REIDEMEISTER TORSION, SURGERY INVARIANTS AND SPHERICAL SPACE FORMS

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1. *Introduction*

The paper gives methods for calculating the surgery obstruction of normal maps with target a finite PD complex with finite fundamental group. The results are applied to the spherical space form problem and give fairly complete answers to the questions of dimensional bounds in terms of equivalent specific statements in algebraic number theory.

First we give a description of the general results for evaluating surgery obstructions. To save space we outline the odd-dimensional cases only. Let $f\colon M \to X$ be a degree 1 normal map in (odd) dimension n and let $\pi = \pi_1(X)$. By duality, X is weakly simple and there is a well-defined invariant $\lambda'(f) \in L'_n(\mathbb{Z}\pi)$. It is mapped into the actual surgery invariant $\lambda^h(f) \in L^h_n(\mathbb{Z}\pi)$ by the map J in the Rothenberg exact sequence

$$(1.1)\qquad \mathrm{Wh}'(\mathbb{Z}\pi) \otimes \mathbb{Z}/2 \xrightarrow{t} L'_n(\mathbb{Z}\pi) \xrightarrow{J} L^h_n(\mathbb{Z}\pi) \longrightarrow 0.$$

Here $L'_n(\mathbb{Z}\pi)$ denotes the so-called intermediate surgery obstruction group, and

$$\mathrm{Wh}'(\mathbb{Z}\pi) = \mathrm{Wh}(\mathbb{Z}\pi)/\mathrm{Torsion}.$$

The groups $L'_n(\mathbb{Z}\pi)$ have been determined for finite groups π by Wall in a series of papers (cf. [**29**]).

Inductively, we can assume that $\lambda'(f_\sigma) \in L'_n(\mathbb{Z}\sigma)$ is known for all subgroups $\sigma \subset \pi$, where f_σ is the covering of f corresponding to σ. This amounts to knowing the image of $\lambda'(f)$ under the restriction map

$$\mathrm{Res}\colon L'_n(\mathbb{Z}\pi) \longrightarrow \prod_{\substack{\sigma \subset \pi \\ \sigma \neq \pi}} L'_n(\mathbb{Z}\sigma).$$

However, Res is not always injective. In general, there are three sources for elements in the kernel of Res. This follows from the Mayer–Vietoris exact sequence

$$(1.2)\qquad \dots \longrightarrow CL_{n+1}(\mathbb{Q}\pi) \xrightarrow{\partial} L'_n(\mathbb{Z}\pi) \xrightarrow{\hat{\rho}} L'_n(\hat{\mathbb{Z}}_2\pi) \oplus L'_n(\hat{\mathbb{Z}}_{\mathrm{od}}\pi) \longrightarrow \dots.$$

For the groups considered in connection with spherical space forms, $CL_{n+1}(\mathbb{Q}\pi)$ and $L'_n(\hat{\mathbb{Z}}_2\pi)$ are mapped injectively by Res, but this is not the case for $L'_n(\hat{\mathbb{Z}}_{\mathrm{od}}\pi)$, so we need an extra invariant.

Let F be a field of char$(F) \neq 2$. To every (simple) PD complex X we have the 'higher signature' invariant $\Delta_F(X) \in L^n_s(F\pi)$, [**19, 21**]. Since $\frac{1}{2} \in F$, we have $L^s_n(F\pi) \cong L^n_s(F\pi)$. We show, in favourable cases (e.g. when F is finite), that Δ_F is the image under the canonical map $t\colon H^{n+1}(K_1(F\pi)) \to L^s_n(F\pi)$ of the Reidemeister torsion of the universal cover $\tilde{X}$ with respect to a suitable choice of 'bases' for the homology groups $H_*(\tilde{X}; F)$.

Of course, the homology groups need not be free over $F\pi$, but modulo the radical they split into modules which are Morita-equivalent to vector spaces over skew fields. This permits us to define the Reidemeister torsion in general, using Milnor's definition of torsion.

The higher signature $\Delta_F(X)$ is an invariant of the bordism type. It gives a homomorphism of the singular PD bordism groups

$$\Delta_F\colon \Omega_n^{\mathrm{PD}}(B\pi) \to L_n^s(F\pi),$$

natural with respect to both the covariant and contravariant structure. In particular, the restriction of Δ_F to the PL-bordism groups $\Omega_n^{\mathrm{PL}}(B\pi)$ can be completely calculated from the restriction to Sylow subgroups of π. This follows from Dress's induction theorem for L-groups and Corollary 3.7 of the present paper. Normal cobordism classes of normal maps are detected on Sylow subgroups, so $\Delta_F(M)$ is calculable when M is the source of a normal map. In §3 we prove

THEOREM A. *Let $f\colon M \to X$ be a normal map over an odd-dimensional PD complex. If*

(i) $\Delta_F(M) = \Delta_F(X)$ *for all* $F = \mathbb{F}_p, \mathbb{R}$, *with* $p \neq 2$, *and*

(ii) $\hat{\rho}_2(\lambda'(f)) = 0$ *in* $L_n'(\hat{\mathbb{Z}}_2\pi)$,

then $\lambda'(f) \in \mathrm{Image}(\partial\colon CL_{n+1}(\mathbb{Q}\pi) \to L_n'(\mathbb{Z}\pi))$.

There is a similar result in even dimensions, cf. Theorem 3.12. The point of Theorem A is that the subgroup of $L_n'(\mathbb{Z}\pi)$ coming from $CL_{n+1}(\mathbb{Q}\pi)$ has much better 'induction' properties than the full group $L_n'(\mathbb{Z}\pi)$. However, Condition (ii) of Theorem A is not satisfying. One would like to modify Δ to an invariant which, on the one hand, is calculable and, on the other hand, detects $\hat{\rho}_2(\lambda'(f))$, at least in favourable situations. This indeed seems possible. The basic 2-adic case is not a field, however, but a 2-adic Dedekind domain. I hope to return to this question in a future paper. It is connected to Conjecture D below.

Next we review the applications to the spherical space form problem. Suppose π is a group with periodic cohomology groups of period d and suppose each involution in π is central. Then it acts freely on some sphere by the result in **[13]**, and we ask for the dimensions in which a free action can take place.

Results from **[28]** give free actions in all dimensions $2rd-1$. However, for certain groups of cohomological period 4, the question of whether a free action can exist on spheres of dimension $8k+3$ was left open. The question was taken up by R. J. Milgram who gave the first examples with non-vanishing finiteness obstruction. The author then calculated the surgery obstruction in a few examples in **[11]**. Slightly later, using different methods, Milgram gave an alternative calculation of the surgery invariant in some other cases.

The results presented in the present paper complete the work in **[11]**. Our results overlap with the recent account in **[16]** of Milgram's calculations.

The basic groups to consider are the semi-direct products

$$Q(8p, q) = \mathbb{Z}/pq \tilde{\times} Q(8),$$

where p and q are distinct odd primes, $Q(8)$ is the quaternion group of order 8, and where the homomorphism $\varphi\colon Q(8) \to (\mathbb{Z}/pq)^\times$ determined by $Q(8p, q)$ has kernel $\mathbb{Z}/2$. These groups act freely on spheres in dimensions $8k+7$ by orthogonal maps; the

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question is whether they can act freely on S^{8k+3}, where $k \geqslant 1$, by homeomorphisms or diffeomorphisms.

Recall that the homotopy type of an action of $\pi = Q(8p, q)$ on S^{8k+3} is determined by the first k-invariant of the orbit space

$$e(S^{8k+3}, \pi) \in H^{8k+4}(\pi; \mathbb{Z}).$$

If the action is linear, e is the Euler class and in general e is always a generator of $H^{8k+4}(\pi; \mathbb{Z}) \cong \mathbb{Z}/|\pi|$.

The Sylow subgroups $\mathbb{Z}/p$, $\mathbb{Z}/q$, and $Q(8)$ all admit free orthogonal actions on S^{8k+3}. Let χ_p, χ_q be any non-trivial characters of $\mathbb{Z}/p$, $\mathbb{Z}/q$, and let Γ be the standard representation on $\mathbb{C}^2$. Let $e_k \in H^{8k+4}(\pi; \mathbb{Z})$ be the generator with

$$\begin{aligned} e_k \,|\, \mathbb{Z}/l &= e(\chi_l + \chi_l^{-1})^k \quad \text{for } l = p, q, \\ e_k \,|\, Q(8) &= e(\Gamma)^k. \end{aligned} \tag{1.3}$$

We ask whether there exists a free action of $Q(8p, q)$ on S^{8k+3} with k-invariant e_k. The answer depends on arithmetic questions. We need some notation. Let ζ_p, ζ_q be primitive roots of 1, let $\zeta_{pq} = \zeta_p\zeta_q$, and write $\eta_r = \zeta_r + \zeta_r^{-1}$. Consider the cyclotomic fields $\mathbb{Q}(\eta_p, \eta_q)$ and $\mathbb{Q}(\eta_{pq})$ with integers

$$A = \mathbb{Z}[\eta_p, \eta_q], \quad B = \mathbb{Z}[\eta_{pq}], \quad |B : A| = 2.$$

Let $\Phi_A \colon A^\times \to (A/p)^\times_{(2)} \times (A/q)^\times_{(2)}$ and $\varphi_A \colon A^\times \to (A/4A)^\times_{(2)}$ denote the reduction maps onto the 2-primary components of $(A/p)^\times \times (A/q)^\times$ and $(A/4A)^\times$. Our main result is the following restatement of Theorem 7.5.

THEOREM B. *The group $Q(8p, q)$ acts freely on S^{8k+3}, for $k \geqslant 1$, with k-invariant e_k if and only if the following three conditions are satisfied:*

(i) $(1, -1) \in \operatorname{Image}(\Phi_A)$;

(ii) $(\eta_q - 2, \eta_p - 2) \in \operatorname{Image}(\Phi_A \,|\, \operatorname{Ker} \varphi_A)$;

(iii) $(2, 2) \in \operatorname{Image}(\Phi_A)$.

The first and third conditions imply that the finiteness obstruction $\sigma(e_k)$ is zero in $\tilde{K}_0(\mathbb{Z}[Q(8p, q)])$. The second condition gives that the surgery obstruction $\lambda^h(f)$ belongs to the image of the composition

$$CL_0(\mathbb{Q}\pi) \xrightarrow{\partial} L'_3(\mathbb{Z}\pi) \xrightarrow{J} L^h_3(\mathbb{Z}\pi),$$

say $J \circ \partial(V) = \lambda^h(f)$. The third condition is equivalent to the assertion that $\partial(V) \subseteq \operatorname{Image} t$ in (1.1), whence $\lambda^h(f) = 0$.

In Theorem B we have not specified in which category the action should take place. Actually, the results are the strongest possible in the sense that if (i), (ii), and (iii) are satisfied, then the action can be taken to be smooth. If the conditions are not satisfied, the claim is that there is no free topological action.

In special cases we can explicate the conditions of Theorem B somewhat.

COROLLARY C. *Suppose $p \equiv 3 \pmod 4$. There exists free action of $Q(8p, q)$ with k-invariant e_k if and only if*

(i) $q \equiv 1 \pmod 8$ *and*

(ii) $(\eta_q - 2, \eta_p - 2) \in \operatorname{Image}(\Phi_A \,|\, \operatorname{Ker} \varphi_A)$.

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Taking norms, over the subgroup $(\mathbb{Z}/q)^\times/\langle -1\rangle$ of the Galois group of $A/\mathbb{Z}$, we note that Condition (ii) of Corollary C implies that the Legendre symbol $\left(\frac{p}{q}\right)$ equals $+1$, but in general this condition is not sufficient to imply Corollary C (ii). Let $\mathrm{Ord}_p(q)$ denote the (multiplicative) order of q in $\mathbb{F}_p^\times$. If we strengthen the necessary condition on the Legendre symbol to the condition that $\mathrm{Ord}_p(q)$ and $\mathrm{Ord}_q(p)$ be of maximal odd order, then Corollary C (ii) follows. For example, $Q(8p, q)$ acts freely for $p = 3$, $q = 313$.

The finiteness obstruction depends on the k-invariant (alias the generator of $H^{8k+4}(Q(8p,q);\mathbb{Z})$), as specified in [**24**]. In favourable cases there is essentially only one k-invariant with vanishing finiteness obstruction. Indeed this happens precisely when the Swan homomorphism

$$S\colon (\mathbb{Z}/8pq)^\times \to \tilde{K}_0(\mathbb{Z}[Q(8p,q)])$$

has maximal image, equal to $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In general

$$\mathrm{Image}\, S = \mathbb{Z}/2 \oplus \tilde{S}(p) \oplus \tilde{S}(q) \oplus \tilde{S}(pq)$$

and from [**1**] we have $\tilde{S}(p) = 0$ only in the following two cases:

$$(*) \qquad p \equiv -1\ (8) \quad \text{or} \quad p \equiv 1\ (8) \quad \text{and} \quad \mathrm{Ord}_p(2) \text{ is odd.}$$

If neither p nor q satisfies (*) then e_k is (up to group automorphism) the only generator for $Q(8p,q)$ which has vanishing finiteness obstruction. In this situation Theorem B gives the complete answer to the question of which groups $Q(8p,q)$ can act freely on $(8k+3)$-dimensional spheres.

If p or q satisfies (*) then there might be k-invariants different from e_k with vanishing finiteness obstruction. For example, if $p \equiv -1$ (8) and $q \equiv 3$ (8) then e_k has non-trivial finiteness obstruction but there is a second k-invariant with vanishing obstruction; the methods of this paper cannot decide if this second k-invariant determines a space form or not. However, if Conjecture D below is true then only the e_k may be k-invariants of space forms.

Consider the generalized quaternion group $Q(4p)$ of order $4p$. There are two finite homotopy types $\Sigma/Q(4p)$ with $\Sigma \simeq S^{4k+3}$. One is realized by an orthogonal space form, the other is not.

CONJECTURE D. *If $\Sigma/Q(4p)$ is not homotopy equivalent to an orthogonal space form then it is not homotopy equivalent to a manifold either.*

In order to settle completely the question of dimensional bounds (in dimensions greater than or equal to 5) for free actions on spheres, it suffices to decide which among the groups $\mathbb{Z}/a \times Q(8b,c,d)$ can act freely on S^{8k+3}, cf. [**28**, **8**]. Our groups $Q(8p,q)$ above correspond to $a = 1$, $b = p$, $c = q$, and $d = 1$. In principle the methods of this paper apply to all groups in the family, but the calculations become more difficult.

There is a generalization of Theorem B to the groups $Q(8a,b)$ where a and b are not necessarily prime numbers. We need extra notation before we can state the result.

For a number α we write $\boldsymbol{\alpha}$ for the set of its full prime power divisors, that is, $\boldsymbol{\alpha} = \{p_1^{i_1}, \ldots, p_r^{i_r}\}$ if $\alpha = p_1^{i_1} \ldots p_r^{i_r}$ is the prime decomposition. Given two relatively prime numbers α and β we shall consider subsets $S \subseteq \boldsymbol{\alpha} \cup \boldsymbol{\beta}$. The cardinality of S is

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denoted by $|S|$. Define elements

$$\eta_S = \zeta_r + \zeta_r^{-1} \quad \text{where } r = \prod_{q \in S} q,$$
$$\eta_\varnothing = 2.$$

Let $A = \mathbb{Z}[\eta_\alpha, \eta_\beta]$. We have

$$(A/a)^\times_{(2)} = \prod_{i=1}^{r} (\mathbb{F}_{p_i} \otimes A)^\times_{(2)}, \quad (A/b)^\times_{(2)} = \prod_{i=r+1}^{n} (\mathbb{F}_{p_i} \otimes A)^\times_{(2)},$$

where $p_1, \ldots, p_r$ are the prime divisors in a, and $p_{r+1}, \ldots, p_n$ the prime divisors in b. Define

$$\bar{V}_\nu(a,b) = \prod \{2 - \eta_q \mid q \in \mathbf{a} \cup \mathbf{b} - \{p_\nu^{i_\nu}\}\} \in (\mathbb{F}_{p_\nu} \otimes A)^\times_{(2)}.$$

Let $\bar{V}(a,b) \in (A/a)^\times_{(2)} \times (A/b)^\times_{(2)}$ denote the n-tuple of the elements $\bar{V}_\nu(a,b)$.

THEOREM E. *The group $Q(8a,b)$ acts freely on S^{8k+3} with k-invariant e_k if and only if, for all divisors $\alpha \mid a$ and $\beta \mid b$ and $A = \mathbb{Z}[\eta_\alpha, \eta_\beta]$, we have*

(i) $(1, -1) \in \operatorname{Image}(\Phi_A)$,

(ii) $(-1)^{n+1}\bar{V}(\alpha, \beta) \in \operatorname{Image}(\Phi_A \mid \operatorname{Ker} \varphi_A)$, *where $n = |\boldsymbol{\alpha} \cup \boldsymbol{\beta}|$,*

(iii) $(2,2) \in \operatorname{Image}(\Phi_A)$.

The proof of Theorem E is quite similar to the proof of Theorem B given in § 7, once one has the necessary information about the Reidemeister torsion and the finiteness obstruction from [**2**], generalizing the present paper's § 6.

For all groups in the family $\mathbb{Z}/a \times Q(8b,c,d)$ the surgery obstruction is detected on the three subgroups $Q(8b,c)$, $Q(8b,d)$, and $Q(8c,d)$, but the finiteness obstruction seems not to be, cf. Corollary 5.12.

The paper is divided into seven sections:

1. Introduction
2. Higher signature and Reidemeister torsion
3. Induction results for surgery invariants
4. Calculation of some L-groups
5. Evaluating the Rothenberg sequence
6. Finiteness obstruction and Reidemeister torsion
7. The surgery obstruction for spherical space forms

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Finally, it is a pleasure to thank Princeton University for its hospitality in connection with a three month stay in the spring of 1980 when much of this work was carried out.

2. *Higher signature and Reidemeister torsion*

Let (R, α, u) be an antistructure in the sense of [**29**, § 1.1]. In this section we give a calculation of the higher signature invariant of symmetric simple Poincaré duality

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(PD) chain complexes over (R, α, u) when R is semisimple and not of characteristic 2. Our main applications are to the cases where R is an algebra over the finite field $\mathbb{F}_q$ with q odd, or an algebra over the real numbers $\mathbb{R}$.

First, we recall the necessary definitions and results from **[19]** and **[21]**. We assume that the unit u in our antistructure is central (usually $u = \pm 1$) and consider chain complexes of finite length C_* consisting of based, finitely generated, right R-modules. The anti-involution α can be used (as usual) to give C_* the structure of a complex of left R-modules ($r \cdot c = c\alpha(r)$). The tensor product complex $C_* \otimes_R C_*$ has an involution T_u given by $T_u(c_1 \otimes c_2) = (-1)^{|c_1||c_2|}u(c_2 \otimes c_1)$. Write $Q_n(C_*)$, respectively $Q^n(C_*)$, for the hyperhomology, respectively hypercohomology, of $\mathbb{Z}/2$ with coefficient in $C_* \otimes_R C_*$. If W is the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/2]$ with $W_n = e_n \cdot \mathbb{Z}[\mathbb{Z}/2]$ and $\partial(e_n) = e_{n-1}(1+(-1)^n T)$, then

$$Q_n(C_*) = H_n(W_* \otimes_{\mathbb{Z}[\mathbb{Z}/2]} C_* \otimes_R C_*),$$
$$Q^n(C_*) = H^n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W_*, C_* \otimes_R C_*)).$$

Every class $\Psi \in Q_n(C_*)$ is represented by a cycle of the form $\sum e_i \otimes \psi_i$, and $(1+T_u)\psi_0$ is a cycle in $C_* \otimes_R C_*$ whose homology class depends only on Ψ.

Let C^* be the complex consisting of the dual R-modules $C^i = \mathrm{Hom}_R(C_i, R)$. Taking the slant product with $(1+T_u)\psi_0$ defines a chain map of R-modules

$$\varphi\colon C^i \to C_{i-n}, \quad \varphi(f) = (1+T_u)\psi_0/f. \tag{2.1}$$

The pair (C_*, Ψ) is called a *quadratic PD chain complex* over (R, α, u) if the map φ in (2.1) is a chain homotopy equivalence. Similarly, if $\Phi \in Q^n(C_*)$ then (C_*, Φ) is called a *symmetric PD chain complex* if the 0th component $\Phi(e_0)$ is represented by a chain homotopy equivalence. If $\Psi \in Q_n(C_*)$ (respectively $\Phi \in Q^n(C_*)$) we say that (C_*, Ψ) (respectively (C_*, Φ)) has *formal dimension n* and in this case we usually assume $C_i = 0$ if $i < 0$ or $i > n$. In **[21]** the terms u-quadratic and u-symmetric Poincaré chain complexes are used, but we prefer to specify the full antistructure.

We have assumed that C_* consists of based R-modules. The dual complex C^* is given the dual basis, and we have the Whitehead torsion $\tau(\varphi) \in K_1(R)$ of the homotopy equivalence in (2.1). If $\tau(\varphi) = 0$ then we say that (C_*, Ψ) or (C_*, Φ) is a *simple* (*quadratic or symmetric*) *PD chain complex*. There are similar definitions for pairs.

Associated to an n-dimensional quadratic PD chain complex over (R, α, u) there is an algebraic surgery invariant $\lambda(C_*)$. It lies in $L_n^S(R, \alpha, u)$ if C_* is simple and in $L_n^K(R, \alpha, u)$ in general. We recall a definition of $\lambda(C_*)$ suitable for our applications. There are three steps to the definition.

First, by algebraic surgery every simple, quadratic PD chain complex is cobordant to a (simple) complex whose homology is concentrated in the 'middle dimensions'. Thus we may assume

$$H_i(C_*) = 0 \text{ for } i \neq \tfrac{1}{2}n \qquad (n \text{ even}),$$
$$H_i(C_*) = 0 \text{ for } i \neq \tfrac{1}{2}(n \pm 1) \quad (n \text{ odd}).$$

Second, such a (C_*, Ψ) can be 'contracted' in the sense that C_* is simple chain homotopy equivalent to a simple, quadratic PD chain complex (C''_*, Ψ'') which is concentrated in the middle dimensions. Indeed, let

$$C_*\colon 0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

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Then ∂_1 is surjective and $C_1 \cong C'_1 \oplus C_0$ as s-based modules, so $C_* \cong C'_* \oplus E'_*$ where

$$E'_*: 0 \to C_0 = C_0 \to 0,$$
$$C'_*: 0 \to C_n \to \ldots \to C_2 \to C'_1 \to 0.$$

Since E'_* is contractible, $Q_n(C_*) = Q_n(C'_*)$. The last differential $\partial_n\colon C_n \to C_{n-1}$ is split injective so $C_{n-1} \cong C'_{n-1} \oplus C_n$ as s-based modules. This follows from duality: the mapping cone of $\varphi\colon (C')^* \to C'_{n-*}$ is contractible, and if s is a contracting homotopy then

$$C_{n-1} \xrightarrow{\binom{0}{1}} C^2 \oplus C_{n-1} \xrightarrow{s} (C')^1 \oplus C_n \xrightarrow{(0,1)} C_n$$

is a section of ∂_n. Let

$$C''_*: 0 \to C'_{n-1} \to C_{n-2} \to \ldots \to C_2 \to C'_1 \to 0.$$

Then $Q_n(C'_*) = Q_n(C''_*)$ and C''_* becomes a simple, quadratic PD chain complex. It is also simple chain homotopy equivalent to C_*.

Third, suppose that C_* is concentrated in the middle dimensions. Then $\lambda(C_*) \in L^S_n(R, \alpha, u)$ is easily defined. We give the details for $n = 2k+1$, leaving the easier case, $n = 2k$, to the reader. The mapping cone of $\varphi\colon C^* \to C_*$ gives an exact sequence ($\varphi = \psi + v\psi^*$, $v = (-1)^k u$):

$$0 \longrightarrow C^k \xrightarrow{\binom{\delta}{\varphi}} C^{k+1} \oplus C_{k+1} \xrightarrow{(v\varphi,\, \partial)} C_k \longrightarrow 0.$$

The sequence is based exact and gives a simple formation $(H_v(C_{k+1}); C_{k+1}, C^k)$ over (R, α, v). Let

$$f\colon H_v(C_{k+1}) \to H_v(C_{k+1})$$

be the isomorphism which takes the based module C_{k+1} isomorphically onto the based module C^k. The class of f is the desired element,

$$\lambda(C_*) = \operatorname{cls}(f) \in L^S_1(R, \alpha, v) = L^S_{2k+1}(R, \alpha, u). \tag{2.2}$$

The cobordism classes of simple symmetric PD chain complexes are denoted $L^n_S(R, \alpha, u)$. The invariant λ gives an identification of the cobordism classes of simple quadratic PD complexes with $L^S_n(R, \alpha, u)$, cf. [**21**]. If $\frac{1}{2} \in R$, there is no distinction between the quadratic and the symmetric cases,

$$H_n(Q_n(C_*)) \cong H_n(Q^n(C_*)) \cong H_n(C_* \otimes_R C_*)^{\mathbb{Z}/2},$$

and $L^S_n(R, \alpha, u) \cong L^n_S(R, \alpha, u)$.

Given a pair $U \subseteq V$ of subgroups of (R, α, u), closed under the involution induced on $K_1(R)$ from the transpose, α-conjugate operation on $Gl(R)$, there is the Rothenberg exact sequence

$$\ldots \longrightarrow H^{n+1}(V/U) \xrightarrow{t_n} L^U_n(R, \alpha, u) \xrightarrow{J} L^V_n(R, \alpha, u) \xrightarrow{d_n} H^n(V/U) \longrightarrow \ldots, \tag{2.3}$$

where $H^i(V/U)$ denotes the Tate cohomology group of $\mathbb{Z}/2$ with coefficients in V/U,

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cf. [27]. On representatives the maps are given by

$$t_0(A) = \left(R^m \oplus R^m, \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\right), \quad t_1(A) = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix},$$
$$d_0(R^{2m}, Q) = \det((-u)^{-1}(Q+uQ^*)), \quad d_1(A) = \det A.$$

Here (R^{2m}, Q) denotes the quadratic form on R^{2m} with matrix Q; the associated hermitian product has matrix $Q+uQ^*$.

