coefficients and the orthogonal system V_n has been normalized

$$\int_{\alpha}^{\beta} g(x)\, V_n(x)\, V_m(x)\, dx = \delta_{m,n}.$$

At the place referred to in the secondary sources [LIOUVILLE 1836d, p. 265] LIOUVILLE proves that if

$$\sigma_n(x) = \sum_{i=1}^{n} C_i V_i(x),$$

$$\varrho_n(x) = \sum_{i=n+1}^{\infty} C_i V_i(x),$$

then

$$\int_{\alpha}^{\beta} g(x) f(x)^2\, dx = \int_{\alpha}^{\beta} g(x)\, (\sigma_n(x)^2 + \varrho_n(x)^2)\, dx,$$

from which

(*) $$\int_{\alpha}^{\beta} g(x) f(x)^2\, dx \geqq \int_{\alpha}^{\beta} g(x)\, \sigma_n(x)^2\, dx.$$

Now it is true, but not pointed out by LIOUVILLE, that the right-hand side of (*) is equal to $\sum_{i=1}^{n} C_n^2$, and so one obtains BESSEL's inequality in the limit $n = \infty$. LIOUVILLE himself made the following comment on (*):

> Cette dernière formule nous prouve que l'intégrale $\int_{\alpha}^{\beta} g\sigma_n^2\, dx$, quelque grand qu'on prenne l'indice n, ne peut jamais avoir une valeur numérique supérieure à la limite $\int_{\alpha}^{\beta} gf(x)^2\, dx$ avec laquelle elle coïncide lorsque $n = \infty$.

Thus for LIOUVILLE (*) was important because it is valid for a finite n, whereas BESSEL's inequality has $n = \infty$. According to LIOUVILLE the two sides of (*) coincide for $n = \infty$; this has made KLINE [1872, p. 716–717] attribute PARSELVAL's equality (= BESSEL's inequality) to LIOUVILLE. For LIOUVILLE, however, the equality was a simple consequence of the expansion theorem, $\sigma_\infty(x) = f(x)$, and therefore he did not consider it a theorem in its own right.

The modern mathematician evaluates the inequality (*) differently from LIOUVILLE because he knows that one cannot always take the limit inside the integral sign whereas LIOUVILLE did not doubt the identity

$$\int_{\alpha}^{\beta} g(x) f(x)^2\, dx = \int_{\alpha}^{\beta} g(x) \lim_{n\to\infty} (\sigma_n(x)^2)\, dx = \lim_{n\to\infty} \int_{\alpha}^{\beta} g(x) \sigma_n\, (x)^2\, dx.$$

To ascribe BESSEL's inequality, and particularly PARSEVAL's equality to LIOUVILLE, therefore, is an overinterpretation.

on the orthogonal set of Legendre polynomials "have the value" $f(x)$ [1837d][44]. Three years later LIOUVILLE complemented CAUCHY's convergence criteria of [1821] with a new "condition de convergence d'une classe générale de séries" [1840]. The series in question were power series of the form

$$\varphi(x) + \alpha\varphi_1(x) + \ldots + \alpha^n\varphi_n(x) + \ldots \qquad x \in [\alpha, \beta] \tag{106}$$

where φ_n is successively defined from the positive function $g(x)$ and the arbitrary function $\varphi(x)$ by

$$\varphi_{n+1}(x) = \int_\alpha^x dx \int_\beta^x g(x)\,\varphi_n(x)\,dx. \tag{107}$$

He obtained the convergence criterion by remarking that the series (106) is of the form $\varphi + \alpha s$ where s arises from solving, by successive approximation, the boundary value problem

$$\frac{d^2 s}{dx^2} - \alpha g s = g\varphi \tag{108}$$

with

$$s = 0 \quad \text{for} \quad x = \alpha, \quad \frac{ds}{dx} = 0 \quad \text{for} \quad x = \beta. \tag{109}$$

This application of the method of successive approximation to the case where the boundary values are given at two points is an improvement over the formulas from [1830/31] (*cf.* §§ 33 and 34).

Finally Sturm-Liouville theory provided the background for LIOUVILLE's short paper on spectral theory of general integral operators with symmetric kernel [1845], though this time his inspiration came from some formulas for Lamé functions (*cf.* [LÜTZEN 1981]). LIOUVILLE did remark the resemblance between the results he obtained for the integral operators and those found in Sturm-Liouville theory but it was left to HILBERT [1904/10] to unveil the profound connection between the two theories provided by the Green's function.

Apart from these few short papers LIOUVILLE stopped publishing on second-order Sturm-Liouville theory in 1837. His further contributions to spectral theory provided generalizations of Sturm-Liouville theory to different types of problems, particularly to higher-order differential equations, to be discussed in the next chapter.

VII. Liouville's Generalization of Sturm-Liouville Theory to Higher-Order Equations

52.

"Le principe sur lequel reposent les théorèmes que je développe n'a jamais, si je ne me trompe, été employé dans l'analyse et il ne me paraît pas susceptible de s'étendre à d'autres équations différentielles." [STURM 1836a, p. 107]

[44] The proof has the same weaknesses as the original deduction and there is no proof of convergence.

LIOUVILLE's greatest effort in Sturm-Liouville theory was an only partially successful attempt to disprove the last part of STURM's conjecture by generalizing his and STURM's theory to other types of equations. Thus in LIOUVILLE's notebooks [LIOUVILLE Ms.] the first recorded note inspired by STURM's work, dating from April or May 1835 [Ms. 3615 (4), pp. 79v–80r], dealt with the equation

$$\frac{du}{dt}=\frac{d^3u}{dx^3}, \tag{110}$$

"qui nous a éte proposé par Mr. POISSON" seven years earlier "comme échappant à plusieurs des procédés connus" [LIOUVILLE 1828, III, § 7][45]. He presented his new results to the Academy the following year (November 14^{th}, 1836) and had them published in [1837a]. In some of his notes (*e.g.* [Ms. 3615 (5)]) LIOUVILLE attempted to apply STURM's general methods to investigate the separated equation

$$V^{(3)}(x)+rV(x)=0 \quad \text{for} \quad x\in[0,1] \tag{111}$$

with the boundary conditions

$$V(x)=V'(x)=0 \quad \text{for} \quad x=0, \tag{112}$$

$$V(x)=0 \quad \text{for} \quad x=1. \tag{113}$$

However, in the published account [1837a] he based his arguments on the explicit expression for the solution of (111) and (112):

$$V=\frac{1}{3\varrho^2}\left(e^{-x\varrho}+\mu e^{-\mu x\varrho}+\mu^2 e^{-\mu^2 x\varrho}\right),$$

where $\varrho^3=r$ and $\mu^3=1$, $\mu\neq 1$. He showed that all of STURM's theorems were valid for the boundary-value problem (111)–(113) except for the orthogonality (60). This relation was replaced by the biorthogonality

$$\int_\alpha^\beta V_m(x)\,U_n(x)\,dx=0 \quad \text{for} \quad m\neq n, \tag{114}$$

where $\alpha=0$, $\beta=1$ and U_n is the n^{th} eigenfunction of the related boundary-value problem

$$U^{(3)}(x)-rU(x)=0 \text{ for } x\in\,]0,1[, \tag{115}$$

$$U(x)=0 \text{ for } x=0, \tag{116}$$

$$U(x)=U'(x)=0 \text{ for } x=1. \tag{117}$$

[45] The three unpublished "Mémoires sur le calcul aux différences partielles", presented to the Academy on December 1^{st}, 1828, are the earliest purely mathematical papers written by LIOUVILLE. With a theorem due to PARSEVAL on the term-by-term product of two infinite series as his starting point the young LIOUVILLE herein presented a peculiar way to express solutions of partial differential equations as integrals. Both aim and method differed from those employed in Sturm-Liouville theory. The problem of 1828 is not a boundary-value problem and when applied to (110) the variables are separated by considering a solution of the form $e^{bx}F(t)$ and not of the form $e^{-rt}V(x)$ as in [1837a].

Therefore the Fourier series of a function $f(x)$ has the altered form

$$F(x) = \sum_{n=1}^{\infty} \frac{V_n \int\limits_{\alpha}^{\beta} f(x)\, U_n(x)\, dx}{\int\limits_{\alpha}^{\beta} V_n(x)\, U_n(x)\, dx}. \tag{118}$$

LIOUVILLE showed, with a proof similar to the one given in [1836d], that if the Fourier series converges it has the value $F(x) = f(x)$, but he did not prove convergence (see Appendix).

