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The block structure spaces of real projective spaces and orthogonal calculus of functors II

Tibor Macko and Michael Weiss

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Abstract. For a finite dimensional real vector space V with inner product, let $F(V)$ be the block structure space, in the sense of surgery theory, of the projective space of V. Continuing a program launched in [Ma], we investigate F as a functor on vector spaces with inner product, relying on functor calculus ideas. It was shown in [Ma] that F agrees with its first Taylor approximation T_1F (which is a polynomial functor of degree 1) on vector spaces V with $\dim(V) \geq 6$. To convert this theorem into a functorial homotopy-theoretic description of $F(V)$, one needs to know in addition what $T_1F(V)$ is when $V = 0$. Here we show that $T_1F(0)$ is the standard L-theory space associated with the group $\mathbb{Z}/2$, except for a deviation in π_0. The main corollary is a functorial two-stage decomposition of $F(V)$ for $\dim(V) \geq 6$ which has the L-theory of the group $\mathbb{Z}/2$ as one layer, and a form of unreduced homology of $\mathbb{R}P(V)$ with coefficients in the L-theory of the trivial group as the other layer. Except for dimension shifts, these are also the layers in the traditional Sullivan-Wall-Quinn-Ranicki decomposition of $F(V)$. But the dimension shifts are serious and the SWQR decomposition of $F(V)$ is not functorial in V. Because of the functoriality, our analysis of $F(V)$ remains meaningful and valid when $V = \mathbb{R}^\infty$.

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1 Introduction

This paper is a continuation of [Ma]. In [Ma] a certain continuous functor from the category $\mathcal{J}$ of finite-dimensional real vector spaces with inner product to the category Spaces$_*$ of pointed spaces was introduced. In the present paper we denote this functor by F^g. For V an object of $\mathcal{J}$, the value $F^g(V)$ is the block structure space

$$\widetilde{\mathcal{S}}(\mathbb{R}P(V))$$

of the projective space of V. For a morphism ξ in $\mathcal{J}$ the map $F^g(\xi)$ is a generalization of the join construction of Wall. See [Ma] or [Qu] for the definition of the block structure space of a manifold, [Ma] and [Wa] for more on the join construction.

Each space $F^g(V)$, alias block structure space of $\mathbb{R}P(V)$, is individually well understood as the n-fold loop space of the homotopy fiber of a standard assembly map in L-theory, where

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$n = \dim(V) - 1$ and we assume $n \geq 5$. See [Qu]. The assembly map has good naturality properties, but the prefix Ω^n tends to corrupt these when n becomes a variable. Hence the standard methods for calculating the values $F^g(V)$ do not lead to very satisfying homotopy theoretic descriptions of the induced maps $F^g(\xi)$.

The goal of the project presented in [Ma] and here is to provide a homotopy theoretic description of the spaces $F^g(V)$ natural in V, i.e. to describe the functor F^g. This will allow us to let $\dim(V)$ tend to infinity. Hence it gives us a homotopy theoretic description of the (homotopy) colimit of the spaces $F^g(\mathbb{R}^n)$, a space which might be considered as the block structure space of $\mathbb{R}P^\infty$ and which, as explained in the introduction to [Ma], is closely related to certain spaces of stable equivariant (honest, i.e. not blockwise) automorphisms of spheres. For this purpose another tool is employed, the orthogonal calculus of functors of Weiss [We]. The desired description should be obtained from the orthogonal calculus Taylor tower of the functor F^g. This tower yields in particular a first Taylor approximation $T_1 F^g$ of F^g "at infinity", which is another functor from $\mathcal{J}$ to Spaces$_*$ and comes with a canonical transformation $F^g \to T_1 F^g$. The degree 1 property of $T_1 F^g$ implies a homotopy fibration sequence, natural in V:

$$\text{(1.1)} \qquad \Omega^\infty[(S(V)_+ \wedge \Theta^{(1)} F^g)_{h\mathrm{O}(1)}] \to T_1 F^g(0) \to T_1 F^g(V).$$

Here $S(V)$ is the unit sphere in V with the antipodal involution, the subscript $+$ denotes an added base point, $\Theta^{(1)} F^g$ denotes the first derivative spectrum of F^g and the subscript $h\mathrm{O}(1)$ denotes a homotopy orbit construction for the symmetry group $\mathrm{O}(1) \cong \mathbb{Z}_2$. See [We] or [Ma] for the definitions and more on the Taylor tower of a continuous functor from $\mathcal{J}$ to Spaces$_*$.

A first step in the project was made in [Ma]. Namely, it was shown that for V such that $\dim(V) \geq 6$ the canonical map $F^g(V) \to T_1 F^g(V)$ is a homotopy equivalence. Therefore, if $\dim(V) \geq 6$, the homotopy fibration sequence (1.1) can be rewritten as a homotopy fibration sequence

$$\text{(1.2)} \qquad \Omega^\infty[(S(V)_+ \wedge \Theta^{(1)} F^g)_{h\mathrm{O}(1)}] \to T_1 F^g(0) \to F^g(V).$$

It was also shown in [Ma] that $\pi_k \Theta^{(1)} F^g$ is the k-th L-group of the trivial group.

In this paper we make a second step towards a complete analysis of F^g by giving a fairly complete description of $T_1 F^g(0)$, the term in the middle of the homotopy fibration sequence (1.2). This step turns out to be far from easy.

Let $\mathbb{A}$ be the additive category of finitely generated free abelian groups and let $\mathbb{A}[\mathbb{Z}_2]$ be the additive category of finitely generated free modules over the group ring $\mathbb{Z}[\mathbb{Z}_2]$. Both of these have duality functors and hence determine L-groups and L-theory spaces. Our main result is as follows.

Theorem 2. *There is a homotopy fibration sequence*

$$T_1 F^g(0) \longrightarrow \mathsf{L}_0(\mathbb{A}[\mathbb{Z}_2]^+) \xrightarrow{\tilde{\sigma}/8} \mathbb{Z}\,.$$

The space $\mathsf{L}_0(\mathbb{A}[\mathbb{Z}_2]^+)$ is the standard L-theory space for the group $\mathbb{Z}_2$ with the trivial orientation character. Its homotopy groups are $\pi_k\mathsf{L}_0(\mathbb{A}[\mathbb{Z}_2]^+) = L_k(\mathbb{Z}_2^+)$. The map $\tilde{\sigma}/8$ is the composition of the transfer $L_0(\mathbb{A}[\mathbb{Z}_2]^+) \to L_0(\mathbb{A})$ and the isomorphism $L_0(\mathbb{A}) \cong \mathbb{Z}$ defined by signature over 8. It is onto. See [Wa, Chapter 13A].

In [Ma] mainly the geometric surgery theory of Wall [Wa] was used, but for the proof of Theorem 2 we switch to the algebraic theory of surgery of Ranicki [Ra]. In the geometric surgery setup, the block structure space of an n-dimensional closed manifold X fits into the homotopy fibration sequence, due to Quinn [Qu],

$$(1.3) \qquad \widetilde{\mathcal{S}}(X) \to \mathcal{N}(X) \to \mathcal{L}_n(X),$$

where $\mathcal{N}(X)$ is the space of normal invariants of X and $\mathcal{L}_n(X)$ is the surgery obstruction space associated with X. The homotopy groups of $\mathcal{L}_n(X)$ are the L-groups of $\pi = \pi_1(X)$ with associated orientation character and with an appropriate dimension shift. Quinn's homotopy fibration sequence is the space version of a long exact sequence of (homotopy) groups usually attributed to Sullivan in the simply connected case, and to Wall in the nonsimply connected case.

In the algebraic surgery setup the input consists typically of a connected simplicial complex X, an integer n, a universal covering space for X with deck transformation group π and a homomorphism $w\colon \pi \to \mathbb{Z}_2$. The output is the homotopy fibration sequence, due to Ranicki [Ra],

$$(1.4) \qquad \mathsf{S}(X,n,w) \to \mathsf{L}_n(\mathbb{A}_*(X,w)) \to \mathsf{L}_n(\mathbb{A}[\pi]^w).$$

Here all three spaces are L-theoretic spaces constructed from certain additive categories with chain duality. The homotopy groups $\pi_k\mathsf{L}_n(\mathbb{A}[\pi]^w)$ are again the groups L_{k+n} of the group ring $\mathbb{Z}[\pi]$ with the w-twisted involution. The homotopy groups $\pi_k\mathsf{L}_n(\mathbb{A}_*(X,w))$ are better known as the generalized homology groups H_{k+n} of X with w-twisted coefficients in the L-theory spectrum of the trivial group. We should perhaps add that our conventions here are such that we have strict 4-periodicity, $\mathsf{S}(X,n,w) \simeq \Omega^4\mathsf{S}(X,n,w)$ for all $n \in \mathbb{Z}$, in addition to the spectrum property $\mathsf{S}(X,n+1,w) \simeq \Omega\mathsf{S}(X,n,w)$ which Ranicki also insists on in most circumstances.

If X is a triangulated closed n-manifold and w is its orientation character, then modulo a small modification the sequence (1.4) can be identified with the sequence (1.3) by a result of [Ra]. (In this situation we usually write $\mathsf{S}(X)$ instead of $\mathsf{S}(X,n,w)$.) To be more precise the first terms in the sequences (the two versions of the block structure space) are related via a homotopy fibration sequence

$$(1.5) \qquad \widetilde{\mathcal{S}}(X) \to \mathsf{S}(X) \to \mathbb{Z}.$$

The advantage of the algebraic setup is that $\mathsf{S}(X)$ is much more tractable from the point of view of algebraic topology: it is an infinite loop space and it is 4-periodic as such, almost by definition. More specifically, this setup enables us to state and prove in sufficient naturality two crucial results, called 4-periodicity and Thom isomorphism in Section 2, which we are unable to state or prove in the geometric setting.

But for us the algebraic setup has some disadvantages, too. No triangulation is invariant under the action of the orthogonal group $\mathrm{O}(V)$ on V. Therefore it is not possible to define a continuous functor from $\mathcal{J}$ to Spaces$_*$ by a formula such as $V \mapsto \mathsf{S}(\mathbb{R}P(V))$. Instead we construct a continuous functor F^a from $\mathcal{J}$ to Spaces$_*$ along the following lines. For V in $\mathcal{J}$, the value $F^a(V)$ is a colimit of spaces $\mathsf{S}(X)$, where X runs through a directed system of generalized simplicial complexes obtained from certain generalized triangulations of $\mathbb{R}P(V)$. Of course, the space $F^a(V)$ will have the homotopy type of $\mathsf{S}(\mathbb{R}P(V))$.

We emphasize that, although F^a is better behaved than F^g from the orthogonal calculus point of view, its behavior on objects V in $\mathcal{J}$ of dimension < 3 is still not good. This is due to the fact that the map $\mathbb{R}P(V) \to \mathbb{R}P(W)$ induced by a morphism $V \to W$ in $\mathcal{J}$ need not be 1-connected if $\dim(V) < 3$. However, that difficulty can be overcome and we have the following result which easily implies Theorem 2:

Theorem 1. *We have* $T_1 F^a(0) \simeq \mathsf{L}_0(\mathbb{A}[\mathbb{Z}_2]^+)$.

The algebraic surgery approach also gives us the following:

Remark. *The sequence* (1.2) *is a homotopy fibration sequence of infinite loop spaces.*

Returning to the questions raised at the beginning of this introduction, recall that in order to use (1.2) for a natural description of $F^g(V)$ we would still need to understand (1) the spectrum $\Theta^{(1)}F^g$ with the action of $\mathrm{O}(1)$, and (2) the first map in the sequence (1.2). We are optimistic in regard to (1), because we already know the homotopy groups of $\Theta^{(1)}F^g$ and because the task has a neat formulation within algebraic surgery theory. Some new ideas are needed for (2).

The paper is organized as follows. In Section 2 we give the proof of Theorem 1, modulo certain Theorems A, B, C and D. Theorem C was proved in [Ma]. In the rest of the present paper we prove Theorems A, B and D. Specifically, in Section 3 we give a review of the tools from algebraic surgery we need. Section 4 contains a somewhat abstract preview of the functor F^a while Sections 5 and 6 deliver the technical details. Section 7 contains the proofs of Theorems A and B and Section 8 the proof of Theorem D. This completes the proof of Theorem 1 and it also reduces Theorem 2 to Theorem 1. At the very end of Section 8 we also explain the remark above on infinite loop space structures.

2 Proof of Theorem 1

This section contains the statements of Theorems A, B, C and D and the proof of Theorem 1 assuming these theorems. The functor F^a sketched in the introduction appears in the statements. Although it has a complicated definition, for the purposes of this section we may pretend that it is given by $F^a : V \mapsto \mathsf{S}(\mathbb{R}P(V))$. The precise definition of F^a and the proofs of the theorems, except for Theorem C which was proved in [Ma], are themes of the subsequent sections.

Theorem A. *For an oriented* $W \in \mathcal{J}$ *such that* 4 *divides* $\dim(W)$*, there is a homotopy equivalence* $F^a(V) \to \Omega^W F^a(V)$*, natural in* V.

This is just the usual 4-periodicity in L-theory. — Our next statement is about a Thom isomorphism in algebraic L-theory. Let V and W be objects in $\mathcal{J}$. As one would expect, there is a join map $\mathsf{S}(\mathbb{R}P(W)) \to \mathsf{S}(\mathbb{R}P(V \oplus W))$ and we do not generally have a way of extending that to a map between the two homotopy fiber sequences (1.4) for $X = \mathbb{R}P(W)$ and $X = \mathbb{R}P(V \oplus W)$, respectively. But it is relatively easy to supply the lower horizontal arrow in a homotopy commutative diagram

$$\begin{array}{ccc} \mathsf{S}(\mathbb{R}P(W)) & \longrightarrow & \mathsf{S}(\mathbb{R}P(V \oplus W)) \\ \downarrow & & \downarrow \\ \mathsf{L}_m(\mathbb{A}_*(\mathbb{R}P(W))) & \longleftarrow & \mathsf{L}_n(\mathbb{A}_*(\mathbb{R}P(V \oplus W))) \end{array}$$

where $m = \dim(W) - 1$ and $n = \dim(V \oplus W) - 1$. It is also easy to promote the resulting composite map

$$(2.1) \qquad \zeta : \mathsf{S}(\mathbb{R}P(V \oplus W)) \to \mathsf{L}_m(\mathbb{A}_*(\mathbb{R}P(W)))$$

to a natural transformation between functors in the variable V (note that the target functor is constant). Our Thom isomorphism result reads as follows.

Theorem B. *Let W in $\mathcal{J}$ be oriented, of even dimension. Then there exists a functor Φ_W on $\mathcal{J}$, which is polynomial of degree ≤ 0, a natural transformation $\zeta : F^a(- \oplus W) \to \Phi_W(-)$, and a natural map*

$$\Omega^W F^a(V) \to \text{hofiber}[F^a(V \oplus W) \xrightarrow{\zeta} \Phi_W(V)]$$

which is a homotopy equivalence for $\dim(V) \geq 3$.

When $V = 0$ the map $\zeta : F^a(V \oplus W)) \to \Phi_W(V)$ in Theorem B specializes to Ranicki's map $\zeta : \mathsf{S}(\mathbb{R}P(W)) \to \mathsf{L}_m(\mathbb{A}_*(\mathbb{R}P(W)))$ in (2.1).

As the referee has pointed out, there is a considerable overlap between Theorems A and B. Theorem B implies that the first derivative spectrum

$$\Theta^{(1)}\Omega^W F^a \ \simeq \ \Omega^W \Theta^{(1)} F^a$$

is homotopy equivalent to the first derivative spectrum

$$\Theta^{(1)} F^a(— \oplus W) \ \simeq \ \Sigma^W \Theta^{(1)} F^a.$$

When $W = \mathbb{R}^2$, this implies 4-periodicity of $\Theta^{(1)} F^a$ in the form of a weak equivalence of spectra with O(1)-action

$$\Theta^{(1)} F^a \ \simeq \ S^4 \wedge \Theta^{(1)} F^a.$$

But here, in the Theorem B setting, the S^4 is the one-point-compactification of a direct sum of two regular representations of O(1), whereas in theorem A it would be an S^4 with the trivial action of O(1).

The following is a simple reformulation of the main result of [Ma].

Theorem C. *Let* $W \in \mathcal{J}$ *be such that* $\dim(W) \geq 6$. *Then the functor*

$$V \mapsto F^g(V \oplus W)$$

on $\mathcal{J}$ *is polynomial of degree* ≤ 1.

The following theorem relates the two functors F^g and F^a.

Theorem D. *Let* $V \in \mathcal{J}$ *be such that* $\dim(V) \geq 6$. *Then there is a natural homotopy fibration sequence*

$$F^g(V) \to F^a(V) \to \mathbb{Z}.$$

On the level of spaces this homotopy fibration sequence is just the well-known relationship (1.5). The issue that needs to be addressed is the naturality in V.

Proof of Theorem 1. Theorems C and D imply that for W in $\mathcal{J}$ with $\dim(W) \geq 6$, the functor $V \mapsto F^a(V \oplus W)$ is polynomial of degree ≤ 1 without any low-dimensional deviations. Now suppose in addition that W is even-dimensional and oriented. Let F^a_W be the functor taking V in $\mathcal{J}$ to the homotopy fiber of $\varphi\colon F^a(V \oplus W) \to \Phi_W(V)$ in Theorem B. Then F^a_W is polynomial of degree ≤ 1 without any low-dimensional deviations, because it is the homotopy fiber of a natural transformation between a functor which is polynomial of degree ≤ 1 and another functor which is polynomial of degree ≤ 0. Therefore we have

$$F^a_W(V) \simeq T_1 F^a_W(V)$$

for all V in $\mathcal{J}$. From Theorem B we obtain

$$T_1 F^a_W(V) \simeq T_1 \Omega^W F^a(V)$$

for all $V \in \mathcal{J}$, since F^a_W and $\Omega^W F^a$ "agree" on objects V of sufficiently large dimension. Now suppose in addition that 4 divides $\dim(W)$. Then we get from Theorem A that

$$T_1 \Omega^W F^a(V) \simeq T_1 F^a(V)$$

for all V in $\mathcal{J}$. Composing these three natural homotopy equivalences, we get $F^a_W(V) \simeq T_1 F^a(V)$ for all V in $\mathcal{J}$. Specializing to $V = 0$ and unraveling the definition of $F^a_W(0)$ we obtain the statement of Theorem 1. □

3 Overview of algebraic surgery

The aim of this section is to recall the homotopy fibration sequence of algebraic surgery due to Ranicki [Ra]. It has the form

$$\text{(3.1)} \qquad \mathsf{S}(X, w) \to \mathsf{L}_n(\mathbb{A}_*(X, w)) \to \mathsf{L}_n(\mathbb{A}[\pi]^w),$$

where X is a connected simplicial complex (equipped with a universal covering, with deck transformation group π) and $w\colon \pi \to \mathbb{Z}_2$ is a homomorphism. More generally, X can be a Δ-complex (see [Hat] and Definition 3.3 below). If X is an n-dimensional manifold and $w\colon \pi \to \mathbb{Z}_2$ is the orientation character, then up to a small modification the homotopy fibration sequence (3.1) can be identified with the geometric homotopy fibration sequence of surgery

$$(3.2)\qquad \widetilde{\mathcal{S}}(X) \to \mathcal{N}(X) \to \mathcal{L}_n(X).$$

This section contains essentially no new results. It is a review of definitions and tools we need from [Ra] (see also [RaWe] and [We2]). We focus mostly on the case where w is trivial, but towards the end of the section we indicate the modifications needed if w is not trivial (the non-orientable case). All spaces in (3.1) are certain L-theory spaces associated with various additive categories with chain duality and all the maps in (3.1) are induced by functors between these categories.