We are mostly interested in the extreme case of (2.3) with $U = \{0\}$ and $V = K_1(R)$, where is it customary to write L_n^S, L_n^K instead of $L_n^{\{0\}}$, $L_n^{K_1(R)}$. We have

THEOREM 2.4. *Let (C_*, Ψ) be a simple quadratic PD chain complex with $J(\lambda(C_*)) = 0$. Then C_* is algebraically cobordant to an acyclic complex $\tilde{C}_*$, and*

$$\lambda(C_*) = t_n(\Delta(\tilde{C}_*)),$$

where $\Delta(\tilde{C}_*)$ is the class of the torsion $\tau(\tilde{C}_*)$.

Proof. Suppose the formal dimension is $2k+1$. Since $J(\lambda(C_*)) = 0$, C_* is cobordant to an acyclic complex C'_* concentrated in the middle dimensions. We show for a suitable choice of bases that the bordism $(D_*, C_* \oplus (-C'_*))$ is a simple (quadratic) PD pair.

Let $i\colon C_* \to D_*$, $i'\colon C'_* \to D_*$ be the inclusions. Consider the mapping cones $D_*(i)$, $D_*(i')$ and the kernel complexes $D^*(i)$, $D^*(i')$ of the dual maps $i^*\colon D^* \to C^*$, $(i')^*\colon D^* \to (C')^*$. There are commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & D^*(i \oplus i') \arrow[r] \arrow[d, "\hat{\Phi}"] & D^*(i') \arrow[r] \arrow[d, "\Phi"] & C^* \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & D_* \arrow[r] & D_*(i) \arrow[r] & C_* \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & D^*(i \oplus i') \arrow[r] \arrow[d, "\hat{\Phi}"] & D^*(i) \arrow[r] \arrow[d, "\Phi^*"] & (C')^* \arrow[r] \arrow[d, "\varphi'"] & 0 \\ 0 \arrow[r] & D_* \arrow[r] & D_*(i') \arrow[r] & C'_* \arrow[r] & 0 \end{tikzcd}$$

of chain homotopy equivalences with φ simple and Φ, Φ^* essentially dual chain maps. It follows from these diagrams that there exist bases for D_* and C'_* such that the horizontal sequences are based and the vertical maps are simple. Then $(D_*, C \oplus (-C'_*))$ is simple and $\lambda(C_*) = \lambda(C'_*)$.

We have $C'_i = 0$ for $i \neq k$, $k+1$, and $\partial\colon C'_{k+1} \to C'_k$ is an isomorphism. The formation $(H_v(C'_{k+1}); C'_{k+1}, C'^k)$ associated to (C'_*, Ψ') is isomorphic to the standard hyperbolic formation by an isomorphism

$$\begin{pmatrix} 1 & v\chi^* - \chi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \partial & 0 \\ 0 & \delta^{-1} \end{pmatrix}\colon (H_v(C'_{k+1}); C'_{k+1}, C'^k) \to (H_v(C'_k); C'_k, C'^k).$$

It follows that $\lambda(C'_*) = t_n(\det(\partial\colon C'_{k+1} \to C'_k))$, and hence that $\lambda(C'_*) = t_n(\Delta(C'_*))$. This completes the proof for quadratic PD complexes of odd formal dimension. The even-dimensional case is similar but easier.

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REMARK 2.5. In the above we have used simple PD chain complexes and have considered the simple L-groups L_n^S. Technically, it is often convenient instead to use PD chain complexes whose torsion vanishes in some natural quotient group $K_1(R)/U(R)$. The related surgery groups are then L_n^U. All results above apply to this situation when we make the obvious substitutions. For group rings $R = \mathbb{Z}\pi$, we shall use $U = \pi/[\pi, \pi] \oplus \langle \pm 1 \rangle$, $U = SK_1(\mathbb{Z}\pi)$, and $U = SK_1(\mathbb{Z}\pi) \oplus \pi/[\pi, \pi] \oplus \langle \pm 1 \rangle$. The quotient groups are $\mathrm{Wh}(\mathbb{Z}\pi)$, $K_1'(\mathbb{Z}\pi)$, and $\mathrm{Wh}'(\mathbb{Z}\pi)$, respectively.

We now recall from [**27**, §6; **29**, §1.1] some calculational results about the structure of (2.3) in the cases where R is a semi-simple algebra over a field whose characteristic is not equal to 2. There is a decomposition

$$(2.6)\qquad (R, \alpha, u) = \prod_{i=1}^{k} (M_{n_i}(E_i), \tilde{\alpha}_i, \tilde{u}_i) \times \prod_{j=1}^{l} (M_{n_j}(E_j) \times M_{n_j}(E_j), \alpha_j, u_j),$$

where in the second product α acts by permuting the factors (Type GL anti-structures). In the first product, $\tilde{\alpha}_i(A) = (A^{\alpha_i})^t$ and $\tilde{u}_i = u_i I$, where α_i is some anti-involution on the division algebra E_i. The sequence in (2.3) decomposes into a direct sum of sequences corresponding to the factors in (2.6).

For the Type GL factors, (2.3) completely vanishes. For each of the other factors, by Morita equivalence, the sequence for $(M_{n_i}(E_i), \tilde{\alpha}_i, \tilde{u}_i)$ is isomorphic to the sequence for (E_i, α_i, u_i).

Suppose $(R, \alpha, u) = (E, \alpha, u)$ with E a division algebra with centre F (char $F \neq 2$). Then (R, α, u) is divided into types: Type O, Type Sp, and Type U. Since Types O and Sp are interchanged when u is replaced by $-u$ and $L_n(R, \alpha, u) = L_{n+2}(R, \alpha, -u)$, it suffices to consider Type O and Type U. We restrict attention to the case of (2.3) with $U = \{0\}$ and $V = K_1(E)$.

From [**27**, Theorem 5],

$$J_n \colon L_n^S(E, \alpha, u) \to L_n^K(E, \alpha, u)$$

is zero when $n \equiv 1 \pmod 2$, so

$$(2.7)\qquad L_{2n-1}^S(E, \alpha, u) \cong H^{2n}(K_1(E))/\mathrm{Image}\, d_{2n}.$$

For n even, the behaviour of J_n depends on the structure of the centre field F. We consider three cases:

$$(2.8)\qquad F \text{ finite (char } F \neq 2), \quad F \text{ global}, \quad F \text{ continuous} \quad (= \mathbb{R}, \mathbb{C}).$$

(By global we mean here an abelian extension of $\mathbb{Q}$.)

If F is finite then $J_{2n} = 0$ (and $E = F$), so

$$(2.9)\qquad L_{2n}^S(F, \alpha, u) = H^{2n+1}(F^\times)/\mathrm{Image}\, d_{2n+1}, \quad F \text{ finite}.$$

In the rest of the cases with one exception, either $J_{2n} = 0$ or d_{2n+1} is surjective. The exceptional case is

$$(2.10)\qquad (E, \alpha, u) \text{ Type O}, \quad F \text{ global}, \quad E \text{ split at infinite primes}.$$

TABLE 2.11. (i) If (E, α, u) has Type O, then $H^{n+1}(K_1(E))/\mathrm{Image}\, d_{n+1}$ is given by

n	finite	F infinite $E \neq F$	infinite $E = F$
1	$F^\times/(F^\times)^2$	$F^*/(F^\times)^2$	$F^\times/(F^\times)^2$
2	$\langle \pm 1 \rangle$	${}_2F^*$	$\langle \pm 1 \rangle$
3	0	0	0
4	0	${}_2F^*$	0

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Here $F^* = F^\times$ unless E is non-split at infinite primes when F^* denotes the subset of elements which become positive at all real embeddings of F.

(ii) If (E, α, u) has Type U, then $H^*(K_1(E))/\text{Image } d_* = 0$.

The discussion above shows that

$$0 \longrightarrow H^{n+1}(K_1(E))/\text{Image}\, d_{n+1} \xrightarrow{t_n} L_n^S(E, \alpha, u) \longrightarrow \text{Image}\, J_n \longrightarrow 0$$

is trivially split, except in the exceptional case (2.10) with $n \equiv 0$ (2). Hence, given a (symmetric) PD chain complex C_* over (E, α, u) one expects an a priori defined invariant

$$\Delta(C_*) \in H^{n+1}(K_1(E))/\text{Image}\, d_{n+1}, \tag{2.12}$$

except in case (2.10) with n even. The invariant we need is the Reidemeister torsion from [**17**, §3]. We recall some notation. Let $\underline{b}_1$ and $\underline{b}_2$ be bases for a finitely generated E-module M. Then $\underline{b}_1 = \underline{b}_2 \cdot B$ for some non-singular E-matrix B. The element of $K_1(E)$ associated to B is denoted by $[\underline{b}_1/\underline{b}_2]$.

Let $M^* = \text{Hom}_E(M, E)$ be the dual E-module with $(f \cdot e)(m) = \alpha(e)f(m)$, and let $\underline{b}_i^*$ be the base which is dual to $\underline{b}_i$. Then

$$[\underline{b}_1^*/\underline{b}_2^*] = -\overline{[\underline{b}_1/\underline{b}_2]}.$$

Here the bar indicates the involution on $K_1(E)$ induced from α; on representing matrices it is given by the α-conjugate, transpose operation.

Consider the situation on homology. Let $H^p = H^p(C^*)$ and $H_p = H_p(C_*)$. The chain homotopy equivalence φ from (2.1) induces isomorphisms

$$\varphi_{n-p} \colon H^p \to H_{n-p}.$$

We also have the canonical isomorphisms $\kappa_p \colon H^p \cong H_p^*$ and a simple calculation with the slant product gives

$$\varphi_p^* \circ \kappa_p = (-1)^{p(n-p)} u \kappa_{n-p}^* \circ \varphi_{n-p}. \tag{2.13}$$

It would be troublesome (if not impossible) to keep track of the elements in $K_1(E)$ determined by such sign homomorphisms. However, $\det((-1)^{p(n-p)}u \colon H^p \to H^p) = 0$ in $K_1(E)$ if $\dim_E H^p$ is even. Therefore we make the following assumption.

ASSUMPTION 2.14. All chain complexes are even-dimensional in the sense that $\dim_E C_p \equiv 0 \pmod 2$ and $\dim_E H_p \equiv 0 \pmod 2$ for all p.

Let C_* be a based PD chain complex of dimension n; the given base for C_p is denoted $\underline{c}_p$. There is a standard decomposition of chain groups $C_p \cong B_p \oplus H_p \oplus B_{p-1}$ which becomes a chain isomorphism when we use the differential

$$\partial(b_p, h_p, b_{p-1}) = (b_{p-1}, 0, 0)$$

on the right-hand side. Since we assume C_i and H_i have even dimension over E, so does B_i for all i.

Let $\underline{h}_p$ be a basis for $H_p = H_p(C_*)$. Milnor defines the torsion

$$\tau(C_*, \underline{h}_*) = \sum (-1)^i [\underline{b}_i \underline{h}_i \underline{b}_{i-1}/\underline{c}_i] \in K_1(E). \tag{2.15}$$

Actually, in [**17**] τ is considered as an element of $K_1(E)/\det(-1)$, but it is relevant for our use of τ that it is lifted to an invariant of $K_1(E)$. This would cause problems in

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general, but with Assumption 2.14, all properties proved in [17] can be proved for the invariant in (2.15). In particular, we need the formula below for the torsion in an exact sequence of chain complexes.

Consider an exact sequence of based chain complexes

$$0 \to C'_* \to C_* \to C''_* \to 0$$

and assume the bases are compatible in the sense that $[\underline{c}'_i\underline{c}''_i/\underline{c}_i] = 0$ for all i. Let $\underline{h}'_i$, $\underline{h}_i$, and $\underline{h}''_i$ be bases for the homology groups, giving a based, exact homology sequence

$$\mathscr{H}_*\colon \dots \to H'_n \to H_n \to H''_n \to \dots \to H'_0 \to H_0 \to H''_0 \to 0.$$

We have, from [17, § 3],

$$\tau(C_*, \underline{h}_*) = \tau(C'_*, h'_*) + \tau(C''_*, h''_*) + \tau(\mathscr{H}_*). \tag{2.16}$$

Suppose C_* has even formal dimension, $n = 2m$. Then $\varphi_m\colon H^m \to H_m$ defines a $(-1)^m u$-hermitian form (H_m, φ_m) on H_m. Such a form is called *hyperbolic* if it admits a symplectic base. In this case $\dim_E H_m = 2p$, and expressed in the symplectic base, $\varphi_m = J_p$ where

$$J_p = \begin{pmatrix} 0 & I_p \\ \varepsilon I_p & 0 \end{pmatrix}, \quad \text{with } \varepsilon = (-1)^m u.$$

DEFINITION 2.17. Let C_* be a symmetric PD chain complex of formal dimension n over (E, α, u). A family $\underline{h}_* = \{\underline{h}_i\}$ of bases $\underline{h}_i$ for $H_i(C_*)$ is called a *PD base* if $\varphi_i\colon H^{n-i} \to H_i$ is simple for $i \neq \frac{1}{2}n$ with respect to the bases $\underline{h}^{n-i}$ $(= \underline{h}^*_{n-i})$ and $\underline{h}_i$. If $n = 2m$ we suppose further that $\underline{h}_m$ is a symplectic base for (H_m, φ_m).

LEMMA 2.18. *Let C_* be a simple (symmetric) PD chain complex over (E, α, u). If C_* has odd formal dimension then it has a PD base. If C_* has even formal dimension $2m$ and $J_{2m}\colon L^S_{2m}(E, \alpha, u) \to L^K_{2m}(E, \alpha, u)$ is trivial, then C_* has a PD base.*

Proof. For odd formal dimension Lemma 2.18 follows from (2.13). If the dimension is even, (2.16) gives

$$\begin{aligned} 0 = \tau(C_*(\varphi)) &= \tau(C_*, \underline{h}_*) + \tau(C^*, \underline{h}^*) + (-1)^m u \det(\varphi_m) \\ &= \tau(C_*, \underline{h}_*) + \overline{\tau(C_*, \underline{h}_*)} + (-1)^m u \det(\varphi_m). \end{aligned}$$

Hence the hermitian form φ_m has vanishing discriminant in $H^0(K_1(E))$. It follows from (2.3) that the class of (H^m, φ_m) in $L^K_{2m}(E, \alpha, u)$ belongs to Image $J_{2m} = \{0\}$, so (H^m, φ_m) is stably hyperbolic. This implies that it is hyperbolic in our cases.

Note, in particular, that if $E = F$ is a finite field then PD bases always exist.

Let C_* be simple, and let $\underline{h}_*$ be a PD base. Then

$$\tau(C_*, \underline{h}_*) = (-1)^{n+1}\overline{\tau(C_*, \underline{h}_*)},$$

where n is the formal dimension. Moreover, if $\underline{k}_*$ is a second PD base, then

$$\tau(C_*, \underline{k}_*) - \tau(C_*, \underline{h}_*) = e - (-1)^n \bar{e},$$

with $e = \sum (-1)^i [\underline{k}_i/\underline{h}_i]$, for $i = 1, \dots, [\frac{1}{2}n]$. Hence $\tau(C_*, \underline{h}_*)$ defines a class in $H^{n+1}(K_1(E))$ which is independent of the choice of PD base for H_*. The resulting invariant is denoted $\tau(C_*)$.

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Next we examine the (algebraic) bordism properties of this invariant. Since it is additive it suffices to consider $\tau(C_*)$ when (D_*, C_*) is a PD chain pair. There is a diagram of chain maps

(2.19)
$$\begin{tikzcd}
0 \arrow[r] & D^*(i) \arrow[r] \arrow[d, "\Phi"] & D^* \arrow[r, "i^*"] \arrow[d, "\tilde{\Phi}"] & C^* \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & D_* \arrow[r] & C_*(i) \arrow[r] & C_{*-1} \arrow[r] & 0
\end{tikzcd}$$

where $C_*(i)$ is the mapping cone of the inclusion $i\colon C_* \to D_*$ and $D^*(i) = \operatorname{Ker} i^*$. The pair (D_*, C_*) is called *simple* if, in (2.19), the three vertical chain homotopy equivalences are simple.

LEMMA 2.20. *Suppose (D_*, C_*) is a simple PD pair of formal dimension $n+1$, and if n is even, that $H_*(C_*)$ admits a PD base. Then*

$$\tau(C_*) \in \operatorname{Image}\,[d_{n+1}\colon L^K_{n+1}(E, \alpha, u) \to H^{n+1}(K_1(E))].$$

Proof. Suppose $n+1 = 2m$, and consider the homology of (2.19),

(*)
$$\begin{tikzcd}
 & & & K^m \arrow[dr] \arrow[dd, dashed, "\theta"] & & & \\
\cdots \arrow[r] & H^{m-1}(C_*) \arrow[r] \arrow[ddd, "\varphi_m"] & H^m(i) \arrow[ur] \arrow[rr] \arrow[ddd, "\Phi_m"] & & H^m(D_*) \arrow[r] \arrow[ddd, "\tilde{\Phi}_m"] & H^m(C_*) \arrow[r] \arrow[ddd, "\varphi_{m-1}"] & \cdots \\
 & & & K_m \arrow[ddr] & & & \\
 & & & & & & \\
\cdots \arrow[r] & H_m(C_*) \arrow[r] & H_m(D_*) \arrow[uur] \arrow[rr] & & H_m(i) \arrow[r] & H_{m-1}(C_*) \arrow[r] & \cdots
\end{tikzcd}$$

Let $K^m = \operatorname{Ker}\{H^m(D_*) \to H^m(C_*)\}$ and dually $K_m = \operatorname{cok}\{H_m(C_*) \to H_m(D_*)\}$. We assume we are given a PD base for $H_*(C_*)$, and construct bases for $H_p(D_*)$ and $H_p(i)$ such that

(i) Φ_p and $\tilde{\Phi}_p$ are simple for $p \neq m$,

(ii) $\det(\Phi_m) = \det(\tilde{\Phi}_m) = \det(\theta)$,

and such that the bottom sequence of homology groups in (*) has vanishing torsion.

With this choice of bases we have, from (2.16),

$$\tau(C_*(i)) = \tau(D_*) - \tau(C_*),$$
$$\tau(C_*(\tilde{\Phi})) = \tau(C_*(i)) - \tau(D^*) + \det(\theta).$$

But $\tau(C_*(\tilde{\Phi})) = 0$ and $\tau(D^*) = -\overline{\tau(D_*)}$, so $\tau(C_*) = \det(\theta) \in \operatorname{Image} d_{n+1}$. The proof in the case where $n+1 = 2m+1$ is similar: the diagram of homology groups above implies that $(H_m(C_*), \varphi_m)$ is hyperbolic, and the rest of the argument is completely analogous to the case where $n+1 = 2m$.

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DEFINITION 2.21. Let C_* be a simple PD chain complex over (E, α, u). If the formal dimension n is even, suppose H_* admits a PD base. The *Reidemeister torsion* $\Delta(C_*)$ is the class of $\tau(C_*)$ in $H^{n+1}(K_1(E))/\mathrm{Image}\, d_{n+1} \subseteq L_n^S(E, \alpha, u)$.

REMARK 2.22. We have assumed above that C_* is even-dimensional in the sense of Assumption 2.14. In fact this is an innocent assumption. If it is not satisfied for C_*, then there exists a suitable elementary PD chain complex X_* such that $C_* \oplus X_*$ is even-dimensional, and one can define $\Delta(C_*) = \Delta(C_* \oplus X_*)$. We give a list of the elementary complexes needed for constructing X_*.

(I) $p \leqslant q$; $C_p = C_q \cong E$; $\partial = 0$; $\Psi = e_p \otimes e_q$ where e_i is the base for C_i.

(II) $p < q-2$; $C_p = C_{p+1} = C_{q-1} = C_q \cong E \oplus E$. The base for C_i is $\{e_i, f_i\}$. The only non-zero differentials are $\partial(e_q) = e_{q-1}$ and $\partial(f_{p+1}) = f_p$, and

$$\Psi = e_p \otimes f_q + f_p \otimes e_q + e_{p+1} \otimes f_{q-1} + (-1)^{p+1} f_{p+1} \otimes e_{q-1}.$$

(III) $p = q-2$; the same as (II), except that $C_{p+1} = C_{q-1} \cong E^{\oplus 4}$.

(IV) $C_p = C_{p+1} \cong E \oplus E$; bases $\{e_i, f_i\}$; $\partial(e_{p+1}) = e_p$ and

$$\partial(f_{q+1}) = 0;\ \Psi = e_p \otimes f_{q+1}.$$

In all four cases the PD chain complexes have vanishing torsion in $K_1(E)/\langle -1 \rangle$ where $\langle -1 \rangle = \det(-1: E \to E)$. Moreover, their Reidemeister torsions vanish in $H^*(K_1(E)/\langle -1 \rangle)$. It follows that any direct sum X_* of the complexes above which satisfies the dimensional hypothesis Assumption 2.14 is a simple PD chain complex with $\Delta(X_*) = 0$. Using similar constructions of elementary pairs, one can check that the invariant $\Delta(C_*)$ is a cobordism invariant in general.

3. *Induction results for surgery invariants*

Let X be an oriented finite PD space of formal dimension n. It consists of a finite cell complex X together with an orientation class $[X] \in H_n(X; \mathbb{Z})$ such that cap-product with a cycle representing $[X]$ induces a chain homotopy equivalence $\varphi: C^*(\tilde{X}) \to C_{n-*}(\tilde{X})$ of $\mathbb{Z}\pi$-complexes, $\pi = \pi_1(X)$. We call X *simple* if φ has vanishing torsion (in $\mathrm{Wh}(\mathbb{Z}\pi)$).

Let F be a field with $\mathrm{char}(F) \neq 2$ and let $\alpha: F\pi \to F\pi$ be the usual anti-involution. Associated to X one defines a quadratic PD chain complex over $(F\pi, \alpha, 1)$ as follows.

The diagonal induces a natural chain map of chain groups with coefficients in F,

$$\mu: C_*(X) \to C_*(\tilde{X}) \otimes_{F\pi} C_*(\tilde{X}),$$

and $T \circ \mu$ is chain homotopic to μ, where $T(\xi \otimes \eta) = (-1)^{|\xi||\eta|} \eta \otimes \xi$. The image of $[X] \in H_n(X; F)$ defines an element in $H_n(C_*(\tilde{X}) \otimes_{F\pi} C_*(\tilde{X}))^{\mathbb{Z}/2}$. Since $\mathrm{char}(F) \neq 2$,

$$Q_n(C_*(\tilde{X})) \cong Q^n(C_*(\tilde{X})) \cong H_n(C_*(\tilde{X}) \otimes_{F\pi} C_*(\tilde{X}))^{\mathbb{Z}/2},$$

so we get a quadratic PD chain complex $(C_*(\tilde{X}), \Psi_X)$ over $(F\pi, \alpha, 1)$ of formal dimension n. Here $\Psi_X + \Psi_X^* = \mu_*([X])$.

Let $U(\pi) = \pi/[\pi, \pi] \oplus \langle -1 \rangle \subseteq K_1(\Lambda\pi)$. The quotient group $K_1(\Lambda\pi)/U(\pi)$ will be denoted by $\bar{K}_1(\Lambda\pi)$. Note that if $\Lambda = \mathbb{Z}$, then $\bar{K}_1(\mathbb{Z}\pi) = \mathrm{Wh}(\mathbb{Z}\pi)$.

For each simple PD chain complex X and map $h: X \to B\pi$ classifying the universal cover $\tilde{X}$, we have from §2 an invariant

$$\Delta_F(X, h) \in H^{n+1}(\bar{K}_1(F\pi))/\mathrm{Image}\, d_{n+1} \subseteq L_n^U(F\pi, \alpha, 1).$$

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(If n is even and F is global we assume that $(F\pi, \alpha, (-1)^m)$ contains no simple summand of the exceptional type (2.10).) If char(F) divides the order of π we use that $H^*(\bar{K}_1(F\pi)) \cong H^*(\bar{K}_1(F\pi/J))$ where J is the radical. We define in all cases

$$\Delta_F(X, h) = \Delta_F(C_*(\tilde{X}) \otimes_{F\pi} F\pi/J),$$

where $J = 0$ if F is already semi-simple.

For PL manifolds we can consider Δ_F as a homomorphism

$$\Delta_F\colon \Omega_n^{\mathrm{PL}}(B\pi) \to H^{n+1}(\bar{K}_1(F\pi))/\mathrm{Image}\, d_{n+1},$$

and we can use induction techniques for calculating it.

Given a pair of subgroups $\tau_1 \subset \tau_2$ of π, let $i = i(\tau_1, \tau_2)$ denote the inclusion. There are homomorphisms

$$i(\tau_1, \tau_2)_*\colon \Omega_n^{\mathrm{PL}}(B\tau_1) \to \Omega_n^{\mathrm{PL}}(B\tau_2),$$
$$i(\tau_1, \tau_2)^*\colon \Omega_n^{\mathrm{PL}}(B\tau_2) \to \Omega_n^{\mathrm{PL}}(B\tau_1),$$

where the first one is the usual covariant map and the second one is the transfer homomorphism. Precisely,

$$i_*([M_1, f_1]) = [M_1, B(i) \circ f_1],$$
$$i^*([M_2, f_2]) = [\tilde{M}_1, \tilde{f}_1],$$

where (M_1, f_1) is the singular manifold determined from the pull-back diagram

(3.1)
$$\begin{tikzcd} \tilde{M}_1 \arrow[r, "\tilde{f}_1"] \arrow[d] & B\tau_1 \arrow[d, "Bi"] \\ M_2 \arrow[r, "f_2"] & B\tau_2 \end{tikzcd}$$

The bi-functor $\Omega_*^{\mathrm{PL}}(B\tau)$ so defined is actually a Mackey functor over the Green functor $\pi_S^*(B\tau)$ (stable cohomotopy). In fact, the same is the case for any homology (or cohomology) theory applied to $B\tau$, cf. [**10**, § 1]. We point out two consequences.