53. The boundary-value problem (115)–(117) is today called the adjoint of the original problem (111)–(113). At first LIOUVILLE did not appreciate the profound difference between self-adjoint problems (*e.g.* (1)–(3)) and non-self-adjoint problems like (111)–(113). For example early in 1835 he apparently tried to prove ordinary orthogonality for the third-order differential equation (111)[46]. He seems to have become aware of the importance of the adjoint equation during the winter of 1837–1838, when he taught his first course at the Collège de France as a substitute for BIOT. This "Cours de Physique générale et Mathématique" was mainly a course on differential equations and covered for example LAGRANGE'S methods of integration using the adjoint equation [LIOUVILLE Ms. 3615 (5), pp. 42v–54r]. It stimulated LIOUVILLE to take up his researches from 1835 on higher-order Sturm-Liouville theory, this time for equations with variable coefficients.

In [1838c] LIOUVILLE published the investigations of the most general of these equations:

$$(K(L \dots (M(NV')')' \dots)')' + rV = 0 \text{ for } x \in [\alpha, \beta] \tag{119}$$

with the boundary conditions:

$$V = A,\, NV' = B, \dots, K(L \dots (M(NV')')' \dots)' = D \text{ for } x = \alpha \tag{120}$$

and

$$aU + b(NV') + \dots + cK(L \dots (M(NV')' \dots)' = 0 \text{ for } x = \beta, \tag{121}$$

where $K(x), L(x), \dots, M(x), N(x) > 0$ for $x \in [\alpha, \beta]$ and $A, B, \dots D, a, b, \dots, c > 0$.

He found that the biorthogonality (114) would hold if U_n is the n^{th} eigen-

[46] In the very first note in the first notebook [Ms. 3615 (1)] LIOUVILLE wrote $\frac{d^3V}{dx^3} + rV = 0$, $\frac{d^3V'}{dx^3} + r'V' = 0$, and tried without success to find an expression for $(r - r') \int VV'$. Next to this calculation he succesfully carried out the corresponding calculation for $\frac{d^4V}{dx^4} + rV = 0$. Most of the notes in [Ms. 3615 (1)] seem to stem from around 1830, but the above mentioned note is probably from a later date.

function of the adjoint problem:

$$(N(M \ldots (L(KU')')' \ldots)')' + (-1)^{\mu} rU = 0 \text{ for } x \in [\alpha, \beta], \tag{122}$$

$$DU - \ldots + (-1)^{\mu-2} B(M \ldots (KU')' \ldots) + (-1)^{\mu-1} AN(M \ldots (KU')' \ldots)' = 0 \quad \text{for } x = \alpha, \tag{123}$$

$$U = c, \ldots, N(M \ldots (L(KU')')' \ldots)' = (-1)^{\mu-1} a \text{ for } x = \beta, \tag{124}$$

where μ is the order of the differential equation[47]. Thus LIOUVILLE extended LAGRANGE'S concept of an adjoint (conjugué) differential equation to include the boundary values as well, though he did not introduce a term for the adjoint boundary values. During the spring of 1838 he found many other remarkable results pertaining to the general boundary-value problem (119)–(121) some of which he presented to his students at the Collège de France (*cf.* [LIOUVILLE 1838c, Introduction]).

54. One can follow LIOUVILLE'S successive progress with these questions in the approximately 200 pages of disorganized notes jotted down in his notebooks [Ms. 3615 (5), Ms. 3616 (2)] from February 1838 and later. In order to facilitate the understanding of the questions considered pell mell in the notes I shall discuss the ideas in their logical order sacrificing partly the chronology.

LIOUVILLE'S goal was still to expand arbitrary functions in series of eigenfunctions of the boundary value problem. In order to find the Fourier coefficients he needed the biorthogonality (114). In some of his notes, for example [Ms. 3616 (2)] from February 19th, 1838, he experimented with suitable boundary conditions and corresponding adjoint conditions, but the problem does not seem to have caused him much trouble. Next he wished Lemma II (§ 47) to hold so that he could show that the Fourier series (118) if convergent has the value $F(x) = f(x)$. The proof of this lemma could be taken over from [1836d] if only the problem would have an infinity of (positive) eigenvalues and if STURM'S oscillation theorem would hold, *i.e.* if V_n has in $]\alpha, \beta[$ exactly $n - 1$ roots, all of which are simple. The existence of infinitely many positive eigenvalues is not dealt with in the notes. In the published paper [1838c, § 7] LIOUVILLE based his proof on a comparison with the equation with constant coefficients for which he could prove the theorem as he had done in (1837a). However, since LIOUVILLE had nothing like STURM'S comparison theorem (§ 24) at his disposal, the proof necessarily differed from STURM'S, and it is in fact insufficient.

55. The proof of STURM'S oscillation theorem caused LIOUVILLE the greatest troubles. The vast majority of the notes from the first half of 1838 are related to this problem. He indicated several ways to prove that the solutions of the equation with suitable boundary values at α (for example (120)) had no multiple roots. Some proofs are wrong, for example the second given on February 22nd, 1838

[47] LIOUVILLE proved, both for equations with constant coefficients [1837a, p. 102] and with variable coefficients [1838c, pp. 604–606] that the adjoint problem has the same eigenvalues as the original problem.

[Ms. 3616 (2), pp. 21r, 21v], whereas the first proof found that day [Ms. 3616 (2), p. 18v] and repeated on April 14th, 1838 is essentially right.

This proof, which he published in [1838c, § 9], amounted to a rather simple accounting for successive roots of the quantities $V, NV', \ldots$, which are supposed to be positive for $x = \alpha$ (120). Such an investigation shows that no two of these quantities can vanish simultaneously; in particular $V = 0$ and $V' = 0$ can have no roots in common. As LIOUVILLE indicated in a note from April or May, 1848 [Ms. 3616 (2), pp. 68v, 77r] and published in [1838c, § 10–11] such an argument also shows that the quantity on the left-hand side of (121) has a root between two consecutive roots of V. Therefore, following LIOUVILLE, we shall concentrate on problems where the boundary condition in β is of the form

$$V_r(x) = 0 \quad \text{for} \quad x = \beta.\text{[48]} \tag{125}$$

Following STURM (see § 20), LIOUVILLE argued as follows [1838c, § 14]: Since V_r has no multiple roots, the number of roots of $V_r(x)$ in $]\alpha, \beta[$ can change only if a root passes one of the two end points, *i.e.* when r has a value for which $V_r(\alpha) = 0$ or $V_r(\beta) = 0$. Now LIOUVILLE always imposed boundary conditions on V_r at α, which allow no root $V_r(\alpha) = 0$ (*e.g.* conditions (120)). Therefore it is clear that if V_n and V_{n+1} are two consecutive eigenfunctions of (119), (120), and (125), the numbers of their roots in $]\alpha, \beta[$ differ by at most one. Since $V_1(x)$ can easily be seen to have no roots in $]\alpha, \beta[$, this implies that V_n has at most $n - 1$ roots in $]\alpha, \beta[$.

56. Thus we are led to the problem which troubled LIOUVILLE more than any of the other problems: to show that V_n has precisely $n - 1$ roots. In his notebooks one can distinguish at least three different methods of proof.

LIOUVILLE called the first method "la méthode Sturmienne" [Ms. 3616 (2), p. 19v] because it was adapted from the method used in [STURM 1836a]. LIOUVILLE had already applied this method with success in July 1836 [Ms. 3615 (5), pp. 18v–20r] to an alternative treatment of the equation (111) with constant coefficients. Therefore it is natural that in his very first note on the equation with variable coefficients from February 1838 [Ms. 3615 (5), pp. 30v–35r] he tried this method again. Recall (§ 21, 22) that STURM'S idea was to show that the roots of $V_r(x)$ decreased with increasing r. This fact would obviously complete LIOUVILLE'S proof since it implies that V_{n+1} has one more root (and not one less) in $]\alpha, \beta[$ than V_n. STURM had proved the decrease of the roots by showing that $\delta V_r(x) \dfrac{dV_r(x)}{dx} > 0$ when $V(x) = 0$, which is in turn implied by

$$\theta = \delta V \frac{dV}{dx} - V \frac{d\,\delta V}{dx} > 0, \tag{126}$$

[48] Consult [LIOUVILLE 1838c] for the treatment of the more general boundary condition.

or corresponding to (49)[49]

$$\delta\left(\frac{V'}{V}\right) > 0. \tag{127}$$

Therefore LIOUVILLE started out investigating the quantity θ defined by (126). After some pages of calculations he believed he had established the inequality (126) and wrote enthusiastically:

> "Il est bien prouvé par ce qui précède que la théorie de Mr Sturm s'étend aux équations linéaires de tous les ordres par la considération des équations conjuguées [adjoint] de Lagrange dont j'ai parlé dans mes leçons au collège de France. Ce résultat que Sturm, d'après les premières lignes de son Mémoire [quoted in § 52] doit regarder comme tout à fait inattendu, est au moins très remarquable. Mais il importe de le généraliser autant que possible[50]". [Ms. 3615 (5), p. 32r]

However his continued calculations led to properties which he found strange and after 10 pages he finally arrived at a contradiction. He felt so uncomfortable with this result that he began to reflect upon the continuous dependence of $V_r(x)$ on r, which he everywhere else considered as self-evident:

> "L'explication serait-elle dans le défaut de continuité de la fonction [V] par rapport à r. Cela parait absurde aussi. Toutefois je n'imagine pas d'autre cause. Les exemples y sont favorables." [Ms. 3615 (5), p. 35r]

In the end he saw that the beginning of the note contained a simple miscalculation which invalidated the whole argument.