Let $\mathbb{A}$ be an additive category and let $\mathbb{B}(\mathbb{A})$ denote the category of chain complexes of $\mathbb{A}$-objects, graded by $\mathbb{Z}$ and bounded below and above. A contravariant additive functor $T\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ can be extended to a contravariant additive functor $T\colon \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A})$ as follows. Let C be a chain complex in $\mathbb{B}(\mathbb{A})$. Then we can define a double complex

$$T(C)_{p,q} = T(C_{-p})_q.$$

The chain complex $T(C) \in \mathbb{B}(\mathbb{A})$ is the total complex of this double complex.

We use the notion of a chain duality $T\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ on $\mathbb{A}$, which is recalled below, to define symmetric and quadratic structures on objects of $\mathbb{B}(\mathbb{A})$.

Definition 3.1. A *chain duality* on an additive category $\mathbb{A}$ is a contravariant additive functor $T\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ together with a natural transformation e from $T^2\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ to $\mathrm{id}\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ such that for each M in $\mathbb{A}$
(1) $e_{T(M)} \cdot T(e_M) = \mathrm{id}\colon T(M) \to T^3(M) \to T(M)$,
(2) $e_M\colon T^2(M) \to M$ is a chain homotopy equivalence.

For a chain complex $C \in \mathbb{B}(\mathbb{A})$ its chain dual $T(C)$ is defined by the extension of T described before the definition. — A chain duality $T\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ can be used to define a tensor product of two objects M, N in $\mathbb{A}$ over $\mathbb{A}$ as

$$(3.3)\qquad M \otimes_{\mathbb{A}} N = \mathrm{Hom}_{\mathbb{A}}(T(M), N).$$

This is a chain complex of abelian groups.

The main examples of additive categories with chain duality we will consider are the following.

Example 3.2. Let R be a ring with involution $r \mapsto \bar{r}$ and let $\mathbb{A}(R)$ be the category of finitely generated projective left R-modules. On the category $\mathbb{A}(R)$ we can define a chain duality $\mathbb{A} \to \mathbb{A}$ by $T(M) = \mathrm{Hom}_R(M, R)$. The involution can be used to make $T(M)$ into a f.g. projective left R-module. The dual $T(C)$ of a finite chain complex C in $\mathbb{A}(R)$ is $\mathrm{Hom}_R(C, R)$.

The most important example for us is $R = \mathbb{Z}[\pi]$, the group ring of a group π, with involution given by $\bar{g} = g^{-1}$ for $g \in \pi$.

The category $\mathbb{A}(\mathbb{Z})$ with chain duality will sometimes be denoted just by $\mathbb{A}$, and the category $\mathbb{A}(\mathbb{Z}[\pi])$ will sometimes be denoted $\mathbb{A}[\pi]$.

In this paper we write Δ for the category with objects $\underline{n} = \{0, 1, \ldots, n\}$, for $n = 0, 1, 2, \ldots,$ and order-preserving *injective* maps as morphisms. A Δ-set is a functor from Δ^{op} to sets. A Δ-set Y has a geometric realization $|Y|$. It is the quotient of $\coprod_n Y_n \times \Delta^n$ by the relations $(u^* y, x) \sim (y, u_* x)$ for $y \in Y_n$, $x \in \Delta^m$ and $u : \underline{m} \to \underline{n}$ a morphism in Δ.

Out of a Δ-set Y, we can make a category $\mathrm{cat}(Y)$ with object set $\coprod_n Y_n$, where a morphism from $\sigma \in Y_m$ to $\tau \in Y_n$ is a morphism $u : \underline{m} \to \underline{n}$ in Δ with $u^* \tau = \sigma$. We write $u : \sigma \to \tau$ for short.

Definition 3.3. A Δ-complex is a space X together with a Δ-set sX and a homeomorphism $|sX| \to X$. It is considered *finite* if sX is finite (meaning that the disjoint union of the sets sX_n, $n \in \mathbb{N}$ is finite). When we write *simplex in X*, for a Δ-complex X, we mean a simplex in sX.

Example 3.4. Let $\mathbb{A}$ be an additive category and let X be a finite Δ-complex. Then there are defined two additive categories $\mathbb{A}_*(X)$ and $\mathbb{A}^*(X)$ of X-based objects in $\mathbb{A}$. An object M of $\mathbb{A}$ is X-*based* if it comes as

$$M = \sum_{n \geq 0} \; \sum_{\sigma \in sX_n} M(\sigma).$$

A morphism $f : M \to N$ in $\mathbb{A}^*(X)$, resp. $\mathbb{A}_*(X)$, is a matrix $f = (f_u)$ of morphisms $f_u : M(\sigma) \to N(\tau)$ in $\mathbb{A}$, resp. $f_u : M(\tau) \to N(\sigma)$ in $\mathbb{A}$, with entries corresponding to morphisms $u : \sigma \to \tau$ between simplices of X. Composition of morphisms is given by matrix multiplication.

Such a morphism f can be thought of as an upper triangular, resp. lower triangular matrix. For example, f is an isomorphism if and only if all diagonal entries, the f_u in which u is an identity, are invertible in $\mathbb{A}$.

Given N in $\mathbb{A}$ and σ in X, let N_σ in $\mathbb{A}_*(X)$, resp. $\mathbb{A}^*(X)$, be defined by $N_\sigma(\sigma) = N$ and $N_\sigma(\tau) = 0$ for $\tau \neq \sigma$. Clearly $N \to N_\sigma$ is a functor from $\mathbb{A}$ to $\mathbb{A}_*(X)$, resp. $\mathbb{A}^*(X)$. This functor has a right adjoint $M \to M[\sigma]$ from $\mathbb{A}_*(X)$, resp. $\mathbb{A}^*(X)$, to $\mathbb{A}$. We have $M[\sigma] = \sum_{\sigma \to \tau} M(\tau)$, resp. $M[\sigma] = \sum_{\tau \to \sigma} M(\tau)$ where the direct sum is taken over all morphisms $\sigma \to \tau$, resp. $\tau \to \sigma$, with fixed σ and arbitrary τ. For a morphism $f : M_0 \to M_1$ in $\mathbb{A}_*(X)$, resp. $\mathbb{A}^*(X)$, the induced morphism $M_0[\sigma] \to M_1[\sigma]$ is a sum of terms $f_u : M_0(\tau) \to M_1(\rho)$, one such for every diagram

$$\sigma \to \tau \xrightarrow{u} \rho, \text{ resp. } \rho \xrightarrow{u} \tau \to \sigma$$

of simplices in X.

Now let $\mathbb{A}$ be an additive category with chain duality $T : \mathbb{A} \to \mathbb{B}(\mathbb{A})$. Then T can be extended to chain dualities on $\mathbb{A}_*(X)$ and $\mathbb{A}^*(X)$. However, rather than defining the extension directly, we focus on the tensor products, $\otimes_{\mathbb{A}_*(X)}$ and $\otimes_{\mathbb{A}^*(X)}$, which are easier to motivate. Suppose therefore that M and N are X-based objects of $\mathbb{A}$. Then

$$(M \otimes_{\mathbb{A}_*(X)} N)_r = \sum_{\sigma \in sX} \sum_{\lambda \leftarrow \sigma \rightarrow \mu} (M(\lambda) \otimes_{\mathbb{A}} N(\mu))_{r-|\sigma|}$$
$$(M \otimes_{\mathbb{A}^*(X)} N)_r = \prod_{\sigma \in sX} \sum_{\lambda \rightarrow \sigma \leftarrow \mu} (M(\lambda) \otimes_{\mathbb{A}} N(\mu))_{r+|\sigma|}.$$

In the case where the Δ-complex X is a simplicial complex (with ordered vertex set, say), these graded abelian groups can be regarded as chain subcomplexes of $C_*X \otimes_{\mathbb{Z}} (M \otimes_{\mathbb{A}} N)$ and $\mathrm{Hom}_{\mathbb{Z}}(C_*X, M \otimes_{\mathbb{A}} N)$ respectively, where C_*X is the cellular chain complex of X. In the general case, we can still say that

$$\sigma \mapsto \sum_{\lambda \leftarrow \sigma \rightarrow \mu} M(\lambda) \otimes_{\mathbb{A}} N(\mu)$$
$$\sigma \mapsto \sum_{\lambda \rightarrow \sigma \leftarrow \mu} M(\lambda) \otimes_{\mathbb{A}} N(\mu)$$

is a contravariant (resp. covariant) functor, with chain complex values, on the category of simplices of X. This determines in the usual way a double chain complex of abelian groups. The corresponding total chain complex is $M \otimes_{\mathbb{A}_*(X)} N$, resp. $M \otimes_{\mathbb{A}^*(X)} N$. The adjunction (3.3) then determines the chain duality functors $\mathbb{A}_*(X) \to \mathbb{B}(\mathbb{A}_*(X))$ and $\mathbb{A}^*(X) \to \mathbb{B}(\mathbb{A}^*(X))$ as follows. Let M be an object in $\mathbb{A}_*(X)$, resp. $\mathbb{A}^*(X)$. Then

$$T(M)_r(\sigma) = \begin{cases} T(M[\sigma])_{r+|\sigma|}, \\ T(M[\sigma])_{r-|\sigma|}, \end{cases}$$

with differential

$$d_{T(M)}(u\colon \sigma \to \tau) : \quad \begin{cases} T(M[\tau]) \to T(M[\sigma]) \\ T(M[\sigma]) \to T(M[\tau]) \end{cases}$$

equal to $d_{T(M[\sigma])}$ if $\sigma = \tau$, equal to

$$\begin{cases} (-1)^i T\left(u^* : M[\tau] \to M[\sigma]\right) \\ (-1)^i T\left(u_* : M[\sigma] \to M[\tau]\right) \end{cases}$$

if $|\tau| = |\sigma| + 1$ and u omits the i-th vertex, and equal to 0 for all other $\sigma \to \tau$.

Remark 3.5. An object M in $\mathbb{A}_*(X)$ determines a contravariant functor $M_\natural$ from the category of simplices of X to $\mathbb{A}$ by $M_\natural(\sigma) = M[\sigma]$. We call such a contravariant functor from the category of simplices of X to $\mathbb{A}$, or any isomorphic one, *cofibrant*. Similarly an object M in $\mathbb{A}^*(X)$ determines a covariant functor $M^\natural$ from the category of simplices of X to $\mathbb{A}$ by $M^\natural(\sigma) = M[\sigma]$. Again we call such a covariant functor, or any isomorphic one, cofibrant.

A morphism $f : M \to N$ in $\mathbb{A}_*(X)$ induces a natural transformation $f_\natural : M_\natural \to N_\natural$, and vice versa. A morphism $f : M \to N$ in $\mathbb{A}^*(X)$ induces a natural transformation $f^\natural : M^\natural \to N^\natural$, and vice versa. In this way $\mathbb{A}_*(X)$ and $\mathbb{A}^*(X)$ are equivalent to, and could be re-defined as, certain categories of functors on the category of simplices of X. There are situations when we have to resort to these alternative definitions.

Example 3.6. For more motivation of the duality on the categories of X-based objects here is an example. To start with let X be a finite simplicial complex with ordered vertex set and let X' be its barycentric subdivision. The simplices σ of X correspond to the vertices $\hat{\sigma}$ of X'. For a simplex σ of X its dual cell $D(\sigma, X)$ is the subcomplex of X' spanned by the simplices with vertex set of the form

$$\{\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_p\}$$

where τ_0 contains σ and $\tau_0 \subset \tau_1 \subset \cdots \subset \tau_p$. Its "boundary" is spanned by all simplices in $D(\sigma, X)$ which do not have $\hat{\sigma}$ as a vertex. The dual cell $D(\sigma, X)$ is contractible. Apart from that it does not always have the properties that we would expect from a cell (such as being homeomorphic to a euclidean space), but it has the dual properties. In particular, $D(\sigma, X) \smallsetminus \partial D(\sigma, X)$ has a trivial normal bundle in X with fibers homeomorphic to $\mathbb{R}^{|\sigma|}$.

Let M be a closed n-dimensional topological manifold, and let $f : M \to X$ be a map transverse to the dual cells of X. Then

$$(M[\sigma], \partial M[\sigma]) = f^{-1}(D(\sigma, X), \partial D(\sigma, X))$$

is an $(n - |\sigma|)$-dimensional manifold ([Ra, Proof of Theorem 16.16]). The collection $\{M[\sigma] \mid \sigma \text{ in } X\}$, or more precisely, the contravariant functor $\sigma \mapsto M[\sigma]$, is called an X-dissection of M. In this situation there exists a structure of a CW-space on M such that each $M[\sigma]$ is a CW-subspace. The cellular chain complex C_*M can then be understood as a chain complex in $\mathbb{B}(\mathbb{A}_*(X))$ via the decomposition

$$C_*M = \sum_{\sigma} C_*(M[\sigma], \partial M[\sigma]).$$

We may expect this to be self-dual, with a shift of n, since M is a closed manifold. The dual of C_*M in $\mathbb{A}_*(X)$ is by definition

$$T(C_*M) = \sum_{\sigma} C^{-|\sigma|-*}(M[\sigma]).$$

The ordinary Poincaré duality homotopy equivalences

$$C_*(M[\sigma], \partial M[\sigma]) \simeq C^{n-|\sigma|-*}(M[\sigma])$$

suggest that $\Sigma^n T(C_*M)$ is indeed homotopy equivalent in $\mathbb{A}_*(X)$ to C_*M. This will be confirmed later.

Now we need to generalize these observations from the setting of simplicial complexes to that of Δ-complexes. For a Δ-complex X and a simplex σ in X, we have the category of simplices of X under σ. Its objects are morphisms $u\colon \sigma \to \tau$ where σ is fixed and τ in X is variable. Its nerve is a Δ-set and the corresponding Δ-complex is, by definition, the dual cell $D(\sigma, X)$. The boundary $\partial D(\sigma, X)$ corresponds to the nerve of the full subcategory with objects $u\colon \sigma \to \tau$ where u is not an identity.

The dual cell $D(\sigma, X)$ is contractible, because the category of simplices of X under σ has an initial object. There is a canonical map

$$c_\sigma\colon D(\sigma, X) \longrightarrow X$$

defined as follows. A k-simplex of $D(\sigma, X)$ corresponds to a diagram

$$\sigma \to \tau_0 \to \tau_1 \to \cdots \to \tau_k$$

of simplices in X. The vertices of that k-simplex are the resulting $\sigma \to \tau_i$ for $i = 0, 1, \ldots, k$. The restriction of c_σ to the k-simplex is the "linear" map taking the vertex $\sigma \to \tau_i$ to the barycenter of τ_i in X.

The map c_σ need not be injective. However, it is locally injective, it embeds $D(\sigma, X) \smallsetminus \partial D(\sigma, X)$, and the image of that restricted embedding has a trivialized normal bundle in X, with fibers homeomorphic to $\mathbb{R}^{|\sigma|}$. This results in a stratification of X where the strata have the form

$$X(\sigma) = c_\sigma(D(\sigma, X) \smallsetminus \partial D(\sigma, X))$$

and each stratum has a trivialized normal bundle with fiber homeomorphic to a euclidean space.[1] The closure $X[\sigma]$ of $X(\sigma)$ in X is the union of all $X(\tau)$ for which there exists a morphism $\sigma \to \tau$. (Zeeman's dunce hat, the two-dimensional Δ-complex with a single 0-simplex, a single 1-simplex and a single 2-simplex, is an instructive example.)

Let M be an n-dimensional topological manifold. Any map $f : M \to X$ is homotopic to a map transverse to the stratification of X by subsets $X(\sigma)$. See [Ra, Proof of Theorem 16.16]. If f is transverse to the stratification, then the pullback of f and c_σ is a manifold $M[\sigma]$ with boundary $\partial M[\sigma]$. Any morphism $\sigma \to \tau$ in X determines a map $M[\tau] \to M[\sigma]$ which, if $|\tau| > |\sigma|$, factors through $\partial M[\sigma]$. That map is locally an embedding and it embeds $M[\tau] \smallsetminus \partial M[\tau]$. The functor $\sigma \mapsto M[\sigma]$ together with the identification $\mathrm{colim}_\sigma M[\sigma] \cong M$ is called an X-dissection of M. If M is a smooth or PL manifold it is possible to equip the functor $\sigma \mapsto M[\sigma]$ with a CW-structure. (A CW-structure on a contravariant functor F from a small category to spaces is a filtration of F by subfunctors F_i for $i = -1, 0, 1, 2, 3, \ldots$, where $F_{-1} = \emptyset$ and F_i is obtained from F_{i-1} by "attaching" functors of the form $a \mapsto D^i \times \coprod_\lambda \hom(a, b_\lambda)$ using natural attaching maps $S^i \times \coprod_\lambda \hom(a, b_\lambda) \to F_{i-1}(a)$. If F comes with a CW-structure, we also say that F is a CW-functor. See [Dro] for more details.)

1 A stratification of a topological space X is a partition into locally closed subspaces $X(i)$, $i \in I$, some index set, such that for each $i \in I$ the closure of $X(i)$ is the union of $X(j)$ for $j \in J$ for some $J \subset I$.

The cellular chain complex C_*M can then be understood as a chain complex in $\mathbb{B}(\mathbb{A}_*(X))$ via the decomposition

$$C_*M = \sum_\sigma C_*(M[\sigma], \partial M[\sigma])$$

where $C_*(M[\sigma], \partial M[\sigma])$ is the cellular chain complex of $M[\sigma]/\partial M[\sigma]$. The dual of C_*M in $\mathbb{A}_*(X)$ is

$$T(C_*M) = \sum_\sigma C^{-|\sigma|-*}(M[\sigma]).$$

If M is (only) a topological manifold, and $f: M \to X$ is transverse to the stratification of X by subsets $X(\sigma)$, then we can still construct a contravariant functor $\sigma \mapsto F[\sigma]$ with CW-structure from the category of simplices of X to the category of spaces, and a natural homotopy equivalence $F[\sigma] \to M[\sigma]$, for σ in X. Then $\partial F[\sigma]$ is well defined: it is the CW-subspace of $F[\sigma]$ containing all the cells which come from some $F[\tau]$ via some $u\colon \sigma \to \tau$. The object

$$C_*F = \sum_\sigma C_*(F[\sigma], \partial F[\sigma])$$

in $\mathbb{B}(\mathbb{A}_*(X))$ is a good substitute for a possibly nonexistent C_*M.