Choose a Sylow p-subgroup π_p for each prime divisor of $|\pi|$. The contravariant part defines an injection

(3.2)
$$\Omega_n^{\mathrm{PL}}(B\pi) \to \sum \Omega_n^{\mathrm{PL}}(B\pi_p).$$

Let γ, τ be subgroups of π, and let $g_i \in \pi$ be double coset representatives, $\pi = \coprod_{i=1}^r \gamma g_i \tau$. Then

(3.3)
$$i(\gamma, \pi)^* \circ i(\tau, \pi)_* = \sum_{i=1}^r i(g_i \tau g_i^{-1} \cap \gamma, \gamma)_* \circ c_{g_i} \circ i(\tau \cap g_i^{-1} \gamma g_i, \tau)^*,$$

where c_{g_i} is induced from conjugation, $\tau \cap g_i^{-1}\gamma g_i \to g_i \tau g_i^{-1} \cap \gamma$.

Choose integers $\lambda_p \in \mathbb{Z}$ for each prime divisor in $|\pi|$ such that the following two conditions are satisfied:

(3.4)
$$\text{(i)}\ \sum \lambda_p |\pi : \pi_p| = 1,$$
$$\text{(ii)}\ \lambda_p |\pi : \pi_p| \equiv 1 \pmod{p^R},$$

where, in (ii), R is a large positive number (for example, $R > n|\pi_p : 1|$).

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THEOREM 3.5. *Suppose π is a group whose Sylow p-subgroups are normal in π for p odd. For each $[M,f] \in \Omega_n^{PL}(B\pi)$,*

$$[M,f] = \sum \lambda_p i(\pi_p,\pi)_*([M_p,f_p]),$$

where $[M_p,f_p] = i(\pi_p,\pi)^*([M,f])$.

Proof. Suppose first that M is a boundary. Then $i(1,\pi)^*[M,f] = 0$ and (3.3) gives

$$i(\pi_q,\pi)^* \circ i(\pi_p,\pi)_*([M_p,f_p]) = 0$$

for $p \neq q$. Let p be odd, so that $\pi_p \lhd \pi$. Then (3.3) reduces to

$$i(\pi_p,\pi)^* \circ i(\pi_p,\pi)_*([M_p,f_p]) = \sum c_g([M_p,f_p]),$$

where $g \in \pi/\pi_p$. Moreover, $[M_p,f_p] = i(\pi_p,\pi)^*([M,f])$ and $c_g \circ i(\pi_p,\pi)^* = i(\pi_p,\pi)^*$ since conjugation with g induces the identity on $B\pi$ (up to homotopy). Hence

$$(*) \qquad i(\pi_p,\pi)^* \circ i(\pi_p,\pi)_*([M_p,f_p]) = |\pi:\pi_p|\cdot[M_p,f_p], \quad \text{for } p \text{ odd}.$$

For $p = 2$ we do not assume $\pi_p \lhd \pi$, but use instead that bordism 2-locally reduces to ordinary homology,

$$\Omega_*(X) \otimes \mathbb{Z}_{(2)} \cong H_*(X;\Omega_*(pt)\otimes\mathbb{Z}_{(2)}).$$

In particular, if $\gamma \subset \tau$ we have that $i(\gamma,\tau)_* \circ i(\gamma,\tau)^*$ is equal to multiplication with $|\tau:\gamma|$ on $\Omega_*(B\tau)\otimes\mathbb{Z}_{(2)}$.

We have $[M_2,f_2] \in \tilde{\Omega}_n(B\pi_2) \cong \tilde{\Omega}_n(B\pi_2)\otimes\mathbb{Z}_{(2)}$ and hence

$$\begin{aligned} i(\pi_2,\pi)^* \circ i(\pi_2,\pi)_*([M_2,f_2]) &= \sum_{i=1}^{r} i(\pi_2^{g_i}\cap\pi_2,\pi_2)_* \circ i(\pi_2^{g_i}\cap\pi_2,\pi_2)^*([M_2,f_2]) \\ &= \sum_{i=1}^{r} |\pi_2:\pi_2^{g_i}\cap\pi_2|\cdot[M_2,f_2], \end{aligned}$$

where $\pi = \coprod_{i=1}^{r}\pi_2 g_i \pi_2$. Since $\sum_{i=1}^{r}|\pi_2:\pi_2^{g_i}\cap\pi_2| = |\pi:\pi_2|$ the formula (*) is also satisfied for $p = 2$.

We can now apply $i(\pi_q,\pi)^*$ to the right-hand side of the formula in Theorem 3.5. This gives

$$\begin{aligned} i(\pi_q,\pi)^*(\textstyle\sum \lambda_p i(\pi_p,\pi)_*([M_p,f_p])) &= \lambda_q|\pi:\pi_q|\cdot[M_q,f_q] \\ &= [M_q,f_q] + q^R\cdot\tilde{\Omega}_n(B\pi_q). \end{aligned}$$

Suppose R is chosen so that q^R annihilates $\tilde{\Omega}_n(B\pi_q)$. Then $i(\pi_q,\pi)^*$ maps both sides of the formula to $[M_q,f_q]$, and (3.2) implies the result.

We have assumed that $[M,f] \in \tilde{\Omega}_n(B\pi)$ and must finally consider $[M,c]$, with $c\colon M \to B\pi$ the constant map. But then $i(\pi_p,\pi)^*([M,c]) = |\pi:\pi_p|\cdot[M,c]$ for all p and the formula follows directly.

Let $\gamma \subset \tau$, and let $f\colon M \to B\gamma$ be a singular n-manifold. The element $i(\gamma,\tau)_*([M,f])$ is represented by the composition $B(i)\circ f\colon M \to B\tau$; and if $\tilde{M}$ is the cover induced from f, then $\tilde{M}\times_\gamma\tau$ is the cover induced from $B(i)\circ f$. Thus we have a commutative

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diagram

$$(3.6)\qquad \begin{array}{ccc} \Omega_n^{\mathrm{PL}}(B\gamma) & \xrightarrow{\Delta_F} & H^{n+1}(\bar{K}_1(F\gamma))/\mathrm{Image}\, d_{n+1} \\ \downarrow{\scriptstyle i(\gamma,\tau)_*} & & \downarrow{\scriptstyle i(\gamma,\tau)_*} \\ \Omega_n^{\mathrm{PL}}(B\tau) & \xrightarrow{\Delta_F} & H^{n+1}(\bar{K}_1(F\tau))/\mathrm{Image}\, d_{n+1} \end{array}$$

COROLLARY 3.7. *With the assumptions of Theorem* 3.5,

$$\Delta_F([M, f]) = \sum \lambda_p i(\pi_p, \pi)_*(\Delta_F[M_p, f_p]).$$

Let X be an oriented *weakly simple* PD space, that is, a PD space where the torsion of $\varphi\colon C^*(X) \to C_*(X)$ vanishes in $\mathrm{Wh}'(\mathbb{Z}\pi) = \mathrm{Wh}(\mathbb{Z}\pi)/SK_1(\mathbb{Z}\pi)$. If $\dim X$ is odd every (finite) PD space is weakly simple by [**31**, Proposition 7.1]. Let

$$f\colon M^n \to X, \quad \hat{f}\colon \nu_M \to \zeta$$

be a degree 1 normal map. The obstruction for converting $(f, \hat{f})$ into a weakly simple homotopy equivalence by surgeries is the element $\lambda'(f, \hat{f}) \in L'_n(\mathbb{Z}\pi)$. For $n \equiv 0\ (2)$, $L'_n(\mathbb{Z}\pi) = L^Y_n(\mathbb{Z}\pi)$, and for $n \equiv 1\ (2)$, $L'_n(\mathbb{Z}\pi) = L^Y_n(\mathbb{Z}\pi)/\langle\tau\rangle$, where $\tau = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$, cf. [**29**, §5.4]. Here we have written

$$L^Y_n(\mathbb{Z}\pi) = L^Y_n(\mathbb{Z}\pi, \alpha, 1) \quad \text{and} \quad Y = SK_1(\mathbb{Z}\pi) \oplus \pi/[\pi, \pi] \oplus \langle -1 \rangle.$$

According to [**21**, §6], $\lambda'(f, \hat{f})$ can be defined as an algebraic surgery obstruction as in (2.2). Indeed, $f\colon M \to X$ induces an exact sequence of PD chain complexes over $(\mathbb{Z}\pi, \alpha, 1)$, $\pi = \pi_1(X)$,

$$(3.8)\qquad 0 \to C_*(\tilde{X}) \to C_*(f) \to C_{*-1}(\tilde{M}) \to 0,$$

where $\tilde{M}$ is the cover induced by f from the universal cover $\tilde{X}$. The S-dual of $\hat{f}\colon \nu_M^+ \to \zeta^+$ induces a quadratic structure on $C_*(f)$, and Ranicki shows that $\lambda^Y(f, \hat{f}) = \lambda^Y(C_*(f)) \in L^Y_n(\mathbb{Z}\pi)$ maps onto $\lambda'(f, \hat{f})$.

Let p be an odd prime. There are isomorphisms

$$L^Y_*(\hat{\mathbb{Z}}_p\pi) \xrightarrow{\cong} L^Y_*(\mathbb{F}_p\pi) \xrightarrow{\cong} L^Y_*(\mathbb{F}_p\pi/J),$$

where $Y(\mathbb{F}_p\pi) = \pi/[\pi, \pi] \oplus \langle -1 \rangle$, and where we have suppressed the antistructure $(\hat{\mathbb{Z}}_p\pi, \alpha, 1)$ etc.

Let π be a 2-hyperelementary group, that is, $\pi = \mathbb{Z}/m \tilde{\times} \sigma$ where m is odd and σ is a 2-group. The projection $F\colon \pi \to \sigma \to \pi$ induces a splitting of any (covariant) abelian-valued functor

$$(3.9)\qquad A(\pi) = F_*A(\pi) \oplus (1 - F_*)A(\pi) = A(\sigma) \oplus A(\pi)_{\mathrm{od}}.$$

From (2.3) and (2.7) we get

$$L^Y_n(\mathbb{F}_p\pi)_{\mathrm{od}} \cong L^S_n(\mathbb{F}_p\pi)_{\mathrm{od}} \cong H^{n+1}(K_1(\mathbb{F}_p\pi)_{\mathrm{od}})/\mathrm{Image}\, d_{n+1}.$$

Let $\hat{\rho}_p\colon L^Y_n(\mathbb{Z}\pi) \to L^Y_n(\hat{\mathbb{Z}}_p\pi)$ be the natural map.

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PROPOSITION 3.10. *Let $h\colon X \to B\pi$ classify the universal covering. If π is 2-hyperelementary and p is odd then*

$$\hat{\rho}_p(\lambda^Y(f,\hat{f})_{\mathrm{od}}) = \Delta_{\mathbb{F}_p}[M, h\circ f]_{\mathrm{od}} - \Delta_{\mathbb{F}_p}[X, h]_{\mathrm{od}}$$

under the identification $L_n^Y(\hat{\mathbb{Z}}_p\pi)_{\mathrm{od}} \cong L_n^Y(\mathbb{F}_p\pi)_{\mathrm{od}}$.

Proof. The map $L_n^Y(\mathbb{F}_p\pi)_{\mathrm{od}} \xrightarrow{J} L_n^K(\mathbb{F}_p\pi)_{\mathrm{od}}$ is trivial, so $C_*(f)_{\mathrm{od}} \otimes \mathbb{F}_p$ is a quadratic PD chain complex over $(\mathbb{F}_p\pi, \alpha, 1)$ which is trivial (up to cobordism) as an unbased complex. It follows from Theorem 2.4 that

$$\lambda^Y(C_*(f)_{\mathrm{od}} \otimes \mathbb{F}_p) = t_n\Delta(C_*(f)_{\mathrm{od}} \otimes \mathbb{F}_p),$$

and (2.16) applied to the sequence (3.8) gives the result.

Similarly, one proves

PROPOSITION 3.11. *The mapping $\rho_\infty\colon L_n^Y(\mathbb{Z}\pi)_{\mathrm{od}} \to L_n^Y(\mathbb{R}\pi)_{\mathrm{od}}$ maps $\lambda^Y(f,\hat{f})_{\mathrm{od}}$ to $\Delta_{\mathbb{R}}[M, f\circ h]_{\mathrm{od}} - \Delta_{\mathbb{R}}[M, h]_{\mathrm{od}}$, for odd dimensions n.*

Recall from [**29**] the exact sequence of L-groups

$$\ldots \to CL_{n+1}^S(\mathbb{Q}\pi) \to L_n^X(\mathbb{Z}\pi) \to L_n^S(\mathbb{R}\pi) \times \prod L_n^X(\hat{\mathbb{Z}}_p\pi) \to CL_n^S(\mathbb{Q}\pi) \to \ldots,$$

where $X(R) = SK_1(R)$. The following theorem is the main result of the paragraph.

THEOREM 3.12. *Suppose the degree 1 normal map $(f,\hat{f})$ satisfies*

(i) $\hat{\rho}_2(\lambda^Y(f,\hat{f})) = 0$ *in* $L_n^Y(\hat{\mathbb{Z}}_2\pi)$,
(ii) $\Delta_F([M, h\circ f]) = \Delta_F([X, h])$ *for* $F = \mathbb{F}_p, \mathbb{R}$ $(p \neq 2)$,
(iii) $\mathrm{sign}_\pi(\tilde{M}) = \mathrm{sign}_\pi(\tilde{X})$ *in* $R(\pi)$ (*if n is even*).

Then $\lambda^Y(f,\hat{f}) \in \mathrm{Image}(CL_{n+1}^S(\mathbb{Q}\pi) \to L_n^Y(\mathbb{Z}\pi))$.

Proof. We can use Dress's induction theorem and may assume π is 2-hyperelementary. From (3.9) we have

$$L_n^Y(\mathbb{Z}\pi)_{\mathrm{od}} = L_n^X(\mathbb{Z}\pi)_{\mathrm{od}}.$$

The kernel of the multisignature

$$\mathrm{sign}_\pi\colon L_{2k}^S(\mathbb{R}\pi) \to R(\pi)$$

is a direct sum of $L_2^S(\mathbb{R}, 1, 1)$ and $L_2^S(\mathbb{C}, 1, 1)$ and is detected by the $\Delta_{\mathbb{R}}$-invariant. It follows from Propositions 3.10 and 3.11 that

$$\lambda^Y(f,\hat{f})_{\mathrm{od}} \in L_n^Y(\mathbb{Z}\pi)_{\mathrm{od}} = L_n^X(\mathbb{Z}\pi)_{\mathrm{od}}$$

maps trivially to $L_n^Y(\hat{\mathbb{Z}}_p\pi)_{\mathrm{od}}$ and to $L_n^Y(\mathbb{R}\pi)_{\mathrm{od}}$, so

$$\lambda^Y(f,\hat{f})_{\mathrm{od}} \in \mathrm{Image}(CL_{n+1}^S(\mathbb{Q}\pi)_{\mathrm{od}} \to L_n^X(\mathbb{Z}\pi)_{\mathrm{od}}).$$

We still have to prove the theorem for a 2-group σ. The exact sequence for calculating $L_n^X(\mathbb{Z}\sigma)$ takes the form

$$\ldots \to CL_{n+1}^S(\mathbb{Q}\sigma) \to L_n^X(\mathbb{Z}\sigma) \to \Sigma \oplus L_n^X(\hat{\mathbb{Z}}_2\sigma) \to \ldots,$$

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where $\Sigma = 0$ for n odd and Σ is mapped injectively by the multisignature for n even. Using (2.3) to compare $L_n^X(\mathbb{Z}\sigma)$, $L_n^X(\hat{\mathbb{Z}}_2\sigma)$ with $L_n^Y(\mathbb{Z}\sigma)$, $L_n^Y(\hat{\mathbb{Z}}_2\sigma)$, we get the exact sequence

$$CL_{n+1}^S(\mathbb{Q}\sigma) \to L_n^Y(\mathbb{Z}\sigma) \to \Sigma \oplus L_n^Y(\hat{\mathbb{Z}}_2\sigma).$$

The theorem follows.

4. *Calculation of some L-groups*

The semi-direct products $\mathbb{Z}/n \tilde{\times} Q(8)$ of an odd-order cyclic group and the quaternion group were tabulated as $\mathbb{Z}/d \times Q(8a, b, c)$ in **[18]**. The integers a, b, and c can be permuted without changing the isomorphism type of the group, so we can write $Q(8a, b)$ for any of the isomorphic groups

$$Q(8a, b, 1) \cong Q(8, a, b) \cong Q(8a, 1, b).$$

In this paragraph we calculate L-theory of the groups above. More precisely, we calculate

$$L_*^Y(\mathbb{Z}\pi) = L_*^Y(\mathbb{Z}\pi, \alpha, 1),$$

where $Y = SK_1(\mathbb{Z}\pi) \oplus \langle -1 \rangle \oplus \pi/[\pi, \pi]$ and α is the usual (oriented) anti-involution. We follow the procedure from **[29]** closely. First recall from **[29**, §4**]** that

$$(4.1) \qquad \begin{aligned} L_*^Y(\mathbb{Z}[\mathbb{Z}/n \tilde{\times} Q(8)]) &= \prod_{d|n} L_*^Y(\mathbb{Z}[\mathbb{Z}/n \tilde{\times} Q(8)])(d) \\ &= \prod_{d|n} L_*^Y(\mathbb{Z}[\mathbb{Z}/d \tilde{\times} Q(8)])(d). \end{aligned}$$

Thus the task is to calculate the top component corresponding to $d = n$. Consider the twisted group rings

$$(4.2) \qquad R(n) = \mathbb{Z}[\zeta_n]^t[Q(8)], \quad S(n) = \mathbb{Q}(\zeta_n)^t[Q(8)], \quad T(n) = \mathbb{R} \otimes_Q S(n),$$

where ζ_n is a primitive nth root of 1, and where the twisting is induced from the homomorphism φ: $Q(8) \to (\mathbb{Z}/n)^\times = \mathrm{Gal}(\mathbb{Z}[\zeta_n]/\mathbb{Z})$ which specifies $\pi = \mathbb{Z}/n \tilde{\times} Q(8)$. Each ring has the anti-involution induced from α. For $n > 1$,

$$L_*^Y(\mathbb{Z}\pi)(n) = L_*^X(\mathbb{Z}\pi)(n),$$

and we have the exact sequence from **[29**, §4.1**]**:

$$(4.3) \qquad \cdots \longrightarrow CL_{i+1}^X(S(n)) \xrightarrow{\partial} L_i^X(\mathbb{Z}\pi)(n) \longrightarrow L_i^X(T(n)) \oplus \prod_{p\nmid n} L_i^X(\hat{R}_p(n)) \longrightarrow \cdots.$$

Here $X(S(n)) = \{0\}$ and $X(T(n)) = \{0\}$ in general, and for the rings in (4.2) we even have $X(\hat{R}_p(n)) = \{0\}$ by results from **[20]**. We follow the notation in **[29]** and write L_*^S instead of $L_*^{\{0\}}$. In (4.3) the suppressed antistructure is induced from $(\mathbb{Z}\pi, \alpha, 1)$ in each case, so $L_i^X(\hat{R}_p(n)) = L_i^X(\hat{R}_p(n), \alpha, 1)$, etc.

We begin the explicit calculations with the groups $\pi = Q(8a, b)$. They can be presented as

$$(4.4) \qquad \begin{aligned} Q(8a, b) = \langle A, B, X, Y \mid\ & A^a = 1,\ B^b = 1,\ X^2 = Y^2,\ YXY^{-1} = X^{-1}, \\ & XAX^{-1} = A^{-1},\ XBX^{-1} = B,\ YAY^{-1} = A,\ YBY^{-1} = B^{-1} \rangle. \end{aligned}$$

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The ring $S = S(ab) = \mathbb{Q}(\zeta_{ab})^t[Q(8)]$ contains the central idempotent $\frac{1}{2}(1+X^2)$ and splits into a product of simple algebras $S = S_+ \times S_-$ each having centre

(4.5) $$F = F_{a,b} = \mathbb{Q}(\zeta_a + \zeta_a^{-1}, \zeta_b + \zeta_b^{-1}).$$

More precisely,

$$S_+ = \mathbb{Q}(\zeta_{ab})^t[X, Y \mid X^2 = Y^2 = 1, XY = YX],$$
$$S_- = \mathbb{Q}(\zeta_{ab})^t[X, Y \mid X^2 = Y^2 = -1, XY = -YX].$$

The antistructure $(S_+, \alpha, 1)$ has Type O, the other one $(S_-, \alpha, 1)$ has type Sp. At infinite (real) primes S_+ is split, S_- is not:

$$S_+ \otimes_F \mathbb{R} \cong M_4(\mathbb{R}), \quad S_- \otimes_F \mathbb{R} \cong M_2(\mathbb{H}),$$

where $F \subset \mathbb{R}$ is any embedding and $\mathbb{H}$ is the usual quaternion algebra. This is easily checked directly, and is all we need for the L-group calculations. Actually, $S_+ \cong M_4(F)$ and $S_- \cong M_2(D)$ where D is a division algebra non-split at infinite primes and at p and q.

For a simple antistructure $(S, \alpha, 1)$ of Type O with centre K we have

(4.6) $$CL_n^X(S) = \mathbb{Z}/2; \quad C(K)_2; \quad {}_2C(K); \quad 0 \quad \text{for } n \equiv 0, 1, 2, 3 \pmod 4,$$

cf. [29, § 1.2]. Here $C(K)$ is the idéle class group and $C(K)_2 = C(K) \otimes \mathbb{Z}/2$, ${}_2C(K) = \{c \in C(K) \mid 2c = 0\}$. If $(S, \alpha, 1)$ has Type Sp we get the same groups but with grading shifted by 2:

$$CL_n^X(\text{Type Sp}) = CL_{n+2}^X(\text{Type O}).$$

All together we have calculated the terms $CL_*^X(S(ab))$ in (4.3).

Next we consider the terms $L_*^X(\hat{R}_p(ab))$, where

$$R = R(ab) = \mathbb{Z}[\zeta_{ab}]^t[Q(8)].$$

The centre consists of two copies of the integers $\mathcal{O}(F_{a,b})$ in $F_{a,b}$. We have

$$\hat{R}_p = \hat{\mathbb{Z}}_p \otimes R \cong \prod \hat{R}_y,$$

where y runs over the set of primes $P_p(F_{a,b})$ in $F_{a,b}$ which divide p and $\hat{R}_y$ is the y-adic completion. To save space we introduce the notation

$$A = \mathcal{O}(F_{a,b}) = \mathbb{Z}[\zeta_a + \zeta_a^{-1}, \zeta_b + \zeta_b^{-1}].$$

We begin calculations of $L_*^X(\hat{R}_p) = \prod L_*^X(\hat{R}_y)$ with the easy case where p is odd (that is, $p \nmid 2ab$). Then

$$\hat{R}_y \cong M_4(\hat{A}_y) \times M_4(\hat{A}_y).$$

Indeed, this isomorphism holds on the residue level and since $\mathbb{Q}(\zeta_{ab})/F_{a,b}$ is unramified at y, the y-integral isomorphism follows by standard methods.

In the decomposition of $\hat{R}_y$ the first factor has Type O, and the second has Type Sp. Thus (by Morita equivalence)

$$L_n^X(\hat{R}_y) = L_n^X(\hat{R}_y, \alpha, 1) = L_n^X(\hat{A}_y, 1, 1) \oplus L_n^X(\hat{A}_y, 1, -1) = L_n^X(\hat{A}_y) \oplus L_{n+2}^X(\hat{A}_y).$$

Explicitly, we have

(4.7) $$L_n^X(\hat{A}_y) = 0; \quad H^0(\hat{A}_y^\times); \quad H^1(\hat{A}_y^\times); \quad 0$$

for $n \equiv 0$, 1, 2, and 3 (mod 4), cf. [29, § 1.2].

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We have used $H^i(\ )$ to denote the Tate cohomology groups with respect to the involution associated with the underlying anti-structure (always clear from the context). In the case above, where the involution is trivial,

$$H^0(\hat{A}_y^\times) = \hat{A}_y^\times/(\hat{A}_y^\times)^2, \quad H^1(\hat{A}_y^\times) = {}_2(\hat{A}_y^\times) = \langle -1 \rangle.$$

For $p = 2$, $\hat{R}_2 = \hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_{ab}]^t[Q(8)]$ has centre $\hat{A}_2[\mathbb{Z}/2] = \hat{A}_2 \oplus \hat{A}_2 \cdot X^2$ where $\hat{A}_2 = \prod\{\hat{A}_y \mid y \in P_2(F_{a,b})\}$. Moreover, $\hat{R}_2$ is an Azumaya algebra, so by [**31**, Theorem 8.3] or [**9**, § 3],

$$\hat{\mathbb{Z}}_2 \otimes \mathbb{Z}[\zeta_{ab}]^t[Q(8)] \cong M_2(\hat{A}_2[\mathbb{Z}/2]).$$

There is an induced Morita-equivalence of anti-structures,

$$(\hat{\mathbb{Z}}[\zeta_{ab}]_2^t[Q(8)], \alpha, 1) \sim (\hat{A}_2[\mathbb{Z}/2], 1, u)$$

for some unit $u \in \hat{A}_2[\mathbb{Z}/2]^\times$. We have $u = T$, the generator of $\mathbb{Z}/2$, since upon tensoring with $\mathbb{Q}$ this is the element which gives the correct $\hat{\mathbb{Q}}_2$-types according to the remarks following (4.4). In conclusion,

$$L_n^X(\hat{R}_2) = L_n^X(\hat{A}_2[\mathbb{Z}/2], 1, T). \tag{4.8}$$

It is well known that $K_1(\hat{A}_2[\mathbb{Z}/2]) = \hat{A}_2[\mathbb{Z}/2]^\times$. The involution is trivial, and we have

LEMMA 4.9. (i) $H^0(\hat{A}_2[\mathbb{Z}/2]^\times) = H^0(\hat{A}_2^\times) \oplus H^0(\hat{A}_2^\times)$.
(ii) $H^1(\hat{A}_2[\mathbb{Z}/2]^\times) = g_2 \cdot (\langle -1 \rangle \oplus \langle T \rangle)$ *where* $g_2 = g_2(F_{a,b})$ *is the number of primes in* $F_{a,b}$ *dividing* 2.