57. LIOUVILLE did not lose courage after this initial failure. On the contrary, it seems to have become almost an obsession for him to make the Sturmian method work. Thus during February, March and April he made over a dozen mostly fruitless attempts to apply the method to different higher-order equations with variable coefficients (*cf.* [Ms. 3615 (5), Ms. 3616 (2)]). I shall summarize the one partly successful application of the Sturmian method in order to show how far LIOUVILLE got with this approach. In this calculation he showed that the first root of $V_r(x) = 0$ greater than α decreases with r when $V_r(x)$ is a solution of the equation

$$V_r^{(3)}(x) + g(x)\, r V_r(x) = 0 \tag{128}$$

satisfying suitable boundary conditions at $x = \alpha$. He proved it on February 10$^{\text{th}}$, 1838, in the case where $V_r(\alpha) = 0$ [3615 (5), pp. 38v–40r] and one month later [3616 (2), pp. 58v–61r] with the general boundary-value condition

$$V_r''(x) = a V_r'(x) - b V_r(x) \quad \text{for } x = \alpha, \tag{129}$$

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[49] In LIOUVILLE's mémoire $K(x)$ does not depend on r.

[50] In this note LIOUVILLE had only treated the equation

$$V^{(3)}(x) + G(r, x)\, V(x) = 0.$$

where $a, b > 0$, and $V_r(\alpha) \neq 0$ say $V_r(\alpha) = c > 0$. After having arrived at the main lines of a proof in a messy way on March 10th, 1838, he immediately drew up a tidy version:

> "Revenons sur la méthode précédente que nous avions si souvent essayé d'employer et qui se trouve obtenir tout à coup un succès inattendu." [Ms. 3616 (2), p. 59v]

He combined (128) with the adjoint equation

$$U_r^{(3)}(x) - g(x)\, rU_r(x) = 0 \tag{130}$$

to find that

$$U_r(x)\, V_r''(x) - U_r'(x)\, V_r'(x) + U_r''(x)\, V_r(x) = \text{const.} \tag{131}$$

Under the conditions imposed on U_r:

$$U_r'(x) = aU(x), \qquad U_r''(x) = bU(x) \text{ for } x = \alpha, \tag{132}$$

the constant in (131) is zero. By differentiation of (128) he obtained

$$\delta V_r'''(x) + g(x)\, r\, \delta V_r(x) + g(x)\, V_r(x)\, \delta r = 0, \tag{133}$$

which combined with (130) yields

$$U_r(x)\, \delta V_r''(x) - U_r'(x)\, \delta V_r'(x) + U_r''(x)\, \delta V_r(x) = -\delta r \int_\alpha^x g(x)\, U_r(x)\, V_r(x)\, dx. \tag{134}$$

LIOUVILLE finally combined (131) and (134) and obtained

$$U_r(x)\, \theta_r'(x) - \theta_r(x)\, U_r'(x) = \delta r V_r(x) \int_\alpha^\beta g(x)\, U_r(x)\, V_r(x)\, dx \tag{135}$$

or

$$\left(\frac{\theta_r(x)}{U_r(x)}\right)' = \frac{V_r(x)\, \delta r}{(U_r(x))^2} \int_\alpha^x g(x)\, U_r(x)\, V_r(x)\, dx. \tag{136}$$

If now we choose $U(\alpha) > 0$ and $r > 0$, it is easily seen that $U(x) > 0$ and $U'(x) > 0$ for $x \in [\alpha, \beta]$. If further $\delta r > 0$, (136) implies that $\theta_r(x)$ increases with x in the interval $[\alpha, x_1(r)]$ where $x_1(r)$ is the first root of $V_r(x) = 0$. Since $\theta_r(\alpha) = 0$, we have

$$\theta_r(x_1(r)) > 0; \tag{137}$$

hence by the analysis above (126), $x_1(r)$ decreases when r increases.

LIOUVILLE tried to apply the method to the second root as well, as he had done for the equation (111) in July of 1836 [Ms. 3615 (5), p. 19v], but this time he could not carry out the generalization. Thus the "succès inattendu" was very limited[51].

58. In spite of these discouraging results LIOUVILLE apparently continued to have confidence in the Sturmian method. It was clearly the method he liked the best. Thus he continued to use it after he had found the more successful approach (February 18th) which he later chose to publish in [1838c][52]. He even returned to

[51] On February 22nd, 1838, LIOUVILLE [Ms. 3616 (2), p. 22–23r] believed that he had another success with the Sturmian method. He considered the equation

$$(K(x)\,(L(x)\,(V'(x))')' + g(x)\,rV(x) = 0, \tag{138}$$

where

$$K(x), L(x), g(x) > 0 \quad \text{for} \quad x \in [\alpha, \beta].$$

By successive differentiation he obtained the differential equation

$$(L(K(L\theta)')')' = gr\theta \tag{139}$$

which he integrated three times to give

$$\theta(x) = F(x) + \frac{r}{L(x)} \int_\alpha^x \frac{dx}{K(x)} \int_\alpha^x \frac{dx}{L(x)} \int_\alpha^x g(x)\,\theta(x)\,dx. \tag{140}$$

The function $F(x)$ depends on the initial conditions at $x = \alpha$. LIOUVILLE chose the conditions:

$$LV' = a(r)\,V, \quad K(LV')' = b(r)\,LV' \quad \text{for} \quad x = \alpha \tag{141}$$

and showed that if

$$a, b > 0, \quad \frac{da}{dr} < 0, \quad \frac{db}{dr} < 0, \tag{142}$$

then $F(x) > 0$. In that case (140) implies that $\theta > 0$ which was the desired result.

However, his derivation of (139) contains an error, for in the beginning of the proof he chose V, V_1 to be two solutions to (138), with the same value r of the parameter. At the end of the proof, however, he set $V_1 = V + \delta V$ where $\delta V = \frac{dV}{dr}\,\delta r$. In that case V_1 in fact satisfies (138) for the value $r + \delta r$ instead of the value r of the parameter.

LIOUVILLE did not cross out this proof as he usually did with erroneous calculations in his notebooks. That indicates that he did not discover the flaw. Yet his continuous search for alternative derivations shows that he was not content with the result. The reason can be that the proof does not work in the most interesting case of constant boundary conditions. In fact if $\frac{da}{dr} = \frac{db}{dr} = 0$ LIOUVILLE would get $F(x) \equiv 0$, and so nothing could be concluded from (140.)

[52] LIOUVILLE also tried a related method in notes from February 8th–10th, March 8th, and August 5th, 1838 [Ms. 3615 (5), 3616 (2)]. It consisted in studying solutions V of (138) of the form

$$V = UW' - WU',$$

where U and W are solutions to the adjoint equation with suitable boundary conditions. The investigations end without result.

the method in two short notes as late as 1843/44 and 1845/46 [Ms. 3617 (4), Ms. 3618 (3)].

Before I discuss the published approach I shall mention a third method, which is in a way a geometrical version of the Sturmian method. This method, thought out on February 19th, 1838 and developed in three notes during the following few days (February 19th–22nd [Ms. 3616 (2), pp. 3v–7r, 11v–12r], consisted in a geometric investigation of the curves $V(r, x) = 0$. Recall that the main problem is to show it impossible that a root of $V(r, x)$ in $]\alpha, \beta[$ can leave the interval at β for increasing r. LIOUVILLE neatly argued that the only way that could happen was if a branch of the curve $V(r, x) = 0$ had the form (Fig. 4):

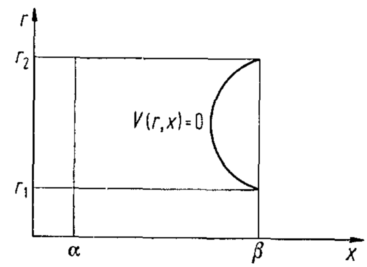

Fig. 4

But that would mean that $V(r_1, x)$ and $V(r_2, x)$ has the same number of roots in $]\alpha, \beta[$, and LIOUVILLE considered that to be absurd. LIOUVILLE found this geometric argument "très claire mais difficile à rédiger" [Ms. 3616 (2), Feb. 19th], but obviously the "absurdity" found at the end of the proof only arises when STURM'S oscillation theorem is used. Thus he argued in a circle.