A chain duality $T\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ can be used to define symmetric and quadratic chain complexes in $\mathbb{B}(\mathbb{A})$ as follows. Firstly, notice that given two objects M and N of $\mathbb{A}$, their tensor product $M \otimes_{\mathbb{A}} N$ possesses a symmetry isomorphism

$$T_{M,N}\colon M \otimes_{\mathbb{A}} N \to N \otimes_{\mathbb{A}} M$$

given by taking

$$f \in (M \otimes_{\mathbb{A}} N)_n = \mathrm{Hom}_{\mathbb{A}}(T(M)_{-n}, N)$$

to

$$T_{M,N}(f) \in (N \otimes_{\mathbb{A}} M)_n = \mathrm{Hom}_{\mathbb{A}}(T(N)_{-n}, M)$$

where

$$T_{M,N}(f) = e_M \cdot T(f)\colon T(N)_n \to T(T(M)_{-n})_{-n} \subseteq T^2(M)_0 \to M.$$

This tensor products extends to a tensor product $C \otimes_{\mathbb{A}} D$ of chain complexes C, D in $\mathbb{B}(\mathbb{A})$ and there is also a symmetry isomorphism

$$T_{C,D}\colon C \otimes_{\mathbb{A}} D \to D \otimes_{\mathbb{A}} C.$$

If $C = D$ this makes $C \otimes_{\mathbb{A}} C$ into a finite chain complex of $\mathbb{Z}[\mathbb{Z}_2]$-modules. Now let W be the standard $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$, i.e. it is a chain complex of $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$W = \cdots \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \to 0$$

concentrated in non-negative degrees. Then there are the following two chain complexes of abelian groups

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C),$$

$$W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathbb{A}} C).$$

Definition 3.7. An n-dimensional *symmetric algebraic complex* in $\mathbb{B}(\mathbb{A})$ is a pair (C, φ) with C a chain complex in $\mathbb{B}(\mathbb{A})$ and φ an n-cycle in $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_{\mathbb{A}} C)$. An n-dimensional *quadratic algebraic complex* in $\mathbb{B}(\mathbb{A})$ is a pair (C, ψ) with C a chain complex in $\mathbb{B}(\mathbb{A})$ and ψ an n-cycle in $W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_{\mathbb{A}} C)$.

We note that in the above definition it is not required that the chain complex C is concentrated in dimensions from 0 to n, it is only required that it is bounded below and above. The dimension n is associated with the symmetric structure φ or with the quadratic structure ψ.

An n-dimensional symmetric structure φ on a chain complex C can be described as a collection of chains in $\mathrm{Hom}_{\mathbb{A}}(T(C), C)$,

$$\varphi = \{\varphi_s \colon T(C)_{-*} \to C_{n-*+s} \mid s \geq 0\}$$

satisfying certain relations.

An n-dimensional quadratic structure ψ on a chain complex C can be described as a collection of chains in $\mathrm{Hom}_{\mathbb{A}}(T(C), C)$,

$$\psi = \{\psi_s \colon T(C)_{-*} \to C_{n-*-s} \mid s \geq 0\}$$

satisfying certain relations.

An n-dimensional quadratic structure ψ on C determines an n-dimensional symmetric structure φ on C by $\varphi_0 = (1+T)\psi_0$ and $\varphi_s = 0$ for $s > 0$. We describe this relationship by writing $\varphi = (1+T)\psi$.

Definition 3.8. For C in $\mathbb{B}(\mathbb{A})$, an n-cycle in $C \otimes_{\mathbb{A}} C \cong \mathrm{Hom}_{\mathbb{A}}(TC, C)$ is *nondegenerate* if the corresponding chain map $TC \to C$ of degree n is a chain homotopy equivalence. An n-dimensional *symmetric algebraic Poincaré complex* (SAPC) in $\mathbb{B}(\mathbb{A})$ is a symmetric algebraic complex (C, φ) such that φ_0 is nondegenerate. An n-dimensional *quadratic algebraic Poincaré complex* (QAPC) in $\mathbb{B}(\mathbb{A})$ is a quadratic algebraic complex (C, ψ) such that $(1+T)\psi_0$ is nondegenerate.

Example 3.9. Let X be a connected finite CW-complex and $\widetilde{X} \to X$ a universal covering with deck transformation group π. The diagonal map

$$\nabla \colon X \to \widetilde{X} \times_{\pi} \widetilde{X}$$

is a $\mathbb{Z}_2$-map for the trivial action of $\mathbb{Z}_2$ on the source and the permutation action on the target. It is not cellular in general. However it is easy to construct a *cellular* $\mathbb{Z}_2$-map

$$\nabla^\sharp : E\mathbb{Z}_2 \times X \to \widetilde{X} \times_\pi \widetilde{X}$$

which is $\mathbb{Z}_2$-homotopic to the composition of ∇ just above with the projection $E\mathbb{Z}_2 \times X \to X$. Here $E\mathbb{Z}_2$ can be taken as the universal (=double) cover of $B\mathbb{Z}_2 = \mathbb{R}P^\infty$, with the standard CW-structure on $\mathbb{R}P^\infty$. Hence the map of cellular chain complexes induced by $\nabla^\sharp$ takes the form

$$W \otimes C_*X \longrightarrow C_*\widetilde{X} \otimes_{\mathbb{Z}[\pi]} C_*\widetilde{X}$$

with adjoint

$$C_*X \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C_*\widetilde{X} \otimes_{\mathbb{Z}[\pi]} C_*\widetilde{X}).$$

Regard now $C_*\widetilde{X}$ as an object in $\mathbb{B}(\mathbb{A}[\pi])$, with $\mathbb{A}[\pi]$ as in Example 3.2. Then by all the above, any n-cycle μ in C_*X determines an n-dimensional symmetric structure $\varphi(X)$ on $C_*\widetilde{X}$. If X is an orientable Poincaré duality space and μ represents a fundamental class $[X]$, then φ_0 is nondegenerate and so

$$(C_*\widetilde{X}, \varphi(X))$$

is an n-dimensional SAPC.

Example 3.10. Let $(f,b)\colon M \to X$ be a degree one normal map of n-dimensional closed manifolds or Poincaré duality spaces, where X is connected and equipped with a universal covering. Denote by $K(f)$ the algebraic mapping cone of the Umkehr map of chain complexes

$$f^!\colon C_*\widetilde{X} \simeq C^{n-*}\widetilde{X} \xrightarrow{\ f^{n-*}\ } C^{n-*}\widetilde{M} \simeq C_*\widetilde{M}.$$

As explained just above, $C_*\widetilde{X}$ comes with a structure of n-dimensional SAPC over $\mathbb{Z}[\pi]$. This projects to a structure of n-dimensional SAPC on $K(f)$. Ranicki in [RaLMS2], [Ra] refines the latter to an n-dimensional QAPC on $(K(f), \psi(f))$.

In the next definition, the standard simplex Δ^n is regarded as a simplicial complex in the usual way. Each face inclusion $\Delta^{n-1} \to \Delta^n$ induced by the monotone injection $\{0,1,\dots,n-1\} \to \{0,1,\dots,n\}$ induces an additive functor $d_i\colon \mathbb{A}^*(\Delta^n) \to \mathbb{A}^*(\Delta^{n-1})$ which commutes with the chain dualities. We identify $\mathbb{A}^*(\Delta^0)$ with $\mathbb{A}$.

Definition 3.11. Two n-dimensional SAPC (QAPC) in $\mathbb{B}(\mathbb{A})$, say (C,φ) and (C',φ'), are called *cobordant* if there exists an n-dimensional SAPC (QAPC), say (D,ψ) in $\mathbb{B}(\mathbb{A}^*(\Delta^1))$, such that $d_0(D,\psi) \cong (C,\varphi)$ and $d_1(D,\psi) \cong (C',\varphi')$.

With the alternative definition of $\mathbb{A}^*(\Delta^1)$ outlined in Remark 3.5, the cobordism relation is an equivalence relation on n-dimensional SAPC (QAPC). The direct sum makes the cobordism classes of SAPC (QAPC) into an abelian group, where the inverse of $[(C,\varphi)]$ is given by $[(C,-\varphi)]$.

Definition 3.12. The group of cobordism classes of n-dimensional SAPC in $\mathbb{B}(\mathbb{A})$ is denoted by $L^n(\mathbb{A})$. The group of cobordism classes of n-dimensional QAPC in $\mathbb{B}(\mathbb{A})$ is denoted by $L_n(\mathbb{A})$.

Example 3.13. For the category $\mathbb{A}[\pi]$ with chain duality as in Example 3.2, the L-groups $L^n(\mathbb{A}[\pi])$ are the usual symmetric L-groups $L^n(\pi)$ of Mishchenko and the L-groups $L_n(\mathbb{A}[\pi])$ are the quadratic L-groups $L_n(\pi)$ of Wall (see [Ra]).

The L-groups are in fact the homotopy groups of certain spaces. These are defined as Δ-sets in the following way.

Definition 3.14. Let $\mathsf{L}^n(\mathbb{A})$, resp. $\mathsf{L}_n(\mathbb{A})$, denote the Δ-set whose k-simplices are n-dimensional SAPC, resp. QAPC in the category $\mathbb{A}^*(\Delta^k)$. The face maps are induced by the functors $d_i\colon \mathbb{A}^*(\Delta^k) \to \mathbb{A}^*(\Delta^{k-1})$. We use the alternative definitions of $\mathbb{A}^*(\Delta^k)$ given in Remark 3.5.

Example 3.15. With $\mathbb{A}[\pi]$ as in Example 3.2, the L-theory space $\mathsf{L}_n(\mathbb{A}[\pi])$ is the L-theory space $\mathsf{L}_n(\pi)$ of Quinn, with homotopy groups $\pi_k\mathsf{L}_n(\pi) = L_{k+n}(\pi)$.

Remark 3.16. The assignment $\psi \mapsto (1+T)\cdot\psi$ for (C,ψ) an n-dimensional QAPC in $\mathbb{B}(\mathbb{A})$ defines a symmetrization map $(1+T)\colon \mathsf{L}_n(\mathbb{A}) \to \mathsf{L}^n(\mathbb{A})$.

Let X be a finite Δ-complex. We now have the definitions of the spaces $\mathsf{L}_n(\mathbb{A}_*(X))$ and $\mathsf{L}_n(\mathbb{A}[\pi])$ from the homotopy fibration sequence (3.1), ignoring orientation matters which will be discussed later. Assuming $\mathbb{A} = \mathbb{A}(\mathbb{Z})$ for simplicity, we proceed to describe the map α from $\mathsf{L}_n(\mathbb{A}_*(X))$ to $\mathsf{L}_n(\mathbb{A}[\pi])$ which is called *assembly*.

Suppose that X comes with a principal π-bundle $p: X^\natural \to X$. (In most applications this will be a universal covering for X, and X will be connected, but we do not have to assume that now.) The map α is induced by an additive functor, also denoted α. Define

$$\alpha\colon \mathbb{A}_*(X) \to \mathbb{A}[\pi]$$

on objects by

$$\alpha(M) = \sum_{\sigma \in sX^\natural} M(p(\sigma))$$

with π acting on the right-hand side by permuting summands in the obvious way. A morphism $f: M \to N$ in $\mathbb{A}_*(X)$ induces $\alpha(f)\colon \alpha(M) \to \alpha(N)$, which we define in matrix notation by

$$\alpha(f)_{(\sigma,\tau)} = \sum_{u\colon \sigma\to\tau} f_{p(u)}$$

where $u\colon \sigma \to \tau$ in $X^\natural$ and $p(u)\colon p(\sigma) \to p(\tau)$ is the induced morphism in X.

In order to see that the assembly functor α induces a map of the L-spaces one has to see that it "commutes" with the chain dualities as in Examples 3.2, 3.4, or with the corresponding tensor products. We choose the tensor product option. The coefficient system $\sigma \mapsto \sum_{\lambda \leftarrow \sigma \rightarrow \mu} M(\lambda) \otimes_{\mathbb{A}} N(\mu)$ on X comes with an evident natural transformation to the constant coefficient system $\sigma \mapsto \alpha(M) \otimes_{\mathbb{A}[\pi]} \alpha(N)$, due to the fact that a diagram such as $\lambda \leftarrow \sigma \rightarrow \mu$ in X determines a preferred path class in X connecting the barycenters of λ and μ. Passing to the cellular chain complexes associated with these coefficient systems gives

$$M \otimes_{\mathbb{A}_*(X)} N \longrightarrow C_*X \otimes \big(\alpha(M) \otimes_{\mathbb{A}[\pi]} \alpha(N)\big).$$

We compose with the augmentation $C_*(X) \to \mathbb{Z}$ to get

$$M \otimes_{\mathbb{A}_*(X)} N \longrightarrow \alpha(M) \otimes_{\mathbb{A}[\pi]} \alpha(N)$$

and more generally

$$C \otimes_{\mathbb{A}_*(X)} D \longrightarrow \alpha(C) \otimes_{\mathbb{A}[\pi]} \alpha(D)$$

for objects C and D in $\mathbb{B}(\mathbb{A}_*(X))$. We use this to transport symmetric and quadratic structures. Nondegeneracy is preserved, so that QAPC are mapped to QAPC. It follows that we have a well defined map of L-spaces

$$\alpha\colon \mathsf{L}_n(\mathbb{A}_*(X)) \to \mathsf{L}_n(\mathbb{A}[\pi]), \tag{3.4}$$

which is also called assembly. It is an algebraic version, due to Ranicki, of the assembly map of Quinn [Qu]. Apart from being algebraic, it also incorporates Poincaré duality to switch from a cohomological setup to a homological one.

Remark 3.17. In the above construction we indicated how an n-dimensional QAPC in $\mathbb{B}(\mathbb{A}_*(X))$, say (C,ψ), determines an assembled n-dimensional QAPC in $\mathbb{B}(\mathbb{A}[\pi])$, denoted $\alpha(C,\psi)$. For the sake of readability we will sometimes omit the prefix α in the sequel, provided it is clear enough in which category we are working.

Remark 3.18. The spaces $\mathsf{L}_n(\mathbb{A})$ can be arranged into an Ω-spectrum $\mathsf{L}_\bullet(\mathbb{A})$, with homotopy groups $\pi_k \mathsf{L}_\bullet(\mathbb{A}) \cong L_k(\mathbb{A})$ for $k \in \mathbb{Z}$. Beware that $\mathsf{L}_n(\mathbb{A})$ is the $(-n)$-th space in the Ω-spectrum $\mathsf{L}_\bullet(\mathbb{A})$. With our conventions, $L_k(\mathbb{A})$ is isomorphic to $L_{k+4}(\mathbb{A})$ for all $k \in \mathbb{Z}$, and indeed $\mathsf{L}_\bullet(\mathbb{A}) \simeq \Omega^4 \mathsf{L}_\bullet(\mathbb{A})$. We also have Ranicki's law

$$\pi_k(\mathsf{L}_n(\mathbb{A}_*(X))) \cong H_{n+k}(X; \mathsf{L}_\bullet(\mathbb{A}))$$

for $k, n \in \mathbb{Z}$, and to be more precise $\mathsf{L}_\bullet(\mathbb{A}_*(X)) \simeq X_+ \wedge \mathsf{L}_\bullet(\mathbb{A})$. See [Ra] for details. When $\mathbb{A}$ is the category of f.g. free $\mathbb{Z}$-modules with the standard chain duality, then we write $\mathsf{L}_\bullet$ for $\mathsf{L}_\bullet(\mathbb{A})$.

Example 3.19. Let $(f,b)\colon M \to N$ be a degree one normal map of closed n-dimensional manifolds and $g\colon N \to X$ be a map to a simplicial complex X such that both gf and g are transverse to the dual cells of X. By Example 3.6 we have X-dissections $M \cong \mathrm{colim} M[\sigma]$ and $N \cong \mathrm{colim} N[\sigma]$, so that C_*M and C_*N can be regarded as objects in $\mathbb{B}(\mathbb{A}_*(X))$, for suitable CW-structures on M and N. (As it stands this holds only if M and N are smooth or PL manifolds. For topological manifolds use the CW approximations as described in 3.6). By analogy with Example 3.9, there are preferred structures of n-dimensional SAPC on C_*M and C_*N, as objects of $\mathbb{B}(\mathbb{A}_*(X))$. By analogy with Example 3.10, there is an algebraic Umkehr map

$$f^! : C_*N \longrightarrow C_*M$$

in $\mathbb{B}(\mathbb{A}_*(X))$ with mapping cone $K(f)$, say. The resulting structure of n-dimensional SAPC on $K(f)$, as an object of $\mathbb{B}(\mathbb{A}_*(X))$, has a preferred refinement to a QAPC structure ψ. We remark that $K(f)(\sigma)$ for a simplex σ in X can be identified with the mapping cone of an algebraic Umkehr map

$$C_*(N[\sigma], \partial N[\sigma]) \longrightarrow C_*(M[\sigma], \partial M[\sigma])$$

which is the diagonal entry $f^!(\sigma,\sigma)$ of $f^!$. See again [Ra] for details. Under assembly, these constructions match and recover those in Examples 3.9 and 3.10.

Suppose that X is a connected Δ-complex equipped with a universal covering, with deck transformation group π. One way to define the space $\mathsf{S}(X,n)$ is to say that it is the homotopy fiber of the assembly map

$$\alpha : \mathsf{L}_n(\mathbb{A}_*(X)) \to \mathsf{L}_n(\mathbb{A}[\pi]).$$

However, the theory of algebraic bordism categories of [Ra] can be used to provide a more direct description of the space $\mathsf{S}(X,n)$ as the L-theory space associated to certain additive category with chain duality (with certain restrictions on the objects). The description is as follows.

Definition 3.20. A k-simplex of the space $\mathsf{S}(X,n)$ is an n-dimensional QAPC in the category $\mathbb{B}((\mathbb{A}_*(X))^*(\Delta^k))$ which assembles to a contractible QAPC in $\mathbb{B}((\mathbb{A}[\pi])^*(\Delta^k))$.

Then there is an obvious inclusion map $\mathsf{S}(X,n) \to \mathsf{L}_n(\mathbb{A}_*(X))$. By the result of [Ra, Proposition 3.9] the sequence (3.1) consisting of this map and the assembly map α is a homotopy fibration sequence.

Theorem 4.5 of [We2] provides the following alternative to Definition 3.20:

Definition 3.21. A k-simplex of $\mathsf{S}(X,n)$ is an n-dimensional SAPC in the category $\mathbb{B}((\mathbb{A}_*(X))^*(\Delta^k))$ which assembles to a contractible SAPC in $\mathbb{B}((\mathbb{A}[\pi])^*(\Delta^k))$.

3.1 Twisted versions

Now we recall modifications in the above machinery needed to treat the general case of nonorientable or just nonoriented manifolds. It will be necessary to modify the definition of the tensor product of X-based objects and the tensor product of $\mathbb{Z}[\pi]$-modules and thus also the assembly map.

Definition 3.22. A *twist* on a group π is a π-module Γ whose underlying abelian group is infinite cyclic. A homomorphism of twisted groups, say from (π,Γ) to (π',Γ'), is a homomorphism $f\colon \pi \to \pi'$ together with an isomorphism $\Gamma \to f^*\Gamma'$ of π-modules.

Example 3.23. For a connected n-manifold X and a universal covering of X with deck transformation group π, there is a canonical way to define a twist on π. Let Γ be the n-th integer homology with locally finite coefficients of the universal covering. The action of π on Γ is obvious.