Proof. Consider the exact sequence

$$1 \longrightarrow (1+2\hat{A}_2)^\times \xrightarrow{j} \hat{A}_2[\mathbb{Z}/2]^\times \xrightarrow{p} \hat{A}_2^\times \longrightarrow 1,$$

where $p(a+bT) = a-b$ and $j(1+2a) = 1+(1+T)a$. The long-exact sequence of Tate cohomology groups (with respect to the trivial involution) splits into short exact sequences

$$0 \to H^1(1+2\hat{A}_2^\times) \to H^1(\hat{A}_2[\mathbb{Z}/2]^\times) \to H^1(\hat{A}_2^\times) \to 0,$$
$$0 \to H^0(1+2\hat{A}_2^\times) \to H^0(\hat{A}_2[\mathbb{Z}/2]^\times) \to H^0(\hat{A}_2^\times) \to 0.$$

The group $\hat{A}_2^\times/1+2\hat{A}_2^\times$ has odd order. Hence $H^0(A_2^\times) \cong H^0(1+2A_2^\times)$, and (i) follows. To prove (ii) we use the cohomology sequence of

$$1 \longrightarrow (1+4\hat{A}_2)^\times \longrightarrow (1+2\hat{A}_2)^\times \xrightarrow{\varphi} A/2A \longrightarrow 0, \tag{4.10}$$

where $\varphi(1+2a) = \bar{a}$ is the reduction of a modulo 2. The logarithm shows that the multiplicative group $(1+4\hat{A}_2)^\times$ is isomorphic to the additive (torsion free) group $\hat{A}_2$. Hence $H^1(1+4\hat{A}_2^\times) = 0$. It follows that $H^1(1+2\hat{A}_2^\times) \cong H^1(A/2A) \cong g_2 \cdot \mathbb{Z}/2$, generated by the g_2 elements $-1 \in 1+2\hat{A}_y^\times$ for $y \mid 2$. This completes the proof.

LEMMA 4.11. (i) $L_{2i}^X(\hat{A}_2[\mathbb{Z}/2], 1, T) \cong g_2 \cdot ((\langle -1 \rangle \oplus \langle T \rangle)/\langle (-1)^{i+1}T \rangle)$.
(ii) $L_{2i+1}^X(\hat{A}_2[\mathbb{Z}/2], 1, T) \cong H^0(\hat{A}_2^\times) \oplus A/2A$.

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Proof. The proof is based on two facts. The first is the Rothenberg exact sequence

$$H^{i+1}(\hat{A}_2[\mathbb{Z}/2]^\times) \xrightarrow{t_i} L_i^X(\hat{A}_2[\mathbb{Z}/2], 1, T) \longrightarrow$$
$$L_i^K(\hat{A}_2[\mathbb{Z}/2], 1, T) \xrightarrow{d_i} H^i(\hat{A}_2[\mathbb{Z}/2]^\times).$$

The second is the Hensel type reduction theorem for L^K-groups:

$$L_i^K(R, \alpha, u) \cong L_i^K(R/J, \bar{\alpha}, \bar{u})$$

whenever $J \subset R$ is an ideal preserved by α, such that R is J-adic complete ($R = \varprojlim R/J^n$), cf. [**29**, § 1.2].

In our case $R = \hat{A}_2[\mathbb{Z}/2]$ and we use the ideals $J^- = \langle 1-T\rangle$ and $J^+ = \langle 1+T\rangle$. The problem is to evaluate

$$d_i\colon L_i^K(\hat{A}_2[\mathbb{Z}/2], 1, T) \to H^i(\hat{A}_2[\mathbb{Z}/2]^\times).$$

There are two commutative diagrams

$$(4.12^+)\qquad \begin{array}{ccc} L_i^K(\hat{A}_2[\mathbb{Z}/2], 1, T) & \xrightarrow{d_i} & H^i(\hat{A}_2[\mathbb{Z}/2]^\times) \\ \downarrow \cong & & \downarrow p_+ \\ L_i^K(\hat{A}_2, 1, 1) & \xrightarrow{d_i^+} & H^i(\hat{A}_2^\times) \end{array}$$

$$(4.12^-)\qquad \begin{array}{ccc} L_i^K(\hat{A}_2[\mathbb{Z}/2], 1, T) & \xrightarrow{d_i} & H^i(\hat{A}_2[\mathbb{Z}/2]^\times) \\ \downarrow \cong & & \downarrow p_- \\ L_i^K(\hat{A}_2, 1, -1) & \xrightarrow{d_i^-} & H^i(\hat{A}_2^\times) \end{array}$$

associated to reductions modulo J^+ and J^-. We have $L_i^K(\hat{A}_2, 1, -1) = L_{i+2}^K(\hat{A}_2)$, and for each i, $L_i^K(\hat{A}_2) \cong L_i^K(A/2A) \cong g_2 \cdot \mathbb{Z}/2$. Indeed, $A/2A \cong \prod\{\bar{F}_y \mid y \in P_2(F)\}$ where $\bar{F}_y$ denotes the residue field of $F = F_{a,b}$ at the dyadic prime y, and $L_i^K(\bar{F}_y) = \mathbb{Z}/2$ for all i.

From [**30**, Theorem 11] the image of d_i^+ is given by

$$\langle 1+4\beta\rangle;\quad \langle -1\rangle;\quad 0;\quad 0 \quad \text{for } i \equiv 0, 1, 2, \text{ and } 3 \pmod 4,$$

where $1+4\beta \in 1+4\hat{A}_y^\times \subseteq \hat{A}_y^\times$ is an element with $\bar{\beta} \in \bar{F}_y$ not of the form $x+x^2$ $(y|2)$.

We use (4.12^+) for $i = 1$ and (4.12^-) for $i = 3$ to conclude that d_i is injective for i odd. For i even, first note that the homology sequence of (4.10) takes the form

$$A/2A \xrightarrow{\delta^*} H^0(1+4\hat{A}_2^\times) \longrightarrow H^0(1+2\hat{A}_2^\times) \longrightarrow A/2A \longrightarrow 0.$$

When we use the 2-adic log-function to identify $H^0(1+4\hat{A}_2^\times)$ with $A/2A$, δ^* becomes the map of $A/2A = \prod \bar{F}_y$ which takes x to $x+x^2$. Combining with the above description of Image($d_i^\pm$), we note that d_{2i} is injective and that

$$\operatorname{cok} d_{2i} = A/2A \oplus H^0(\hat{A}_2^\times).$$

For $i = 0$, $A/2A$ corresponds to the Type O factor and $H^0(\hat{A}_2^\times)$ to the Type Sp factor, and vice versa if $i = 1$.

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Finally, the terms $L_i^X(T)$, where $T = T(ab)$, in (4.3) are easy to list:

$$L_0^X(T) \cong g_\infty \cdot (L_0^S(\mathbb{R}, 1, 1) \oplus L_0^S(\mathbb{H}, c, 1)) \cong g_\infty \cdot (4\mathbb{Z} \oplus 2\mathbb{Z}),$$
$$L_3^X(T) = 0, \quad L_1^X(T) = g_\infty \cdot \mathbb{Z}/2.$$

Here $g_\infty = g_\infty(F)$ is the number of infinite (real) primes of $F_{a,b}$.

We are ready to evaluate the exact sequence (4.3) for the groups $\pi = Q(8a, b)$ in (4.1) with $a \geqslant 1$, $b \geqslant 1$. The available information is contained in the exact sequence

$$0 \longrightarrow \operatorname{cok}\psi_{i+1} \longrightarrow L_i^X(\mathbb{Z}\pi)(ab) \xrightarrow{\rho} \operatorname{Ker}\psi_i \longrightarrow 0,$$

where

$$\text{(4.13)} \quad \begin{aligned} \psi_1&: \prod_{y \nmid pq} H^0(\hat{A}_y^\times) \times A/2A \times H^0(F_\infty^\times) \to H^0(C(F)),\\ \psi_3&: \prod_{y \nmid pq} H^0(\hat{A}_y^\times) \times A/2A \to H^0(C(F)),\\ \psi_0&: \prod_{y \nmid pq} H^1(\hat{A}_y^\times) \times g_\infty \cdot (4\mathbb{Z} \oplus 2\mathbb{Z}) \to H^1(C(F)) \times \mathbb{Z}/2,\\ \psi_2&: \prod_{y \nmid pq} H^1(\hat{A}_y^\times) \times H^1(F_\infty^\times) \to H^1(C(F)) \times \mathbb{Z}/2. \end{aligned}$$

Here $F = F_{a,b}$, and $F_\infty = \mathbb{R} \oplus_Q F \cong g_\infty \cdot \mathbb{R}$ is the product of the infinite (real) completions. The idele class group is $C(F) = F_A^\times / F^\times$ where

$$F_A^\times = \varinjlim_{\Omega} F^\times(\Omega), \quad F^\times(\Omega) = \prod_{y \in \Omega} \hat{F}_y^\times \times \hat{A}_y^\times,$$

with Ω running over a finite set of primes in F such that $\Omega \supseteq P_\infty(F)$.

The mappings in (4.13) can be described as follows. The summand $A/2A$ belongs to the kernel of ψ_{2i+1}. Each factor $H^0(\hat{A}_y^\times)$ is mapped via the inclusion $A_y^\times \subseteq F_y^\times \subseteq F_A^\times$, and $H^0(F_\infty^\times)$ is mapped via $F_\infty^\times \subseteq F_A^\times$. For ψ_0, each factor $4\mathbb{Z}$ is mapped onto the $\mathbb{Z}/2$ with no components in $H^1(C(F))$ and the g_∞ factor $2\mathbb{Z}$ surjects onto $H^1(F_\infty^\times) \subseteq H^1(C(F))$. Finally, the factors $H^1(\hat{A}_y^\times)$ and $H^1(F_\infty^\times)$ are mapped into $H^1(C(F))$ via the natural map. These results all follow from the proof of the identifications (4.6), compare [**29**, §§4.4, 4.5].

In describing the kernels and cokernels in (4.13) it is convenient to compare the maps ψ_1, ψ_3 with the natural homomorphism

$$H^0(\hat{A}^\times) \times H^0(F_\infty^\times) \times H^0(F^\times) \to H^0(F_A^\times).$$

It has kernel $F^{(2)}/(F^\times)^2$ where $F^{(2)} \subset F^\times$ consists of the elements with even valuation at all finite primes. Its cokernel is equal to $H^0(\Gamma(F))$, where $\Gamma(F) = I(F)/F^\times$ is the ideal class group of F, and $I(F) = F_A^\times / F_\infty^\times \cdot \hat{A}^\times$ is the ideal group.

It follows that

$$\text{(4.14)} \qquad \operatorname{Ker}\psi_{2i+1}^F = A/2 \oplus \operatorname{Ker}\tilde{\psi}_{2i+1}^F,$$

and that $\operatorname{Ker}\tilde{\psi}_{2i+1}^F$ and $\operatorname{cok}\psi_{2i+1}$ can be determined from the exact sequences

$$\text{(4.15)} \quad \begin{aligned} &0 \to \operatorname{Ker}\tilde{\psi}_1^F \to F^{(2)}/F^2 \to H^0((A/ab)^\times) \to \operatorname{cok}\psi_1^F \to H^0(\Gamma(F)) \to 0,\\ &0 \to \operatorname{Ker}\tilde{\psi}_3^F \to F^{(2)}/F^2 \to H^0((A/ab)^\times) \times H^0(F_\infty^\times) \to \operatorname{cok}\psi_3^F \to H^0(\Gamma(F)) \to 0. \end{aligned}$$

In deriving (4.14) we have used the global square theorem for F and the fact that the reduction homomorphism $\prod_{y|ab} A_y^\times \to (A/ab)^\times$ induces an isomorphism on H^0.

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Let $F^* \subset F^\times$ be the subgroup of elements with positive valuation at all infinite (real) primes. Write $\Gamma(F)^*$ for the strict (or narrow) class group ($\Gamma(F)^* = I(F)/F^*$) and let $F^{*(2)} = F^* \cap F^{(2)}$. The second sequence in (4.14) can be rewritten in the form

$$0 \to \operatorname{Ker} \tilde{\psi}_3^F \to F^{*(2)}/F^{\times 2} \to H_0((A/ab)^\times) \to \operatorname{cok} \psi_3^F \to H^0(\Gamma(F)^*) \to 0.$$

We collect the calculations in

THEOREM 4.16. *Let $\pi = Q(8a, b)$. There is an exact sequence*

$$0 \to \operatorname{cok} \psi_{i+1}^F \to L_i^X(\mathbb{Z}\pi)(ab) \to \operatorname{Ker} \psi_i^F \to 0.$$

The terms $\operatorname{cok} \psi_{2i+1}^F$ *and* $\operatorname{Ker} \psi_{2i+1}^F$ *are given in* (4.14) *and* (4.15) *with*

$$F = \mathbb{Q}[\zeta_a + \zeta_a^{-1}, \zeta_b + \zeta_b^{-1}]$$

and $A = \mathcal{O}(F)$. *The rest of the terms are as follows:* $\operatorname{cok} \psi_0 = (g_{ab} - 1) \cdot \mathbb{Z}/2$, $\operatorname{cok} \psi_2 = \mathbb{Z}/2 \oplus \operatorname{cok} \psi_0$, $\operatorname{Ker} \psi_0 = \Sigma$, *and* $\operatorname{Ker} \psi_2 = 0$. *Here Σ is a free abelian signature group and g_{ab} is the number of primes in F dividing ab.*

At one point in the next paragraph we shall also need $L_{2i+1}^X(\Lambda[\pi])$ where $\Lambda = \mathbb{Z}[i]$ and $\pi = Q(8a, b)$. This calculation is quite analogous to the one above: $L_{2i+1}^X(\Lambda[\pi])$ is an extension of $\operatorname{cok} \psi_{2i+2}^\Lambda$ by $\operatorname{Ker} \psi_{2i+1}^\Lambda$ where

(4.17)
$$\operatorname{cok} \psi_{2i+2}^\Lambda = g_{ab}(F[i]) \cdot \mathbb{Z}/2,$$
$$\operatorname{Ker} \psi_{2i+1}^\Lambda = A[i]/2A[i] \oplus \operatorname{Ker}\{F[i]^{(2)}/(F[i]^\times)^2 \to H^0((\hat{A}[i]/ab)^\times)\}.$$

Here $F[i]$ and $A[i]$ are the quadratic extensions by $i = \sqrt{-1}$. Note that $g_2(F[i]) = 2g_2(F)$ so that $\hat{\mathbb{Z}}_2 \otimes A[i] = 2(\hat{\mathbb{Z}}_2 \otimes A)$; thus the 2-adic calculations follow from Lemma 4.11.

The (top component) of L-theory for $Q(8a, b)$ will be compared with L-theory of the generalized quaternion group $Q(4n) = \mathbb{Z}/n \tilde{\times} \mathbb{Z}/4$ specified by the map $\varphi\colon \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2 \subseteq (\mathbb{Z}/n)^\times$. Indeed, $Q(4ab)$ is included in $Q(8a, b, 1)$ as the subgroup generated by $\mathbb{Z}/ab$ and XY.

The rational group ring of $Q(4ab)$ is similar in structure to the rational group ring of $Q(8a, b)$. Specifically, its top component decomposes as

$$\mathbb{Q}(\zeta_{ab})^t[\mathbb{Z}/4] \cong M_2(F_{ab}) \times D, \quad F_{ab} = \mathbb{Q}(\zeta_{ab} + \zeta_{ab}^{-1}),$$

where both factors have centre F_{ab}, and the types are O, Sp. Arguing as above, we get

THEOREM 4.18. *For $\pi = Q(4ab)$ there is an exact sequence*

$$0 \to \operatorname{cok} \psi_{i+1}^F \to L_i^X(\mathbb{Z}\pi)(ab) \to \operatorname{Ker} \psi_i^F \to 0.$$

The terms $\operatorname{cok} \psi_i^F$, $\operatorname{Ker} \psi_i^F$ *follow from the description given in* (4.15) *but with* $F = \mathbb{Q}(\zeta_{ab} + \zeta_{ab}^{-1})$ *and* $A = \mathbb{Z}[\zeta_{ab} + \zeta_{ab}^{-1}]$.

We shall compare $L_n^X(\mathbb{Z}[Q(4ab)])(ab)$ and $L_n^X(\mathbb{Z}[Q(8a, b)])(ab)$ via the contravariant structure

$$i^*\colon L_n^X(\mathbb{Z}[Q(8a, b)]) \to L_n^X(\mathbb{Z}[Q(4ab)]).$$

The sequence (4.3) is functorial, so information about i^* can be obtained from information about the restriction of i^* to $\operatorname{cok} \psi_{n+1}$ and $\operatorname{Ker} \psi_{n+1}$; thus we shall compare Theorems 4.16 and 4.18.

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This requires the fact (easily checked) that

$$i^*\colon K_1(\mathbb{Q}(\zeta_{ab})^t[Q8]) \to K_1(\mathbb{Q}(\zeta_{ab})^t[\mathbb{Z}/4])$$

becomes the inclusion of centres when the K_1-groups are identified as subgroups of the centres by the reduced norm homomorphism. There is a similar result for p-adic group rings: i^* becomes the inclusion of centres combined with the diagonal inclusions (when there are more p-adic primes in the centres on the right-hand side than in the centres on the left-hand side).

We have $g_{ab}(F_{ab}) \geqslant g_{ab}(F_{a,b})$, with equality if a and b are primes. It follows that i^* maps the cok ψ_0 term injectively. The extra $\mathbb{Z}/2$ in cok ψ_2 comes from the reciprocity law. It can be identified with the right-hand term in the exact sequence

$$0 \to \mathrm{Br}_2(F) \to \mathrm{Br}_2(F_A) \to \mathbb{Z}/2 \to 0,$$

where Br_2 denotes the elements of order 2 in the Brauer group (the part generated by quaternion algebras). Since both $F_{a,b}$ and F_{ab} are real fields, $\mathrm{Br}_2(F_{a,b})_\infty \subseteq \mathrm{Br}_2(F_{ab})_\infty$. Hence the extra $\mathbb{Z}/2 \subseteq \operatorname{cok}\psi_2^F$ is mapped injectively under

$$i^*\colon L_1^X(\mathbb{Z}[Q(8a,b)]) \to L_1^X(\mathbb{Z}[Q(4ab)]).$$

The kernel and cokernel of ψ_{2i+1} are more intricate to chase under i^*, as they depend on the ideal class groups. In special cases, e.g. if $a = 3$ and b is a prime such that the relative class number of $\mathbb{Q}(\zeta_b)/\mathbb{Q}(\zeta_b+\zeta_b{}^{-1})$ is odd, it follows from [7, satz 45] that $\operatorname{Ker}\psi_{2i+1}^K$ injects into $\operatorname{Ker}\psi_{2i+1}^E$. In this situation

$$i^*\colon L_{2i+1}^X(\mathbb{Z}[Q(8a,b)])(ab) \to L_{2i+1}^X(\mathbb{Z}[Q(4ab)])(ab)$$

is injective. In general, however, it seems difficult to determine the kernels of $i^*|\operatorname{Ker}\psi_{2i+1}$ and $i^*|\operatorname{cok}\psi_{2i+1}$.

Instead we can combine the above with Theorem 3.12 and use that the image of

$$\hat{\rho}_{\mathrm{od}}\colon L_n^Y(\mathbb{Z}[\pi]) \to L_n^Y(\hat{\mathbb{Z}}_{\mathrm{od}}[\pi])$$

can be detected by Reidemeister torsion invariants.

THEOREM 4.19. *Let $\pi = Q(8a,b)$. Write $i(\tau,\pi)$ for the inclusion of a subgroup τ in π. Then*

$$\prod i(\tau,\pi)^* \times \hat{\rho}_{\mathrm{od}}\colon L_{2i+1}^Y(\mathbb{Z}[\pi]) \to \prod_{\substack{\tau\subset\pi\\ \tau\neq\pi}} L_{2i+1}^Y(\mathbb{Z}[\tau]) \times L_{2i+1}^Y(\hat{\mathbb{Z}}_{\mathrm{od}}[\pi])$$

is injective, where $\hat{\mathbb{Z}}_{\mathrm{od}} = \hat{\mathbb{Z}}/\hat{\mathbb{Z}}_2$.

Proof. Consider the decomposition

$$(*)\qquad L_n^Y(\mathbb{Z}[\pi]) \cong \prod_{d\mid ab} L_n^Y(\mathbb{Z}[\pi])(d).$$

If $d < ab$ then $d = a'b'$, and there is an inclusion $j\colon Q(8a',b') \subset Q(8a,b)$. We claim that

$$L_n^Y(\mathbb{Z}[Q(8a',b')])(d) \xrightarrow{j^*} L_n^Y(\mathbb{Z}[Q(8a,b)])(d)$$

is injective. Since the decomposition (*) uses the covariant structure of the bi-functor L_n^Y, the claim is not obvious. We know that j_* is an isomorphism and must calculate

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$j^* \circ j_*$ from the double coset formula, cf. [5] or [10]:

$$j^* \circ j_* = \mathrm{Id} + N, \quad N = \sum (k_i)_* \circ k_i^*.$$

Here the maps k_i are inclusions of subgroups π'' in $\pi' = Q(8a', b')$ such that $|\pi' : \pi''|$ is a power of 2. One more application of the double coset formula shows that $N \circ N = 2N'$. Since the torsion subgroup of L_n^Y consists of 2-torsion, it follows that j_* is injective. Thus all we have left to consider is the top component $L_n^Y(\mathbb{Z}[\pi])(ab)$.

The kernel of $\hat{\rho}_{\mathrm{od}}$: $L_n^Y(\mathbb{Z}[\pi])(ab) \to L_n^Y(\hat{\mathbb{Z}}_{\mathrm{od}}[\pi])(ab)$ is equal to an extension of $\operatorname{cok} \psi_{n+1}$ by

$$A/2 \text{ for } n \equiv 1\ (2), \quad \Sigma \text{ for } n \equiv 0\ (4), \quad 0 \text{ for } n \equiv 2\ (4),$$

where Σ is torsion free (and detected by signatures). This follows from Theorem 4.16 and the global square theorem.

The field $F_{ab} = \mathbb{Q}(\zeta_{ab} + \zeta_{ab}{}^{-1})$ is a quadratic extension of

$$F_{a,b} = \mathbb{Q}(\zeta_a + \zeta_a{}^{-1}, \zeta_b + \zeta_b{}^{-1})$$

and $A/2 = \mathcal{O}(F_{a,b})/2$ injects into $\mathcal{O}(F_{ab})/2$. We argued above that

$$i^*\colon L_{2i+1}^X(\mathbb{Z}[Q(8a, b)])(ab) \to L_{2i+1}^X(\mathbb{Z}[Q(4ab)])(ab)$$

maps the subgroup $\operatorname{cok} \psi_{2i+1}^F$ injectively. Hence i^* maps $\operatorname{Ker}(\hat{\rho}_{\mathrm{od}})$ injectively.

For later reference we note, from the proof above, the following corollary.

COROLLARY 4.20. *The image of* $\hat{\rho}_2$: $L_{2i+1}^Y(\mathbb{Z}[Q(8a, b)])(ab) \to L_{2i+1}^Y(\hat{R}_2(ab))$ *is mapped injectively into* $L_{2i+1}^Y(\hat{\mathbb{Z}}_2[Q(4ab)])(ab)$ *under* i^*.

The L-theory of the groups $Q(8a, b, c) \times \mathbb{Z}/d$ can be read off from Theorem 4.16, except for the torsion-free part Σ of L_{2i}^Y. Indeed,

$$\operatorname{Torsion} L_*^Y(\mathbb{Z}[Q(8a, b, c) \times \mathbb{Z}/d]) \cong \operatorname{Torsion} L_*^Y(\mathbb{Z}[Q(8a, b, c)]),$$

by [29, §2.4], and from [9, §3] it follows that in the decomposition (4.1),

$$\operatorname{Torsion} L_*^Y(\mathbb{Z}[Q(8a, b, c)])(d) = 0,$$

unless d is prime to at least 1 on the integers a, b, and c. (The point is that the centre fields of $S(d)$ are complex in the other cases.) We get

THEOREM 4.21. *The natural inclusions induce a monomorphism between the torsion subgroup of* $L_*^Y(\mathbb{Z}[Q(8a, b, c) \times \mathbb{Z}/d])$ *and the torsion subgroup of*

$$L_*^Y(\mathbb{Z}[Q(8a, b)]) \oplus L_*^Y(\mathbb{Z}[Q(8a, c)]) \oplus L_*^Y(\mathbb{Z}[Q(8b, c)]).$$

5. *Evaluating the Rothenberg sequence*

In the application of surgery theory to solve geometric existence problems, e.g. the existence of spherical space forms, one needs the groups $L_i^K(\mathbb{Z}\pi)$. They are connected via the exact sequence (2.3) to the groups $L_i^Y(\mathbb{Z}\pi)$ treated in the previous paragraph. The extra terms in (2.3) are the Tate cohomology groups of $\mathrm{Wh}'(\mathbb{Z}\pi) = K_1(\mathbb{Z}\pi)/Y(\mathbb{Z}\pi)$. The group $\mathrm{Wh}'(\mathbb{Z}\pi)$ is torsion free and has trivial $\mathbb{Z}/2$-

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action by [**31**, §7], so we get the exact sequence

$$\text{(5.1)} \qquad \mathrm{Wh}'(\mathbb{Z}\pi)\otimes \mathbb{Z}/2 \xrightarrow{t} L^Y_{2i-1}(\mathbb{Z}\pi) \xrightarrow{J} L^K_{2i-1}(\mathbb{Z}\pi) \longrightarrow 0.$$

This paragraph gives a calculation of (5.1) for the groups $\pi = Q(8a, b)$ and for their subgroups $Q(4ab)$. The results enable us to calculate surgery obstructions in $L^K_{2i-1}(\mathbb{Z}\pi)$ of normal maps which satisfy the conditions in Theorem 3.12.