59. One day before LIOUVILLE conceived the geometrical argument he had thought out the central idea of the proof of STURM'S oscillation theorem that he eventually published in [1838c]. The inspiration clearly came from STURM'S proposition N (§ 28), more specifically from STURM'S alternative version of LIOUVILLE'S proof of the theorem (§ 50). Thus his aim was to prove

Lemma III.

$$A_m r_m V_m + \dots + A_n r_n V_n \tag{105}$$

has at least as many roots in $]\alpha, \beta[$ *as*

$$\psi = A_m V_m + \dots + A_n V_n{}^{53}. \tag{61}$$

[53] At first LIOUVILLE had $m = 1$ but when he began his investigations of (146) he applied his results in the general case without comment.

In the note of February 18th, 1838 [Ms. 3616 (2), pp. 1r, v] he gave the following rigorous proof of this theorem when V_n, $n = 1, 2, \dots$ are the eigenfunctions of the equation (128) $(g(x) > 0)$ with the boundary conditions

$$V' = aV,\ V'' = bV \quad \text{for} \quad x = \alpha, \tag{143}$$

$$V'' + AV' + BV = 0 \quad \text{for} \quad x = \beta, \tag{144}$$

where $a, b, A, B > 0$.

Since ψ satisfies (144), either

1) both $\psi(\beta)$ and $\psi'(\beta)$ are of the opposite sign of $\psi''(\beta)$, or
2) only one of $\psi(\beta)$ and $\psi'(\beta)$ is of the opposite sign of $\psi''(\beta)$.

In the first case LIOUVILLE argued as follows (Fig. 5):

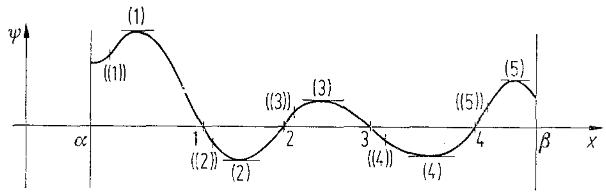

Fig. 5

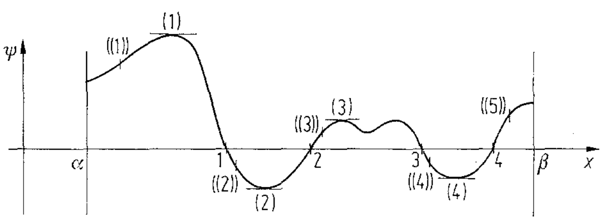

Fig. 6

Let μ denote the number of roots of ψ in $]\alpha, \beta[$. Then there are at least μ roots of ψ' in $]\alpha, \beta[$, namely one between α and the first root of ψ, and at least one between each of the consecutive roots of ψ according to ROLLE's theorem. Next he showed that ψ'' has at least $\mu + 1$ roots in $]\alpha, \beta[$, namely the $\mu - 1$ roots secured by ROLLE's theorem plus one to the left of the first root of ψ' and one to the right of the last root of ψ'. A similar proof applies to the second case (Fig. 6). Therefore, according to ROLLE's theorem, ψ''' has at least μ roots in $]\alpha, \beta[$ and by the differential equation (128) we have

$$\psi'''(x) = -g(x)\,(A_m r_m V_m + \dots + A_n r_n V_n)\,.$$

This establishes the theorem. As in [STURM 1836b] and [LIOUVILLE 1836e], LIOUVILLE concluded that V_n has at least as many roots in $]\alpha, \beta[$ as ψ, which has in turn at least as many roots as V_m.

60. Let us follow LIOUVILLE [1838c] and introduce the notation

$$p = \text{number of roots of } V_m,$$
$$q = \text{number of roots of } V_n,$$
$$\mu = \text{number of roots of } \psi.$$

Then the above theorem states that

$$p \leqq \mu \leqq q. \tag{145}$$

In the next note from the same day LIOUVILLE considered the special case where $m = n - 1$ and $A_{n-1} = V_n(\alpha)$, $A_n = -V_{n-1}(\alpha)$[54]:

$$\psi(x) = V_n(\alpha)\, V_{n-1}(x) - V_{n-1}(\alpha)\, V_n(x). \tag{146}$$

According to (143) this particular function has $\psi''(\alpha) = 0$ and therefore ψ''' has an extra root between α and the first root of ψ'' in $]\alpha, \beta[$. Hence

$$\nu(x) = V_n(\alpha)\, r_{n-1} V_{n-1}(x) - V_{n-1}(\alpha)\, r_n V_n(x)$$

has at least $\mu + 1$ roots. The second inequality of (145) applied to $\nu(x)$ thus yields $\mu + 1 \leqq q$ and the first inequality applied to ψ yields $p \leqq \mu$. Therefore $p + 1 \leqq q$. Since V_1 has no roots in $]\alpha, \beta[$ V_n has at least $n - 1$ roots in $]\alpha, \beta[$. On the other hand LIOUVILLE knew that V_n has at most $n - 1$ roots in $]\alpha, \beta[$, and he had completed the desired proof of STURM's oscillation theorem.

61. Four days after having designed the above proof LIOUVILLE generalized it to equation (138) with variable coefficients and with the boundary conditions (143)–(144) and $V(\alpha) = 1$ [Ms. 3616 (2), Feb. 22nd]. In this general case LIOUVILLE counted roots of U, U' and $(LU')'$ by reasoning as he had done a few days earlier. However this time there is a flaw in the argument because one cannot conclude that $(LU')'$ has a root between α and the first root of U'. At first LIOUVILLE did not see the mistake, but later he seems to have discovered it. At least he altered the boundary conditions in the published version of the proof [1838c] from the conditions (143), (144) and $V(\alpha) = 1$, which made V, V', V'' positive at α to the conditions (120) and (121), which made V, V', $(NV')'$ positive at α. Under the new conditions LIOUVILLE's argument of February 22nd, 1838, is correct. Before publishing the mémoire [1838c] LIOUVILLE showed parts of it to STURM who suggested that the above simple proof of Lemma III by ROLLE's theorem should be replaced by a more elegant argument resting on a generalization of FOURIER's theorem on the number of roots of a function [LIOUVILLE 1838c § 17]. With this revision the proof of February 18th–22nd of STURM's oscillation theorem was published in [LIOUVILLE 1838c, § 17].

62. The only published result of LIOUVILLE's hard work during February, March, and April of 1838 in the field of higher-order Sturm-Liouville theory was the paper "Premier Mémoire sur la Théorie des Équations différentielles linéaires et sur le développement des Fonctions en séries" from December [1838c] and a

[54] LIOUVILLE wrongly wrote $A_n = -V_n(\alpha)$.

brief note published in May that year [1838a] announcing the theory later to appear. Let me briefly summarize the contents of [LIOUVILLE 1838c]. First the solutions of (119) and (120) are found by successive approximation, and it is shown, by comparison with the equation with constant coefficients that (119)–(121) have infinitely many positive eigenvalues. Then it is shown that V_r has only simple roots, and the crucial proof of STURM'S oscillation theorem is given. Next comes the argument for Lemma II (see § 47) and the introduction of the dual problem (122)–(124) followed by the proof of the biorthogonality (114). Finally it is proved that the Fourier series (118) of f has the value f if it converges, and LIOUVILLE argues that all eigenvalues are positive.

63. The title of the mémoire "Premier Mémoire ..." indicates that LIOUVILLE had planned a more comprehensive treatment of the generalized Sturm-Liouville theory. A gigantic project appears from the announcement of his new results in [1838a]:

> "Dès que l'abondance des matières me permettra de prendre dans ce Journal une place suffisante, je m'empresserai d'y publier le Mémoire dont je viens d'indiquer les principeaux résultats, et qui n'est du reste à mes yeux qu'une petite partie d'un très long travail que j'ai entrepris sur la théorie générale des équations différentielles et sur le développement des fonctions en séries". [LIOUVILLE 1838a, p. 256]

In some of LIOUVILLE's notes one can even get some idea of his general plan for this larger project. For example on July 3rd, 1838, he began to reflect about the presentation of the material and remarked that in the first mémoire the principal aim must be very clear; therefore unnecessary difficulties such as problematic boundary conditions should be avoided. Here he aluded to the boundary values mentioned in a note from March 31st, where he suggested setting U or U' equal to zero at three different points. Such subtleties were apparently postponed to the later papers in the sequence. On August 5th–10th, 1838, LIOUVILLE drew up a few more points for his programme of research:

> "Nos recherches sur les propriétés générales des intégrales des Équations linéaires prendent tous les jours plus d'importance, il est nécessaire de s'occuper avec soin
>
> 1°. de la convergence des séries auxquelles nous conduit notre analyse
>
> 2°. de l'étude des Équations simultaneés &c.&c.
>
> $$\frac{dU}{dx} + rV = 0$$
>
> $$\frac{dV}{dx} - rU = 0.$$
>
> [LIOUVILLE Ms. 3616 (2), p. 87v]

The question of convergence of the Fourier series (118), mentioned in 1°, was left open in both [1837a] and [1838c]. LIOUVILLE had already stressed its importance in a note from February 22nd, 1838 saying: "Il est tout à fait indispensable de s'occuper très sérieusement de la convergence de ces séries" [Ms. 3616 (2), p. 25r], but he left only one (or perhaps two) inconclusive notes on the question from February 23rd and March 5th of 1838 [Ms. 3616 (2), pp. 32, 36]. In fact, U. Haagerup has pointed out that even for the boundary-value problem (111)–(113) with constant coefficients the Fourier series converges only for very special functions f (*cf.* Appendix).