We now fix a finite Δ-complex X with a principal π-bundle $p: X^\natural \to X$ and a twist Γ on π. The twist determines a homomorphism $w\colon \pi \to \{\pm 1\}$ such that $gz = w(g)\cdot z$ for all $g \in \pi$ and $z \in \Gamma$. On the group ring $\mathbb{Z}[\pi]$ we have the w-twisted involution given by

$$g \mapsto w(g)\cdot g^{-1}.$$

The group ring with this involution will be denoted by $\mathbb{Z}[\pi]^w$ and the category of f.g. free left $\mathbb{Z}[\pi]^w$-modules will be denoted by $\mathbb{A}[\pi]^w$. We already have a chain duality T on $\mathbb{A}[\pi]^w$ from Example 3.2; unfortunately this is no longer considered quite right and to correct it we compose with the functor $\Gamma\otimes_{\mathbb{Z}}$. Note that this causes a small change in $\otimes_{\mathbb{A}[\pi]^w}$ as well. Namely, for objects M and N in $\mathbb{A}[\pi]^w$ there is the following isomorphism of abelian groups:

$$(3.5) \qquad M \otimes_{\mathbb{A}[\pi]^w} N = \Gamma \otimes_{\mathbb{Z}[\pi]} (M \otimes_{\mathbb{Z}} N).$$

Similar remarks apply to $\mathbb{A}_*(X)$, which we now rename $\mathbb{A}_*(X,w)$ to indicate a modified chain duality. We already have a chain duality T on $\mathbb{A}_*(X)$ from Example 3.4; this is no longer considered quite right for $\mathbb{A}_*(X,w)$. To correct it we define a "local coefficient system" $\Gamma_!$ of infinite cyclic groups on X by $\Gamma_!(\sigma) = \Gamma \times_\pi p^{-1}(\hat{\sigma})$, for simplices $\sigma \in sX$ with barycenter $\hat{\sigma}$. Then we compose the old chain duality T with the endofunctor given by

$$\sum_{\sigma\in sX} M(\sigma) \;\mapsto\; \sum_{\sigma\in sX} M(\sigma)\otimes\Gamma_!(\sigma)$$

to obtain the new duality. Hence, for objects M and N of $\mathbb{A}_*(X)$, there is an embedding

$$M \otimes_{\mathbb{A}_*(X)} N \longrightarrow C_*(X;\Gamma_!)\otimes(M\otimes_{\mathbb{Z}} N).$$

The assembly functor $\alpha\colon \mathbb{A}_*(X,w) \to \mathbb{A}[\pi]^w$ is defined exactly as in the untwisted setting by

$$M \mapsto \alpha(M) = \sum_{\sigma^\natural \in sX^\natural} M(\sigma^\natural)$$

with $M(\sigma^\natural) := M(p(\sigma^\natural))$, but the old definition of the comparison maps

$$C \otimes_{\mathbb{A}_*(X)} D \longrightarrow \alpha(C) \otimes_{\mathbb{A}[\pi]} \alpha(D)$$

for C, D in $\mathbb{B}(\mathbb{A}_*(X))$ has to be modified since its source and target are not what they were then. This is straightforward. As before, the assembly functor induces a map between L-theory spaces

$$\alpha\colon \mathsf{L}_n(\mathbb{A}_*(X,w)) \to \mathsf{L}_n(\mathbb{A}[\pi]^w)$$

also called assembly. Assuming that X is connected and $p : X^\natural \to X$ is a universal covering, one can define the space $\mathsf{S}(X,n,w)$ as the homotopy fiber of the assembly map. There is also a description of this space as an L-theory space of the category of chain complexes in $\mathbb{B}(\mathbb{A}_*(X,w))$ with contractible assembly in $\mathbb{B}(\mathbb{A}[\pi]^w)$.

3.2 Truncated version

Let X be a finite Δ-complex with subcomplexes X_1 and X_2 such that $X_1 \cup X_2 = X$. Ranicki's law mentioned earlier in Remark 3.18 can also be formulated by saying that the square of Ω-spectra

$$\begin{array}{ccc} \mathsf{L}_\bullet(\mathbb{A}_*(X_1 \cap X_2)) & \longrightarrow & \mathsf{L}_\bullet(\mathbb{A}_*(X_1)) \\ \downarrow & & \downarrow \\ \mathsf{L}_\bullet(\mathbb{A}_*(X_2)) & \longrightarrow & \mathsf{L}_\bullet(\mathbb{A}_*(X_1 \cup X_2)) \end{array}$$

is a homotopy pushout square. This implies that $X \mapsto \pi_*\mathsf{L}_\bullet(\mathbb{A}_*(X))$ is a generalized (unreduced) homology theory. We now recall some of Postnikov's method for making truncated variants of generalized homology theories.

Let $X \mapsto Q_*(X)$ be a generalized homology theory (from finite Δ-complexes to graded abelian groups, say). Fix $r \in \mathbb{Z}$. We define the Postnikov fiber truncation $\wp^r$ by

$$\wp^r Q_n(X) = \mathrm{im}[Q_n(X^{n-r}) \to Q_n(X^{n-r+1})].$$

Then $\wp^r Q_*$ is again a generalized homology theory and there are long exact sequences

$$\cdots \to H_{n-r+1}(X;Q_r) \to \wp^{r+1}Q_n(X) \to \wp^r Q_n(X) \to H_{n-r}(X;Q_r) \to \cdots$$

where $Q_r = Q_r(*)$. For fixed $r \le n - \dim(X)$, we clearly have $\wp^r Q_n(X) = Q_n(X)$.

In our situation there is a space version of this construction which we outline very briefly. Let (C,ψ) be a k-simplex in $\mathsf{L}_n(\mathbb{A}_*(X))$, alias n-dimensional QAPC in $(\mathbb{A}_*(X))^*(\Delta^k)$. We say that (C,ψ) is of type $\wp^r$ if, for simplices $\sigma \in sX$, $\tau \in \Delta^k$, the chain complex $C(\sigma,\tau)$ is zero if $|\sigma| - |\tau| > n - r + 1$, and nullbordant as a QAPC of dimension $n - |\sigma| + |\tau|$ in $\mathbb{B}(\mathbb{A})$ if $|\sigma| - |\tau| = n - r + 1$ (with the QAPC structure determined by ψ). The simplices of type $\wp^r$ form a Δ-subset $\mathsf{L}_n(\mathbb{A}_*(X);\, \wp^r)$ of $\mathsf{L}_n(\mathbb{A}_*(X))$. Letting n vary, these Δ-subsets can be arranged into a subspectrum

$$\mathsf{L}_\bullet(\mathbb{A}_*(X);\, \wp^r) \subset \mathsf{L}_\bullet(\mathbb{A}_*(X))$$

and we have $\mathsf{L}_\bullet(\mathbb{A}_*(X);\, \wp^r) \simeq X_+ \wedge \wp^r \mathsf{L}_\bullet(\mathbb{A})$. The verification, along the lines of Ranicki's reasoning for $r = -\infty$, is left to the reader.

For us, only the case of $\mathsf{L}_n(\mathbb{A}_*(X);\, \wp^1)$ with $\dim(X) = n$ is of interest. In that case $\mathsf{L}_n(\mathbb{A}_*(X);\, \wp^1)$ is a Δ-subset of $\mathsf{L}_n(\mathbb{A}_*(X))$ determined by a condition on the 0-simplices only. A 0-simplex (C,ψ) of $\mathsf{L}_n(\mathbb{A}_*(X))$, alias n-dimensional QAPC in $\mathbb{A}_*(X)$, belongs to $\mathsf{L}_n(\mathbb{A}_*(X);\, \wp^1)$ if and only if $(C(\sigma), \psi|\sigma)$ is a nullbordant 0-dimensional QAPC in $\mathbb{B}(\mathbb{A})$ for every n-simplex σ in X. (The higher-dimensional simplices in $\mathsf{L}_n(\mathbb{A}_*(X))$ belong to $\mathsf{L}_n(\mathbb{A}_*(X);\, \wp^1)$ precisely if all their vertices do.) Note that if X is connected, then the phrase *for every n-simplex* σ can be replaced by *for some n-simplex* σ.

3.3 Geometric versus algebraic surgery sequence.

For a Δ-complex X which is an n-dimensional connected oriented manifold and a specified universal covering of X with deck transformation group π, there is the following diagram of homotopy fibration sequences

(3.6)
$$\begin{array}{ccccc}
\widetilde{\mathcal{S}}(X) & \longrightarrow & \mathcal{N}(X) & \longrightarrow & \mathcal{L}_n(X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{S}(X;\, \wp^1) & \longrightarrow & \mathsf{L}_n(\mathbb{A}_*(X);\, \wp^1) & \longrightarrow & \mathsf{L}_n(\mathbb{A}[\pi])
\end{array}$$

where the vertical arrows are homotopy equivalences, and $\mathsf{S}(X;\, \wp^1)$ is *defined* as the homotopy fiber of $\mathsf{L}_n(\mathbb{A}_*(X);\, \wp^1) \to \mathsf{L}_n(\mathbb{A}[\pi])$.

Note that in Example 3.19 we essentially described a map $\mathcal{N}(X) \to \mathsf{L}_n(\mathbb{A}_*(X))$. This factors through $\mathsf{L}_n(\mathbb{A}_*(X);\, \wp^1)$. Indeed for an n-dimensional simplex σ in X the dual cell $D(\sigma, X)$ is a point $\hat{\sigma}$ and for any degree one map $f\colon M \to X$ transverse to $\hat{\sigma}$, the inverse image $f^{-1}(\hat{\sigma})$ is a closed 0-manifold of signature 1. (Hence the signature of $f^{-1}(\hat{\sigma})$ minus the signature of $\hat{\sigma}$ is 0.) Again there are versions of the identification for the cases when X is a non-orientable or just non-oriented manifold. The details are again left to the reader.

Remark 3.24. Strictly speaking, in order to define a map $\mathcal{N}(X) \to \mathsf{L}_n(\mathbb{A}_*(X))$ we should have added CW-structures as in Example 3.6 and "geometric symmetric structures" (maps $\nabla^\sharp$ as in Example 3.9) on the manifolds or CW-spaces involved to the geometric data, since choices of these must be made before the algebraic data can be extracted. However, these choices are "contractible" choices. Adding them or neglecting them does not change the homotopy type of $\mathcal{N}(X)$.

By a result of Ranicki [Ra, Theorem 18.5] the vertical maps in the diagram (3.6) are homotopy equivalences. Further note that in our case $Q_0(*) = \mathbb{Z}$ and hence we have a homotopy fibration sequence $\mathsf{L}_n(\mathbb{A}_*(X); \wp^1) \to \mathsf{L}_n(\mathbb{A}_*(X)) \to \mathbb{Z}$ and, as a consequence also a homotopy fibration sequence

(3.7) $\quad \widetilde{\mathcal{S}}(X) \to \mathsf{S}(X) \to \mathbb{Z}.$

3.4 Products

For (C, φ) an m-dimensional SAPC in $\mathbb{B}(\mathbb{A}[\pi])$, and (C', φ') an n-dimensional SAPC in $\mathbb{B}(\mathbb{A}[\pi'])$, we have $(C \otimes C', \varphi \otimes \varphi')$, an $(m+n)$-dimensional SAPC in $\mathbb{B}(\mathbb{A}[\pi \times \pi'])$. For (C, φ) an m-dimensional SAPC in $\mathbb{B}(\mathbb{A}[\pi])$, and (D, ψ) an n-dimensional QAPC in $\mathbb{B}(\mathbb{A}[\pi'])$, we have $(C \otimes D, \varphi \otimes \psi)$, an $(m+n)$-dimensional QAPC in $\mathbb{B}(\mathbb{A}[\pi \times \pi'])$. See [RaLMS1, Section 8].

Let X be an m-dimensional, and Y an n-dimensional Poincaré duality CW-complexes with chosen orientation classes. Then we have a natural isomorphism

(3.8) $\quad (C_*(\widetilde{X} \times \widetilde{Y}), \varphi(X \times Y)) \cong (C_*\widetilde{X} \otimes C_*\widetilde{Y}, \varphi(X) \otimes \varphi(Y))$

of $(m+n)$-dimensional SAPCs in $\mathbb{B}(\mathbb{A}[\pi_1(X \times Y)])$. See [RaLMS2, Section 8]. We are assuming that g+eometric symmetric structures on X and Y as in Example 3.9 have been selected, and use the product geometric symmetric structure on $X \times Y$. A similar but slightly more complicated statement for normal maps and quadratic structures is available. We do not formulate this because we will not need it, thanks to the stated equivalence of Definitions 3.21 and 3.20.

4 Orthogonal calculus and products

The orthogonal calculus [We] is about continuous functors from a certain category $\mathcal{J}$ of real vector spaces to the category of spaces. For details and definitions, see also [Ma]. Here we take another look at orthogonal calculus from a "multiplicative" viewpoint.

Let $\mathcal{J}^{\mathrm{iso}}$ be the subcategory of the isomorphisms in $\mathcal{J}$. The objects of $\mathcal{J}^{\mathrm{iso}}$ are the finite dimensional real vector spaces $V, W, \ldots$ with inner product, and the space of morphisms from V to W in $\mathcal{J}^{\mathrm{iso}}$ is the space of invertible linear isometries from V to W.

Definition 4.1. Let E and F be continuous functors from $\mathcal{J}^{\mathrm{iso}}$ to based spaces. A *multiplication* on E is a binatural based map $m : E(V) \wedge E(W) \to E(V \oplus W)$, defined for V and W in $\mathcal{J}^{\mathrm{iso}}$, which satisfies the appropriate associativity law. A *unit* for the multiplication is a distinguished element $1 \in E(0)$ which is a neutral element for the multiplication m. For E equipped with a multiplication m and a unit, an *action* of E on F is a binatural based map

$$a : E(V) \wedge F(W) \to F(V \oplus W),$$

again defined for V and W in $\mathcal{J}^{\mathrm{iso}}$, which satisfies the appropriate associativity law involving m and a, and has $1 \in E(0)$ acting by identity maps.

Example 4.2. Let $E(V) = \mathbb{S}^0$ for all V, with $m : E(V) \wedge E(W) \cong E(V \oplus W)$ for all V, W. For a continuous F from $\mathcal{J}^{\mathrm{iso}}$ to based spaces, an action of E on F amounts to an extension of F from $\mathcal{J}^{\mathrm{iso}}$ to $\mathcal{J}$.

Example 4.3. Let E be given as in Definition 4.1, with multiplication m. Fix U in $\mathcal{J}^{\mathrm{iso}}$. We are going to define an F from $\mathcal{J}^{\mathrm{iso}}$ to spaces, with an action of E, in such a way that F is *free* on one generator $\iota \in F(U)$. We set $F(W) = *$ if $\dim(W) < \dim(U)$. For W with $\dim(W) - \dim(U) = k \geq 0$ we define

$$F(W) = \mathrm{mor}(U \oplus \mathbb{R}^k, W)_+ \wedge_{\mathrm{O}(k)} E(\mathbb{R}^k).$$

Here "mor" refers to a space of morphisms in $\mathcal{J}^{\mathrm{iso}}$, and the $\wedge_{\mathrm{O}(k)}$ notation means that we are dividing by the equivalence relation which identifies (gh, x) with (g, hx) whenever $h \in \mathrm{O}(k) \subset \mathrm{O}(U \oplus \mathbb{R}^k)$. It is clear that F is a functor on $\mathcal{J}^{\mathrm{iso}}$. The multiplication m on E determines an action of E on F as follows. For $x \in E(V)$ we have the left multiplication $m_x : E(\mathbb{R}^k) \to E(V \oplus \mathbb{R}^k)$ and we define the action $a_x : F(W) \to F(V \oplus W)$ by

$$\begin{array}{c} \mathrm{mor}(U \oplus \mathbb{R}^k, W)_+ \wedge_{\mathrm{O}(k)} E(\mathbb{R}^k) \\ \Big\downarrow \mathrm{incl.} \wedge m_x \\ \mathrm{mor}(U \oplus V \oplus \mathbb{R}^k, V \oplus W)_+ \wedge_{\mathrm{O}(V \oplus \mathbb{R}^k)} E(V \oplus \mathbb{R}^k) \xrightarrow{\cong} F(V \oplus W). \end{array}$$

The generator of F is $\iota = (\mathrm{id}, 1) \in F(U)$.

Let F_1 be another continuous functor from $\mathcal{J}^{\mathrm{iso}}$ to spaces with an action of E. Then a map $v : F \to F_1$ which respects the actions of E is completely determined by $v(\iota) \in F_1(U)$, which can be prescribed arbitrarily.

Definition 4.4. Given E and F as in Definition 4.1, with multiplication m and action a, we say that F is *free* if it has a wedge decomposition $F \cong \bigvee F_\lambda$ where each F_λ is free on one generator, as in Example 4.3.

Definition 4.5. Let E, F be as in Definition 4.1, with multiplication m on E and action a of E on F. An *E-CW-structure* on F is a collection of subfunctors $F^i \subset F$ for $i = -1, 0, 1, 2, \ldots$, subject to a few conditions:

- $F^{-1} = *$ and $F^i \subset F^{i+1}$ for $i \geq -1$;
- $F(V) = \bigcup_i F^i(V)$ with the colimit topology, for all V;
- the action of E on F respects each F^i;
- for every $i \geq -1$, there exists a pushout square

$$\begin{array}{ccc} Z^i \wedge \mathbb{S}^i_+ & \xrightarrow{\subset} & Z^i \wedge \mathbb{D}^{i+1}_+ \\ \downarrow & & \downarrow \\ F^i & \xrightarrow{\subset} & F^{i+1} \end{array}$$

where Z^i is another functor from $\mathcal{J}^{\rm iso}$ to spaces, with a free action of E, and the (vertical) arrows respect the actions of E.

The subfunctor F^i is sometimes called the i-skeleton of F.

Lemma 4.6. *Let E, F be as in Definition 4.1, with multiplication m on E and action a of E on F. There exists an E-CW-approximation for F. That is, there exists a weak equivalence $\hat{F} \to F$ of continuous functors on $\mathcal{J}^{\rm iso}$ with E-action, where $\hat{F}$ has an E-CW-structure as in 4.5.*

Definition 4.7. Let E_1, E_2 be functors as in Definition 4.1, with multiplications m_1 on E_1 and m_2 on E_2. Let $h\colon E_1 \to E_2$ be a natural transformation respecting the units and multiplications. We consider continuous functors F from $\mathcal{J}^{\rm iso}$ to pointed spaces, either with an action of E_1 or with an action of E_2. Composition with h gives a functor from the category of functors F as above with an action of E_2 to the category of functors F as above with an action of E_1. This functor has a left adjoint, which we call *induction along h* and denote by ind_h. Thus, for F from $\mathcal{J}^{\rm iso}$ to based spaces with an action of E_1, we have $\mathrm{ind}_h F$ from $\mathcal{J}^{\rm iso}$ to based spaces with an action of E_2. There is a canonical transformation $F \to \mathrm{ind}_h F$ which "intertwines" the actions and has a universal property.

The induction functor ind_h as defined above tends to produce pathological results. However, there are situations where it is well behaved:

Lemma 4.8. *Keep the notation of Definition 4.7. Suppose that F from $\mathcal{J}^{\rm iso}$ to based spaces comes with an action of E_1. If F has an E_1-CW-structure with skeletons F^i, then* $\mathrm{ind}_h F$ *has an E_2-CW-structure with skeletons* $\mathrm{ind}_h F^i$.

We therefore have something like a "derived induction" procedure which is as follows. Fix $h\colon E_1 \to E_2$ as above and some F from $\mathcal{J}^{\rm iso}$ to based spaces, with an action of E_1. Replace F by an E_1-CW-approximation as in Lemma 4.6. Then apply ind_h to the CW-approximation, assuming that a multiplicative $h\colon E_1 \to E_2$ is given.

Our interest here is mainly in the case where $E_2(V) = \mathbb{S}^0$ for all V as in Example 4.2. Then ind_h of an E_1-CW-approximation to F is a continuous functor on $\mathcal{J}$, and that is (still) the sort of object we are after.