Although, as we shall see at the end of the paragraph, our main conclusions are true for all groups $Q(8a, b)$, we now make the *simplifying assumption* that ab is square free. Then $Q(8a, b)$ has sufficiently many automorphisms to imply that each abelian-valued functor decomposes over the divisors of ab, cf. [**29**, §4]. In particular, for $\pi = Q(8a, b)$, (5.1) is a direct sum of exact sequences

$$\mathrm{Wh}'(\mathbb{Z}\pi)(d) \otimes \mathbb{Z}/2 \to L^Y_{2i-1}(\mathbb{Z}\pi)(d) \to L^K_{2i-1}(\mathbb{Z}\pi)(d) \to 0,$$

where d divides ab. The Meyer–Vietoris sequence of the arithmetic square gives the exact sequence

$$\text{(5.2)} \qquad 0 \to K'_1(\mathbb{Z}\pi) \to K'_1(\hat{\mathbb{Z}}\pi) \oplus K_1(\mathbb{Q}\pi) \to K_1(\hat{\mathbb{Q}}\pi) \to \tilde{K}_0(\mathbb{Z}\pi) \to 0.$$

If $\pi = Q(8a, b)$ then $SK_1(\hat{\mathbb{Z}}\pi) = 0$ and $SK_1(\mathbb{Z}\pi) = 0$ by results from [**20**], so $K_1 = K'_1$ for $\hat{\mathbb{Z}}\pi$ and $\mathbb{Z}\pi$. For components with $d > 1$, $\mathrm{Wh}'(\mathbb{Z}\pi)(d) = K'_1(\mathbb{Z}\pi)(d)$. We are interested in only the top component, $d = ab$. To simplify notation we leave out the indication of the (top) component in the rest of the paragraph when there is no danger of confusion and write $K_1(\mathbb{Z}\pi) = K_1(\mathbb{Z}\pi)(ab)$, etc. We use the same notation as in §4:

$$F = F_{a,b} = \mathbb{Q}(\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1}), \quad A = \mathcal{O}(F_{a,b}) = \mathbb{Z}[\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1}].$$

PROPOSITION 5.3. *For* $\pi = Q(8a, b)$,

$$K_i(\mathbb{Z}\pi) = K_i(\mathbb{Z}\pi)_+ \oplus K_i(\mathbb{Z}\pi)_-, \quad \textit{with } i = 0, 1,$$

where the summands are calculated from the exact sequences

$$0 \longrightarrow K_1(\mathbb{Z}\pi)_+ \longrightarrow A^\times \xrightarrow{\rho} A/2ab^\times \longrightarrow \tilde{K}_0(\mathbb{Z}\pi)_+ \longrightarrow \Gamma(F) \longrightarrow 0,$$

$$0 \longrightarrow K_1(\mathbb{Z}\pi)_- \longrightarrow A^* \xrightarrow{\rho} A/ab^\times \longrightarrow \tilde{K}_0(\mathbb{Z}\pi)_- \longrightarrow \Gamma(F)^* \longrightarrow 0.$$

Proof. The result follows from (5.2) when we calculate the individual terms via the reduced norm homomorphism. We begin with the rational terms where we have isomorphisms

$$\mathrm{Nrd}\colon K_i(\mathbb{Q}\pi) \xrightarrow{\cong} F^\times \times F^*, \quad \mathrm{Nrd}\colon K_1(\hat{\mathbb{Q}}\pi) \xrightarrow{\cong} \hat{F}^\times \times \hat{F}^\times.$$

The first factor corresponds to the Type O component of $\mathbb{Q}(\zeta_{ab})^t[Q(8)]$ and the second to the Type Sp component, cf. (4.5).

The term $K_1(\hat{\mathbb{Z}}\pi) = \prod K_1(\hat{\mathbb{Z}}_p\pi)$ is more delicate. If p is a prime not dividing ab, then $K_1(\hat{\mathbb{Z}}_p\pi) = K_1(\hat{R}_p)$ where $\hat{R}_p = \hat{\mathbb{Z}}_p \otimes \mathbb{Z}[\zeta_{ab}]^t[Q(8)]$, and we have, from §4,

$$K_1(\hat{\mathbb{Z}}_p\pi) = \hat{A}_p^\times \times \hat{A}_p^\times \cdot \quad \text{for } p \nmid 2ab,$$
$$K_1(\hat{\mathbb{Z}}_2\pi) = K_1(\hat{A}_2[\mathbb{Z}/2]) = (1+2\hat{A}_2)^\times \times \hat{A}_2^\times.$$

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Suppose $p \mid a$ and write $ab = pc$ with $(c,p) = 1$. Consider the twisted group ring $R = \mathbb{Z}[\zeta_c][\mathbb{Z}/p]^t[Q(8)]$ and let $\hat{R}_p = \hat{\mathbb{Z}}_p \otimes R$. Write T for the generator of $\mathbb{Z}/p \subset R$ and consider the ideal $J_p \subset \hat{R}_p$ generated by $1-T$. Since $\hat{R}_p$ is complete in the J_p-adic topology there is an exact sequence

$$1+J_p \to K_1(\hat{\mathbb{Z}}_p \otimes R) \to K_1(\hat{\mathbb{Z}}_p \otimes \mathbb{Z}[\zeta_c]^t[Q(8)]) \to 0.$$

For the top component we have $K_1(\hat{\mathbb{Z}}_p\pi)(ab) = \mathrm{Nrd}(1+J_p)$. Further,

$$\mathrm{Nrd}(1+J_p) = \mathrm{Nrd}(1+(1-\zeta_p)(\hat{\mathbb{Z}}_p \otimes \tilde{R})),$$

with $\tilde{R} = \mathbb{Z}[\zeta_{pc}]^t[Q(8)]$. It is direct to check that

$$\mathrm{Nrd}(1+J_p) \subseteq U_p^1(F) \times U_p^1(F)$$

where

$$U_p^1(F) = \prod_{y \mid p} U^1(\hat{F}_y) = \prod_{y \mid p} (1+y\hat{A}_y)^\times.$$

On the other hand, the restriction of the reduced norm to $1+(1-\zeta_p)\hat{\mathbb{Z}}[\zeta_{pc}]_y$ is equal to the usual norm

$$N\colon \hat{\mathbb{Z}}[\zeta_{pc}]_y^\times \to \hat{A}_y^\times.$$

It follows from [23, V, §3] that N maps $1+(1-\zeta_p)\hat{\mathbb{Z}}[\zeta_{pc}]_y$ onto $U^1(\hat{F}_y)$, so in conclusion we have

$$K_1(\hat{\mathbb{Z}}_p\pi) = U_p^1(F) \times U_p^1(F). \tag{5.4}$$

Using (5.2) we have $K_1(\mathbb{Z}\pi) = K_1(\mathbb{Z}\pi)_+ \oplus K_1(\mathbb{Z}\pi)_-$, and exact sequences

$$0 \to K_1(\mathbb{Z}\pi)_+ \to \hat{A}^\times / \hat{A}_{2ab}^\times \times U_{2ab}^1(\hat{F}) \times F^\times \to \hat{F}^\times.$$

$$0 \to K_1(\mathbb{Z}\pi)_- \to \hat{A}^\times / \hat{A}_{ab}^\times \times U_{ab}^1(F) \times F^* \to \hat{F}^\times.$$

Finally, the exact sequences

$$0 \to A^\times \to \hat{A}^\times \times F^\times \to \hat{F}^\times \to \Gamma(F) \to 0,$$

$$0 \to A^* \to \hat{A}^\times \times F^* \to \hat{F}^\times \to \Gamma(F)^* \to 0,$$

together with a 'snake' lemma argument, complete the proof.

Later in the paragraph we shall need the groups $K_1'(\Lambda\pi)$ where $\Lambda = \mathbb{Z}[i]$ is the Gaussian integers. The calculations are fairly similar to those used for Theorem 5.3, but the result is a little different because $S \otimes_{\mathbb{Q}} \mathbb{Q}[i]$ is split at infinite primes for $S = \mathbb{Q}(\zeta_{ab})^t[Q(8)]$. There are exact sequences

$$\begin{aligned} &0 \to K_1'(\mathbb{Z}[i]\pi)_+ \to A[i]^\times \to A[i]/2ab^\times, \\ &0 \to K_1'(\mathbb{Z}[i]\pi)_- \to A[i]^\times \to A[i]/ab^\times, \end{aligned} \tag{5.5}$$

and $K_1'(\mathbb{Z}[i]\pi) = K_1'(\mathbb{Z}[i]\pi)_+ \oplus K_1'(\mathbb{Z}[i]\pi)_-$.

We have determined the source and target of

$$t\colon K_1'(\mathbb{Z}\pi) \otimes \mathbb{Z}/2 \to L_{2i-1}^X(\mathbb{Z}\pi), \quad \text{where } \pi = Q(8a,b),$$

and it remains to determine the map itself.

To this end we need the following naturality result for the Rothenberg sequence.

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Let Λ be a Dedekind domain and L its field of fractions. The Mayer–Vietoris sequence of the arithmetic square breaks up into exact sequences

$$\begin{aligned}&0 \to K_1'(\Lambda\pi) \to K_1'(\hat{\Lambda}\pi) \oplus K_1(L\pi) \to \bar{K}_1(\hat{L}\pi) \to 0,\\ &0 \to \bar{K}_1(\hat{L}\pi) \to K_1(\hat{L}\pi) \to \tilde{K}_0(\Delta\pi) \to 0.\end{aligned} \tag{5.6}$$

The terms have involutions, trivial in our cases, and there is a natural commutative ladder of exact sequences

(5.7)

$$\begin{tikzcd} \cdots \arrow[r] & H^{i+1}(\bar{K}_1(\hat{L}\pi)) \arrow[r, "\delta"] \arrow[d, "t"] & H^i(K_1'(\Lambda\pi)) \arrow[r] \arrow[d, "t"] & H^i(K_1'(\hat{\Lambda}\pi)) \oplus H^i(K_1(L\pi)) \arrow[r] \arrow[d, "t"] & \cdots \\ \cdots \arrow[r] & L_i^X(\hat{L}\pi) \arrow[r, "\partial"] & L_{i-1}^X(\Lambda\pi) \arrow[r, "\rho"] & L_i^X(\hat{\Lambda}\pi) \oplus L_i^X(L\pi) \arrow[r] & \cdots \end{tikzcd}$$

where the top sequence is the exact sequence of Tate cohomology groups of the first sequence in (5.6), and the vertical maps are from the Rothenberg sequence. The commutativity of (5.7) was originally observed in [**11**]; a proof is included in [**22**, §6].

Recall from (4.15) the extension

$$0 \to \operatorname{cok}\psi_{2i} \to L_{2i-1}^X(\mathbb{Z}\pi) \to \operatorname{Ker}\psi_{2i-1} \to 0$$

where

$$\begin{aligned}&\operatorname{cok}\psi_0 = H^1(A/ab^\times)/\langle -1\rangle, \quad \operatorname{cok}\psi_2 = \operatorname{cok}\psi_0 \oplus \mathbb{Z}/2,\\ &\operatorname{Ker}\psi_1 = A/2A \oplus \operatorname{Ker}\{F^{(2)}/(F^\times)^2 \to H^0(A/ab^\times)\},\\ &\operatorname{Ker}\psi_3 = A/2A \oplus \operatorname{Ker}\{F^{*(2)}/(F^\times)^2 \to H^0(\hat{A}_{ab}^\times)\}.\end{aligned}$$

We note that $H^1(M) = {}_2M$ and $H^0(M) = M/2M$ in each case since the (suppressed) involution is trivial.

Define

$$\sqrt{\rho_{ab}}\colon \operatorname{Ker}\{(A^\times)^2 \to A/ab^\times\} \to H^1(A/ab^\times)/\langle -1\rangle$$

to be the square root of the reduction homomorphism, that is

$$(\sqrt{\rho_{ab}})(\lambda^2) = \rho_{ab}(\lambda),$$

where $\rho_{ab}\colon A \to A/ab$ is the ordinary reduction.

PROPOSITION 5.8. (i) *The composition* $\bar{t}\colon K_1'(\mathbb{Z}\pi)\otimes\mathbb{Z}/2 \to \operatorname{Ker}\psi_{2i-1}$ *maps* $K_1'(\mathbb{Z}\pi)_+ \subseteq A^\times$ *to* $A/2A = (A/4A)_{(2)}^\times$ *by reduction modulo* 4, *and maps* $K_1'(\mathbb{Z}\pi)_-$ *trivially. It maps the summand* $K_1'(\mathbb{Z}\pi)_{(-)^{i+1}}$ *into* $\operatorname{Ker}\{F^{(2)}/(F^\times)^2 \to H^0(A/ab^\times)\}$ *by the natural inclusion, and* $\operatorname{Ker}(\bar{t}_{(-)^{i+1}}) = \operatorname{Ker}\{(A^\times)^2 \to (A/ab)^\times\}$.

(ii) *The induced homomorphism* $t\colon \operatorname{Ker}\bar{t} \to \operatorname{cok}\psi_{2i}$ *is trivial on* $\operatorname{Ker}\bar{t}_{(-)^i}$ *and is equal to* $\sqrt{\rho_{ab}}$ *on* $\operatorname{Ker}\bar{t}_{(-)^{i+1}}$.

Proof. The first part, Proposition 5.8(i), follows directly from the description of $\operatorname{Ker}\psi_{2i-1}$ given in §4 and from the commutativity of the second square in (5.7). We prove Proposition 5.8(ii) when i is even.

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To prove that $t \mid \operatorname{Ker} \bar{t}_+$ is trivial we consider the projection of $\mathbb{Z}\pi$ onto the algebra

$$R = \mathbb{Z}[\mathbb{Z}/ab]'[X, Y \mid X^2 = -1, \quad Y^2 = -1, XY = -YX].$$

There is an induced homomorphism of L- groups, in fact of the whole diagram (5.7) into an analogous diagram for R. In particular,

$$L^X_{2i-1}(\mathbb{Z}\pi) \to L^X_{2i-1}(R)$$

maps the subgroup $\operatorname{cok}\psi_{2i}$ injectively. But the subgroup $K'_1(\mathbb{Z}\pi)_+$ of $K'_1(\mathbb{Z}\pi)$ is mapped trivially, so $t \mid \operatorname{Ker} \bar{t}_+ = 0$.

The restriction of t to $\operatorname{Ker} \bar{t}_-$ is also determined by naturality, but this time we extend scalars and consider the natural map

$$L^X_{2i-1}(\mathbb{Z}\pi) \to L^X_{2i-1}(\Lambda\pi)$$

where $\Lambda = \mathbb{Z}[i]$ and $\Lambda\pi$ has the usual anti-involution given by $\alpha(\sum \lambda_i g_i) = \sum \lambda_i \bar{g}_i$ (no conjugation on Λ!). It follows from Theorems 4.16 and 4.18 that $\operatorname{cok}\psi_{2i}$ is mapped monomorphically into $L^X_{2i-1}(\Lambda\pi)$.

Let $\operatorname{Im} A^\times$ (respectively $\operatorname{Im} A[i]^\times$) denote the image of the reduction homomorphism $\rho_{ab}\colon A^\times \to A/ab^\times$ (respectively $A[i]^\times \to A[i]/ab^\times$). The subgroup $H^1(A[i]/ab^\times)$ of elements of order 2 is included naturally in $H^1(\hat{F}[i]^\times)$. This yields a homomorphism φ and a commutative triangle.

$$\begin{tikzcd} H^1(\operatorname{Im} A[i]^\times) \arrow[rr, "\varphi"] \arrow[dr, "{\delta[i]}"'] & & H^1(\bar{K}_1(\hat{\mathbb{Q}}[i]\pi)_-) \arrow[dl, "\delta"] \\ & H^0(K'_1(\mathbb{Z}[i]\pi)_-) & \end{tikzcd}$$

Here δ is the coboundary in the long exact sequence of Tate cohomology groups induced from the first sequence in (5.6) (with $\Lambda = \mathbb{Z}[i]$) and $\delta[i]$ is the corresponding boundary associated to the exact cohomology sequence for

$$0 \to K'_1(\mathbb{Z}[i]\pi)_- \to A[i]^\times \to \operatorname{Im} A[i]^\times \to 0.$$

We have $\operatorname{Im} A^\times \subseteq \operatorname{Im} A[i]^\times$ and therefore

$$\operatorname{Ker}\{(A^\times)^2 \to A/ab^\times\} \subseteq \operatorname{Im}(\delta[i]).$$

The triangle above, together with the first commutative square in (5.7), yields the commutative diagram

$$\begin{tikzcd} & H^1(A^\times) \arrow[r] \arrow[dl] & H^1(A/ab^\times) \arrow[dl] \arrow[dddd] \\ H^1(\operatorname{Im} A[i]^\times) \arrow[r] \arrow[dd, "{\delta[i]}"] & L^X_{2i}(\hat{\mathbb{Q}}[i]\pi) \arrow[dd] & \\ & & \\ H^0(K'_1(\mathbb{Z}[i]\pi)_-) \arrow[r, "t^\Lambda"] & L^X_{2i-1}(\mathbb{Z}[i]\pi) & \\ & \operatorname{Ker} \bar{t}_- \arrow[r, "t"] \arrow[ul] \arrow[u] & L^X_{2i-1}(\mathbb{Z}\pi) \arrow[ul] \end{tikzcd}$$

It follows that $t = \sqrt{\rho_{ab}}$.

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REMARK 5.9. The results in Propositions 5.3 and 5.8 remain true for the group $Q(4ab)$ when one substitutes $F_{ab} = \mathbb{Q}(\zeta_{ab}+\zeta_{ab}^{-1})$ for F, and $B = \mathbb{Z}[\zeta_{ab}+\zeta_{ab}^{-1}]$ for A.

We can now combine the results of § 5 with the main conclusions of § 3 and § 4 to obtain a (theoretical) calculation of the surgery obstruction in the situation of space forms. Let $f\colon M \to X$, $\hat{f}\colon \nu_M \to \zeta$ be a surgery problem whose target is an odd-dimensional PD space with fundamental group equal to $\pi = Q(8a, b)$. Let $h\colon X \to B\pi$ classify the universal cover of X. For each $\sigma \subset \pi$, let X_σ be the cover induced from $B\sigma \to B\pi$ and let $(f_\sigma, \hat{f}_\sigma)$ be the induced surgery problem over X_σ.

Given a homotopy equivalence $g_\tau\colon N_\tau \to X_\tau$ where $\tau = Q(4ab)$, the top component of its Whitehead torsion is denoted $\mathrm{wh}(g_\tau)(ab)$; it has two components $\mathrm{wh}(g_\tau)_+(ab) \in K_1'(\mathbb{Z}\tau)_+(ab)$ and $\mathrm{wh}(g_\tau)_-(ab) \in K_1'(\mathbb{Z}\tau)_-(ab)$.

THEOREM 5.10. *Let $(f, \hat{f})$ be as above. Let $A = \mathbb{Z}[\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1}]$ and let $B = \mathbb{Z}[\zeta_{ab}+\zeta_{ab}^{-1}]$. Suppose that $(f_\sigma, \hat{f}_\sigma)$ is normally cobordant to a homotopy equivalence for each proper subgroup σ of $Q(8a, b)$. Then $(f, \hat{f})$ is itself normally cobordant to a homotopy equivalence if the three conditions below are satisfied:*

(i) $\Delta_{\mathbf{F}_l}([M, h\circ f])(ab) = \Delta_{\mathbf{F}_l}([X, h])(ab)$ *for all primes* $l \nmid 2ab$;

(ii) $\mathrm{wh}(g_\tau)_+(ab) \in \mathrm{Ker}\{\varphi\colon B^\times \to (B/4)^\times_{(2)}\}$;

(iii) $\mathrm{wh}(g_\tau)_-(ab) \in (B^\times)^2$ *and* $\sqrt{\rho_{ab}}(\mathrm{wh}(g_\tau)_-(ab))$ *belongs to the image of the composition*

$$\mathrm{Ker}\{(A^\times)^2 \to A/ab^\times\} \xrightarrow{\sqrt{\rho_{ab}}} H^1(A/ab^\times)/\langle -1\rangle \longrightarrow H^1(B/ab^\times)/\langle -1\rangle.$$

Proof. We write $\pi = Q(8a, b)$ and $\tau = Q(4ab)$. Since X is odd dimensional, it is weakly simple by [31, 7.1]. Hence $(f, \hat{f})$ has a surgery obstruction $\lambda^Y(f, \hat{f})$ in $L^Y_{2i-1}(\mathbb{Z}\pi)$ and we are interested in its image $\lambda^K(f, \hat{f}) \in L^K_{2i-1}(\mathbb{Z}\pi)$. Since $\lambda^K(f_\sigma, \hat{f}_\sigma) = 0$ for all proper subgroups $\sigma \subseteq \pi$, we have that $\lambda^K(f, \hat{f})(d) = 0$ in $L^K_{2i-1}(\mathbb{Z}\pi)(d)$ for all divisors d of ab with $d < ab$. Indeed, the proof of Theorem 4.19 for L^Y-groups works equally well for L^K-groups. Thus $\lambda^K(f, \hat{f})$ belongs entirely to the top component,

$$\lambda^K(f, \hat{f}) = \lambda^K(f, \hat{f})(ab) \in L^K_{2i-1}(\mathbb{Z}\pi)(ab).$$

From Proposition 3.10 and Theorem 5.10(i) it follows that $\lambda^Y(f, \hat{f})(ab)$ maps trivially to $L^X_{2i-1}(\hat{R}_p(ab))$ for $p \nmid 2ab$, and Theorem 5.10(ii) shows that $\hat{\rho}_2\lambda^Y(f_\tau, \hat{f}_\tau) = 0$. Hence by Corollary 4.20,

$$\hat{\rho}_2\lambda^Y(f, \hat{f}) = 0 \quad \text{in } L^X_{2i-1}(\hat{R}_2(ab)),$$

and Theorem 3.12 shows that

$$\lambda^Y(f, \hat{f})(ab) \in \operatorname{cok}\psi_{2i} \subseteq L^X_{2i-1}(\mathbb{Z}\pi)(ab).$$

The term $\operatorname{cok}\psi_{2i}$ is listed in Theorem 4.16. We suppose first that i is even. Then $\operatorname{cok}\psi_{2i} \subseteq L^X_{2i-1}(\mathbb{Z}\pi)_-(ab)$ and we consider the diagram

$$(*) \qquad \begin{tikzcd} \mathrm{Ker}\{B^* \to B/ab^\times\} \arrow[r, "t_\tau"] & L^X_{2i-1}(\mathbb{Z}\tau)_-(ab) \arrow[r, "J"] & L^K_{2i-1}(\mathbb{Z}\tau)_-(ab) \arrow[r] & 0 \\ \mathrm{Ker}\{A^* \to A/ab^\times\} \arrow[r, "t_\pi"] \arrow[u, "i^*"] & L^X_{2i-1}(\mathbb{Z}\pi)_-(ab) \arrow[r, "J"] \arrow[u, "i^*_X"] & L^K_{2i-1}(\mathbb{Z}\pi)_-(ab) \arrow[r] \arrow[u, "i^*_K"] & 0 \end{tikzcd}$$

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where we have $i_X^*(\lambda^Y(f,\hat{f})(a,b)) = t_\tau(\mathrm{wh}(g_\tau)_-(ab))$. Since $\lambda^Y(f,\hat{f})(ab)$, and hence $\lambda^Y(f_\tau,\hat{f}_\tau)(ab)$, belongs to $\operatorname{cok}\psi_{2i}$, $\lambda^Y(f_\tau,\hat{f}_\tau)(ab)$ maps to zero in $\operatorname{Ker}\psi_{2i-1}$. Thus by Proposition 5.8(i),

$$\mathrm{wh}(g_\tau)_-(ab) \in \operatorname{Ker}\{(B^\times)^2 \to B/ab^\times\},$$

and from Proposition 5.8(ii) we get

$$\lambda^Y(f_\tau,\hat{f}_\tau)(ab) = \sqrt{\rho_{ab}}(\mathrm{wh}(g_\tau)_-(ab)).$$

Write $cL^X_{2i-1}(\mathbb{Z}\pi)$ and $cL^X_{2i-1}(\mathbb{Z}\tau)$ for the subgroups $\operatorname{cok}\psi_{2i}$ of $L^X_{2i-1}(\mathbb{Z}\pi)_-(ab)$ and $L^X_{2i-1}(\mathbb{Z}\tau)_-(ab)$. Define the subgroup $cL^K_{2i-1}(\mathbb{Z}\pi) \subseteq L^K_{2i-1}(\mathbb{Z}\pi)_-(ab)$ by the exact sequence

$$\operatorname{Ker}\{(A^\times)^2 \to A/ab^\times\} \xrightarrow{t_\pi} cL^X_{2i-1}(\mathbb{Z}\pi) \longrightarrow cL^K_{2i-1}(\mathbb{Z}\pi) \longrightarrow 0,$$

and similarly for the group ring $\mathbb{Z}\tau$.