The theory of simultaneous equations mentioned in 2° was studied in several particular cases in notes from March 11th, 1838 (two second-order equations), August 5th–10th, 1838 (two and three first-order equations) [Ms. 3616 (2), pp. 51v, 87v, 88] and one as late as March 1839 (three second-order equatlons) [Ms. 3616 (5), pp. 8v–9v].

64. On July 3rd, 1838, Liouville planned another "grande extension" [3616 (2), p. 80v]. In the note he seems to suggest that the expressions NV', $M(NV')'$ ... of (119) and (120) be replaced by

$$\begin{aligned} \nabla V &= MV' + NV, \\ \nabla^2 V &= M_1(\nabla V)' + N_1 \nabla V - P_1 V, \\ \nabla^3 V &= M_2(\nabla^2 U)' + N_2 \nabla^2 U - P_2 \nabla U \\ &\vdots \end{aligned}$$

Half a year later, on January 3rd, 1839, he began to study Sturm-Liouville theory for complex functions, and he intended to "suivre cette théorie qui peut devenir très importante, introduire dans nos fonctions une paramètre; voir les propriétés correspondantes des fonctions et de leurs intégrales dans un contour &c.&c". [Ms. 3616 (3), p. 6r].

Finally, a note probably from 1842 [Ms. 3617 (2), pp. 66v–67r] shows that Liouville continued to be occupied with Sturm-Liouville theory many years after 1838. This note contains a few results on "les propriétés des puissances ou des produits des fonctions V de Mr. Sturm". Again he put off further investigations though "tout cela parait mériter d'être étudié avec soin". (Liouville's underlining).

65. In spite of his extensive research programme for the generalized Sturm-Liouville theory Liouville never composed the subsequent mémoires in the series he had planned in 1838. What was the reason his interruption of the series? An explanation cannot be found in a new absorbing interest taking all of Liouville's time; he did turn to many other problems of analysis, algebra, mechanics and celestial mechanics but from his published papers and his notebooks it appears as if none of these interests was so strong that it could have beaten him off the track[55]. A more likely explanation will present itself after a view of the subsequent development of Sturm-Liouville theory. Such a view will be given in the concluding section.

[55] To judge from his notebooks potential theory, particularly for ellipsoids, was the next problem to occupy him as much as Sturm-Liouville theory had done. But this new problem did not really catch his attention until 1841 or 1842.

VIII. Concluding Remarks

66. The subsequent development of the Sturm-Liouville theory can be divided roughly into two mutually interacting categories: generalizations and rigorizations. The following brief summary of the late 19th century and early 20th century advances in these two directions and makes no claim to be complete. More details and references to primary sources can be found in the relevant articles of the "Encyklopädie der Mathematischen Wissenschaften" [Bôcher 1899/1916] and [Hilb & Szász 1922].

Around 1880 the problem of vibrating rods and plates led Lord Rayleigh [1877], G. Kirchhoff [1879] and others to develop theorems similar to Sturm's for higher-order boundary value problems. At the same time the Sturm-Liouville theory of singular differential equations, where for example $k(x)$ in (1) has zeroes in the interval $[\alpha, \beta]$, was studied first in special cases such as the Bessel equation (16) (*e.g.* [Schläfli 1876]) and later in more generality by Bôcher and others. A third kind of generalization was undertaken by F. Klein [1881], who studied ordinary differential equations with several parameters. By allowing the solutions to have prescribed types of infinities at the boundary of the intervals considered he was able to adapt his theory to the Lamé functions. Thereby he combined the two hitherto entirely separate theories of boundary-value problems and polynomial solutions of ordinary differential equations. Finally Poincaré initiated spectral theory of partial differential operators with his study of the Laplace operator [1894].

67. The rigorization of Sturm-Liouville theory took place on two levels. To adjoin to Sturm's and Liouville's essentially correct arguments the lacking proofs of continuity, differentiability, and uniformity constituted the simpler task. It was undertaken piecewise in several papers from the 1890's and systematically carried through for the Sturmian theorems by Bôcher [1898, 1899]. Other mathematicians took up the much more difficult problem of finding a rigorous replacement for Liouville's essentially wrong proof that a Fourier series converges to the function that gives rise to it. H. Heine [1880] and U. Dini [1880] applied Cauchy's theorem of residues to this problem and Poincaré [1894] developed this idea in his proof of a general theorem on expansion in eigenfunctions for the Laplace operator. Poincaré also used complex function theory. An improvement of this method was carried over to the Sturm-Liouville problem by Stekloff [1898] to get the first rigorous proof that a twice differentiable function satisfying the boundary conditions (2) and (3) could be expanded in a Fourier series. By combining the method of Poincaré and Stekloff, based on the theory of functions, with Liouville's original second proof of convergence [1837e] (see § 46) A. Kneser [1904] succeeded in proving the expansion theorem for any piecewise continuous function of bounded variation (though taking the values $\frac{1}{2}(f(x+0) + f(x-0))$ at points of discontinuity). He here provided a parallel to the classical result of Dirichlet [1829] for trigonometric Fourier series.

68. An entirely new approach to Sturm-Liouville theory was opened up by the rise of the general theory of integral equations, beginning at the end of the

1890's. The correspondence between spectral theory for differential and integral equations is established by the Green's function. If r is not an eigenvalue of (1)–(3) and $\varphi(x) > 0$ in $[\alpha, \beta]$, there is a solution u of

$$(k(x)\, u'(x))' + (g(x)\, r - l(x))\, u(x) = -\varphi(x) \text{ for } x \in\,]\alpha, \beta[\tag{149}$$

satisfying the boundary conditions (2) and (3). This solution can be expressed in the form

$$u(x) = \int_{\alpha}^{\beta} G(r, x, \xi)\, \varphi(\xi)\, d\xi. \tag{150}$$

$G(r, x, \xi)$ is called Green's "function". If r_n is an eigenvalue to the boundary-value problem (1)–(3) and if V_n is the corresponding eigenfunction, it follows from (149) and (150) that

$$V_n = (r_n - r) \int_{\alpha}^{\beta} g(\xi)\, G(r, x, \xi)\, V_n(\xi)\, d\xi.$$

Therefore the eigenvalues and eigenfunctions of (1)–(3) correspond to eigenvalues and eigenfunctions of the integral operator

$$\varphi \to \int_{\alpha}^{\beta} g(\xi)\, G(r, x, \xi)\, \varphi(\xi). \tag{151}$$

This idea is found in the work of POINCARÉ [1894] on the Laplace equation. However its full importance was not revealed until HILBERT [1904/1910] and SCHMIDT [1907] had proved that any continuous function of the form

$$f(x) = \int_{\alpha}^{\beta} g(\xi)\, G(r, x, \xi)\, \varphi(\xi)$$

can be expanded in a Fourier series of eigenfunctions to the integral operator (151). When applied to the Sturm-Liouville problem this theorem almost immediately gives the expansion theorem for twice differentiable functions which satisfies the boundary conditions. Thus, as long as mainly pointwise convergence was considered, HILBERT and SCHMIDT's method could not compete with KNESER'S, but later in the 20th century when mathematicians became interested in L^2-convergence the new methods proved more valuable. Nowadays the Sturm-Liouville theory is treated in close connection to the general theory of operators in Hilbert space which developed from HILBERT and SCHMIDT'S investigations.

69. Of the generalizations listed in § 67 only the study of higher-order equations belongs to LIOUVILLE'S research programme. He had also touched upon spectral theory for the Laplacian in his remark of 1830/31 about heat conduction in bodies of more than one dimension, but he had given it up because he could not even find the stationary temperature distribution explicitly (*cf.* note 27). A further comparison of LIOUVILLE'S research programme with the late 19th century development is interesting because it may throw some light on LIOUVILLE'S failure to carry out his plans.

What distinguishes LIOUVILLE from his successors is not so much the difference between their problems as their different motives for studying them. The generalizations of the Sturm-Liouville theory obtained in the late 19th century were motivated by physics or by a desire to link up different trends of analysis. LIOUVILLE, on the other hand, appears in this case as a blind generalizer who wanted to extend his and his friend's new theory as far as possible just for the sake of generality. Of course he may have had other motives, which he did not reveal to his readers or to his notebooks, but at least in the case of third-order equations his only motivation seems to have been POISSON'S remark about the difficulty of the problem (§ 52). Thus lack of genuine motivation may have been a reason why LIOUVILLE lost interest in the problems.