Lemma 4.9. *Keep the assumptions of Lemma 4.8. Suppose in addition that the map $h\colon E_1(V) \to E_2(V)$ is a based homotopy equivalence for every V. Then the canonical map $F(W) \to$* $\mathrm{ind}_h F(W)$ *is a based homotopy equivalence for every W.*

Definition 4.10. Let $\mathcal{P}$ be the following monoidal category. Objects of $\mathcal{P}$ are pairs (X, u) where X is a finite Δ-complex homotopy equivalent to a sphere and $u\colon X \to X$ is a free involution respecting the Δ-complex structure. Morphisms are Δ-maps, respecting the involutions, which are weak homotopy equivalences. The monoidal operation $*$ is the join: $(X,u) * (Y,v) = (X * Y, u * v)$. Here $X * Y$ is given as follows. For a k-simplices $x \in X$ and an l-simplex $y \in Y$ there is a k-simplex x in $X * Y$, an l-simplex $y \in X * Y$ and a $(k+l+1)$-simplex $x * y \in X * Y$. The definition of the face maps and of $u * v$ is left to the reader.

Comment. We will often pretend that $\mathcal{P}$ is a *small* monoidal category. The reader is invited to add appropriate conditions to Definition 4.10 which ensure that.

Comment. If X is a finite Δ-complex homotopy equivalent to S^m and u is a free involution on X, then by the Lefschetz trace formula u acts on the reduced m-th homology of X by $(-1)^{m+1}$. Using that observation and obstruction theory, one can easily show that the orbit space X/u is homotopy equivalent to $\mathbb{R}P^m$. The case $m = -1$ is *not* an exception: in that case $X = \emptyset$ and the reduced (-1)-th homology is $\cong \mathbb{Z}$. Obviously $\emptyset$ is a very important object of $\mathcal{P}$ because it is a unit for the join operation.

We think of $\mathcal{P}$ as a combinatorial variant of the monoidal category $\mathcal{J}^{\mathrm{iso}}$ (the subcategory of $\mathcal{J}$ in which only isomorphisms are allowed as morphisms, with the monoidal operation *product* alias direct sum). The following definitions introduce a construction, essentially a homotopy Kan extension, which "transforms" a space-valued functor on $\mathcal{P}$ into a space-valued continuous functor on $\mathcal{J}^{\mathrm{iso}}$.

Definition 4.11. For V in $\mathcal{J}^{\mathrm{iso}}$, let $\mathcal{P}_V$ be the following topological category. An object is a triple (X,u,λ) where (X,u) is an object of $\mathcal{P}$ and $\lambda\colon X \to V \smallsetminus 0$ is a map which is

- a homotopy equivalence,
- linear on each simplex of X, and
- satisfies $\lambda u(x) = -\lambda(x)$.

A morphism from (X,u,λ) to (Y,v,ζ) is a morphism $f\colon (X,u) \to (Y,v)$ in $\mathcal{P}$ satisfying $\zeta f = \lambda$. The topology on the object class of $\mathcal{P}_V$ comes from the fact that the last "coordinate" λ in objects (X,u,λ) can vary continuously (within a finite dimensional space, since λ is determined by its values on vertices of X). The topology on the morphism class of $\mathcal{P}_V$ is defined in such a way that the square

$$\begin{array}{ccc} \mathrm{mor}(\mathcal{P}_V) & \xrightarrow{\text{target}} & \mathrm{ob}(\mathcal{P}_V) \\ {\scriptstyle\text{forgetful}}\downarrow & & \downarrow{\scriptstyle\text{forgetful}} \\ \mathrm{mor}(\mathcal{P}) & \xrightarrow{\text{target}} & \mathrm{ob}(\mathcal{P}) \end{array}$$

is a pullback square. Thus the projection functor $\mathcal{P}_V \to \mathcal{P}$ is continuous.

Lemma 4.12. *The classifying space $B\mathcal{P}_V$ is contractible.*

Proof. We replace the topological category $\mathcal{P}_V$ by a simplicial category $k \mapsto \mathcal{P}_{V,k}$. An object of $\mathcal{P}_{V,k}$ is an object (X,u) of $\mathcal{P}$ together with a map

$$\lambda\colon X \times \Delta^k \longrightarrow V$$

such that, for every $z \in \Delta^k$, the map $\lambda_z : X \to V$ defined by $\lambda_z(x) = \lambda(x, z)$ defines an object in $\mathcal{P}_V$. A morphism from (X, u, λ) to (Y, v, ζ) is a morphism $(X, u) \to (Y, v)$ in $\mathcal{P}$ making a certain triangle commute. It is easy to show that, for fixed $j \geq 0$, the canonical map

$$\left|k \mapsto N_j \mathcal{P}_{V,k}\right| \longrightarrow N_j \mathcal{P}_V$$

is a homotopy equivalence, where $N_\bullet$ denotes the nerve construction. Integrating over j, we conclude that the canonical map from $\left|k \mapsto B\mathcal{P}_{V,k}\right|$ to $B\mathcal{P}_V$ is a homotopy equivalence.

It remains to show that $B\mathcal{P}_{V,k}$ is contractible for fixed $k \geq 0$. This is a consequence of the fact that $\mathcal{P}_{V,k}$ is a *directed* category, in the strong sense that every finitely generated diagram in $\mathcal{P}_{V,k}$ admits a co-cone. Namely, suppose that $\mathcal{D}$ is a finitely generated category and $f : \mathcal{D} \to \mathcal{P}_{V,k}$ is any functor. Then there exists a constant functor $c : \mathcal{D} \to \mathcal{P}_{V,k}$ and a natural transformation $f \Rightarrow c$. To construct c, form the "direct limit" of f, not necessarily an object of $\mathcal{P}_{V,k}$ but a well defined finite Δ-complex Z with a free involution, and with a certain map from $Z \times \Delta^k$ to $V \smallsetminus 0$ respecting involutions. By attaching simplices to Z as appropriate (details follow), embed Z in an object Z' of $\mathcal{P}_{V,k}$.

We explain the last step in detail. The map $Z \times \Delta^k \to V$ can be written in adjoint form as $g : Z \longrightarrow U$ where $U = \mathrm{map}(\Delta^k, V \smallsetminus 0)$ is an open subset of the topological vector space $\mathrm{map}(\Delta^k, V)$ with the compact-open topology. The map g is involution-preserving and linear on each simplex of Z. It is easy to find a CW-space Z' relative to Z, with a free involution extending the one on Z and permuting the cells of $Z' \smallsetminus Z$ freely, and an involution-preserving map

$$g' : Z' \longrightarrow U$$

extending g. Now it remains to "improve" Z' to a Δ-complex with free involution containing Z as a Δ-subcomplex with free involution, and g' to a map which is linear on each simplex. This can be done by induction on the number of cell orbits in $Z' \smallsetminus Z$. Therefore we are left only with the case where $Z' \smallsetminus Z$ has exactly two cells, freely interchanged by the involution. Choosing a characteristic map for one the two cells, we get a commutative diagram

$$\begin{array}{ccc} S^{i-1} & \xrightarrow{\subset} & D^i \\ \downarrow{\scriptstyle \partial a} & & \downarrow{\scriptstyle a} \\ Z & \xrightarrow{g} & U. \end{array}$$

The next step is to construct a pair of Δ-complexes (K, L) and a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\subset} & K \\ \downarrow{\scriptstyle \partial b} & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow{g} & U \end{array}$$

such that $\partial b : K \to Z$ is a Δ-map, and the homotopy class of $(b, \partial b)$ is conjugate to the homotopy class of $(a, \partial a)$ under some homeomorphism $(K, L) \cong (D^i, S^{i-1})$. This is easy. Now it only remains to improve b to a map which is linear on each simplex of K. We do not want to corrupt the homotopy class of b relative to L and ∂b, but we may consider a change of the Δ-complex structure of K relative to L. Indeed, by choosing a sufficiently fine subdivision of K relative to L, and replacing b by the simplexwise linear map which has the same values as b on the vertices of the subdivided K, we obtain a solution. □

Definition 4.13. For a functor $\bar{G}$ from $\mathcal{P}$ to (well-)based spaces, let G be the continuous functor on $\mathcal{J}^{\mathrm{iso}}$ defined by

$$G(V) = \operatorname*{hocolim}_{\lambda\colon X\to V} \bar{G}(X,u)$$

where $\lambda\colon X \to V$ runs through $\mathcal{P}_V$ (and the homotopy colimit is "reduced" so that it is again a based space).

Remark. The topology on $\mathcal{P}_V$ does influence the topology on $G(V)$. Here we can avoid a more detailed discussion by saying that there is a pullback square

$$\begin{array}{ccc} G(V) & \longrightarrow & B\mathcal{P}_V \\ \downarrow & & \downarrow \\ \operatorname{hocolim} \bar{G} & \longrightarrow & B\mathcal{P}. \end{array}$$

Lemma 4.14. *Suppose that $\bar{G}$ takes all morphisms in $\mathcal{P}$ to homotopy equivalences. Then for any $\lambda\colon X \to V$ in $\mathcal{P}_V$ the inclusion $\bar{G}(X,u) \to G(V)$ determined by λ is a homotopy equivalence.*

Proof. Our hypothesis implies that $\operatorname{hocolim} \bar{G} \to B\mathcal{P}$ is a quasi-fibration. Because of the pullback square just mentioned, it follows that the projection $G(V) \to B\mathcal{P}_V$ is also a quasi-fibration. But $B\mathcal{P}_V$ is contractible. Therefore the inclusion of any of the fibers of $G(V) \to B\mathcal{P}_V$ into $G(V)$ is a homotopy equivalence. □

Lemma 4.15. *Let $\bar{G}_0$, $\bar{G}_1$ and $\bar{G}_2$ be functors from $\mathcal{P}$ to (well-)based spaces. Any natural multiplication*

$$\bar{G}_0(X,u) \wedge \bar{G}_1(Y,v) \longrightarrow \bar{G}_2(X * Y, u * v)$$

induces a natural multiplication $G_0(V) \wedge G_1) \longrightarrow G_2(V \times W)$.

5 Joins in dissected L-theory

Let $\mathbb{A}$ be the additive category of finitely generated free abelian groups, with the standard chain duality. Let (C,D) and (C',D') be chain complex pairs in $\mathbb{B}(\mathbb{A})$. We assume the boundary inclusions $D \to C$ and $D' \to C'$ are cofibrations, i.e., degreewise split, and to be quite precise we assume that such degreewise splittings have been specified. Then there is the product pair $(C,D) \otimes (C',D')$ consisting of $C \otimes C'$ and the subcomplex $(C \otimes D') \oplus_{(D\otimes D')} (D \otimes C')$ as the "boundary".

Let X and Y be finite Δ-complexes. By a *dissection of D over X* we mean a splitting $D = \sum_{\sigma\in sX} D(\sigma)$ of D as a graded abelian group which promotes D to an object of $\mathbb{B}(\mathbb{A}_*(X))$. (Of course, the dissection of D assembles back to D as an object of $\mathbb{B}(\mathbb{A})$.)

Lemma 5.1. *A dissection of D over X and a dissection of D' over Y together determine a dissection of the boundary of $(C,D) \otimes (C',D')$ over $X * Y$.*

Proof. Suppose that the dissections of D and D' are given by graded group splittings $D = \bigoplus_\sigma D(\sigma)$ and $D' = \bigoplus_\tau D(\tau)$. The specified splittings of $D \to C$ and $D' \to C'$ also give us identifications $C \cong D \oplus C/D$ and $C' \cong D' \oplus C'/D'$ of graded groups. Hence the boundary complex of $(C,D) \otimes (C',D')$ splits (as a graded group) into summands

$$D(\sigma) \otimes D'(\tau), \quad D(\sigma) \otimes C'/D', \quad C/D \otimes D'(\tau).$$

We now label these summands by simplices of $X * Y$. A summand of the form $D(\sigma) \otimes D'(\tau)$ gets the label $\sigma * \tau$. A summand of type $D(\sigma) \otimes C'/D'$ gets the label σ, and we note that $X \subset X * Y$. A summand of type $C/D \otimes D'(\tau)$ gets the label τ. It is easy to verify that this labeling defines a dissection. ☐

We next discuss a few variations on the theme of Lemma 5.1 where the pairs (C,D) and (C',D') come with symmetric structures. *Notation*: We write for example

$$(D \otimes D)^{h\mathbb{Z}/2}, \qquad (D \otimes D)_{h\mathbb{Z}/2}$$

for $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, D \otimes D)$ and $W \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (D \otimes D)$, respectively, where W is the standard $\mathbb{Z}[\mathbb{Z}/2]$-resolution of $\mathbb{Z}$. A symmetric structure on a pair (C,D) of chain complexes will be described as a chain $\varphi \in (C \otimes C)^{h\mathbb{Z}/2}$ whose boundary $\partial\varphi$ is in the subcomplex $(D \otimes D)^{h\mathbb{Z}/2}$. Similarly a quadratic structure on (C,D) will be described as a chain $\psi \in (C \otimes C)_{h\mathbb{Z}/2}$ whose boundary $\partial\varphi$ is in the subcomplex $(D \otimes D)_{h\mathbb{Z}/2}$.

Lemma 5.2. *Keep the assumptions of Lemma 5.1. Suppose also that the pairs (C,D) and (C',D') are equipped with symmetric structures φ and ψ. If the boundary symmetric structures $\partial\varphi$ on D and $\partial\psi$ on D' are dissected over X and Y, respectively, then the symmetric structure $\partial(\varphi \otimes \psi)$ on the boundary chain complex of $(C,D) \otimes (C',D')$ is dissected over $X * Y$. If in addition $\partial\varphi$ and $\partial\psi$ are dissected Poincaré, then $\partial(\varphi \otimes \psi)$ is dissected Poincaré.*

Lemma 5.3. *Keep the assumptions of Lemma 5.2. Suppose also that the Δ-complexes X and Y come with free actions of a finite group π, and the chain complex pairs (C,D), (C',D') come with actions of π so that the dissections of D and D' are π-invariant (in the sense that $gD(\sigma) = D(g\sigma)$ and $gD'(\tau) = D'(g\tau)$ for $g \in \pi$ and simplices σ in X, τ in Y). Suppose further that φ and ψ are π-invariant. Then $\varphi \otimes \psi$ is π-invariant, and the dissection of $\partial(\varphi \otimes \psi)$ over $X * Y$ is π-invariant for the diagonal action of π on $X * Y$.*

We will need slight generalizations of Lemmas 5.1, 5.2 and 5.3. Let $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$ be additive categories with chain duality. We assume given a functor

$$\begin{array}{ccc} \mathbb{B}(\mathbb{A}_1) \times \mathbb{B}(\mathbb{A}_2) & \longrightarrow & \mathbb{B}(\mathbb{A}_3) \\ (C, C') & \mapsto & C \boxtimes C' \end{array}$$

which is bi-additive and respects cofibration sequences in any of the two input variables. (This means that if C appears in a degreewise split short exact sequence $K \to C \to Q$, then $K \boxtimes C' \to C \boxtimes C' \to Q \boxtimes C'$ is also degreewise split short exact, and similarly if C' appears in a degreewise split short exact sequence.) We also need some compatibility between $\boxtimes$ and the tensor products $\otimes_{\mathbb{A}_i}$ for $i = 1,2,3$. We assume therefore that a natural chain map

$$u\colon (B \otimes_{\mathbb{A}_1} C) \otimes (B' \otimes_{\mathbb{A}_2} C') \longrightarrow (B \boxtimes B') \otimes_{\mathbb{A}_3} (C \boxtimes C')$$

is given, depending on variables B, C in $\mathbb{B}(\mathbb{A}_1)$ and B', C' in $\mathbb{B}(\mathbb{A}_2)$. This is supposed to respect nondegenerate cycles. That is, if x and y are nondegenerate cycles in $B \otimes_{\mathbb{A}_1} C$ and $B' \otimes_{\mathbb{A}_2} C'$, respectively, then $u(x,y)$ is a nondegenerate cycle in $(B \boxtimes B') \otimes_{\mathbb{A}_3} (C \boxtimes C')$.

Given C in $\mathbb{B}(\mathbb{A}_1)$ and C' in $\mathbb{B}(\mathbb{A}_2)$ and symmetric structures φ, ψ on C and C', respectively (of degrees m and n, respectively) we have a symmetric structure $\varphi \otimes \psi$ on $C \boxtimes C'$, of degree $m+n$, by composing

$$\begin{array}{ccc} W \xrightarrow{\text{diagonal}} & W \otimes W & \\ & \Big\downarrow{\scriptstyle \varphi\times\psi} & \\ & (C \otimes_{\mathbb{A}_1} C) \otimes (C' \otimes_{\mathbb{A}_2} C') & \xrightarrow{\ u\ } (C \boxtimes C') \otimes_{\mathbb{A}_3} C \boxtimes C'). \end{array}$$

There is a similar construction for pairs (C,D) and (C',D') with symmetric structures φ and ψ, respectively.

Lemma 5.4. *Lemmas 5.1, 5.2 and 5.3 remain valid for pairs* (C,D) *and* (C',D') *in* $\mathbb{B}(\mathbb{A}_1)$ *and* $\mathbb{B}(\mathbb{A}_2)$*, respectively, in which case the product pair is to be taken as* $(C,D) \boxtimes (C',D')$ *in* $\mathbb{B}(\mathbb{A}_3)$.

Example 5.5. We use the above ideas on dissection and joins to produce examples of functors $\bar{F}$ and $\bar{E}$ on $\mathcal{P}$ satisfying the condition of Lemma 4.14 and related by multiplications as in lemma 4.15. Fix X in $\mathcal{P}$. Determine m so that $X \simeq S^{m-1}$. Let $\mathbb{A}$ be the category of finitely generated free abelian groups, as before. In outline, $\bar{E}^a(X)$ will be defined as the algebraic cobordism space of formally m-dimensional symmetric Poincaré pairs (C,D,φ) in $\mathbb{B}(\mathbb{A})$ where the boundary $(D,\partial\varphi)$ is equipped with an involution and a (symmetric Poincaré) dissection over X which respects the involutions. The definition of $\bar{F}^a(X)$ is the same, except for an additional condition on the symmetric Poincaré pairs (C,D,φ), which is that both C and D have to be contractible as objects of $\mathbb{B}(\mathbb{A})$. A tensor product construction, where we use the above lemmas on dissection and joins, gives us maps

$$\bar{E}^a(X) \wedge \bar{E}^a(Y) \longrightarrow \bar{E}^a(X * Y), \quad \bar{E}^a(X) \wedge \bar{F}^a(Y) \longrightarrow \bar{F}^a(X * Y).$$

These maps have the usual associativity properties. There is also a unit in $\bar{E}^a(\emptyset)$. For all details, see the next section.

Remark 5.6. Lemma 4.14 gives us some information about the associated functors F^a and E^a on $\mathcal{J}^{\text{iso}}$. We will use that in the next section to understand F^a in homotopy theoretic terms. But we will not attempt to describe the homotopy type of $E^a(V)$ for all or some V. Instead we will use geometric ideas in Section 8 (proof of Theorem D) to construct a "smaller" functor $E^{a,\eta}$ on $\mathcal{J}^{\text{iso}}$, with multiplication as in Definition 4.1, and a multiplicative transformation $E^{a,\eta} \to E^a$. The point of the smallness is that $E^{a,\eta}$ admits a weak equivalence h to the constant functor $V \mapsto \mathbb{S}^0$ with the standard product. We regard F^a as a functor with an action of $E^{a,\eta}$ and do a derived induction along h to obtain a functor defined on all of $\mathcal{J}$. See Example 4.2 and Lemma 4.9.

6 Structure spaces in the algebraic setting

We begin with a remark on the additive categories $\mathbb{A}_*(X)$ and $\mathbb{A}^*(X)$ with chain duality. They have been defined for any Δ-complex X. However, it should be clear that the definitions extend to more general cases where the faces of X are "convex polytopes". We will only need this extension in the case of $\mathbb{A}^*(X)$, and then only when X is a "multisimplex", that is, a product of finitely many standard simplices.