We have the exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}(t_\tau) \arrow[r] & cL^X_{2i-1}(\mathbb{Z}\tau) \arrow[r, "J"] & cL^K_{2i-1}(\mathbb{Z}\tau) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Im}(t_\pi) \arrow[r] \arrow[u, "i^*"] & cL^X_{2i-1}(\mathbb{Z}\pi) \arrow[r, "J"] \arrow[u, "i_X^*"] & cL^K_{2i-1}(\mathbb{Z}\pi) \arrow[r] \arrow[u, "i_K^*"] & 0
\end{tikzcd}$$

derived from (*) above. It induces a six-term exact sequence

(**) $$0 \longrightarrow \operatorname{Ker} i^* \longrightarrow \operatorname{Ker} i_X^* \xrightarrow{J} \operatorname{Ker} i_K^* \xrightarrow{\nabla}$$
$$\operatorname{cok} i^* \longrightarrow \operatorname{cok} i_X^* \longrightarrow \operatorname{cok} i_K^* \longrightarrow 0.$$

We have argued that $\lambda^Y(f,\hat{f}) \in cL^X_{2i-1}(\mathbb{Z}\pi)$. Since $\lambda^K(f,\hat{f}) \in \operatorname{Ker} i_K^*$, we get

$$\nabla(\lambda^K(f,\hat{f})) = \sqrt{\rho_{ab}}(\mathrm{wh}(g_\tau)_-(ab)).$$

In particular, if $\sqrt{\rho_{ab}}(\mathrm{wh}(g_\tau)_-(ab)) \neq 0$ in $\operatorname{cok} i^*$, then $\lambda^K(f,\hat{f}) \neq 0$.

On the other hand, if $\sqrt{\rho_{ab}}(\mathrm{wh}(g_\tau)_-(ab)) \in \operatorname{Im} i^*$ then it follows from (**) that

$$(\lambda^Y(f,\hat{f}) + \operatorname{Im}(t_\pi)) \cap \operatorname{Ker} i_X^* \neq \{0\}.$$

Since $L^X_{2i-1}(\hat{\mathbb{Z}}_p\pi)(ab) = 0$ when p divides ab, Theorem 4.19 shows that

$$\hat{\rho}_{\mathrm{od}}\colon L^X_{2i-1}(\mathbb{Z}\pi)_-(ab) \to \prod_{p \nmid 2ab} L^X_{2i-1}(\hat{\mathbb{Z}}_p\pi)(ab)$$

maps $\operatorname{Ker} i_X^*$ injectively. But $\hat{\rho}_{\mathrm{od}}(\lambda^Y(f,\hat{f})) = 0$ and $\hat{\rho}_{\mathrm{od}} \circ t_\pi$ annihilates $\operatorname{Ker}\{(A^\times)^2 \to A/ab^\times\}$, so it follows that $\lambda^Y(f,\hat{f}) \in \operatorname{Im}(t_\pi)$. Hence $\lambda^K(f,\hat{f}) = 0$.

The case where i is odd and the obstruction lies in L_1 is similar. We leave the details to the reader.

In the above we have considered only the groups $\pi = Q(8a, b)$ for ab square free. Without this assumption the automorphisms of π give only a partial decomposition, namely

$$K_1'(\mathbb{Z}\pi) = \prod K_1'(\mathbb{Z}\pi)(P),$$

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where P runs over the prime divisors (not all divisors) of ab. However, each factor $K_1'(\mathbb{Z}\pi)(P)$ can be analysed using the methods above. We sketch the procedure.

Write $ab = p^i c$ with $(p, c) = 1$. We have algebras $R_j = \mathbb{Z}[\zeta_c][\mathbb{Z}/p^j]^t[Q(8)]$ and in place of (5.4) we consider the string of epimorphisms

$$K_1'(\hat{\mathbb{Z}}_p \otimes R_i) \to K_1'(\hat{\mathbb{Z}}_p \otimes R_{i-1}) \to \dots \to K_1'(\hat{\mathbb{Z}}_p \otimes R_0).$$

This leads to a decomposition

$$K_1'(\hat{\mathbb{Z}}_p \otimes R_i)_{(2)} = \left(\prod_{j=0}^{i-1} U_p^{p^j}(F) \times U_p^{p^j}(F)\right)_{(2)} \times K_1'(\hat{\mathbb{Z}}_p \otimes R_0)_{(2)}.$$

We note that the index $|U^1 : U^r|$ is a power of p when $r > 1$. It follows that each functor $K_1'(\mathbb{Z}\pi)(P) \otimes \hat{\mathbb{Z}}_2$ does indeed split further into a product of functors indexed by those divisors of ab which belong to P. This decomposition fits together with the decomposition of $L_*^Y(\mathbb{Z}\pi)$ from (4.2) and gives a decomposition of the Rothenberg sequence (5.1) in general. The results above are in fact true for all groups $Q(8a, b)$.

Addendum 5.11. *The conclusions of Theorem* 5.10 *remain valid for all groups* $Q(8a, b)$.

For the groups $\pi = Q(8a, b, c) \times \mathbb{Z}/d$, we can use Theorem 4.21 to get

Corollary 5.12. *Let* $\pi = Q(8a, b, c) \times \mathbb{Z}/d$. *Let* $(f, \hat{f})$ *be a surgery problem as in Theorem* 5.10, *such that* $\lambda^K(f_\sigma, \hat{f}_\sigma) = 0$ *for* $\sigma = Q(8a, b)$, $\sigma = Q(8b, c)$, *and* $\sigma = Q(8a, c)$. *Then* $\lambda^K(f, \hat{f}) = 0$.

6. *Finiteness obstruction and Reidemeister torsion*

Let π be a group with periodic cohomology groups of period N. It has a periodic resolution P_* of finitely generated projective $\mathbb{Z}\pi$-modules of length N,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} P_{N-1} \longrightarrow \dots \longrightarrow P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0. \tag{6.1}$$

The homotopy type of P_* is uniquely determined by the 'k-invariant' $e(P_*) \in H^N(\pi; \mathbb{Z})$. It is a generator of $H^N(\pi; \mathbb{Z}) \cong \mathbb{Z}/|\pi|$, and each generator can be realized by some P_*, cf. **[24]**.

We shall use the result from **[25]** that projective modules are locally free to construct a torsion invariant $\hat{\tau}(P_*)$ of (6.1). For each i and each prime l we pick a $\hat{\mathbb{Z}}_l\pi$-base for $P_i \otimes \hat{\mathbb{Z}}_l$ and use the $\mathbb{Z}$-bases for $H_0(P_*)$ and $H_{N-1}(P_*)$ induced by ε and η. Let ν be the usual idempotent for $\mathbb{Q}\pi$, that is $\nu = 1/|\pi| \sum \{g \mid g \in \pi\}$. From (6.1) we ge two based sequences

$$0 \to (1-\nu)(P_{N-1} \otimes \hat{\mathbb{Q}}_l) \to \dots \to (1-\nu)(P_0 \otimes \hat{\mathbb{Q}}_l) \to 0,$$
$$0 \to \mathbb{Q}_l \to \nu(P_{N-1} \otimes \hat{\mathbb{Q}}_l) \to \dots \to \nu(P_0 \otimes \hat{\mathbb{Q}}_l) \to \hat{\mathbb{Q}}_l \to 0,$$

and hence a pair of Reidemeister torsion invariants in $K_1(\nu\hat{\mathbb{Q}}_l\pi)$, $K_1((1-\nu)\hat{\mathbb{Q}}_l\pi)$. Since $K_1(\hat{\mathbb{Q}}_l\pi) = K_1(\nu\hat{\mathbb{Q}}_l\pi) \oplus K_1((1-\nu)\hat{\mathbb{Q}}_l\pi)$, we obtain an element $\tau_N(P_* \otimes \hat{\mathbb{Q}}_l) \in K_1(\hat{\mathbb{Q}}_l\pi)$, depending on the choices of bases, cf. **[28]**. This is equivalent to the Reidemeister torsion invariant from §2.

A different set of choices gives the same coset modulo $K_1'(\hat{\mathbb{Z}}_l\pi)$, so there is a well-defined invariant in $K_1(\hat{\mathbb{Q}}_l\pi)/K_1'(\hat{\mathbb{Z}}_l\pi)$. If l is not a divisor of $|\pi|$ then

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$\tau(P_* \otimes \hat{\mathbb{Q}}_l) \in K_1'(\mathbb{Z}_l\pi)$, so the torsion invariants $\tau(P_* \otimes \hat{\mathbb{Q}}_l)$ combine to define

$$\hat{\tau}_N(P_*) \in K_1(\hat{\mathbb{Q}}\pi)/K_1'(\hat{\mathbb{Z}}\pi). \tag{6.2}$$

The element $\hat{\tau}_N(P_*)$ depends only on the homotopy type of P_*, or, what is the same, on $e(P_*)$. Indeed, if $P_* \simeq P'_*$ and $f\colon P_* \to P'_*$ is a homotopy equivalence, inducing the identity on $H_0(P_*)$ and $H_N(P_*)$, then

$$\tau(P'_* \otimes \hat{\mathbb{Q}}_l) - \tau(P_* \otimes \hat{\mathbb{Q}}_l) = \tau(f \otimes \hat{\mathbb{Q}}_l).$$

Hence $\tau(f \otimes \hat{\mathbb{Q}}_l)$ is the image of the $\hat{\mathbb{Z}}_l$-integral Whitehead torsion

$$\tau(f \otimes \hat{\mathbb{Z}}_l) \in K_1(\hat{\mathbb{Z}}_l\pi),$$

that is, $\tau(f \otimes \hat{\mathbb{Q}}_l) \in K_1'(\hat{\mathbb{Z}}_l\pi)$. We have defined an invariant (the *idèle Reidemeister torsion*)

$$\hat{\tau}_N\colon \mathscr{G}H^N(\pi;\mathbb{Z}) \to K_1(\hat{\mathbb{Q}}\pi)/K_1'(\hat{\mathbb{Z}}\pi), \tag{6.3}$$

where $\mathscr{G}H^N$ denotes the generators of H^N. From (5.2) we get the exact sequence

$$0 \longrightarrow K_1(\mathbb{Q}\pi)/K_1'(\mathbb{Z}\pi) \xrightarrow{\ j\ } K_1(\hat{\mathbb{Q}}\pi)/K_1(\hat{\mathbb{Z}}\pi) \xrightarrow{\ \partial\ } \tilde{K}_0(\mathbb{Z}\pi) \longrightarrow 0. \tag{6.4}$$

If P_* in (6.1) is homotopy equivalent to a free resolution C_*, we have a well-defined Reidemeister torsion

$$\tau_N'(C_*) \in K_1(\mathbb{Q}\pi)/K_1'(\mathbb{Z}\pi)$$

and $j(\tau_N') = \hat{\tau}_N$. Thus $\partial(\hat{\tau}_N(e)) = 0$ if and only if e can be realized by a free periodic resolution of length N. Let

$$\sigma_N\colon \mathscr{G}H^N(\pi;\mathbb{Z}) \to \tilde{K}_0(\mathbb{Z}\pi)$$

be the finiteness obstruction from [**24**], $\sigma_N(e(P_*)) = \sum(-1)^i[P_i]$. A slight reformulation of [**28**, Lemma 9.2] gives the following interpretation of exactness in (6.4):

LEMMA 6.5. *For each generator $e \in H^N(\pi;\mathbb{Z})$, we have $\sigma_N(e) = \partial(\hat{\tau}_N(e))$. If $\sigma_N(e) = 0$ then there exists a based free resolution C_* whose Reidemeister torsion $\tau(C_*)$ maps onto $\hat{\tau}_N(e)$.*

We shall make calculations with (6.4) and recall some (standard) notation. Let S be a simple $\mathbb{Q}$-algebra with centre K which we assume is Galois over $\mathbb{Q}$. Let A be its algebraic integers. Then

$$U_y^i(S) = U_y^i(K) = \{u \in A_y^\times \mid u \equiv 1 \pmod{y^i}\} \quad (U_y^0 = U_y = A_y^\times),$$
$$U_l^i(S) = U_l^i(K) = \prod_{y|l} U_y^i(K),$$
$$E(K) = A^\times, \quad E(K)^* = A^* = \{a \in A^\times \mid a_j > 0 \text{ for all real primes } y\}, \tag{6.6}$$
$$E(S) = \begin{cases} E(K) & \text{if } S \otimes_K \mathbb{R} = M_r(\mathbb{R}), \\ E(K)^* & \text{if } S \otimes_K \mathbb{R} \neq M_r(\mathbb{R}). \end{cases}$$

If S is semi-simple then $E(S)$, $U_y^i(S)$ etc. denote the products over the simple factors.

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Further, define

$$D_l(\pi) = \mathrm{cok}\{K_1(\hat{\mathbb{Z}}_l\pi) \to U_l(\mathbb{Q}\pi)\}, \quad \hat{D}(\pi) = \prod_{l \mid |\pi|} D_l(\pi),$$

$$I(\pi) = Z(\hat{\mathbb{Q}}\pi)^\times / \prod U_l(\mathbb{Q}\pi), \quad \Gamma(\pi) = I(\pi)/E(\mathbb{Q}\pi),$$

where Z denotes the centre.

Note that $I(\pi)$ is the direct sum of the ideal groups of the centre fields in $\mathbb{Q}\pi$, and that $\Gamma(\pi)$ is the product of ideal class groups corresponding to the split factors of $\mathbb{Q}\pi$ with the strict ideal class groups corresponding to the non-split factors. Here 'split' refers to split at the infinite primes.

LEMMA 6.7. *There are exact sequences*

$$0 \longrightarrow \hat{D}(\pi) \longrightarrow K_1(\hat{\mathbb{Q}}\pi)/K'_1(\hat{\mathbb{Z}}\pi) \longrightarrow I(\pi) \longrightarrow 0,$$

$$0 \longrightarrow K'_1(\mathbb{Z}\pi) \longrightarrow E(\mathbb{Q}\pi) \xrightarrow{\Phi} \hat{D}(\pi) \xrightarrow{\partial} \tilde{K}_0(\mathbb{Z}\pi) \longrightarrow \Gamma(\pi) \longrightarrow 0.$$

Proof. The reduced norm gives isomorphisms

$$K_1(\mathbb{Q}\pi) \cong Z(\mathbb{Q}\pi)^*, \quad K_1(\hat{\mathbb{Q}}\pi) \cong Z(\hat{\mathbb{Q}}\pi)^\times,$$

where $Z(\mathbb{Q}\pi)^* = \prod \{Z(S)^* \mid S \text{ simple}\}$ and $Z(S)^* \subseteq Z(S)^\times$ is the totally positive element in $Z(S)$ if S is non-split at infinite primes and otherwise $Z(S)^* = Z(S)^\times$.

If l is not a divisor of $|\pi|$ then $\hat{\mathbb{Z}}_l\pi$ is a maximal order and $K'_1(\hat{\mathbb{Z}}_l\pi) = U_l(\mathbb{Q}\pi)$. The first exact sequence follows. The second sequence follows from the 'snake lemma' applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\mathbb{Q}\pi)^* \times K'_1(\hat{\mathbb{Z}}\pi) & \longrightarrow & Z(\mathbb{Q}\pi)^* \times \hat{U}(\mathbb{Q}\pi) & \longrightarrow & \hat{D}(\pi) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z(\hat{\mathbb{Q}}\pi)^\times & \longrightarrow & Z(\hat{\mathbb{Q}}\pi)^\times & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

and the exact sequence in (5.2).

The rest of the section is concerned with explicit calculations for the groups $\pi = Q(8p, q)$ where p and q are (odd) primes. The sequences in Lemma 6.7 decompose into four exact sequences corresponding to the four divisors of pq. We are primarily interested in the top component. We have

$$E(\mathbb{Q}\pi)(pq) = E(S_+) \times E(S_-) = E(F) \times E(F)^*,$$

where $S_\pm$ are the simple summands listed in (4.5) and $F = \mathbb{Q}(\eta_p, \eta_q)$, $\eta_r = \zeta_r + \zeta_r^{-1}$. The l-blocks (alias the irreducible components of $\hat{\mathbb{Z}}_l[\pi]$) 'containing' $S_\pm$ are given by

$$B_p(S_\pm) = \hat{\mathbb{Z}}_p[\mathbb{Z}/p][\zeta_q]^t[X, Y \mid X^2 = Y^2 = \pm 1,\ YXY^{-1} = X^{-1}],$$
$$B_q(S_\pm) = \hat{\mathbb{Z}}_q[\mathbb{Z}/q][\zeta_p]^t[X, Y \mid X^2 = Y^2 = \pm 1,\ YXY^{-1} = X^{-1}],$$
$$B_2(S_+) = B_2(S_-) = \hat{\mathbb{Z}}_2[\zeta_{pq}]^t[Q(8)],$$

with self-explanatory notation. We have

$$D_l(\pi)(pq) = D_l(S_+) \times D_l(S_-), \quad \text{for } l = p, q,$$
$$D_2(\pi)(pq) = D_2(S_+) = D_2(S_-).$$

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The actual values of $D_l(\pi)(pq)$ can easily be extracted from the proof of Proposition 5.3. We are interested in only the 2-primary components.

LEMMA 6.8. *For the blocks $B_l(S_\pm)$ the corresponding components of $\hat{D}(\pi)_{(2)}$ are given by*

$$D_p(S_+)_{(2)} = D_p(S_-)_{(2)} = (\mathbb{Z}[\eta_p, \eta_q]/p)^\times_{(2)},$$
$$D_q(S_+)_{(2)} = D_q(S_-)_{(2)} = (\mathbb{Z}[\eta_p, \eta_q]/q)^\times_{(2)},$$
$$D_2(S_\pm)_{(2)} = \{1\}.$$

Let χ_p, χ_q be faithful (one-dimensional) characters of $\mathbb{Z}/p$, $\mathbb{Z}/q$, and let Γ be the standard representation of $Q(8)$ on $\mathbb{C}^2$. We have generators

$$c_2(\chi_l + \chi_l^{-1}) \in H^4(\mathbb{Z}/l; \mathbb{Z}), \quad c_2(\Gamma) \in H^4(Q(8); \mathbb{Z}),$$

where c_2 denotes the second Chern class.

COROLLARY 6.9. *Let $\pi = Q(8p, q)$. Suppose $e \in H^4(\pi; \mathbb{Z})$ is a generator which restricts to $c_2(\chi_p + \chi_p^{-1})$, $c_2(\chi_q + \chi_q^{-1})$, and $c_2(\Gamma)$ on the Sylow subgroups of π. Then the finiteness obstruction lies in the top component and is 2-primary.*

Proof. First we show that $\sigma_4(e)$ is 2-primary. Since $\sigma_8(e^2) = 2\sigma_4(e)$, it suffices to show that e^2 is realized by a free representation. Indeed, we can use the free representation induced up from a suitable faithful character on $C = \mathbb{Z}/2pq$:

$$e^2 = c_4(\mathrm{Ind}_C^\pi(\chi)).$$

The subgroup $\tilde{K}_0(\mathbb{Z}\pi)_{(2)}(1) \oplus \tilde{K}_0(\mathbb{Z}\pi)_{(2)}(p) \oplus \tilde{K}_0(\mathbb{Z}\pi)_{(2)}(q)$ maps injectively into $\tilde{K}_0(\mathbb{Z}[Q(8p)])_{(2)} \oplus \tilde{K}_0(\mathbb{Z}[Q(8q)])_{(2)}$ by an argument formally the same as the argument used in the proof of Theorem 4.19, so it remains to be shown that the restrictions

$$e \,|\, Q(8p) \in H^4(Q(8p); \mathbb{Z}), \quad e \,|\, Q(8q) \in H^4(Q(8q); \mathbb{Z})$$

are k-invariants of free orthogonal actions on S^3.

Let χ be a (one-dimensional) character of the normal subgroup $\mathbb{Z}/4l \subset Q(8l)$, and let V_{8l} be the induced representation of $Q(8l)$ on $\mathbb{C}^2$. Its restriction to $\mathbb{Z}/4l$ is $\chi + \chi^{-1}$ and its restriction to $Q(8)$ is Γ. Hence for suitable $\chi, c_2(V_{8l}) = e \,|\, Q(8l)$. Since $c_2(V_{8l})$ is the k-invariant of the free action on $S^3 = S(V_{8l})$, the claim follows.

The top component of $\sigma_4(e)$ can be conveniently examined by comparing the group $\pi = Q(8p, q)$ with its normal subgroup $C = \mathbb{Z}/2pq$. First we need

LEMMA 6.10. *For $C = \mathbb{Z}/2pq$ we have*

$$E(C)(pq) = E(\mathbb{Q}(\zeta_{pq})_+) \times E(\mathbb{Q}(\zeta_{pq})_-).$$

The l-blocks containing $\mathbb{Q}(\zeta_{pq})_\pm$ are $\hat{\mathbb{Z}}_p[\mathbb{Z}/p][\zeta_q]$, $\hat{\mathbb{Z}}_q[\mathbb{Z}/q][\zeta_p]$, and $\hat{\mathbb{Z}}_2[\zeta_{pq}][\mathbb{Z}/2]$. The corresponding components of $\hat{D}(C)_{(2)}$ are

$$D_p(\mathbb{Q}(\zeta_{pq})_+)_{(2)} = D_p(\mathbb{Q}(\zeta_{pq})_-)_{(2)} = (\mathbb{Z}[\zeta_{pq}]/p)^\times_{(2)},$$
$$D_q(\mathbb{Q}(\zeta_{pq})_+)_{(2)} = D_q(\mathbb{Q}(\zeta_{pq})_-)_{(2)} = (\mathbb{Z}[\zeta_{pq}]/q)^\times_{(2)},$$

and $D_2(\mathbb{Q}(\zeta_{pq})_\pm)_{(2)} = \{1\}$.

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Proof. The only part which is not completely obvious is the calculation of $D_l(\mathbb{Q}(\zeta_{pq})_\pm)$. The exact sequence

$$1+(1-T)\hat{\mathbb{Z}}_p[\zeta_q][\mathbb{Z}/p] \to K_1(\hat{\mathbb{Z}}_p[\zeta_q][\mathbb{Z}/p]) \to K_1(\hat{\mathbb{Z}}_p[\zeta_q]) \to 0$$

shows that

$$K_1(\hat{\mathbb{Z}}_p[\zeta_q][\mathbb{Z}/p])(pq) \cong 1+(1-\zeta_p)\hat{\mathbb{Z}}_p[\zeta_p,\zeta_q] = U_y^1(\mathbb{Q}(\zeta_{pq})),$$

where $y = 1-\zeta_p$. Since p is odd, U_y^1 has odd index in U_y^{p-1}, so

$$(U_y/U_y^1)_{(2)} = (\hat{\mathbb{Z}}_p[\zeta_{pq}]/p)^\times_{(2)}.$$

Finally,

$$K_1(\hat{\mathbb{Z}}_p[\mathbb{Z}/pq])(pq) = g_p \cdot K_1(\hat{\mathbb{Z}}_p[\zeta_q][\mathbb{Z}/p])(pq)$$

where g_p is the number of p-adic primes in $\mathbb{Q}(\zeta_{pq})$. This gives the stated value of $D_p(\mathbb{Q}(\zeta_{pq})_\pm)$.

The 2-block containing $\mathbb{Q}(\zeta_{pq})_+$ (and $\mathbb{Q}(\zeta_{pq})_-$) is $\hat{\mathbb{Z}}_2[\zeta_{pq}][\mathbb{Z}/2]$. The exact sequence

$$1 \to 1+2\hat{\mathbb{Z}}_2[\zeta_{pq}]^\times \to K_1(\hat{\mathbb{Z}}_2[\zeta_{pq}][\mathbb{Z}/2]) \to K_1(\hat{\mathbb{Z}}_2[\zeta_{pq}]) \to 0$$

shows that $D_2(\mathbb{Q}(\zeta_{pq})_\pm)$ has odd order.

We are now ready to evaluate $\hat{\tau}_4(e)$ and $\sigma_4(e)$ for the generators $e \in \mathcal{G}H^4(Q(8p,q);\mathbb{Z})$ considered in Corollary 6.9. The result below can be extracted from Milgram's work on the finiteness obstruction, in particular from the proof of Theorem C in [**15**]—once one understands that proof. For those who do not, I include an alternative argument.

THEOREM 6.11. *Let $\pi = Q(8p,q)$ with p and q odd primes. Write $A = \mathbb{Z}[\eta_p,\eta_q]$ with $\eta_p = \zeta_p+\zeta_p^{-1}$, and let $\Phi\colon A^\times \to (A/p)^\times_{(2)} \times (A/q)^\times_{(2)}$ be the reduction homomorphism. Consider a generator $e \in \mathcal{G}H^4(\pi;\mathbb{Z})$ which satisfies the requirements of Corollary* 6.9. *Then*

(i) *$\sigma_4(e) = 0$ if and only if there exist $u_+ \in A^\times$ and $u_- \in A^*$ (the totally positive units), such that*

$$\Phi(u_+) = (1,-1), \quad \Phi(u_-) = (\tfrac{1}{4},\tfrac{1}{4}),$$

(ii) *given u_+, u_- satisfying* (i), *the element*

$$(u_+(\eta_p-\eta_q)/4p^2q^2(2-\eta_p)(2-\eta_q), 4u_-) \in K_1(\mathbb{Q}\pi)(pq) \otimes \mathbb{Z}_{(2)}$$

represents $\tau_4'(e)_{(2)}(pq) \in (K_1(\mathbb{Q}\pi)(pq)/K_1'(\mathbb{Z}\pi)(pq)) \otimes \mathbb{Z}_{(2)}$.