The missing physical inspiration had a serious consequence, for in analysis, and particularly in the type of analysis cultivated in early 19th century France, theorems and methods had generally been suggested by physical reality. For example, STURM and LIOUVILLE'S research on second-order differential equations had been guided by their knowledge of heat conduction and vibratory motion. In the extended research programme, however, LIOUVILLE did not have such physical guidance for what were the important questions and even what were the correct theorems to prove. A manifestation of LIOUVILLE'S failure of intuition in the broader field was his belief in the convergence theorem for Fourier series.

To conclude, LIOUVILLE'S lack of physical motivation, his resulting failing intuition and more specifically his inability to prove the central expansion theorem for higher-order equations explain at least in part why the first chapter of the history of the Sturm-Liouville theory ended in 1838, only a few years after it had started. Yet during these few years STURM and LIOUVILLE had advanced the theory to such a degree of completeness that no substantial additions were made during the next half century.

Appendix

On the Convergence of the Fourier Series for a Third Order Sturm-Liouville Problem

The following remarks have been communicated to me by Professor UFFE HAAGERUP, Odense. He has made a more detailed investigation of the boundary-value problem (111)–(113) in a course at Odense University [HAAGERUP 1982].

LIOUVILLE noted that the eigenfunctions of the boundary-value problem (111)–(113) were of the form $G(\varrho_n x)$, where

$$G(x) = e^{-x} + \mu e^{-\mu x} + \mu^2 e^{-\mu^2 x}; \qquad \mu = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and the positive eigenvalues $\varrho_1 < \varrho_2 < \varrho_3 < \ldots$ are the positive roots of $G(x)$.

Theorem. A necessary condition that a function f on $]0, 1[$ can be written in the form

$$f(x) = \sum_{n=1}^{\infty} c_n G(\varrho_n x), \qquad c_n \in \mathbb{C}$$

(pointwise convergence) is that f has an analytic continuation $\tilde{f}$ to the open triangle with vertices $1, \mu, \mu^2$, such that the Taylor expansion of $\tilde{f}$ around $z = 0$ is of the form

$$\tilde{f}(z) = a_2 z^2 + a_5 z^5 + a_8 z^8 + \dots . \tag{152}$$

In particular, the polynomials $1, x, x^3, x^4, x^6, \dots$ cannot be expanded in the above form.

Proof. Note that $G(x) = 2e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x - \frac{\pi}{6}\right) + e^{-x}$. The roots ϱ_n are solutions of the equation

$$\sin\left(\frac{\sqrt{3}}{2}x - \frac{\pi}{6}\right) = -\frac{1}{2}e^{-\frac{3x}{2}},$$

which gives the asymptotic expression

$$\varrho_n = \frac{2\pi}{\sqrt{3}}\left(n + \frac{1}{6}\right) + \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. (Note that G has a double root at $x = 0$, but no other roots close to 0.)

Let $p \in \mathbb{N}$, $p \geqq 2$. By the assumptions the series

$$\sum_{n=1}^{\infty} c_n G\left(\varrho_n\left(1 - \frac{1}{p}\right)\right)$$

converges. Hence there is a constant K_p such that

$$|c_n| \left| G\left(\varrho_n\left(1 - \frac{1}{p}\right)\right)\right| \leqq K_p \quad \text{for all } n.$$

By the asymptotic formula for ϱ_n, we get

$$G\left(\varrho_n\left(1 - \frac{1}{p}\right)\right) \sim 2e^{\frac{\varrho_n}{2}\left(1-\frac{1}{p}\right)}\left(\sin\left(n\pi - \frac{1}{p}\left(n + \frac{1}{6}\right)\pi\right)\right),$$

the error of which grows less rapidly than $e^{\frac{\varrho_n}{2}\left(1-\frac{1}{p}\right)}$. Since

$$\left|\sin\left(n\pi - \frac{1}{p}\left(n + \frac{1}{6}\right)\pi\right)\right| = \left|\sin\left(\frac{1}{p}\left(n + \frac{1}{6}\right)\pi\right)\right| \geqq \sin\frac{\pi}{6p}$$

for all n, there are a constant K_p' and a number n_p such that

$$\left|G\left(\varrho_n\left(1-\frac{1}{p}\right)\right)\right| \geqq K_p' \, e^{\frac{\varrho_n}{2}\left(1-\frac{1}{p}\right)} \quad \text{if} \quad n \geqq n_p.$$

Therefore

$$|c_n| \leqq \frac{K_p}{K_p'} e^{-\frac{\varrho_n}{2}\left(1-\frac{1}{p}\right)}, \quad n \geqq n_p,$$

which implies the existence of a constant K_p'', such that

$$|c_n| \leqq K_p'' \, e^{-\frac{\varrho_n}{2}\left(1-\frac{1}{p}\right)}, \quad n = 1, 2, 3, \ldots$$

For $r > 0$ we let T_r denote the triangle with vertices r, μr, $\mu^2 r$. The function

$$G(x) = e^{-x} + \mu e^{-\mu x} + \mu^2 e^{-\mu^2 x}$$

has an extension to the whole complex plane given by the same formula. Since

$$T_r = \left\{z \in \mathbb{C} \mid \operatorname{Re} z \geqq -\frac{r}{2}, \ \operatorname{Re}(\mu z) \geqq -\frac{r}{2}, \ \operatorname{Re}(\mu^2 z) \geqq -\frac{r}{2}\right\}$$

it follows that

$$\sup_{z \in T_r} |G(z)| \leqq 3e^{\frac{r}{2}}.$$

Let $r \in \,]0, 1[$. Choose $p \in \mathbb{N}$ and such that $1 - \dfrac{1}{p} > r$. For $z \in T_r$

$$\sum_{n=1}^{\infty} |c_n| \, |G(\varrho_n z)| \leqq K_p'' \sum_{n=1}^{\infty} e^{-\frac{\varrho_n}{2}\left(1-\frac{1}{p}\right)} 3e^{\frac{\varrho_n r}{2}} = 3K_p'' \sum_{n=1}^{\infty} e^{-\frac{\varrho_n}{2}\left(1-\frac{1}{p}-r\right)}.$$

Since $\varrho_n \sim \dfrac{2\pi}{\sqrt{3}}\left(n + \dfrac{1}{6}\right)$ and $r < 1 - \dfrac{1}{p}$, the last sum is finite. This shows that $\sum_{n=1}^{\infty} c_n G(\varrho_n z)$ converges uniformly on every triangle T_r, $r < 1$. Hence there is an analytic function $\tilde{f}$ defined on the interior of the triangle T_1, such that

$$\tilde{f}(z) = \sum_{n=1}^{\infty} c_n W_n(z), \quad z \in \mathring{T}_1.$$

Since $G(\mu z) = \mu^2 G(z)$, $z \in \mathbb{C}$, we have also

$$\tilde{f}(\mu z) = \mu^2 \tilde{f}(z), \quad z \in \mathring{T}_1.$$

This condition is equivalent to saying that the Taylor expansion of $\tilde{f}$ around $z = 0$ is of the form

$$\tilde{f}(z) = a_2 z^2 + a_5 z^5 + a_8 z^8 + \ldots. \tag{152}$$

Q.E.D.

Remark. By a more detailed analysis of the problem one can obtain a partial converse, namely:

If a function f on $[0, 1]$ has a *bounded* analytic extension to the interior of the triangle with vertices 1, μ, μ^2, such that $\tilde{f}(\mu z) = \mu^2 \tilde{f}(z)$, then $f(x)$ can be written in the form (152). In particular the Fourier series converges when $f(x) = x^2, x^5, x^8$, *etc.* (see [HAAGERUP 1982]).

Bibliography

Abbreviations:

P. V. = Procès-Verbaux des Séances de l'Académie des Sciences, Paris 1795–1835.
C. R. = Comptes Rendus des Séances de l'Académie des Sciences, Paris 1835–1982.

Numbers in brackets refer to year of presentation to l'Académie des Sciences.

J. D'ALEMBERT:

1747 "Recherches sur la courbe que forme une corde tenduë mise en vibration". Hist. Acad. Roy. Sci. et Belles Let., Berlin (1747 publ. 1749), pp. 214–219.

1763 (1770) "Extrait de différentes lettres de Mr. d'Alembert à Mr. de La Grange". Hist. Acad. Roy. Sci. Belles Let., Berlin (1763 publ. 1770), pp. 235–277.

D. BERNOULLI:

1732/33 "Theoremata de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae". Comm. Acad. Sci. Petrop. vol. **6** (1732/33), pp. 108–122.

1734/35 "Demonstrationes theorematum suorum de oscillationibus ..." Comm. Acad. Sci. Petrop. vol. **7** (1734/35), pp. 162–173.