Definition 6.1. Here we give the full definition of $\bar{E}^a(X)$ and $\bar{F}^a(X)$ in Example 5.5. Both are geometric realizations of m-fold Δ-sets. Let $k = (k_1, \dots, k_m)$ be a multi-index. Write Δ^k for $\Delta^{k_1} \times \cdots \times \Delta^{k_m}$ and

$$\bar{E}^a(X,k)$$

for the set of k-multisimplices of $\bar{E}^a(X)$. By definition, an element of that set is a pair (C,D) in the category $\mathbb{B}(\mathbb{A}^*(\Delta^k))$. There are more data:

- We ask for an SAPC pair structure ψ of formal dimension $m+\sum_i k_i$ on the chain complex pair (C,D).
- A dissection of $(D,\partial\psi)$ over X is also part of the data, and we want this to be Poincaré; hence the dissected $(D,\partial\psi)$ is an SAPC in

$$\mathbb{B}((\mathbb{A}^*(\Delta^k))_*(X)) \cong \mathbb{B}((\mathbb{A}_*(X))^*(\Delta^k))\,.$$

- An involution on (C,D) is required, respecting the dissection of D (and compatible with the given free involution on X, as far as D is concerned) and respecting ψ up to a sign $(-1)^m$.

Note that $\bar{E}^a(X,k)$ is a pointed set: the zero object ($C = D = 0$) serves as the base point. When we form the geometric realization

$$\big|\, k \mapsto \bar{E}^a(X,k) \,\big|$$

we collapse all base point simplices to a single point; hence the geometric realization is a pointed space. The product maps

$$\bar{E}^a(X) \wedge \bar{E}^a(Y) \longrightarrow \bar{E}^a(X * Y)$$

are induced by set maps

$$\bar{E}^a(X,k) \wedge \bar{E}^a(Y,\ell) \longrightarrow \bar{E}^a(X, k\#\ell)$$

with $k\#\ell = (k_1, k_2, \dots, k_m, \ell_1, \ell_2, \dots, \ell_n)$, which in turn are given by a generalized (but obvious) tensor product construction $\boxtimes$ as in lemma 5.4.

The definition of $\bar{F}^a(X)$ is almost identical with that of $\bar{E}^a(X)$ just given. Again it is the geometric realization of an m-fold Δ-set and we write $\bar{F}^a(X,k)$ for the set of k-multisimplices. The elements of that set are pairs (C,D) almost exactly as above, but with one added condition, that C and D be *contractible* as objects of $\mathbb{B}(\mathbb{A}^*(\Delta^k))$.

There are “first axis” subspaces of $\bar{E}^a(X)$ and $\bar{F}^a(X)$, obtained by allowing only k-multi-simplices where k has the form $(k_1, 0, 0, \ldots, 0)$.

Lemma 6.2. *The inclusions of the first axes in $\bar{E}^a(X)$ and $\bar{F}^a(X)$, respectively, are homotopy equivalences.*

Lemma 6.3. *Let F^a be the functor on $\mathcal{J}^{\mathrm{iso}}$ associated to $\bar{F}^a$ as in Example 4.13. For V in $\mathcal{J}^{\mathrm{iso}}$ with $\dim(V) = j + 1$ there is a homotopy equivalence*

$$F^a(V) \simeq \mathsf{S}(\mathbb{R}P(V), j).$$

Proof. The functor $\bar{F}^a$ satisfies the condition of Lemma 4.14. Hence $F^a(V) \simeq \bar{F}^a(X)$, assuming that X and V are related as in that lemma. Now use Lemma 6.2 to complete the proof. □

7 Periodicity and Thom isomorphism

This section provides the proofs of Theorems A and B.

Theorem A. *For an oriented $W \in \mathcal{J}$ such that 4 divides $\dim(W)$, there is a homotopy equivalence $F^a(V) \to \Omega^W F^a(V)$, natural in V.*

Proof. In view of Remark 5.6 it is enough to prove that we have $F^a(V) \cong \Omega^4 F^a(V)$ by a natural homeomorphism as functors on $\mathcal{J}^{\mathrm{iso}}$ which respects the action of E^a.

The proof is by inspection, using the well-known periodicity of algebraic L-theory given by the double (skew-)suspension. But it is appropriate to say what exactly $\Omega^4 F^a(V)$ means. First of all, given an m-fold based Δ-set Y, what do we mean by ΩY ? We can define ΩY as the based m-fold Δ-set given by

$$(k_1, k_2, \ldots, k_m) \mapsto \{y \in Y(k_1 + 1, k_2, \ldots, k_m) \mid u_{k_1}^*(y) = *, \; v_{k_1}^*(y) = *\}$$

where $u_{k_1} : \{0\} \to \{0, 1, \ldots, k_1, k_1 + 1\}$ is the inclusion and

$$v_{k_1} : \{0, 1, \ldots, k_1\} \to \{0, 1, \ldots, k_1, k_1 + 1\}$$

is given by $i \mapsto i + 1$. (These monotone maps act as face operators in the “first” coordinate direction.) This definition is justifiable if Y has the Kan extension property.

We defined $F^a(V)$ for m-dimensional V as a reduced homotopy colimit of spaces $\bar{F}^a(X)$ for $X \to V$ in $\mathcal{P}_V$. Since $\bar{F}^a$ satisfies the condition of Lemma 4.14, we may define $\Omega^4 F^a(V)$ as a reduced homotopy colimit

$$\operatorname*{hocolim}_{X \to V} \Omega^4 \bar{F}^a(X)$$

for $X \to V$ in $\mathcal{P}_V$. Moreover in that last expression we may interpret Ω^4 as the fourth power of an operator Ω on based Δ-sets, as defined above, to be applied before geometric realization. Then we have indeed $F^a(V) \cong \Omega^4 F^a(V)$. □

Next we discuss the proof of the following theorem to which we refer as a Thom isomorphism.

Theorem B. *Let W in $\mathcal{J}$ be oriented, of even dimension. Then there exists a functor Φ_W on $\mathcal{J}$, which is polynomial of degree ≤ 0, a natural transformation $\zeta\colon F^a(-\oplus W) \to \Phi_W(-)$, and a natural map*

$$\Omega^W F^a(V) \to \text{hofiber}[F^a(V \oplus W) \xrightarrow{\zeta} \Phi_W(V)]$$

which is a homotopy equivalence for $\dim(V) \geq 3$.

For X in $\mathcal{P}$ we have a definition of $\Omega^d \bar{F}^a(X)$ which looks almost exactly like the definition of $\bar{F}^a(X)$, except for one change which consists in increasing the formal dimensions of all SAPC's in the definition of $\bar{F}^a(X)$ by d. (If $X \simeq S^{m-1}$, then the $(0, 0, \ldots, 0)$-simplices of $\Omega^d \bar{F}^a(X)$ are certain SAPC pairs (C, D, φ) of formal dimension $m + d$ with a dissection of the boundary $(D, \partial\varphi)$ over X.) Using that description of $\Omega^d \bar{F}^a(X)$, we have the following maps:

$$\begin{array}{ccc}
\Omega^d \left(\underset{f\colon X\to V}{\text{hocolim}}\, \bar{F}^a(X)\right) & & \\
\uparrow \simeq & & \\
\underset{\substack{f\colon X\to V\\ g\colon Y\to W}}{\text{hocolim}}\, \Omega^d \bar{F}^a(X) & \xrightarrow{\iota} & \underset{e\colon Z\to V\oplus W}{\text{hocolim}}\ \bar{F}^a(Z).
\end{array}$$

The vertical map is obtained by forgetting the data $g\colon Y \to W$, which run through $\mathcal{P}_W$, and using the inclusion $\text{hocolim}\, \Omega^d \bar{F}^a \to \Omega^d(\text{hocolim}\, \bar{F}^a)$, where the homotopy colimits are taken over $\mathcal{P}_V$ only. It is a homotopy equivalence because its target can be identified up to homotopy equivalence with $\Omega^d \bar{F}^a(X)$ for any $f\colon X \to V$ in $\mathcal{P}_V$ by Lemma 4.14, and its source can also be identified up to homotopy equivalence with $\Omega^d \bar{F}^a(X)$ by the same kind of argument. The horizontal map uses an embedding

$$\Omega^d \bar{F}^a(X) \longrightarrow \bar{F}^a(X * Y)$$

for $f\colon X \to V$ in P_V and $g\colon Y \to W$ in $\mathcal{P}_W$. This is simply induced by the inclusions $X \to X * Y$ and $\mathbb{A}_*(X) \to \mathbb{A}_*(X * Y)$, which respect the chain dualities. Using the vertical arrow as an "identification", we write

$$\iota\colon \Omega^d F^a(V) \longrightarrow F^a(V \oplus W)\,.$$

We now wish to extend ι to a homotopy fiber sequence

$$\Omega^d F^a(V) \xrightarrow{\ \iota\ } F^a(V \oplus W) \xrightarrow{\ \zeta\ } \Phi(V \oplus W, V)$$

where Φ is a certain functor of pairs. The definition of Φ follows the standard pattern. Hence we start by introducing functors on pairs of certain Δ-complexes.

Definition 7.1. Fix a pair (Z, X) of finite Δ-complexes, with a free involution, homotopy equivalent to the pair $(S(V \oplus W), S(V))$ with the antipodal involution. We define

$$\bar{\Phi}(Z, X)$$

essentially by repeating the definition of $\bar{F}^a(Z)$ in terms of Poincaré pairs (C,D,ψ) contractible in $B(\mathbb{A})$ with dissected boundary, but relaxing it in one respect: for the dissection of $\partial\psi$ as a symmetric structure on the dissected D, we require that to be Poincaré modulo X only. (This means that the mapping cone of the appropriate duality map is chain equivalent, as an object dissected over Z, to something dissected over X.) That being done, we put

$$\Phi(V \oplus W, V) := \underset{(Z,X)\to(V\oplus W,V)}{\text{hocolim}} \bar{\Phi}(Z,X).$$

The homotopy colimit is taken over all pairs (Z,X) as above and simplexwise affine $\mathbb{Z}/2$-maps $(Z,X) \to (V \oplus W, V)$ taking X to $V \smallsetminus 0$ and Z to $V \oplus W \smallsetminus 0$, and inducing a homotopy equivalence $(Z,X) \to (V \oplus W \smallsetminus 0, V \smallsetminus 0)$. We have maps

$$\begin{array}{ccc} \underset{(Z,X)\to(V\oplus W,V)}{\text{hocolim}} \bar{F}^a(Z) & \xrightarrow{\ \zeta\ } & \underset{(Z,X)\to(V\oplus W,V)}{\text{hocolim}} \bar{\Phi}(Z,X) \\ \simeq \big\downarrow & & \\ \underset{Z\to V\oplus W}{\text{hocolim}} \bar{F}^a(Z) & & \end{array}$$

where the horizontal arrow is determined by the inclusions $\bar{F}^a(Z) \to \bar{\Phi}(Z,X)$. Using the vertical arrow as an identification, we may write

$$\zeta : F^a(V \oplus W) \longrightarrow \Phi(V \oplus W, W).$$

Proposition 7.2. *For V of dimension ≥ 3 and oriented W of even dimension d, both in $\mathcal{J}^{\text{iso}}$, the following is a homotopy fiber sequence:*

$$\Omega^d F^a(V) \xrightarrow{\ \iota\ } F^a(V \oplus W) \xrightarrow{\ \zeta\ } \Phi(V \oplus W, V).$$

Remark. We need $\dim(V) \geq 3$ to ensure that the inclusion $V \to V \oplus W$ induces an isomorphism of fundamental groups of the associated projective spaces.

Proof. For fixed V and W, after routine reformulations as in Lemma 4.14, this boils down to a homotopy fiber sequence

$$\Omega^d \bar{F}^a(X) \longrightarrow \bar{F}^a(X * Y) \longrightarrow \bar{\Phi}(X * Y, X)$$

where $X \to V$ in $\mathcal{P}_V$ and $Y \to W$ in $\mathcal{P}_W$. Both arrows are inclusion maps. This homotopy fiber sequence is very standard, e.g. from [Ra]. ☐

Remark. In the applications of this proposition we need a fair amount of naturality in the variable V. This calls for a more precise formulation and a better proof. We have the following commutative square:

$$\begin{array}{ccc}
\operatorname*{hocolim}_{\substack{X\to V\\ Y\to W}} \Omega^d \bar{F}^a(X) & \longrightarrow & \operatorname*{hocolim}_{(Z,X)\to(V\oplus W,V)} \bar{F}^a(Z) \\
\downarrow & & \downarrow \\
\operatorname*{hocolim}_{\substack{X\to V\\ Y\to W}} \Omega^d \bar{\Phi}(X,X) & \longrightarrow & \operatorname*{hocolim}_{(Z,X)\to(V\oplus W,V)} \bar{\Phi}(Z,X).
\end{array}$$

Here the upper left-hand term can be identified with $\Omega^d F^a(V)$ by a forgetful homotopy equivalence which is natural in V as an object of $\mathcal{J}^{\text{iso}}$. The upper right-hand term can be identified with $F^a(V \oplus W)$, again by a forgetful homotopy equivalence which is natural in V as an object of $\mathcal{J}^{\text{iso}}$. The lower right-hand term is $\Phi(V \oplus W, V)$. The lower left-hand term is contractible and is functorial in V as an object of $\mathcal{J}^{\text{iso}}$. The square as a whole is homotopy cartesian (by the argument already given).

Let E^a be the functor on $\mathcal{J}^{\text{iso}}$ associated with $\bar{E}^a$ of Definition 6.1, as in Definition 4.13. Then E^a acts on each of the four functors (of the variable V) represented by the four terms of the square just above. For example, there is an action map

$$E^a(U) \wedge \Phi(V \oplus W, V) \longrightarrow \Phi(U \oplus V \oplus W, U \oplus V).$$

It is given by a straightforward tensor product construction and we omit the details.

Recall from Remark 5.6 that the functor F^a on $\mathcal{J}$ was promised to be constructed in Section 8 from the pair E^a, F^a by a derived induction process, during which E^a is replaced by a smaller functor $E^{a,\eta}$ weakly equivalent to the constant functor $V \mapsto \mathbb{S}^0$. Let Φ_W of be the functor on $\mathcal{J}$ constructed from E^a and $\Phi(W \oplus -, -)$ via the same procedure. Proposition 7.3 below proves that Φ_W is of degree ≤ 0.

A connected component of $E^a(U)$, with $\dim(U) = m$, determines (forgetfully) an element in the relative L^m group of the assembly functor $\mathbb{A}_*(X) \to \mathbb{A}_*$, for any $X \to U$ in $\mathcal{P}_U$. This relative L^m group is canonically isomorphic to $L^0(\mathbb{A}) \cong \mathbb{Z}$. In this way, connected components of $E^a(U)$ have a "degree" which is an integer.

Proposition 7.3. *Keep the assumptions of Proposition 7.2 and let U be in $\mathcal{J}^{\text{iso}}$. Let $z \in E^a(U)$ be in a component of degree* 1. *Then multiplication by z is a homotopy equivalence*

$$\Phi(V \oplus W, V) \longrightarrow \Phi(U \oplus V \oplus W, U \oplus V).$$

Proof. Choose $(Z, X) \to (V \oplus W, W)$ as in the definition of $\Phi(V \oplus W, V)$ and choose $Y \to U$ in $\mathcal{P}_U$. For typographic reasons we use the abbreviations $Z^Y = Z * Y$ and $X^Y = X * Y$, and denote passage to $\mathbb{Z}/2$-orbits by a tilde subscript, as in $X_\sim$. We can assume that $z \in E(Y)$, and we have to show that multiplication by z is a homotopy equivalence

$$\bar{\Phi}(Z, X) \longrightarrow \bar{\Phi}(Z^Y, X^Y).$$

Let $k = \dim(V) + d - 1 = \dim(V) + \dim(W) - 1$ and write $m = \dim(U)$ as before. Most of the proof is in the following commutative diagram:

$$\begin{array}{ccc}
\bar{\Phi}(Z,X) & \longrightarrow & \bar{\Phi}(Z^Y, X^Y) \\
\simeq \uparrow & & \simeq \uparrow \\
\mathsf{S}(Z_\sim, X_\sim, k) & \longrightarrow & \mathsf{S}(Z^Y_\sim, X^Y_\sim, k+m) \\
\simeq \downarrow & & \simeq \downarrow \\
\mathsf{L}_k(\mathbb{A}_*(X_\sim) \to \mathbb{A}_*(Z_\sim)) & \longrightarrow & \mathsf{L}_{k+m}\begin{pmatrix} \mathbb{A}_*(Y_\sim) & \xrightarrow[=]{} & \mathbb{A}_*(Y_\sim) \\ \downarrow & & \downarrow \\ \mathbb{A}_*(X^Y_\sim) & \longrightarrow & \mathbb{A}_*(Z^Y_\sim) \end{pmatrix}
\end{array}$$

In this diagram, all horizontal arrows are defined as multiplication by $z \in E(Y)$. The vertical arrows in the upper half of the diagram are forgetful: they forget boundaries in dissected Poincaré pairs. By Definition 3.21 and [Ra, 3.9], they are homotopy equivalences. That is, the forgotten dissected boundaries can always be recovered as "obstructions to nondegeneracy" in $\mathbb{A}_*(Z_\sim)$. To produce the vertical arrow in the lower half of the diagram, we switch to Definition 3.20 of the algebraic structure spaces, i.e., to quadratic structures. (Strictly speaking, we should insert another row into the diagram to do that.) These vertical arrows in the lower half of the diagram can then be defined as inclusion maps. They are homotopy equivalences by the alternative definition of the algebraic structure spaces as homotopy fibers of assembly maps. Here we are also exploiting the fact that the inclusions $X_\sim \to Z_\sim$ and $X^Y_\sim \to Z^Y_\sim$ induce isomorphisms of fundamental groups, i.e., we are using $\dim(V) \geq 3$.

It remains to show that the lower horizontal arrow in the diagram is a homotopy equivalence. By a five lemma argument, this reduces to showing that multiplication by z induces isomorphisms

$$\pi_* \mathsf{L}_j(\mathbb{A}_*(X_\sim)) \quad \longrightarrow \quad \pi_* \mathsf{L}_{j+m}\left(\mathbb{A}_*(Y_\sim) \to \mathbb{A}_*(X^Y_\sim)\right)$$

for all $j \in \mathbb{Z}$, and similarly with X replaced by Z. But this is a case of an ordinary Thom isomorphism. The standard proof uses a spectral sequence comparison argument. The spectral sequences are determined by the skeleton filtration of X. ☐

We conclude this section with a note which will be useful in the next section. Recall that in Section 3 we defined for a simplicial complex X with certain additional data the space $\mathsf{S}(X)$ which we used to construct the functor F^a. Furthermore we showed that there is also a space $\mathsf{S}(X; \wp^1)$, a "truncated version" of $\mathsf{S}(X)$, which is better related to geometry. Let $F^{a,1}$ be the functor on $\mathcal{J}$ defined by the same process as F^a, but using $\mathsf{S}(X, \wp^1)$ instead of $\mathsf{S}(X)$. Using (3.7) we easily obtain a homotopy fibration sequence $F^{a,1}(V) \to F^a(V) \to \mathbb{Z}$ for each $V \in \mathcal{J}^{\mathrm{iso}}$. Lemma 7.4 below says this sequence behaves well with respect to multiplication by E^a and hence the sequence is natural when the functors are considered on $\mathcal{J}$.