Before giving the proof we need to be more precise about the decomposition of $K_1(\mathbb{Z}\pi)$ into components $K_1(\mathbb{Z}\pi)(r)$ than has been necessary so far.

Choose generators T_p and T_q in $Q(8p,q)$ of order p and q, respectively. Let

$$E_p\colon Q(8p,q) \to Q(8p,q), \quad E_q\colon Q(8p,q) \to Q(8p,q) \tag{6.12}$$

be the automorphisms which leave $Q(8)$ fixed, and map

$$E_p(T_p) = 1, \quad E_q(T_q) = 1, \quad E_p(T_q) = T_q, \quad E_q(T_p) = T_p.$$

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With this notation,

$$\begin{aligned}K_1(\Lambda\pi)(1) &= \operatorname{Image}(E_p)_* \circ (E_q)_*,\\ K_1(\Lambda\pi)(p) &= \operatorname{Image}(1-E_p)_* \circ (E_q)_*,\\ K_1(\Lambda\pi)(q) &= \operatorname{Image}(1-E_q)_* \circ (E_p)_*,\\ K_1(\Lambda\pi)(pq) &= \operatorname{Image}(1-E_p)_* \circ (1-E_q)_*,\end{aligned}$$

where $\pi = Q(8p, q)$. There is a similar formula for $K_1(\Lambda C)(r)$ where $C \subset Q(8p, q)$ is the subgroup $C = \mathbb{Z}/2pq$.

It is convenient for calculations to describe elements of $K_1(\mathbb{Q}\pi)$ and $K_1(\mathbb{Q}C)$ as character homomorphisms via the isomorphism

(6.13) $$K_1(\mathbb{Q}\pi) \cong \operatorname{Hom}_\Omega(R(\pi), \bar{\mathbb{Q}}^\times).$$

Here $R(\pi)$ is the complex character ring and $\Omega = \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $\bar{\mathbb{Q}} = \varinjlim \mathbb{Q}(\zeta_n)$. The isomorphism in (6.13) is natural with respect to both the covariant and the contravariant structures on the two sides.

We recall the homomorphism from $K_1(\mathbb{Q}\pi)$ to $\operatorname{Hom}_\Omega(R(\pi), \bar{\mathbb{Q}}^\times)$ which defines the isomorphism in (6.13). Let $x \in K_1(\mathbb{Q}\pi)$; we construct the corresponding character homomorphism f_x. Let χ be an irreducible character and $T_\chi\colon \pi \to Gl_r(\bar{\mathbb{Q}})$ the representation realizing it. Extend T_χ to an algebra homomorphism $T_\chi\colon \mathbb{Q}\pi \to M_r(\bar{\mathbb{Q}})$ and look at the induced homomorphism on the units of the centre $T_\chi\colon Z(\mathbb{Q}\pi)^\times \to \bar{\mathbb{Q}}^\times$. Then

$$f_x(\chi) = T_\chi(\operatorname{Nrd}(x)),$$

where $\operatorname{Nrd}\colon K_1(\mathbb{Q}\pi) \to Z(\mathbb{Q}\pi)^\times$ is the reduced norm.

Let $\{\chi_i\}$ be a set of irreducible characters representing the Ω-conjugacy classes. We shall use the composition

(6.14) $$K_1(\mathbb{Q}\pi) \cong \operatorname{Hom}_\Omega(R(\pi), \bar{\mathbb{Q}}^\times) \cong \prod \mathbb{Q}(\chi_i)^\times$$

to list elements of $K_1(\mathbb{Q}\pi)$, where the ith component of the last isomorphism takes a character homomorphism f into $f(\chi_i) \in \mathbb{Q}(\chi_i)^\times \subset \bar{\mathbb{Q}}^\times$.

We illustrate the use of (6.13) by calculating the top component $K_1(C)(pq)$ in terms of character homomorphisms. Write $C = \langle T_2\rangle \times \langle T_p\rangle \times \langle T_q\rangle$. It has eight Ω-conjugacy classes of irreducible characters denoted $\chi_1^\pm$, $\chi_p^\pm$, $\chi_q^\pm$, and $\chi_{pq}^\pm$; they can be specified by

$$\begin{aligned}&\chi_1^\pm(T_p) = 1, \qquad \chi_1^\pm(T_q) = 1, \qquad \chi_1^\pm(T_2) = \pm 1,\\ &\chi_p^\pm(T_p) = \zeta_p, \qquad \chi_p^\pm(T_q) = 1, \qquad \chi_p^\pm(T_2) = \pm 1,\end{aligned}$$

with a similar definition of $\chi_q^\pm$, and $\chi_{pq}^+ = \chi_p^+ \cdot \chi_q^+$, $\chi_{pq}^- = \chi_1^- \cdot \chi_{pq}^+$.

In (6.13) the covariant structure on the right-hand side is induced from the contravariant structure on $R(-)$. Thus if f is a character homomorphism, we get the following determination of its top component:

(6.15) $$\begin{aligned}&(1-E_p)_*(1-E_q)_*(f)(\chi_{pq}^\pm) = f(\chi_{pq}^\pm)f(\chi_1^\pm)/f(\chi_p^\pm)f(\chi_q^\pm),\\ &(1-E_p)_*(1-E_q)_*(f)(\chi_r) = 1 \quad \text{for } r = 1,\ p, \text{ and } q.\end{aligned}$$

As a final preparation for the proof of Theroem 6.11 we recall from [**17**, **28**] the formula for the Reidemeister torsion of a free representation (or lens space). We fix the generator $t = T_2T_pT_q$ for C and let χ be the faithful character with $\chi(t) = e^{2\pi i/2pq}$. The

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Reidemeister torsion of $S(\chi+\chi^{-1})$ is given by

(6.16) $$\tau(S(\chi+\chi^{-1})) = v\cdot(1/4p^2q^2)+(1-v)(t-1)(t^{-1}-1) \in \mathbb{Q}[C]^{\times},$$

with $v = 1/2pq\sum t^i$.

Proof of Theorem 6.11. We may assume that the generator $e \in H^4(\pi;\mathbb{Z})$ restricts to the k-invariant of the free C-sphere $S(\chi+\chi^{-1})$, that is,

$$e\,|\,C = c_2(\chi+\chi^{-1}).$$

From (6.15) and (6.16) we get via (6.14) the top component of $\tau_4(e\,|\,C)$, considered as an element of $\mathbb{Q}(\chi_{pq}^{+})\times\mathbb{Q}(\chi_{pq}^{-}) = \mathbb{Q}(\zeta_{pq})_+\times\mathbb{Q}(\zeta_{pq})_-$. The result is

$$\tau_4(e\,|\,C)(pq) = [(2-\eta_{pq})/4p^2q^2(2-\eta_p)(2-\eta_q),\, 4(2+\eta_{pq})/(2+\eta_p)(2+\eta_q)].$$

Consider the top component of the (2-local) diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1(\mathbb{Q}C)/K_1'(\mathbb{Z}C)_{(2)} \arrow[r, "j_C"] & K_1(\hat{\mathbb{Q}}C)/K_1'(\hat{\mathbb{Z}}C)_{(2)} \arrow[r, "\partial"] & \tilde{K}_0(\mathbb{Z}C)_{(2)} \arrow[r] & 0 \\
0 \arrow[r] & K_1(\mathbb{Q}\pi)/K_1'(\mathbb{Z}\pi)_{(2)} \arrow[r, "j_\pi"] \arrow[u, "i^*"] & K_1(\hat{\mathbb{Q}}\pi)/K_1'(\hat{\mathbb{Z}}\pi)_{(2)} \arrow[r, "\partial"] \arrow[u, "\hat{\imath}^*"] & \tilde{K}_0(\mathbb{Z}\pi)_{(2)} \arrow[r] \arrow[u, "i_K^*"] & 0
\end{tikzcd}$$

The rows are exact by (6.4), and $\hat{\imath}^*$ is injective. Indeed, the contravariant map

$$\hat{D}(\pi)_{(2)}(pq) \to \hat{D}(C)_{(2)}(pq)$$

is induced from the inclusion of $\mathbb{Z}[\eta_p,\eta_q]^{\times}$ in $\mathbb{Z}[\eta_{pq}]^{\times}$, cf. Lemma 6.10. It follows from Lemma 6.7 that $\hat{\imath}^*$ in the above diagram is injective, whence $\sigma_4(e)_{(2)}(pq)$ maps injectively to cok i^*.

In the top component of $K_1(\mathbb{Q}\pi)/K_1'(\mathbb{Z}\pi)_{(2)}$ we consider the element

$$R = (4p^2q^2(2-\eta_p)(2-\eta_q),\,(2+\eta_p)(2+\eta_q)/4).$$

Then (using multiplicative notation for $K_1(\ )$), we have

$$\tau_4(e\,|\,C)(pq)\hat{\imath}^*j_\pi(R) = (2-\eta_{pq},\,2+\eta_{pq}) \in \hat{D}(\pi)_{(2)}(pq).$$

It follows that $\sigma_4(e)_{(2)}(pq)$ is the image of $(2-\eta_{pq},\,2+\eta_{pq})$ under the mapping

$$\partial\colon \hat{D}(\pi)_{(2)}(pq) \to \tilde{K}_0(\mathbb{Z}\pi)_{(2)}(pq),$$

cf. Lemmas 6.5 and 6.7. The group $\hat{D}(\pi)_{(2)}(pq) = \hat{D}(S_+)_{(2)}\times\hat{D}(S_-)_{(2)}$ was calculated in Lemma 6.8. Under the decomposition

$$(\mathbb{Z}[\eta_{pq}]/pq)^{\times}_{(2)} \cong (\mathbb{Z}[\eta_q]/p)^{\times}_{(2)}\times(\mathbb{Z}[\eta_p]/q)^{\times}_{(2)},$$

$2\pm\eta_{pq}$ corresponds to $(2\pm\eta_q, 2\pm\eta_p)$, and for $\Phi_\pi\colon E(S_\pm)\to\hat{D}(S_\pm)_{(2)}$ we have

$$\Phi_\pi(\eta_p-\eta_q) = (1,-1)\cdot(2-\eta_q,\,2-\eta_p),$$
$$\Phi_\pi((2+\eta_p)(2+\eta_q)) = (4,4)\cdot(2+\eta_q,\,2+\eta_p).$$

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It follows that

$$(*)\qquad \begin{aligned} \hat{\tau}_4(e)_{(2)}(pq)_+ &= \Phi_\pi((\eta_p-\eta_q)/4p^2p^2(2-\eta_p)(2-\eta_q))\cdot(1,-1),\\ \hat{\tau}_4(e)_{(2)}(pq)_- &= \Phi_\pi(4)\cdot(4,4). \end{aligned}$$

Since $\sigma_4(e) = \sigma_4(e)_{(2)}(pq)$ by Corollary 6.9, both (i) and (ii) of Theorem 6.11 follow from (*).

Next, we examine how $\sigma_N(e)$ varies with the choice of generator. The simplest change to make is to vary e by an automorphism of π. This corresponds to changing the projective resolution P_* by an automorphism. Inner automorphisms preserve e, and hence the homotopy type of P_*, so we are interested in the outer automorphism group. For $\pi = Q(8p, q)$,

$$\mathrm{Out}(\pi) = \mathbb{F}_p^\times/\langle -1\rangle \times \mathbb{F}_q^\times/\langle -1\rangle.$$

It acts on

$$\mathcal{G}H^4(\pi;\mathbb{Z}) = \mathbb{F}_p^\times \times \mathbb{F}_q^\times \times (\mathbb{Z}/8)^\times$$

by $(r_p, r_q)\cdot(g_p, g_q, g_2) = (r_p{}^2 g_p, r_q{}^2 g_q, g_2)$ with quotient

$$\mathcal{G}H^4(\pi;\mathbb{Z})/\mathrm{Out}(\pi) = \mathbb{F}_p^\times/\mathbb{F}_p^{\times 2} \times \mathbb{F}_q^\times/\mathbb{F}_q^{\times 2} \times (\mathbb{Z}/8)^\times.$$

One can make one more trivial change of P_*, namely replace $\eta\colon \mathbb{Z} \to P_3$ with $-\eta$. Then $e(P_*)$ is multiplied by -1. The resulting quotient group is

$$(6.18)\qquad \mathcal{G}H^4(\pi;\mathbb{Z})/\mathrm{Out}(\pi)\times\langle -1\rangle \cong \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.$$

Let $S\colon (\mathbb{Z}/|\pi|)^\times \to \tilde{K}_0(\mathbb{Z}\pi)$ be the Swan homomorphism which maps r into the projective ideal (r, Σ). Then $\sigma_N(re) = \sigma_N(e) + S(r)$ for $e \in \mathcal{G}H^N(\pi;\mathbb{Z})$, cf. [**24**]. There is a commutative diagram

$$(6.19)\qquad \begin{array}{ccc} \hat{\mathbb{Z}}_{|\pi|}^\times & \xrightarrow{\ i\ } & \hat{D}(\pi) \\ \Big\downarrow{\scriptstyle \mathrm{pr}} & & \Big\downarrow{\scriptstyle \partial} \\ (\mathbb{Z}/|\pi|)^\times & \xrightarrow{\ S\ } & \tilde{K}_0(\mathbb{Z}\pi) \end{array}$$

where $\hat{\mathbb{Z}}_{|\pi|}^\times = \prod_{l\,|\,|\pi|}\hat{\mathbb{Z}}_l$. In (6.19), i maps $\hat{\mathbb{Z}}_l^\times$ into $U_l(\mathbb{Q}\pi)$ via the simple summand in $\mathbb{Q}\pi$ corresponding to the augmentation.

The discussion above shows that the Swan homomorphism factors over $(\mathbb{Z}/8pq)^\times/\langle \text{Squares}, -1\rangle \cong \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$. The resulting map

$$S\colon \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \to \tilde{K}_0(\mathbb{Z}[Q(8p,q)])$$

is evaluated in [**1**]; we recall the result briefly.

Let $\tilde{S}(r)$ denote Image $S \cap \tilde{K}_0(\mathbb{Z}[Q(8p,q)])(r)$. We have $\tilde{S}(1) = \mathbb{Z}/2$ by a well-known result for $\tilde{K}_0(\mathbb{Z}[Q(8)])$. Write $\mathrm{Ord}_p(2)$ for the order of 2 in $\mathbb{F}_p^\times$. In [**1**] we prove

THEOREM 6.20. (i) *$\tilde{S}(p) = 0$ if $p \equiv -1 \pmod 8$.*
(ii) *$\tilde{S}(p) = 0$ if $p \equiv 1 \pmod 8$ and $\mathrm{Ord}_p(2)$ is odd.*
(iii) *$\tilde{S}(p) = \mathbb{Z}/2$ otherwise.*

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If both p and q belong to Case (iii) then e_k is the only k-invariant which can have vanishing finiteness obstruction.

In [1] we also evaluate $\tilde{S}(pq)$, except when both p and q belong to Case (ii) of Theorem 6.20. For example, if $p \equiv 3 \pmod 8$ and $q \equiv -1 \pmod 8$ then $\tilde{S}(pq) = \mathbb{Z}/2$, and there exists a generator $e'_k \in H^{8k+4}(Q(8p, q); \mathbb{Z})$ with $\sigma(e'_k) = 0$. But e'_k does not restrict to the k-invariant of a free representation on $Q(8q)$.

7. *The surgery obstruction for spherical space forms*

Suppose $\pi = Q(8p, q)$ satisfies Theorem 6.11 (i). Let $C_* \to \mathbb{Z}$ be the resulting free periodic resolution of length 4. We splice it together with itself $2k+1$ times to obtain free periodic resolutions $C_*^{(k)} \to \mathbb{Z}$ of length $8k+4$. For all k, $C_*^{(k)}$ can be topologically realized: there exists a finite cell complex Σ, $\Sigma \simeq S^{8k+3}$, with a free cellular action of π so that the induced complex of cellular chains $C_*(\Sigma)$ is chain homotopy equivalent to $C_*^{(k)}$. ((Σ, π) is called an $(8k+4, \pi)$-polarized space in [**26**, **13**].)

The π-homotopy type of Σ is specified by the k-invariant

$$e(\Sigma, \pi) = e(C_*)^{2k+1} \in H^{8k+4}(\pi; \mathbb{Z}).$$

The homotopy type of Σ/π, on the other hand, is determined by the image of e in $H^{8k+4}(\pi; \mathbb{Z})/\mathrm{Out}(\pi) \times \langle -1 \rangle$.

Let $\tau(\Sigma, \pi) \in K_1(\mathbb{Q}\pi)$ denote the Reidemeister torsion of Σ with respect to some equivariant cell decomposition. Its image in $K_1(\mathbb{Q}\pi)/K'_1(\mathbb{Z}\pi)$ is the invariant $\tau'_{8k+4}(e^{2k+1})$ introduced in the last paragraph. We shall use this invariant to evaluate surgery obstructions, but need to know only the 2-local part of its top component which is given by

$$\tau'(\Sigma, \pi)_{(2)}(pq) = [\tau'_4(e)_{(2)}(pq)]^{2k+1},$$

where τ'_4 is listed in Theorem 6.11 (ii).

There are three maximal subgroups of $Q(8p, q)$:

$$\pi_1 = Q(8p), \quad \pi_2 = Q(8q), \quad \pi_3 = Q(4pq). \tag{7.1}$$

Each of them admits free (complex) representations of real degree 4 (and thus also of degree $8k+4$).

The subgroup π_3 is of particular interest for the results below. We recall from § 5 that the top component $K'_1(\mathbb{Z}\pi_3)(pq)$ is a direct product of the groups

$$\begin{aligned} K'_1(\mathbb{Z}\pi_3)_+(pq) &= \mathrm{Ker}(B^\times \to (B/2pq)^\times), \\ K'_1(\mathbb{Z}\pi_3)_-(pq) &= \mathrm{Ker}(B^\times \to (B/pq)^\times), \end{aligned} \tag{7.2}$$

where $B = \mathbb{Z}[\eta_{pq}]$.

LEMMA 7.3. *There exist free representations V_i of π_i and π_i-homotopy equivalences*

$$g_i\colon S(V_i) \to \Sigma.$$

For a suitable cell structure of Σ/π the 2-local part of the (weak) Whitehead torsion of g_3 has top component given by

$$\mathrm{Nrd}(\mathrm{wh}'(g_3)_{(2)}(pq)) = (w_+, w_-).$$

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Here $w_+ \in \mathbb{Q}(\eta_{pq})_+$, $w_- \in \mathbb{Q}(\eta_{pq})_-$ *are the elements*

$$w_+ = [u_+(\eta_p - \eta_q)/2 - \eta_{pq}]^{2k+1},$$
$$w_- = [u_-(2+\eta_p)(2+\eta_q)/(2+\eta_{pq})]^{2k+1},$$

where u_+ *and* u_- *are the units from Theorem* 6.11(i).

Proof. The free representations of π_i are induced from free representations of index 2 cyclic subgroups. Their restrictions to $\mathbb{Z}/p$, $\mathbb{Z}/q$ are of the form $U \oplus \bar{U}$, where the bar indicates the complex conjugate. Their restriction to $Q(8)$ is a multiple of Γ, the unique complex representation of real degree 4. Since the k-invariant of $S(V)$ is the Euler class of the representation, the first part of the lemma follows when we use the fact that group cohomology is detected on the Sylow subgroups.

The second part uses the relation

$$\mathrm{wh}'(g_3) = \tau(\Sigma, \pi_3)\tau(S(V_3), \pi_3)^{-1},$$

where $\tau(\ ,\pi_3)$ denotes the Reidemeister torsion in $\mathrm{Wh}(\mathbb{Q}\pi_3)$, and the group structure is written multiplicatively. For the top component we have

$$\mathrm{Wh}(\mathbb{Q}\pi_3)(pq) = K_1(\mathbb{Q}\pi_3)(pq) \cong \mathbb{Q}(\eta_{pq})_+^\times \times \mathbb{Q}(\eta_{pq})_-^\times.$$

It follows from Theorem 6.11 that there exists a π-equivariant cell decomposition of Σ with $\tau(\Sigma, \pi)_{(2)}(pq) = (\tau_+, \tau_-)$ and

$$\tau_+ = [u_+(\eta_p - \eta_q)/4p^2q^2(2-\eta_p)(2-\eta_q)]^{2k+1},$$
$$\tau_- = [4u_-]^{2k+1}.$$

On the other hand, $\tau(S(V_3), \pi_3)_{(2)}(pq)$ can be calculated using the method explained in §6 (and used in the proof of Theorem 6.11). It has top components

$$\tau(S(V_3), \pi_3)_{(2)}(pq)_+ = [2 - \eta_{pq}/4p^2q^2(2-\eta_p)(2-\eta_q)]^{2k+1},$$
$$\tau(S(V_3), \pi_3)_{(2)}(pq)_- = [4(2+\eta_{pq})/(2+\eta_p)(2+\eta_q)]^{2k+1},$$

and Lemma 7.3 follows.

Consider the surgery-exact sequence

$$L'_0(\mathbb{Z}\pi) \longrightarrow \mathcal{S}'(\Sigma/\pi) \xrightarrow{\mathcal{N}} [\Sigma/\pi, G/\mathrm{Top}] \xrightarrow{\lambda} L'_3(\mathbb{Z}\pi)$$

in the weakly simple category. Here $L'_0(\mathbb{Z}\pi) = L^Y_0(\mathbb{Z}\pi)$ and $L'_3(\mathbb{Z}\pi) = L^Y_3(\mathbb{Z}\pi)/\langle\tau\rangle$, cf. [**29**, §5.4]. In our cases, $\pi = Q(8p, q)$, $\langle\tau\rangle = \mathbb{Z}/2 \subseteq L^Y_3(\mathbb{Z}\pi)(1)$, and $L'_3(\mathbb{Z}\pi)(d) = L^Y_3(\mathbb{Z}\pi)(d)$ for $d > 1$.

LEMMA 7.4. *There exists a degree* 1 *normal map* $f\colon M^{8k+3} \to \Sigma/\pi$, $\hat{f}\colon \nu_M \to \zeta$ *whose induced covers over* Σ/π_i *are normally cobordant to the homotopy equivalences* g_i *from Lemma* 7.3.

Proof. The existence of $(f, \hat{f})$ with the prescribed properties is equivalent to the existence of an element $\varphi \in [\Sigma/\pi, G/\mathrm{Top}]$ with $i(\pi_i, \pi)^*(\varphi) = \mathcal{N}(g_i)$ for $i = 1, 2$, and 3. Since G/Top is an infinite loop space, we have an isomorphism

$$\varprojlim i(\pi_i, \pi)^*\colon [\Sigma/\pi, G/\mathrm{Top}] \xrightarrow{\cong} \varprojlim [\Sigma/\pi_i, G/\mathrm{Top}],$$

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where $\varprojlim$ denotes the 'stable' element in the sense of [3, Chapter XII] (see also [10]).

Given two $(8k+4, \rho)$-polarized spaces (Σ_1, ρ) and (Σ_2, ρ) with the same k-invariant, $e(\Sigma_1, \rho) = e(\Sigma_2, \rho)$, there exists a unique homotopy class of homotopy equivalences $h\colon \Sigma_1/\rho \to \Sigma_2/\rho$ making the triangle

$$\begin{tikzcd} \Sigma_1/\rho \arrow[rr, "h"] \arrow[dr, "\alpha_1"'] & & \Sigma_2/\rho \arrow[dl, "\alpha_2"] \\ & B\rho & \end{tikzcd}$$

homotopy commute; α_i is the map classifying the cover $\Sigma_i \to \Sigma_i/\rho$. Using this, together with the well-known fact that group cohomology is detected on the Sylow subgroups, we see that the normal invariants $\mathcal{N}(g_i)$ are stable,

$$(\mathcal{N}(g_i)) \in \varprojlim\, [\Sigma/\pi_i, G/\mathrm{Top}].$$

This completes the proof.

We still have to calculate the surgery invariant $\lambda^K(f,\hat{f}) \in L_3^h(\pi)$. Since the covers $(f_\sigma, \hat{f}_\sigma)$ are normally cobordant to homotopy equivalences for all proper subgroups σ,

$$\lambda^K(f,\hat{f}) \in L_3^h(\pi)(pq) = L_3^K(\mathbb{Z}\pi)(pq).$$

It vanishes if and only if there exists a free topological action on S^{8k+3} in the π-homotopy class of the action on Σ. In fact, if a topological action exists then there even exists a differentiable action in the given homotopy class (cf. [10]).

We use the same notation as in §6: $A = \mathbb{Z}[\eta_p, \eta_q]$, $B = \mathbb{Z}[\eta_{pq}]$, and

$$\Phi_A\colon A^\times \to (A/pA)^\times_{(2)} \times (A/qA)^\times_{(2)}, \quad \varphi_A\colon A^\times \to (A/4A)^\times_{(2)} = A/2A,$$

with similar notation for Φ_B and φ_B.

Let $e \in H^4(Q(8p,q); \mathbb{Z})$ be a generator which restricts to $c_2(\chi_p + \chi_p{}^{-1})$, $c_2(\chi_q + \chi_q{}^{-1})$, and $c_2(\Gamma)$ on the Sylow subgroups, where χ_p, χ_q are faithful characters of $\mathbb{Z}/p$ and $\mathbb{Z}/q$, and Γ is the standard representation of $Q(8)$ on $\mathbb{C}^2$. Such an e is specified up to an automorphism of $Q(8p,q)$. Let e_k be the k-fold cup-product power of e.