J. BERNOULLI:

1728 "Meditationes de chordis vibrantibus". Comm. Acad. Sci. Petrop., vol. **3** (1728), pp. 13–28 = Opera, vol 3, pp. 198–210.

G. BIRKHOFF:

1973 "A Source Book in Classical Analysis". Cambridge, Mass. 1973.

M. BÔCHER:

1898/99 "The theorems of oscillation of Sturm and Klein". N. Y. Bull. Amer. Math. Soc., vol. **4** (1898) pp. 295–313, 365–376; vol. **5** (1899) pp. 22–43.

1900 "On Sturm's theorem of comparison". N. Y. Bull. Amer. Math. Soc. vol. **6** (1900) pp. 96, 100.

1899/1916 "Randwertaufgaben bei gewöhnlichen Differentialgleichungen". Encyklopädie der mathematischen Wissenschaften II A 7a, vol. **II** 1.1, pp. 437–463.

1911/1912 "The published and unpublished work of Charles Sturm on algebraic and differential equations". Bull. Amer. Math. Soc., vol. **18** (1911–12), pp. 1–18.

1912 "Boundary problems in one dimension". Proceedings of the Fifth international Congress of Mathematicians. Cambridge, Mass. 1912.
1917 "Leçons sur les méthodes de Sturm dans la théorie des équations différentielles linéaires et leurs développements modernes", Paris 1917.

A.-L. CAUCHY:

1821 "Cours d'Analyse de l'École Royale Polytechnique, 1re partie; Analyse Algébrique", Paris 1821, Oeuvres (2) 3.
1824/1981 "Equations Différentielles Ordinaires". Paris. New York 1981. Fragment of lecture notes for the second year at the École Polytechnique. Introduction by C. GILAIN.
1829 "Leçons sur le calcul différentiel". Paris 1829. Oeuvres (2) 4, pp. 263–609.
1835/1840 "Mémoire sur l'intégration des équations différentielles". Exercices d'analyse et de physique mathématique vol. **1** (1840), pp. 327ff. = Oeuvres (2) XI, pp. 399–465. A small number of lithographed versions of this paper appeared in 1835.

D. COLLADON & C. STURM:

1827/1834 "Mémoire sur la compression des liquides". Mém. Savans Étrang. Acad. Sci. Paris vol. **5** (1834), pp. 267ff. = Ann. Chémie et Phys. vol. **22**, pp. 113ff.

J. DIEUDONNÉ:

1978 "Abrégé d'Histoire des Mathématiques 1700–1900". I + II Paris 1978.
1981 "History of Functional Analysis". Amsterdam 1981.

U. DINI:

1880 "Serie di Fourier ed altre rappresentazioni analitiche delle funzioni di una variabile reale". Pisa 1880.

P. G. L. DIRICHLET:

1829 "Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données". Journ. Reine Angew. Math. vol. **4** (1829), pp. 157–169 = Werke I, pp. 117–132.

L. EULER:

1764/1766 "De motu vibratorio tympanorum". Novi Comm. Acad. Sci. Petrop. **10** (1764/66), pp. 243–60 (publ. 1766) = Opera (2) **10**, pp. 344–59.

H. FAYE:

1882 "Discours prononcé aux funérailles de M. Liouville". Comp. Rend. Acad. Sci. Paris vol. **95** (1882), pp. 468–69.

J. FOURIER

1820 "Sur l'usage du théorème de Descartes dans la recherche des limites des racines". Bull. Sci. Soc. Philomathique Paris, October (1820) pp. 156–65, December (1820) pp. 181–87 = Oeuvres II pp. 289–309.
1822 "Théorie analytique de la chaleur". Paris 1822.
1831 "Remarques Générales sur l'application des Principes de l'analyse algébrique aux équations transcendantes". Mém. Acad. Sci. vol. **10** (1831) pp. 119–146 [1829].

U. HAAGERUP:

1982 Notes on the Sturm-Liouville problem (111)–(113) used in a course on "Sturm-Liouville theory before and now" conducted by J. LÜTZEN & N. J. NIELSEN at the Department of Mathematics, Odense University.

E. HEINE:

1870 "Über trigonometrische Reihen". Journ. Reine Angew. Math. vol. **71** (1870) pp. 353–365.

1880 "Einige Anwendungen der Residuenrechnung von Cauchy". Journ. Reine Angew. Math. vol **89** (1880) pp. 19–39.

E. HILB & O. SZÁSZ:

1922 "Allgemeine Reihenentwicklungen". Encyklopädie der mathematischen Wissenschaften II C 11, vol. **II** 3.2 pp. 1229–1276.

D. HILBERT:

1904/1910 "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen". Nachrichten Königl. Ges. Wiss. Göttingen Math.-Phys. Kl. (1904) pp. 49–91, 213–259; (1905) pp. 307–338; (1906) pp. 157–227, 439–480; (1910) pp. 355–417, 595–618. Collected as monograph Leipzig 1912. References refer to monograph.

G. KIRCHHOFF:

1879 "Über die Transversalschwingungen eines Stabes von veränderlichem Querschnitt". Berlin Acad. Monatsber. (1879) pp. 815–828 = Ann. Phys. Chem. vol. **10** (1880) pp. 501–512.

F. KLEIN:

1881 "Über Körper, welche von confocalen Flächen zweiten Grades begränzt sind". Math. Ann. vol. **18** (1881) pp. 410–427.

M. KLINE

1972 "Mathematical Thought from Ancient to Modern Times". New York 1972.

A. KNESER:

1904 "Untersuchungen über die Darstellung willkürlicher Funktionen in der mathematischen Physik". Math. Ann. vol. **58** (1904) pp. 81–147.

P. S. LAPLACE:

1799 "Traité de Mécanique Céleste, tome premier". Paris an VII = Oeuvres vol 1.

J. LIOUVILLE:

1828 "Mémoire sur le Calcul aux différences partielles (1er–3me)". Unpublished manuscripts presented to the Académie des Sciences on Dec. 1. 1828.

1830/1831 "Recherches sur la théorie physico-mathématique de la Chaleur". Manuscript presented to the Académie des Sciences on August 16. 1830. Most of it printed as "Mémoire sur la théorie analytique de la chaleur". Ann. Math. vol. **21** (1830/31), pp. 133–181.

1832 "Mémoire sur le calcul des différentielles à indices quelconques". Journ. Ec. Polytech. vol. **13** (21. cah.), pp. 71–162.

1836a "Mémoire sur le développement des fonctions ou parties de fonctions en séries de sinus et de cosinus". Journ. Math. Pures Appl. vol. **1** (1836), pp. 14–32.

1836b "Mémoire sur une question d'analyse aux différences partielles". Journ. Math. Pures Appl. (1836), pp. 33–74.

1836c "Note sur une manière de généraliser la formule de Fourier". Journ. Math. Pures Appl. vol **1** (1836), pp. 102–105.

1836d "Mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable". Journ. Math. Pures Appl. vol **1** (1836), pp. 253–265. [1835].

1836e "Démonstration d'un théorème du à M. Sturm relatif à une classe de fonctions transcendantes". Journ. Math. Pures Appl. vol. **1** (1836), pp. 269–277.

1837a "Mémoire sur l'intégration de l'équation $\frac{du}{dt} = \frac{d^3u}{dx^3}$". Journ. Ec. Polytechn. Paris vol. **15** (1837) (25 cah.), pp. 85–117. Extract in Comp. Rend. vol. **3** (1836), pp. 572–575. [1836].

1837b "Solution d'un problème d'analyse". Journ. Math. Pures Appl. vol. **2** (1837), pp. 1–2.

1837c "Second Mémoire sur le développement des fonctions ...". Journ. Math. Pures Appl. vol. **2** (1837), pp. 16–35.

1837d "Sur la sommation d'une série". Journ. Math. Pures Appl. vol. **2** (1837), pp. 107–108.

1837e "Troisième Mémoire sur le développement des fonctions...". Journ. Math. Pures Appl. vol. **2** (1837), pp. 418–437. Extract in Comp. Rend. vol. **5** (1837), pp. 205–207. [1837].

1837f "Solution nouvelle d'un problème d'analyse, relatif aux phénomènes thermo-mécaniques". Journ. Math. Pures Appl. vol. **2** (1837), pp. 439–456.

1838a "Note sur la théorie des équations différentielles". Journ. Math. Pures Appl. vol. **3** (1838), pp. 255–56.

1838b "Observations sur un mémoire de M. Libri, relatif à la théorie de la chaleur". Journ. Math. Pures Appl. vol. **3** (1838), pp. 350–354.

1838c "Premier mémoire sur la théorie des équations différentielles linéaires, et sur le développement des fonctions en séries". Journ. Math. Pures Appl. vol. **3** (1838), pp. 561–614. Extract in Comp. Rend. vol. **7** (1838), pp. 1112–1116.