An element of $F^a(V)$, with $\dim(V) = n$, determines forgetfully an element in the relative L_n group of the assembly functor $\mathbb{A}_*(S(V)) \to \mathbb{A}_*$. This relative L_n group is canonically isomorphic to $L_0(\mathbb{A})$ which we identify with $\mathbb{Z}$ using the isomorphism $\sigma/8$, signature divided by 8. In this way there is a degree function from $F^a(V)$ to $\mathbb{Z}$ which we denote by $\tilde{\sigma}/8$ and which is one of the maps in the homotopy fibration sequence above.

Lemma 7.4. *Let U be in $\mathcal{J}^{\mathrm{iso}}$. Let $z \in E^a(U)$ be in a component of degree* 1. *Then for every V in $\mathcal{J}^{\mathrm{iso}}$, the following is commutative:*

$$\begin{array}{ccc} F^a(V) & \xrightarrow{z\cdot} & F^a(U \oplus V) \\ \downarrow{\scriptstyle \tilde{\sigma}/8} & & \downarrow{\scriptstyle \tilde{\sigma}/8} \\ \mathbb{Z} & \xrightarrow{=} & \mathbb{Z}. \end{array}$$

Proof. The square can be enlarged to a six-term diagram

$$\begin{array}{ccc} F^a(V) & \xrightarrow{z\cdot} & F^a(U \oplus V) \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta} \\ \Phi(V,0) & \xrightarrow{z\cdot} & \Phi(U \oplus V, U) \\ \downarrow{\scriptstyle \tilde{\sigma}/8} & & \downarrow{\scriptstyle \tilde{\sigma}/8} \\ \mathbb{Z} & \xrightarrow{=} & \mathbb{Z}. \end{array}$$

Here the top square commutes by construction. The middle row is an isomorphism by Theorem 7.3, and the reasoning used in the proof of that theorem also shows that the lower square commutes. ☐

8 Structure spaces in the mixed algebraic-geometric setting

This section contains the proof of Theorem D. In Sections 3, 4, 5 and 6, we constructed an algebraic analogue F^a of the geometric functor F^g from [Ma]. In Section 7 we proved Theorems A and B, which are about F^a. But we also need Theorem C which is about F^g and so a translation between algebra and geometry is needed. This is provided by Theorem D. The translation is a tedious business and the method that we have chosen might not be the best. In any case we have in a few places sacrificed completeness for the sake of intelligibility.

Theorem D. *Let $V \in \mathcal{J}$ be such that* $\dim(V) \geq 6$. *Then there is a natural homotopy fibration sequence*

$$F^g(V) \to F^a(V) \to \mathbb{Z}.$$

On the level of spaces this homotopy fibration sequence is just Ranicki's fibration sequence (3.7). The issue addressed here is the naturality in V. Further, in view of Lemma 7.4 it is enough to prove that there is a natural homotopy equivalence $F^g(V) \to F^{a,1}(V)$ for all $V \in \mathcal{J}$ with $\dim(V) \geq 6$.

Scheme of the proof. Recall that the functor F^a was constructed in Section 6 by the following method. We started with two functors $\bar{E}^a, \bar{F}^a$ from the category $\mathcal{P}$ to pointed spaces, equipped with suitable product and multiplication maps. For $V \in \mathcal{J}^{\text{iso}}$ these restrict to functors from $\mathcal{P}_V$ to pointed spaces, which we used to construct functors E^a, F^a from $\mathcal{J}^{\text{iso}}$ to pointed spaces. The derived induction of Section 4 then promotes F^a to a continuous functor from $\mathcal{J}$ to pointed spaces. Exactly the same procedure applied to $\bar{F}^{a,1}$ instead of $\bar{F}^a$ delivers $F^{a,1}$ instead of F^a. (See the end of Section 7 for the definition of $F^{a,1}$.)

Note that the functor F^g can also be constructed by the technology of Section 4, i.e., by starting with two functors E^g and F^g on $\mathcal{J}^{\text{iso}}$ with multiplication and action maps. The details are given in Definition 8.20.

To relate the two functors we construct an "intermediate" functor F^{ga} from $\mathcal{J}$ to pointed spaces, again using the same technology. A major part of the present section is devoted to this aim, which is finally achieved in Definition 8.18. Furthermore there are forgetful natural transformations

$$\begin{array}{ccccc} E^g & \xleftarrow{w_0} & E^{ga} & \xrightarrow{v_0} & E^a \\ F^g & \xleftarrow{w_1} & F^{ga} & \xrightarrow{v_1} & F^{a,1} \end{array}$$

(details in Definitions 8.21 and 8.22) respecting the multiplications and the actions. The two in the lower row, v_1, w_1 are natural homotopy equivalences, giving us a two-step identification of F^g with $F^{a,1}$ as functors on $\mathcal{J}^{\text{iso}}$. Of the two in the upper row, w_0 is again a natural homotopy equivalence, whereas v_0 is not. But since v_0 is multiplicative, we can use it to let E^{ga} act on $F^{a,1}$. We then do a derived induction along w_0 to obtain what is essentially an extension of $F^{a,1}$ to $\mathcal{J} \supset \mathcal{J}^{\text{iso}}$ (Example 4.2 and Lemma 4.9). Note that $E^g(V) = \mathbb{S}^0$ (Definition 8.20), as was promised in Remark 5.6, and so this procedure promotes the identification of F^g and $F^{a,1}$ as functors on $\mathcal{J}^{\text{iso}}$ to an identification as functors on $\mathcal{J}$. $\square$

Notation 8.1. The definition of E^{ga} and F^{ga} comes at the end of a lengthy staircase of definitions. Climbing the staircase, we use a uniform symbol G with mostly numerical decorations, instead of letters E and F with decorations. At the top of the staircase we switch back to the familiar E, F notation (Definition 8.18).

8.1 Transversality

We will rely mainly on the notion of "transversality to a foliation". Let N be a topological space. There is a presheaf on N which to an open subset $W \subset N$ associates the set of equivalence relations on W. A global section ρ of the associated sheaf is called a *local equivalence relation* on N. If N can be covered by open subsets W_α which admit homeomorphisms $(p_\alpha, q_\alpha)\colon W_\alpha \to V_\alpha \times U_\alpha$ with U_α open in $\mathbb{R}^k$ (but no conditions on V_α other than "being a space") such that $\rho|W_\alpha$ is represented by the equivalence relation

$$y \sim z \quad \Leftrightarrow \quad q_\alpha(y) = q_\alpha(z)$$

on W_α, then ρ is a *codimension k foliation* of N. See [KM] for more details. A map f from a topological manifold M to N is *transverse* to a codimension k foliation ρ on N if for every $x \in M$ there exists an open neighborhood W of $f(x) \in N$ and a product structure $(p, q)\colon W \to V \times U$ with $U \subset \mathbb{R}^k$ representing $\rho|W$, as above, such that qf is a topological submersion $f^{-1}(W) \to U$. (To preclude misunderstandings, we point out that submersions don't have to be surjective. A map between topological manifolds is a submersion if it satisfies a certain regularity condition at or near every point of the source manifold.)

Closely related is the following concept of transversality: Suppose that N is a space, $X \subset N$ is a locally closed subspace, U is an open neighborhood of X in N which comes with a codimension k foliation, and X is a leaf of that foliation. We say informally that a map f from a manifold M to N is transverse to X if f restricted to a sufficiently small neighborhood of $f^{-1}(X)$ is transverse to the foliation of U. In the examples that we will be looking at, the foliation of U is determined by a single map $q : U \to \mathbb{R}^k$, so that the leaves of the foliation are the fibers of q.

8.2 Main examples

Definition 8.2. For a Δ-complex X, the open cone $\mathbb{O}(X)$ is $X_+ \wedge [0, \infty)$. We describe a point in $\mathbb{O}(X)$ as $x = ty$ where $y \in X$ and $t \in [0, \infty)$, or by its barycentric coordinates,

$$x = (x_i)_{i \in \sigma} = (ty_i)_{i \in \sigma}$$

where $x_i \geq 0$ and i runs through the vertices of a simplex σ containing y. The open cone $\mathbb{O}(X)$ comes with the *norm* function $x \mapsto \max_i\{x_i\}$. The levels of the norm define a foliation on $\mathbb{O}(X) \smallsetminus 0$ with the leaves $\mathbb{O}(X, c) = \{x \in \mathbb{O}(X) \mid \|x\| = c\}$ for $c > 0$. For (X, u) in $\mathcal{P}$, the open cone $\mathbb{O}(X)$ also comes with an involution $ty \mapsto t \cdot u(y)$, where $y \in X$.

Definition 8.3. We often regard $\mathbb{O}(X)$ as a stratified space, stratified by the interiors of the coned dual cells and the cone point. This is similar to the stratification of X in Example 3.6. But we need a few more details here. We need to declare what it means for a map $f : M \to \mathbb{O}(X) \smallsetminus 0$ to be transverse to the stratification.

For a k-simplex σ of X the stratum $\mathbb{O}(X, \sigma)$ (the interior of the coned dual cell) consists of points $x \in \mathbb{O}(X)$ whose barycentric coordinates satisfy $x_i = \|x\|$ if $i \in \sigma$ and $x_i < \|x\|$ if $i \notin \sigma$. The map q_σ defined on a (sufficiently small) neighborhood of $\mathbb{O}(X, \sigma)$ in $\mathbb{O}(X)$ by

$$x \mapsto \left(\frac{x_i}{\|x\|} \right)_{i \in \sigma} \in \mathbb{R}^{|\sigma|+1}$$

has image contained in the hypersurface

$$Z_k = \left\{ (y_i)_{i \in \sigma} \in \mathbb{R}^{|\sigma|+1} \mid \max_i \{y_i\} = 1 \right\}.$$

The sets $q_\sigma^{-1}(y)$ for $y \in Z_k$ are the leaves of a foliation of the neighborhood. One of these leaves is $\mathbb{O}(X, \sigma)$ itself. We say that a map f from a manifold to $\mathbb{O}(X)$ is *transverse* to $\mathbb{O}(X, \sigma)$ if $q_\sigma f$, as a map with target Z_k, is a topological submersion in a neighborhood of $f^{-1}(\mathbb{O}(X, \sigma))$.

Comment. The codimension k foliation defined by q_σ (of an open neighborhood of $\mathbb{O}(X, \sigma)$ in $\mathbb{O}(X)$) also restricts to a codimension k foliation of an open neighborhood of $\mathbb{O}(X, \sigma) \cap \mathbb{O}(X, c)$ in $\mathbb{O}(X, c)$, for every $c > 0$.

Definition 8.4. For (X,u) in $\mathcal{P}$ with $X \simeq S^{m-1}$ we define $\bar{G}_0(X,u)$ as an m-fold simplicial set. A *nontrivial* $(0,0,\ldots,0)$-simplex consists of a based space W homeomorphic to $\mathbb{R}^m$, with an involution fixing the base point, and an equivariant based proper map $p : W \to \mathbb{O}(X)$ of degree ± 1 which is transverse to the foliation of $\mathbb{O}(X) \smallsetminus 0$ by norm levels, and has preimage of base point equal to base point. There is also a unique trivial $(0,0,\ldots,0)$-simplex which, along with all its degeneracies, constitutes a connected component of $\bar{G}_0(X,u)$ after realization.

Definition 8.5. For (X,u) in $\mathcal{P}$ with $X \simeq S^{m-1}$ we define $\bar{G}_1(X,u)$ as an m-fold simplicial set. A $(0,0,\ldots,0)$-simplex consists of two based spaces W, W' both homeomorphic to $\mathbb{R}^m$, both with an involution fixing the base point, and equivariant proper maps $W \to W' \to \mathbb{O}(X)$ of degree ± 1 such that both $W' \to \mathbb{O}(X)$ and the composite map $W \to \mathbb{O}(X)$ are transverse to the foliation of $\mathbb{O}(X) \smallsetminus 0$ by norm levels, and have preimage of base point equal to base point. We regard such a $(0,0,\ldots,0)$-simplex as *trivial* if $W \to W'$ is a homeomorphism. All trivial $(0,0,\ldots,0)$-simplices are to be identified with each other.

With the above definitions of $\bar{G}_0$ and $\bar{G}_1$, there are products

$$\begin{aligned} \bar{G}_0(X,u) \wedge \bar{G}_0(Y,v) &\longrightarrow \bar{G}_0(X * Y, u * v), \\ \bar{G}_0(X,u) \wedge \bar{G}_1(Y,v) &\longrightarrow \bar{G}_1(X * Y, u * v). \end{aligned}$$

In more detail, on nontrivial simplices the multiplication map is given simply by the product. The action map takes

$$(\,V \xrightarrow{p} \mathbb{O}(X),\ W \xrightarrow{q} W' \xrightarrow{p'} \mathbb{O}(Y)\,)$$

to

$$V \times W \xrightarrow{\mathrm{id}\times q} V \times W' \xrightarrow{p \times p'} \mathbb{O}(X * Y).$$

These products induce similar products involving G_0 and G_1.

Next we introduce certain refinements of $\bar{G}_0$ containing more combinatorial information. These refinements make up a diagram of functors on $\mathcal{P}$ and natural transformations

$$\bar{G}_{0-3} \to \bar{G}_{0-2} \to \bar{G}_{0-1} \hookrightarrow \bar{G}_0.$$

Moving from right to left, we first add a transversality condition, then certain CW-approximations, then cellular diagonal approximations and cellular fundamental cycles/chains for the approximating CW-spaces involved. Fix (X,u) in $\mathcal{P}$, where $X \simeq S^{m-1}$, and a nontrivial $(0,0,\ldots,0)$-simplex $p : W \to \mathbb{O}(X)$ of $\bar{G}_0(X,u)$.

Definition 8.6. To promote $p : W \to \mathbb{O}(X)$ to a $(0,0,\ldots,0)$-simplex in the m-fold Δ-set $\bar{G}_{0-1}(X,u)$, we impose the condition that p be *simultaneously transverse* to the strata $\mathbb{O}(X,\sigma)$ and to the levels of the norm fibration (see the comment just below). This implies that for every $c > 0$, the restriction of p to the sphere $p^{-1}(c)$ is a map to $\mathbb{O}(X,c) \cong X$ which is transverse to the strata $X(\sigma)$.

Comment. Let σ be a k-simplex of X. We test p for transversality to the stratum $\mathcal{O}(X,\sigma)$ by asking whether $q_\sigma p$ is a submersion (see Definition 8.3). We test for transversality to the norm levels by asking whether $\|p\|: W \smallsetminus p^{-1}(0) \to \mathbb{R}$ is a submersion. Here we need a condition which is slightly stronger than these two transversality conditions put together. We require that the map

$$\text{neighborhood of } p^{-1}(\mathcal{O}(X,\sigma)) \longrightarrow Z_k \times \mathbb{R}$$

defined by $w \mapsto (q_\sigma p(w), \|p(w)\|)$ be a submersion. Since $Z_k \times \mathbb{R}$ can be identified with $\mathbb{R}^{|\sigma|+1}$, the formula $w \mapsto (q_\sigma p(w), \|p(w)\|)$ can also be replaced by the much simpler formula

$$w \mapsto \big(p_i(w)\big)_{i\in\sigma} \in \mathbb{R}^{|\sigma|+1}$$

where the $p_i(w)$ are the barycentric coordinates of $p(w)$ corresponding to the vertices i of σ.

We mention the following in passing. Suppose that p satisfies the above strong transversality condition for a particular σ. Let τ be a face of σ. Then, in a sufficiently small neighborhood of $p^{-1}(\mathcal{O}(X,\sigma))$, the strong transversality condition for p in relation to τ and the stratum $\mathcal{O}(X,\tau)$ is automatically satisfied. The reason is, of course, that the barycentric coordinates $p_i(w)$ for w in W and i a vertex of τ are subsumed in the barycentric coordinates $p_i(w)$ for i a vertex of σ.

Lemma 8.7. *The product on $\bar{G}_0$ can be refined to a product on $\bar{G}_{0-1}$.*

Proof. Given (X,u) and (Y,v) in $\mathcal{P}$ and $p: W \to \mathcal{O}(X)$ and $q: W' \to Y$ satisfying the appropriate transversality conditions, we verify that the resulting map from $W \times W'$ to $\mathcal{O}(X) \times \mathcal{O}(Y) \cong \mathcal{O}(X * Y)$ satisfies the appropriate transversality condition. The strata of $\mathcal{O}(X * Y) \cong \mathcal{O}(X) \times \mathcal{O}(Y)$, apart from the cone point, can be described as follows:

(i) for each σ in X, there is a stratum consisting of all (x,y) in $\mathcal{O}(X,\sigma) \times \mathcal{O}(Y)$ where $\|x\| > \|y\|$;

(ii) for each τ in Y, there is a stratum consisting of all (x,y) in $\mathcal{O}(X) \times \mathcal{O}(Y,\tau)$ where $\|x\| < \|y\|$;

(iii) for each simplex of the form $\sigma * \tau$ with σ in X and τ in Y, there is a stratum consisting of all (x,y) in $\mathcal{O}(X,\sigma) \times \mathcal{O}(Y,\tau)$ where $\|x\| = \|y\|$.

Hence the transversality properties that we need follow from the transversality properties of p in case (i), from the transversality properties of q in case (ii), and from the transversality properties of both p and q in case (iii). We omit the details, except for pointing out that in the case (iii), the vertex set of $\sigma * \tau$ is identified with the disjoint union of the vertex sets of σ and τ respectively. It follows that an expression such as

$$\big((p \times q)_i(w,w')\big)_{i\in\sigma*\tau} \in \mathbb{R}^{|\sigma*\tau|+1}$$

for $(w,w') \in W \times W'$ can be re-arranged to look like

$$\Big(\big(p_i(w)\big)_{i\in\sigma}, \big(q_i(w')\big)_{i\in\tau}\Big) \in \mathbb{R}^{|\sigma|+1} \times \mathbb{R}^{|\tau|+1}. \qquad \square$$

We enlarge cat(X), the category of simplices of X, to a category $\{0\} * \mathrm{cat}(X)$ by adding the object 0, its identity morphism, and one morphism $0 \to \sigma$ for each $\sigma \in sX$. The transversality condition in the previous definition yields, for every $(0,0,\ldots,0)$-simplex $p\colon W \to \mathbb{O}(X)$ in $\bar{G}_{0-1}(X,u)$ as above, a contravariant functor $W_\diamond$ from $\{0\} * \mathrm{cat}(X)$ to compact spaces by

$$\begin{cases} W_\diamond(\sigma) = W[\sigma][1] \\ W_\diamond(0) = W[0,1] \end{cases}$$

where $W[\sigma][1]$ is p^* of $\mathbb{O}(X,\sigma) \cap \mathbb{O}(X,1)$, the norm level 1 of the coned dual cell corresponding to σ, and $W[0,1]$ denotes $p^{-1}\mathbb{O}(X,[0,1])$, the inverse image of the portion of $\mathbb{O}(X)$ with the norm ≤ 1. (Here p^* denotes a pullback. We do not write p^{-1} since the dual cell corresponding to σ need not be a subspace of X.)

Definition 8.8. To promote $p : W \to \mathbb{O}(X)$ further to a $(0,\ldots,0)$-simplex in $\bar{G}_{0-2}(X,u)$ we add the following: a contravariant CW-functor $W_{\diamond\diamond}$ from $\{0\} * \mathrm{cat}(X)$ to compact spaces, with a natural transformation $\gamma\colon W_{\diamond\diamond} \to W_\diamond$ which evaluates to a homotopy equivalence for every object of $\{0\} * \mathrm{cat}(X)$.