1838d "Mémoire sur une question d'analyse aux différences partielles". Mém. Savans Étrang. Acad. Sci. Paris vol. **5** (1838), pp. 559–606.

1840 "Sur les conditions de convergence d'une classe générale de séries". Journ. Math. Pures Appl. vol. **5** (1840), pp. 356–359 = Comp. Rend. vol. **11** (1840), pp. 615–618.

1845 "Sur une propriété générale d'une classe de fonctions". Journ. Math. Pures Appl. vol. **10** (1845), pp. 327–328.

1855 "Discours de M. Liouville, Membre de l'Académie, prononcé aux funérailles de M. Sturm". Included in [Prouhet 1856].

Ms. Manuscripts in the Liouville Nachlass at the Library of l'Institut de France. The Nachlass consists in 340 notebooks and a box of papers. It bears the code numbers 3616–3640.

LIOUVILLE & STURM:

1836 "Démonstration d'un théorème de M. Cauchy relatif aux racines imaginaires des équations". Journ. Math. Pures Appl. vol. **1** (1836), pp. 278–289.

1837a "Note sur un théorème de M. Cauchy relatif aux racines des équations simultanées". Comp. Rend. vol. **4** (1837), pp. 720–739.

1837b "Extrait d'un Mémoire sur le développement des fonctions en séries dont les différents termes sont assujettis à satisfaire à une même équation différentielle linéaire, contenant un paramètre variable". Journ. Math. Pures Appl. vol. **2** (1837), pp. 220–233. = Comp. Rend. vol. **4** (1837), pp. 675–677.

J. LÜTZEN:

1981 "Joseph Liouville's contribution to the theory of integral equations". Historia Mathematica vol. **9** (1982), pp. 373–391.

F. N. M. MOIGNO:

1844 "Leçons de calcul différentiel et de calcul intégral rédigées principalement d'après les méthodes de M. A.-L. Cauchy, tome 2" Paris 1844.

E. PICARD:

1890 "Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives". Journ. Math. Pures Appl. ser. 4. vol. **6** (1890), pp. 145–210.

H. POINCARÉ:

1894 "Sur les Équations de la Physique Mathématique". Rend. Circ. Matem. vol. **8** (1894), pp. 57–155.

S. D. POISSON:

1823a "Mémoire sur la Distribution de la Chaleur dans les Corps solides". Journ. Ec. Polytechn. vol. **12** (19. cah.) (1823), pp. 1–144.

1823b "Second Mémoire sur la Distribution de la Chaleur dans les Corps solides". Journ. Ec. Polytechn. vol. **12** (19. cah.) (1823), pp. 249–402 [1821].

1826 "Note sur les racines des équations transcendantes". Bull. Soc. Philomatique (1826), pp. 145–148.

1830 "Note sur les racines des équations transcendantes". Mém. Acad. Sci. Paris vol. **9** (1830), pp. 89–95.

1835 "Théorie mathématique de la chaleur". Paris 1835.

E. PROUHET:

1856 "Notice sur la vie et les travaux de Ch. Sturm". Bull. de bibliographie, d'histoire et de biographie mathématiques, vol. **2** (1856), pp. 72–89; repr. in STURM's "Cours d'analyse" 5 ed. vol. I (1877), pp. XV–XXIX.

Lord RAYLEIGH = J. W. STRUTT:

1877 "The Theory of Sound". London 1877.

L. SCHÄFLLI:

1876 "Über die Convergenz der Entwicklung einer arbiträren Funktion $f(x)$ nach den Bessel'schen Funktionen ...". Math. Ann. vol. **10** (1876), pp. 137–142.

E. SCHMIDT:

1907 "Zur Theorie der linearen und nichtlinearen Integralgleichungen ". Math. Ann. vol. **63** (1907), pp. 433–476.

1908 "Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten". Rend. Circ. Mat. Palermo, vol. **25** (1908), pp. 53–77.

W. STEKLOV:

1898 "Sur le problème de refroidissement d'une barre hétérogène". Comp. Rend. Acad. Sci. Paris, vol. **126** (1898), pp. 215–218.

C. STURM:

1829a "Analyse d'un Mémoire sur la résolution des équations numériques". Bull. Sci. Math. Astr. Phys. vol. **11** (1829), pp. 419–422 [May 25, 1829].

1829b "Extrait d'un Mémoire de M. Sturm". Bull. Sci. Math. Astr. Phys. vol. **11** (1829), pp. 422–425 [June 1., 1829].

1829c "Note présentée à l'Academie par M. Ch. Sturm". Bull. Sci. Math. Astr. Phys. vol. **11** (1829), pp. 425 [June 8, 1829].

1829d "Extrait d'un Mémoire sur l'intégration d'un système d'équations différentielles linéaires". Bull. Sci. Math. Astr. Phys. vol. **12** (1829), pp. 315–322 [July 27, 1829].

1829e "Sur la distribution de la chaleur dans un assemblage de vases". Bull. Sci. Math. Astr. Phys. vol. **12** (1829), p. 322 [August 3, 1829].

1829f "Nouvelle théorie relative à une classe de fonctions transcendantes que l'on rencontre dans la résolution des problèmes de la physique mathématique". Bull. Sci. Math. Astr. Phys. vol. **12** (1829), p. 322. CAUCHY's review in the Academy on July 5, 1830 is printed in the Procès Verbeaux.

1833a "Analyse d'un mémoire sur les propriétés générales des fonctions qui dépendent d'équations différentielles linéaires du second ordre". L'Institut Journ. Acad. et Soc. Sci. nov. 9 (1833), pp. 219–223. (summary of 1836a).

1833b Note without name. L'Institut Journ. Acad. et Soc. Sci. nov. 30 (1833), pp. 247–248. (summary of 1836b).

1835 "Mémoire sur la résolution des équations numériques". Mém. Savants Étrangers, Acad. Sci. Paris vol. **6**, p. 271–318.

1836a "Mémoire sur les Équations différentielles linéaires du second ordre". Journ. Math. Pures Appl. vol. **1** (1836), pp. 106–186. [sept. 28, 1833].

1836b "Mémoire sur une classe d'Équations à différences partielles". Journ. Math. Pures Appl. vol. **1** (1836), pp. 373–444.

J. TANNERY:

1910 "Correspondance entre Lejeune Dirichlet et Liouville" Paris 1910 = Bull. Sci. Math. (2. ser), vol. **32** (1908), pp. 47–62, 88–95 and vol. **33** (1908/09), pp. 47–64.

B. TAYLOR:

1713 "De motu nervi tensi". Phil. Trans. vol. **28** (1713), pp. 26–32.

C. TRUESDELL:

1960 "The Rational Mechanics of Flexible or Elastic Bodies 1638–1788". EULER's Opera Omnia (2) vol. **11**, part 2. Zürich 1960.

Chronological Summary

	1763	– d'Alembert on heterogeneous vibrating strings.
	1803	– Sturm born.
	1807	– Fourier's first theory of heat.
	1809	– Liouville born.
	1822	– Fourier's "Théorie analytique de la Chaleur".
	1823	– Poisson's *a posteriori* proof of reality of eigenvalues.
	1824	– Cauchy's existence proof.
	1826	– Poisson's *a priori* proof of reality of eigenvalues.
	1829	– Dirichlet: Convergence of trigonometric Fourier series.
I	1829	– Sturm's lost mémoires.
	1830	– Liouville's theory of heat. Successive approximation.
II	1833	– Libri elected to the Academy.
	1833	– Summaries of Sturm's two great mémoires.
III	1835	– Liouville: Fourier series has correct value.
	1835/40	– Publication of Cauchy's existence proof.
	1836	– Publication of Sturm's two great mémoires.
	1836	– Liouville: Third order equation with constant coefficients.
	1836	– Sturm elected to the Academy.
	1837	– Liouville: Convergence of Fourier series – 2 proofs.
IV	1837–38	– Liouville: Lectures on Physics and Mathematics.
	1838	– Liouville: Higher-order equations with variable coefficients.
	1838	– Liouville professor and Sturm répétiteur at the Ecole Polytechnique.
	1838–40	– Liouville: Attempts at further generalisations.
	1839	– Liouville elected to the Academy.
	1855	– Sturm †
	1879/80	– Rayleigh and Kirchhoff on vibrating rod and membrane.
	1881	– Klein on Lamé functions.
	1882	– Liouville †
	1894	– Poincaré: Spectral theory of Δ.
	1898/99	– Bôcher: Rigorization of Sturm's methods.
	1898	– Stekloff: Proof of expansion theorem.
	1904	– Kneser: Proof of expansion theorem.
	1904/10	– Hilbert: Spectral theory of integral operators.
	1907	– Schmidt: Spectral theory of integral operators.

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(Received August 23, 1982)