The definition gives us in particular a CW-pair $(W_{\diamond\diamond}(0), W_{\diamond\diamond}[1])$ with dissected boundary, where

$$W_{\diamond\diamond}[1] = \operatorname*{colim}_{\sigma\neq 0} W_{\diamond\diamond}(\sigma).$$

Passage to cellular chain complexes transforms a pair of chain complexes (C,D) with $C = C_*(W_{\diamond\diamond}(0))$ in $\mathbb{B}(\mathbb{A})$ and with dissected boundary $D = C_*(W_{\diamond\diamond}[1])$ in $\mathbb{B}(\mathbb{A}_*(X))$.

Definition 8.9. To promote $p : W \to \mathbb{O}(X)$ further to a $(0,0,\ldots,0)$-simplex in $\bar{G}_{0-3}(X,u)$, we add the following data: cellular diagonal approximations and fundamental cycles/chains in the cellular chain complex(es) of $W_{\diamond\diamond}$.

The additional data in Definition 8.9 imply a preferred structure of an n-dimensional SAPP (where $n = \dim(W)$) on the pair of chain complexes

$$(C_*(W_{\diamond\diamond}(0)), C_*(W_{\diamond\diamond}[1]))$$

in $\mathbb{B}(\mathbb{A})$, refined to a dissected $(n-1)$-dimensional SAPC structure on the boundary in $\mathbb{B}(\mathbb{A}_*(X))$.

All the above refinements of $\bar{G}_0$ have multiplications refining the one on $\bar{G}_0$. The case of $\bar{G}_{0-1}$ has already been discussed. For the case of $\bar{G}_{0-2}$, suppose given (X,u) and (Y,v) in $\mathcal{P}$, as well as $p : W \to \mathbb{O}(X)$ and $q : W' \to \mathbb{O}(Y)$ and

$$W_{\diamond\diamond} \to W_\diamond, \qquad W'_{\diamond\diamond} \to W'_\diamond$$

satisfying the conditions of Definition 8.8. We need to say what $(W \times W')_{\diamond\diamond}$ should be. We have $\{0\} * \mathrm{cat}(X * Y) \cong \{0\} * \mathrm{cat}(X) \times \{0\} * \mathrm{cat}(Y)$. Using this identification we let $(W \times W')_{\diamond\diamond}(i,j) = W_{\diamond\diamond}(i) \times W'_{\diamond\diamond}(j)$ for i in $\{0\} * \mathrm{cat}(X)$ and j in $\{0\} * \mathrm{cat}(Y)$.

Lemma 8.10. *The forgetful maps*

$$\bar{G}_{0-3} \to \bar{G}_{0-2} \to \bar{G}_{0-1} \hookrightarrow \bar{G}_0$$

are (weak) homotopy equivalences.

Next, there are refinements of $\bar{G}_1$ analogous to the above refinements of $\bar{G}_0$. These make up a diagram of functors and natural transformations

$$\bar{G}_{1-3} \to \bar{G}_{1-2} \to \bar{G}_{1-1} \hookrightarrow \bar{G}_1.$$

Let's define them very briefly. We fix (X,u) in $\mathcal{P}$ as before and a $(0,\dots,0)$-simplex $W \to W' \to \mathbb{O}(X)$ in $\bar{G}_1(X,u)$.

Definition 8.11. To promote $W \to W' \to \mathbb{O}(X)$ to a $(0,0,\dots,0)$-simplex in $\bar{G}_{1-1}(X,u)$, we impose the condition that both $W \to \mathbb{O}(X)$ and $W' \to \mathbb{O}(X)$ be simultaneously transverse to the stratification of $\mathbb{O}(X) \smallsetminus 0$ by strata $\mathbb{O}(X,\sigma)$, and to the norm levels. (Compare Definition 8.6.)

Definition 8.12. To promote $W \to W' \to \mathbb{O}(X)$ further to a $(0,\dots,0)$-simplex in $\bar{G}_{1-2}(X,u)$ we add the following data: contravariant CW-functors $W_{\diamond\diamond}$ and $W'_{\diamond\diamond}$ from $\{0\} * \mathrm{cat}(X)$ to compact spaces, a CW-embedding $W_{\diamond\diamond} \to W'_{\diamond\diamond}$ and natural transformations $\gamma\colon W_{\diamond\diamond} \to W_{\diamond}$, $\gamma'\colon W'_{\diamond\diamond} \to W'_{\diamond}$ which evaluate to homotopy equivalences for every object of $\{0\} * \mathrm{cat}(X)$. We require commutativity of

$$\begin{array}{ccc} W_{\diamond\diamond} & \longrightarrow & W'_{\diamond\diamond} \\ \downarrow & & \downarrow \\ W_{\diamond} & \longrightarrow & W'_{\diamond}\,. \end{array}$$

Definition 8.13. To promote $W \to W' \to \mathbb{O}(X)$ further to a $(0,0,\dots,0)$-simplex in $\bar{G}_{1-3}(X,u)$, we add compatible cellular diagonal approximations for $W_{\diamond\diamond}$ and $W'_{\diamond\diamond}$, and fundamental cycles/chains in the cellular chain complex(es) of $W_{\diamond\diamond}$.

As in the case of the functor $\bar{G}_{0-3}$, the additional features allow us to extract certain algebraic data. These are two preferred structures of n-dimensional SAPPs (where $n = \dim(W) = \dim(W')$) and a map

$$q_{\diamond\diamond}\colon (C_*(W_{\diamond\diamond}(0)), C_*(W_{\diamond\diamond}[1])) \to (C_*(W'_{\diamond\diamond}(0)), C_*(W'_{\diamond\diamond}[1]))$$

of SAPPs in $\mathbb{B}(\mathbb{A})$, refined to a map of dissected $(n-1)$-dimensional SAPCs on the boundary in $\mathbb{B}(\mathbb{A}_*(X))$. To obtain a single SAPP with a contractibility property, which is our goal, we need the construction of *symmetric kernels* in the setting of Section 3, Definitions 3.1 and 3.7. This is a purely algebraic and functorial construction and is given after Definition 8.21 below.

Remark. A $(0,0,\dots,0)$-simplex in $\bar{G}_{1-i}(X,u)$ is still considered trivial if the corresponding map $W \to W'$ is a homeomorphism. All trivial $(0,0,\dots,0)$-simplices are to be identified with each other.

Lemma 8.14. *The forgetful maps*

$$\bar{G}_{1-3} \to \bar{G}_{1-2} \to \bar{G}_{1-1} \hookrightarrow \bar{G}_1$$

are weak homotopy equivalences.

All the above refinements of $\bar{G}_1$ admit actions by the corresponding refinements of $\bar{G}_0$ (which refine the action of $\bar{G}_0$ on $\bar{G}_1$). At the top of the range we get multiplication and action maps

$$\bar{G}_{0-3}(X,u) \wedge \bar{G}_{0-3}(Y,v) \longrightarrow \bar{G}_{0-3}(X * Y, u * v)$$

$$\bar{G}_{0-3}(X,u) \wedge \bar{G}_{1-3}(Y,v) \longrightarrow \bar{G}_{1-3}(X * Y, u * v).$$

for any (X,u) and (Y,v) in $\mathcal{P}$.

That completes our efforts to extract chain complex algebra from the functors $\bar{G}_0$ and $\bar{G}_1$. Now we need to do some more work on the geometric side.

Definition 8.15. Let $s \in [0,1]$. For V in $\mathcal{J}^{\text{iso}}$ and $f: X \to V$ in $\mathcal{P}_V$, we define a "norm" function on $\mathcal{O}(X)$ by $tx \mapsto \|tx\|_s = (1-s)\|tx\| + s\|f(tx)\|$, using the Euclidean norm on V.

Keeping the notation of Definition 8.15, we introduce a refinement $r_f\bar{G}_0(X,u)$ of $\bar{G}_0(X,u)$, and a refinement $r_f\bar{G}_1(X,u)$ of $\bar{G}_1(X,u)$, both depending on $f: X \to V$.

Definition 8.16. Let V be in $\mathcal{J}^{\text{iso}}$ and $f: X \to V$ in $\mathcal{P}_V$. To promote a nontrivial $(0,0,\ldots,0)$-simplex $p: W \to \mathcal{O}(X)$ in $\bar{G}_0(X,u)$ to the status of a nontrivial $(0,0,\ldots,0)$-simplex in $r_f\bar{G}_0(X,u)$ we add the assumption $W = V$ and the following data:

- a homotopy $(p_s)_{0\le s\le 1}$ through $\mathbb{Z}/2$-maps $p_s: W \to \mathcal{O}(X)$, each having preimage of base point equal to base point, such that $p_0 = p$ and p_s is transverse to the nonzero levels of the norm function $\|\ldots\|_s$ on $\mathcal{O}(X)$, for $s \in [0,1]$;
- a homotopy $(h_t)_{1\le t\le 2}$ from the composition

$$W \xrightarrow{p_1} \mathcal{O}(X) \xrightarrow{f} V$$

  to the identity, where each $h_t: W \to V$ is equivariant, with preimage of base point equal to base point, and transverse to the nonzero levels of the euclidean norm on V.

Definition 8.17. Let V be in $\mathcal{J}^{\text{iso}}$ and $f: X \to V$ in $\mathcal{P}_V$. To promote a $(0,0,\ldots,0)$-simplex

$$W \xrightarrow{q} W' \xrightarrow{p} \mathcal{O}(X)$$

in $\bar{G}_1(X,u)$ to the status of a $(0,0,\ldots,0)$-simplex in $r_f\bar{G}_1(X,u)$ we add the assumption $W' = V$ and the following data:

- a homotopy $(p_s)_{0\le s\le 1}$ through $\mathbb{Z}/2$-maps $p_s: W' \to \mathcal{O}(X)$, each having preimage of base point equal to base point, such that $p_0 = p$ and p_s is transverse to the nonzero levels of the norm function $\|\ldots\|_s$ on $\mathcal{O}(X)$, for each s;
- a homotopy $(h_t)_{1\le t\le 2}$ from the composition

$$W' \xrightarrow{p_1} \mathcal{O}(X) \xrightarrow{f} V$$

to the identity, where each $h_t : W' \to V$ is equivariant, with preimage of base point equal to base point, and transverse to the nonzero levels of the euclidean norm on V ;
- a homotopy $(q_s)_{0 \le s \le 2}$ through $\mathbb{Z}/2$-maps $W \to W'$, each having preimage of base point equal to base point, with $q_0 = q$, such that $p_s q_s$ is transverse to the nonzero levels of the norm function $\| \ldots \|_s$ on $\mathbb{O}(X)$ for $s \le 1$, and $h_s q_s$ is transverse to the nonzero levels of the euclidean norm on V for $s \ge 1$.

Such a simplex is *trivial* if q_s is a homeomorphism for all $s \in [0, 2]$. All trivial simplices are to be identified with each other.

Next, $r_f \bar{G}_{i-3}(X, u)$ is defined (for $i = 0, 1$) by means of a pullback square

$$\begin{array}{ccc} r_f \bar{G}_{i-3}(X,u) & \longrightarrow & \bar{G}_{i-3}(X,u) \\ \downarrow & & \downarrow \text{forget} \\ r_f \bar{G}_i(X,u) & \xrightarrow{\text{forget}} & \bar{G}_i(X,u). \end{array}$$

To be more precise we form the pullback at the level of multisimplicial sets, i.e., before realization. The square is also a homotopy pullback square (by a direct check on homotopy groups).

Definition 8.18. For V in $\mathcal{J}^{\text{iso}}$ let

$$E^{ga}(V) = \underset{f : X \to V \text{ in } \mathcal{P}_V}{\text{hocolim}} \; r_f \bar{G}_{0-3}(X, u),$$

$$F^{ga}(V) = \underset{f : X \to V \text{ in } \mathcal{P}_V}{\text{hocolim}} \; r_f \bar{G}_{1-3}(X, u).$$

Lemma 8.19. *We still have multiplication and action maps*

$$\begin{array}{ccc} E^{ga}(V) \wedge E^{ga}(W) & \longrightarrow & E^{ga}(V \oplus W) \\ E^{ga}(V) \wedge F^{ga}(W) & \longrightarrow & F^{ga}(V \oplus W) \end{array}$$

for V, W in $\mathcal{J}^{\text{iso}}$.

We define F^g on $\mathcal{J}^{\text{iso}}$ essentially as the restriction of the functor F^g on $\mathcal{J}$ constructed in [Ma], except for the small change that $F^g(W)$ is defined as the geometric realization of an m-fold simplicial set, where $m = \dim(W)$. To make up for the information lost in restricting from $\mathcal{J}$ to $\mathcal{J}^{\text{iso}}$, we introduce a multiplicative functor E^g on $\mathcal{J}^{\text{iso}}$ as in Example 4.2.

Definition 8.20. For V in $\mathcal{J}^{\text{iso}}$, let $E^g(V) = \mathbb{S}^0$. In each multidegree $(k_1, \ldots, k_n)$ we have a base-point and another point which we think of as represented by the space $V \times \prod \Delta^{k_i}$. The action map $E^g \wedge F^g \to F^g$ is given by multiplying a map $q : W \times \prod \Delta^{l_i} \to W' \times \prod \Delta^{l_i}$ with the identity on $V \times \prod \Delta^{k_i}$.

Now we can relate the functor pair $(E^a, F^{a,1})$ to the pair (E^g, F^g), using (E^{ga}, F^{ga}) as a stepping stone. Namely, in Definitions 8.21 and 8.22 below we define the forgetful natural transformations of functors on $\mathcal{J}^{\mathrm{iso}}$

$$\begin{array}{ccccc} E^g & \xleftarrow{w_0} & E^{ga} & \xrightarrow{v_0} & E^a \\ F^g & \xleftarrow{w_1} & F^{ga} & \xrightarrow{v_1} & F^{a,1} \end{array}$$

respecting the multiplications and the actions. It is described in the preview of the proof of Theorem D how these natural transformations make up the identification of F^g and $F^{a,1}$ as functors on $\mathcal{J}$.

Definition 8.21. The transformation w_0 is defined by taking trivial simplices to the basepoint of $\mathbb{S}^0$, and nontrivial simplices to the non-basepoint. The transformation w_1 is induced by forgetful maps

$$r_f \bar{G}_{1-3}(X,u) \to r_f \bar{G}_1(X,u) \to F^g(V).$$

In the notation of Definition 8.17, we proceed by taking a $(0,0,\ldots,0)$-simplex (for example) in $r_f \bar{G}_1(X,u)$, consisting of

$$W \xrightarrow{q} W' \xrightarrow{p} \mathbb{O}(X)$$

and homotopies (p_s), (h_s), (q_s) to $q_2 \colon W \to W' = V$ which is a $(0,0,\ldots,0)$-simplex in $F^g(V)$.

Before describing v_0 and v_1, we review the construction of *symmetric kernels* in the setting of Section 3, Definitions 3.1 and 3.7. Let $\mathbb{A}$ be an additive category with chain duality. Let $f \colon C \to D$ be a morphism in $\mathbb{B}(\mathbb{A})$ and let ψ be an n-dimensional symmetric Poincaré structure on C_* such that $f_*\psi$ is also symmetric Poincaré. It is well known that in such a case (C,ψ) must break up, up to suitable homotopy equivalence, into a sum

$$(K_*, \varphi) \oplus (D, f_*\psi)$$

of SAPC's. What we need is a functorial construction of (K,φ). Let K be the mapping cone of the composite chain map of degree n defined as

$$TD \xrightarrow{Tf} TC \xrightarrow{\psi_0} C.$$

(Think of this as an ordinary chain map of degree zero from a shifted copy of TD to C, where the differentials $d \colon TD_i \to TD_{i-1}$ have been multiplied by $(-1)^{ni}$.) Then there is an inclusion $e \colon C \to K$. We let $\varphi := e_*\psi$, which is a symmetric Poincaré structure on K. The construction works also for pairs.

Hence, in the notation of the comments after Definition 8.13, using the symmetric kernel construction, we obtain from a map of SAPPs

$$q_{\diamond\diamond} \colon (C_*(W_{\diamond\diamond}(0)), C_*(W_{\diamond\diamond}[1])) \to (C_*(W'_{\diamond\diamond}(0)), C_*(W'_{\diamond\diamond}[1]))$$

an n-dimensional SAPP

$$(K(q_{\diamond\diamond}(0)), K(q_{\diamond\diamond}[1]))$$

in $\mathbb{B}(\mathbb{A})$, refined to a dissected SAPC structure on the boundary in $\mathbb{B}(\mathbb{A}_*(X))$. The contractibility condition is satisfied because the map q has degree ± 1.

Definition 8.22. The transformation v_0 is induced by forgetful maps

$$r_f \bar{G}_{0-3}(X,u) \to \bar{G}_{0-3}(X,u) \to E^a(X,u)$$

where the second one extracts the chain complex data (including symmetric structures). The transformation v_1 is induced by maps

$$r_f \bar{G}_{1-3}(X,u) \to \bar{G}_{1-3}(X,u) \to F^{a,1}(X,u)$$

where the first is forgetful and the second extracts the symmetric kernels from the available chain complex data (including symmetric structures).

The definitions of v_0, v_1 are set up to respect the multiplication and action maps.

Remark. There is a slight complication in the definition of v_1, due to the fact that the symmetric kernels determined by multisimplices in $\bar{G}_{1-3}(X,u)$ which we have defined as trivial are not completely trivial. (They are contractible but they are not equal to zero.) It seems best to agree that, wherever a multisimplex in $\bar{G}_{1-3}(X,u)$ has trivial (multi)faces, the corresponding subcomplexes of the symmetric kernel determined by that multisimplex must be collapsed to zero.

This finishes the proof of Theorem D. □

We conclude with an explanation of the remark on infinite loop space structures at the end of the introduction. Each space $\bar{F}^a(X,u)$, in the notation of Section 7, comes equipped with a structure of (underlying space of a) Γ-space in the sense of Segal [Seg], determined by the direct sum operation in the categories $\mathbb{B}(\mathbb{A})$ and $\mathbb{B}(\mathbb{A}_*(X))$. This structure is clearly preserved by the multiplication maps

$$\bar{E}^a(X,u) \wedge \bar{F}^a(Y,v) \longrightarrow \bar{F}^a(X * Y, u * v).$$

(We do not need and we do not use a structure of Γ-space on $\bar{E}^a(X,u)$ here. Informally, one could say that the adjoint map from $\bar{E}^a(X,u)$ to the space of maps from $\bar{F}^a(Y,v)$ to $\bar{F}^a(X * Y, u * v)$ factors canonically through the space of Γ-maps from $\bar{F}^a(Y,v)$ to $\bar{F}^a(X * Y, u * v)$.) It follows that F^a can be refined to a functor (first on $\mathcal{J}^{\mathrm{iso}}$, then on $\mathcal{J}$) with values in the category of group-like Γ-spaces. By [Seg], the "underlying space" functor from group-like Γ-spaces to spaces factors through the category of infinite loop spaces.

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Mathematisches Institut, Universität Münster, Einsteinstraße 62, 48149 Münster, Germany, and Matematický Ústav SAV, Štefánikova 49, Bratislava, SK-81473, Slovakia
macko@math.uni-muenster.de

Department of Mathematical Sciences, University of Aberdeen, Aberdeen, AB24 3UE, Scotland, UK
mweiss@maths.abdn.ac.uk