THEOREM 7.5. *There exists a free topological action of $Q(8p,q)$ on S^{8k+3} with k-invariant e_k if and only if the following conditions are satisfied:*

(i) $(1, -1) \in \operatorname{Image}\Phi_A$ *and* $(4,4) \in \operatorname{Image}(\Phi_A | A^{\times 2})$;

(ii) $(\eta_q - 2, \eta_p - 2) \in \operatorname{Image}(\Phi_A | \operatorname{Ker}\varphi_A)$;

(iii) $\Phi_B((\eta_{pq}/\bar{\eta}_{pq})U) \in \operatorname{Image}(\Phi_A)/\langle -1\rangle$, *where $U \in A^\times$ is such that $\Phi_A(U^2) = (4,4)$.*

Proof. The first condition is stronger than the conditions in Theorem 6.11(i). Thus we have the $(8k+4, \pi)$-polarized finite π-complex (Σ, π) and the homotopy equivalences g_i from Lemma 7.3. Let $(f,\hat{f})$ be the degree 1 normal constructed in Lemma 7.4, and let $\lambda^K(f,\hat{f})$ be its surgery obstruction on $L_3^K(\mathbb{Z}\pi)(pq)$. We want to apply Theorem 5.10 and must check conditions (i)–(iii) of that theorem.

Let $\mathbb{F}$ be the finite field with l elements, where l is a prime not dividing $2pq$. Let $\Sigma/\pi \xrightarrow{h} B\pi$ be the map classifying the cover $\Sigma \to \Sigma/\pi$. We use Table 2.11 and Corollary 3.7 to evaluate $\Delta_{\mathbb{F}}(M, h\circ f)$.

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Since the antistructure $(\mathbb{F}[\mathbb{Z}/p], \alpha, 1)$ contains no Type Sp summands, $\Delta_{\mathbb{F}}(M_p, h_p \circ f_p) = 0$, and similarly $\Delta_{\mathbb{F}}(M_q, h_q \circ f_q) = 0$. The antistructure $(\mathbb{F}[Q(8)], \alpha, 1)$ contains one Type Sp summand, isomorphie to $M_2(\mathbb{F})$, so $\Delta_{\mathbb{F}}[M_2, h_2 \circ f_2] \in \mathbb{F}^\times/(\mathbb{F}^\times)^2$. But $\Delta_{\mathbb{F}}[M_2, h_2 \circ f_2]$ is the l-primary reduction of $\Delta_{\mathbb{Q}}[M_2, h_2 \circ f_2]$. Since the Type Sp factor of the rational group ring of $Q(8)$ is non-split, $\Delta_{\mathbb{Q}}[M_2, h_2 \circ f_2]$ takes values in $\mathbb{Z}^* = \{1\}$ by Table 2.11. Thus $\Delta_{\mathbb{F}}[M_2, h_2 \circ f_2] = 0$. It follows from Corollary 3.7 that $\Delta_{\mathbb{F}}(M, h \circ f) = 0$.

We cannot use induction to calculate $\Delta_{\mathbb{F}}(\Sigma/\pi, h)$ (because PD-bordism is not a homology theory), but we can use the information contained in Theorem 6.11(ii) to calculate $\Delta_{\mathbb{F}}(\Sigma/\pi, h)(pq)$. Indeed, for a suitably chosen simple homotopy type Σ/π, the Reidemeister torsion $\tau(\Sigma/\pi)_{(2)}$ has top components $(\tau_+, \tau_-) \in \mathbb{Q}(\eta_p, \eta_q)_+^\times \times \mathbb{Q}(\eta_p, \eta_q)_-^\times$, where

$$\tau_+ = [u_+(\eta_p - \eta_q)/4p^2q^2(2-\eta_p)(2-\eta_q)]^{2k+1},$$

$$\tau_- = [4u_-]^{2k+1}.$$

The reduction $A \to (\mathbb{F} \otimes A)^\times/(\mathbb{F} \otimes A)^{\times 2}$ maps τ_- into $\Delta_{\mathbb{F}}(\Sigma/\pi, h)(pq)$, so

$$\Delta_{\mathbb{F}}(\Sigma/\pi, h)(pq) = [4u_-]^{2k+1} \in (\mathbb{F} \otimes A)^\times/(\mathbb{F} \otimes A)^{\times 2}.$$

Hence Theorem 5.10(i) is satisfied if and only if u_- is a square at all primes l not dividing $2pq$; this happens if and only if $u_- \in (A^\times)^2$ by the global square theorem.

The second condition in Theorem 5.10 can be checked using the formula for w_+ in Lemma 7.3. We must check if $\varphi_B(w_+) = 0$ for a suitable choice of $u_+ \in A^\times$ with $\Phi_A(u_+) = (1, -1)$. Since $\eta_{pq} - 2$ is a square in $B^\times$, namely $\eta_{pq} - 2 = (\zeta_{pq}{}^r - \zeta_{pq}{}^{-r})^2$ with $r = \frac{1}{2}(1-pq)$, we have $\varphi_B(w_+) = \varphi_B(u_+(\eta_q - \eta_p))$.

Moreover, $\Phi_B(u_+(\eta_q - \eta_p)) = (\eta_q - 2, \eta_p - 2) \in (B/pB)_{(2)}^\times \times (B/qB)_{(2)}^\times$ so by Theorem 7.5(ii) there exists $v_+ \in \operatorname{Ker} \varphi_A$ with $\Phi_A(v_+) = \Phi_B(u_+(\eta_q - \eta_p))$. Hence $v_+/u_+(\eta_q - \eta_p) \in \operatorname{Ker} \Phi_A$ and u_+ can be replaced with $u'_+ = v_+/\eta_q - \eta_p$. Then $\varphi_B(u'_+(\eta_q - \eta_p)) = 0$ and Theorem 5.10(ii) is satisfied.

Finally, the top component of the Whitehead torsion of g_3 is the element w_- from Lemma 7.3, with $u_- = U_-^2$. It has square root

$$\sqrt{w_-} = [(\eta'_{pq}/\eta'_p\eta'_q)U_-]^{2k+1},$$

where $\eta'_s = \zeta_s{}^r + \zeta_s{}^{-r}$, with $r = \frac{1}{2}(1-s)$, is a Galois conjugate of η_s. Hence

$$\sqrt{\rho_{ab}(\mathrm{wh}(g_3)_-(pq))} = \Phi_B((\eta'_{pq}/\eta'_p\eta'_q)U_-)^{2k+1},$$

and Condition (iii) of Theorem 5.10 is equivalent to Condition (iii) of Theorem 7.5.

We have seen that the three conditions in Theorem 7.5 together imply the three conditions of Theorem 5.10; whence the required free action exists.

It remains to be seen that the conditions in Theorem 7.5 are also necessary. Given the free action (S^{8k+3}, π) with k-invariant e_k, the finiteness obstruction $\sigma(e_k)$ must vanish. Since the finiteness obstruction is 2-primary, it follows that $\sigma(e_1) = 0$ and by Theorem 6.11 there exist u_+, $u_- \in A^\times$ with $\Phi_A(u_+) = (1, -1)$, $\Phi_A(u_-) = (\frac{1}{4}, \frac{1}{4})$. We must check that u_- is a square. This follows because $\Delta_{\mathbb{F}}(S^{8k+3}/\pi) = [u_-]$ by Theorem 6.11(ii) and because $\Delta_{\mathbb{F}}(S^{8k+3}/\pi) = 0$ by Corollary 3.7 and Table 2.11, as in the first part of the proof.

The covering S^{8k+3}/π_3 is homotopy equivalent to $S(V_3)/\pi_3$ by g_3 and $\lambda^Y(g_3)(pq) = 0$. This follows, for example, because the surgery invariant is detected on the Sylow 2-subgroup $Q(8)$. Using Proposition 5.8, we see that Conditions 7.5(ii) and 7.5(iii) must be satisfied. This completes the proof.

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We give Theorem 7.5 a corollary when $p \equiv 3$ (4). The smallest order group with $\sigma(e_k) = 0$ is $Q(24, 13)$. The surgery obstruction for $Q(24, 13)$ was settled in [**11**]. The general case is similar, but the arithmetic is now arranged better (inspired by [**15**]).

COROLLARY 7.6. *Suppose $p \equiv 3$ (4). There is a free action of $Q(8p, q)$ on S^{8k+3} with k-invariant e_k, for $k \geqslant 1$, if and only if $q \equiv 1$ (8) and Condition 7.5(ii) is satisfied.*

Proof. The Galois group of $A = \mathbb{Z}[\eta_p, \eta_q]$ over $\mathbb{Z}$ is $(\mathbb{Z}/p)^\times/\langle -1\rangle \times (\mathbb{Z}/q)^\times/\langle -1\rangle$ and since $p \equiv 3$ (4), $(\mathbb{Z}/p)^\times/\langle -1\rangle$ has odd order. Let $G_0 = \mathrm{Gal}(A/\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}]$, and let A_0 be the subring of A fixed by G_0; it is a subring of $\mathbb{Z}[\eta_q]$ in fact. Let

$$\Phi_0\colon A_0^\times \to (A_0/p)^\times_{(2)} \times (A_0/q)^\times_{(2)},$$
$$\varphi_0\colon A_0^\times \to (A_0/4)^\times_{(2)}$$

be the reduction homomorphisms. The elements $(1, -1)$ and $(4, 4)$ considered in Theorem 7.5 lie in $(A_0/p)^\times_{(2)} \times (A_0/q)^\times_{(2)}$ and Condition 7.5(i) is equivalent to the conditions

(a) $(1, -1) \in \mathrm{Image}(\Phi_0 \mid \mathrm{Ker}\,\varphi_0)$,
(b) $(4, 4) \in \mathrm{Image}(\Phi_0 \mid (A_0^\times)^2)$,

because G_0 has odd order. The homomorphism

$$\varphi_0\colon A_0^\times/(A_0^\times)^2 \to (A_0/4)^\times_{(2)} \cong (A_0/2)^+$$

is an isomorphism, cf. Lemma 7.7 below. Hence $\mathrm{Ker}\,\varphi_0 = (A_0^\times)^2$, and it follows that a necessary condition for (a) to be satisfied is that $q \equiv 1$ (4).

Suppose $q \equiv 5$ (8). Then A_0 is a quadratic extension of $\mathbb{Z}$ and

$$(*) \qquad (A_0/p)^\times \times (A_0/q)^\times = \begin{cases} \mathbb{F}_{p^2}^\times \times \mathbb{F}_q^\times, & \text{or} \\ \mathbb{F}_p^\times \times \mathbb{F}_p^\times \times \mathbb{F}_q^\times. \end{cases}$$

In the first case, $\mathbb{F}_{p^2}^\times \times \mathbb{F}_q^\times \supseteq \mathbb{Z}/8 \times \mathbb{Z}/4$ and $(4, 4) = (1, -1)$ in $(\mathbb{F}_{p^2}^\times \times \mathbb{F}_q^\times)_{(2)}$, since 2 is a non-square modulo q. The fundamental unit $\varepsilon \in A_0^\times$ has norm image $-1 \in \mathbb{Z}^\times$, so $\Phi_0(\varepsilon)$ is equal to a pair of generators in $(\mathbb{F}_{p^2}^\times)_{(2)} \times (\mathbb{F}_q^\times)_{(2)}$. It follows that (b) above cannot be satisfied.

In the second case of (*), p has odd order in $(\mathbb{Z}/q)^\times$ so $(A/p)^\times = \mathbb{F}_{p^r}^\times \times \ldots \times \mathbb{F}_{p^r}^\times$ and $(A/q)^\times = \mathbb{F}_{q^s}^\times \times \ldots \times \mathbb{F}_{q^s}^\times$ where r and s are odd. It follows that

$$\Phi_A(\eta_p{}^2\eta_q{}^2) = (4, 4) \quad \text{in } (A/p)^\times_{(2)} \times (A/q)^\times_{(2)},$$

so (b) is satisfied. But in this case, Condition 7.5(iii) is equivalent to

$$\Phi_B(\eta_{pq}) \in \mathrm{Image}(\Phi_A).$$

Let $N\colon A^\times \to A_0^\times$ be the norm homomorphism. It induces a norm

$$N\colon (A/p)^\times_{(2)} \times (A/q)^\times_{(2)} \to (\mathbb{F}_p^\times \times \mathbb{F}_p^\times \times \mathbb{F}_q^\times)_{(2)}$$

and

$$N(\Phi_B(\eta_{pq})) = (N(\eta_q); N(\eta_p)) = (\alpha, -\alpha^{-1}; \pm 1)$$

for some $\alpha \in \mathbb{F}_q^\times$. (This follows because 2 is a non-square mod q, so that η_q has norm image -1 in $\mathbb{Z}^\times$.) On the other hand,

$$(\alpha, -\alpha^{-1}; \pm 1) \notin \mathrm{Image}(\Phi_0)$$

because the fundamental unit $\varepsilon \in A_0^\times$ reduces to $(\alpha, -\alpha^{-1}; \omega)$ with $\omega \in (\mathbb{F}_q^\times)_{(2)}$ a generator. Thus Condition 7.5(iii) is not satisfied in the second case of (*).

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Finally, suppose $q \equiv 1\ (8)$. Let r be an integer which generates $\mathbb{F}_q^\times$ and consider the unit $v_r = \zeta_q^{\,r} - \zeta_q^{\,-r}/\zeta_q - \zeta_q^{\,-1}$ in $A^\times$. Its norm $N(v_r) \in A_0^\times$ reduces to a generator of $(A_0/q)_{(2)}^\times = (\mathbb{F}_q^\times)_{(2)}$. Since both -1 and 2 are squares in $\mathbb{F}_q^\times$, suitable 2-powers U_+ and U_- of $N(v_r)$ have reductions

$$\Phi_0(U_+) = (1, -1), \quad \Phi_0(U_-) = (1, 2).$$

It follows that Condition 7.5 (i) is satisfied and Condition 7.5 (iii) becomes equivalent to $(\frac{1}{2}, 1) \in \mathrm{Image}(\Phi_0)$, which is indeed the case since $\frac{1}{2} = -1$ in $(A_0/p)_{(2)}^\times$.

In the proof above we used the following well-known lemma. For the readers convenience we include a proof of it.

LEMMA 7.7. *Let G_0 be the subgroup of elements of odd order in $(\mathbb{Z}/q)^\times/\langle -1\rangle$ where q is a prime. Let $A_0 = \mathbb{Z}[\eta_q]^{G_0}$. Then $\varphi_0\colon A_0^\times(A_0^\times)^2 \to (A_0/4A_0)_{(2)}^\times = A_0/2A_0$ is an isomorphism.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{F}_q^\times \arrow[dd, "(\ )^{\frac{1}{2}(q-1)}"'] & \mathbb{Z}[\eta_q]^\times \arrow[l] \arrow[r, "\varphi"] \arrow[d, "N_0"] & \mathbb{Z}[\eta_q]/2\mathbb{Z}[\eta_q] \arrow[d, "T_0"] \\
 & A_0^\times \arrow[r, "\varphi_0"] \arrow[d, "N_1"] & A_0/2A_0 \arrow[d, "T_1"] \\
\mathbb{F}_q^\times & \mathbb{Z}^\times \arrow[l, "\rho_1"'] \arrow[r, "\varphi_1"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

where N_0, N_1 indicate norm homomorphisms and T_0, T_1 trace homomorphisms. Choose an integer r which generates $\mathbb{F}_q^\times$. Then $\zeta_q^{\,r} - \zeta_q^{\,-r}/\zeta_q - \zeta_q^{\,-1} \in \mathbb{Z}[\eta_q]^\times$ has norm image -1 in $\mathbb{Z}^\times$, by the left-hand commutative square. It follows that $\varphi_0(N(\zeta_q^{\,r} - \zeta_q^{\,-r}/\zeta_q - \zeta_q^{\,-1}))$ has non-zero trace in $\mathbb{Z}/2$. Since $(A_0/2A_0 : \mathbb{Z}/2\mathbb{Z})$ is a power of 2, $\varphi_0(N(\zeta_q^{\,r} - \zeta_q^{\,-r}/\zeta_q - \zeta_q^{\,-1}))$ is a normal base for $A_0/2A_0$, whence φ_0 is surjective. Since $A_0^\times/(A_0^\times)^2$ and $A_0/2A_0$ have the same rank, the result follows.

The condition of Theorem 7.5 (ii) is somewhat unpleasant, and requires further discussion. The following necessary condition appears in **[15]**.

LEMMA 7.8 (Milgram). *A necessary condition that $Q(8p, q)$ acts freely on S^{8k+3} with k-invariant e_k is that the Legendre symbol $\left(\dfrac{p}{q}\right) = +1$.*

Proof. Suppose $\left(\dfrac{p}{q}\right) = -1$; we prove that Condition 7.5 (ii) cannot be satisfied.

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Consider the diagram

$$\begin{array}{ccccc}
A/2A & \xleftarrow{\varphi_A} & A^\times & \xrightarrow{\rho} & (\mathbb{F}_q\otimes\mathbb{Z}[\eta_p])^\times \\
\downarrow{\scriptstyle T_p} & & \downarrow{\scriptstyle N_p} & & \downarrow{\scriptstyle \bar{N}_p} \\
\mathbb{Z}[\eta_q]/2\mathbb{Z}[\eta_q] & \longleftarrow & \mathbb{Z}[\eta_q] & \longrightarrow & \mathbb{F}_q^\times \\
\downarrow{\scriptstyle T_q} & & \downarrow{\scriptstyle N_q} & & \downarrow{\scriptstyle \bar{N}_q} \\
\mathbb{Z}/2\mathbb{Z} & \xleftarrow{\cong} & \mathbb{Z}^\times & \longrightarrow & \mathbb{F}_q^\times
\end{array}$$

We have $\bar{N}_p(\eta_p-2) = p$ and $\bar{N}_q(p) = p^{\frac{1}{2}(q-1)} = -1$. It follows from the diagram that if $\rho(a) = \eta_p-2$ then $T_q\circ T_p(\varphi_A(a)) \neq 0$, and, in particular, $\varphi_A(a)\neq 0$.

We point out that the Legendre symbol condition in Lemma 7.8 is not equivalent to Condition 7.5 (ii). Consider, for example, the case where $p=5$, $q=19$. We have $\mathbb{Z}[\eta_5] = \mathbb{Z}[\frac{1}{2}(1+\sqrt{5})]$, and

$$2-\eta_5 = \tfrac{1}{2}(5+\sqrt{5}) \quad\text{or}\quad \tfrac{1}{2}(5-\sqrt{5}),$$

$$\mathbb{F}_p\otimes\mathbb{Z}[\eta_q] = \mathbb{F}_{5^q}, \quad \mathbb{F}_q\otimes\mathbb{Z}[\eta_p] = \mathbb{F}_{19}\times\mathbb{F}_{19}.$$

In $\mathbb{F}_{19}$, $(\frac{1}{2}(\sqrt{5}-5), \frac{1}{2}(-\sqrt{5}-5)) = (2-7)$ is a pair of non-squares, and it follows that $\eta_5-2\in(\mathbb{F}_{19}\otimes\mathbb{Z}[\eta_5])^\times$ is not the reduction of an element from

$$\operatorname{Ker}\{\mathbb{Z}[\eta_5]\to\mathbb{Z}[\eta_5]/2\mathbb{Z}[\eta_5]\}$$

and hence is not a reduction of an element from $\operatorname{Ker}\varphi_A$ either.

Recall that $\operatorname{Ord}_p(q)$ denotes the order of q in $\mathbb{F}_p^\times$. On the positive side we have the following.

COROLLARY 7.9. *Suppose $p\equiv 3\ (4)$, $q\equiv 1\ (8)$, and that the orders* $\operatorname{Ord}_p(q)$ *and* $\operatorname{Ord}_q(p)$ *are odd and maximal. Then $Q(8p,q)$ acts freely on S^{8k+3} with k-invariant e_k.*

Proof. Since $\operatorname{Ord}_p(q)$ is an odd number, the quadratic extension $\mathbb{Q}(\eta_p)\subset\mathbb{Q}(\zeta_p)$ is split at each q-adic prime in $\mathbb{Q}(\eta_p)$, whence $\mathbb{F}_q[\eta_p] = \mathbb{F}_q[\zeta_p]$. Since $\eta_p-2 = \zeta_q^{-1}(\zeta_p-1)^2$, it follows that η_p-2 is a square in each field component of $\mathbb{F}_q\otimes\mathbb{Z}[\eta_p]$, whence it is a square in $(\mathbb{F}_q\otimes\mathbb{Z}[\eta_p])^\times$. We also assumed $\operatorname{Ord}_q(p)$ was odd, so η_q-2 is a square in $(\mathbb{F}_p\otimes\mathbb{Z}[\eta_q])^\times$.

We now use the maximality which can be restated as follows: $\operatorname{Ord}_p(q) = \frac{1}{2}(p-1)$ and $\operatorname{Ord}_q(p)$ is the index of the Sylow 2-subgroup in $\mathbb{F}_q^\times$. It follows that $\mathbb{Q}(\eta_p)^{\langle q\rangle} = \mathbb{Q}$ and $(\mathbb{Q}(\eta_q)^{\langle p\rangle}:\mathbb{Q})$ is a power of 2. Let $A_1 = \mathbb{Z}[\eta_q]^{\langle p\rangle}$ and consider the diagram

$$\begin{array}{ccccc}
A/2A & \xleftarrow{\varphi} & A^\times & \xrightarrow{\Phi} & (\mathbb{F}_p\otimes\mathbb{Z}[\eta_q])^\times_{(2)}\times(\mathbb{F}_q\otimes\mathbb{Z}[\eta_p])^\times_{(2)} \\
\downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle N_1} & & \cong\downarrow{\scriptstyle \bar{N}_1} \\
A_1/2A_1 & \xtwoheadleftarrow{\varphi_1} & A_1^\times & \xrightarrow{\Phi_1} & (\mathbb{F}_p\otimes A_1)^\times_{(2)}\times(\mathbb{F}_q^\times)_{(2)}
\end{array}$$

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We note that $\bar{N}_1$ is an isomorphism and that φ_1 induces an isomorphism from $A_1^\times/A_1^{\times 2}$ onto $A_1/2A_1$. We must prove that

$$(\bar{N}_1(\eta_q-2), \bar{N}_1(\eta_p-2)) = (\bar{N}_1(\eta_q-2), p)$$

belongs to Image$(\Phi_1 \mid A_1^{\times 2})$. But $\mathbb{F}_p \otimes A_1 = \mathbb{F}_p \times \ldots \times \mathbb{F}_p$, so the 2-primary component of $(\mathbb{F}_p \otimes A_1)^\times$ is an elementary abelian 2-group. Since $\eta_q - 2$ was a square, $\bar{N}_1(\eta_q-2) = 1$. Since $A_1^\times \to (\mathbb{F}_q^\times)_{(2)}$ is surjective, and $p \in \mathbb{F}_q^\times$ is a square, there exists an $a_1 \in A_1^\times$ with $\Phi_1(a_1{}^2) = (1, p)$. This completes the proof.

The assumptions in Corollary 7.9 are satisfied in the following examples:

$$(p, q) = (3, 313), (7, 809), (11, 1321),$$

and in each case the second component q is minimal for the given p. However, I doubt that the conditions on $\mathrm{Ord}_p(q)$ and $\mathrm{Ord}_q(p)$ in Corollary 7.9 are necessary.

For certain values (p, q) there exist generators $e_k' \in H^{8k+4}(Q(8p, q); \mathbb{Z})$, different from the generators e_k considered above with $\sigma(e_k') = 0$ in $\tilde{K}_0(\mathbb{Z}[Q(8p, q)])$. Such 'exotic' generators do not restrict to Euler classes of free linear representations on the subgroups $Q(8p)$, $Q(8q)$, and $Q(4pq)$. Therefore the methods used above do not apply directly: one may construct the surgery problem over $\Sigma'/Q(8p, q)$ realizing e_k' but cannot get sufficient control of the Reidemeister torsion of the source to carry through the arguments.

In fact, the basic problem occurs for the quaternion group $Q(4p)$. It admits free linear representations in all degrees $8k+4$, but only half the generators in $H^{8k+4}(Q(4p); \mathbb{Z})$ are realized by representations. The Swan homomorphism turns out to be trivial, so all generators have vanishing finiteness obstruction.

The methods used in §4 show that

$$\hat{\rho}\colon L_3^Y(\mathbb{Z}[Q(4p)])(p) \to L_3^Y(\hat{\mathbb{Z}}_2[Q(4p)])(p)$$

is injective, but we need an evaluation of $\hat{\rho}_2(\lambda^Y(f, \hat{f}))$. I shall return to this question in a forthcoming paper.

References

1. S. Bentzen and I. Madsen, 'On the Swan subgroup for certain periodic groups', preprint, Aarhus University, 1982.
2. S. Bentzen, 'On the finiteness obstruction', to appear.
3. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, 1956).
4. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics 10 (Springer, Berlin, 1973).
5. A. Dress, 'Induction and structure theorems for orthogonal representations of finite groups', *Ann. of Math.*, 102 (1975), 291–326.
6. G. Gras and M.-N. Gras, 'Signature des unités cyclotomiques et parité du nombre de classes des extensions cycliques de Q de degré premier impair', *Ann. Inst. Fourier (Grenoble)*, 25 (1975), 1–22.
7. H. Hasse, *Über die Klassenzahl Abelscher Zahlkörper* (Akademic-Verlag, Berlin, 1952).
8. R. Lee, 'Semicharacteristic classes', *Topology*, 12 (1973), 183–199.
9. E. Laitinen and I. Madsen, 'L-theory of groups with periodic cohomology I', preprint, Aarhus University, 1981.
10. I. Madsen, 'Smooth spherical space forms', *Geometric applications of homotopy theory I, Proceedings, Evanston 1977*, Lecture Notes in Mathematics 657 (Springer, Berlin, 1978), pp. 303–353.
11. I. Madsen, 'Spherical space forms in the period dimension I', Topology Conference, Moscow 1979, *Proc. Steklow Inst. Math.*, 154 (1980); preprint, Aarhus University, 1979.
12. I. Madsen, 'Spherical space forms: a survey', *Proceedings of the 18th Scandinavian Congress of Mathematicians* (Birkhäuser, Boston, 1980).
13. I. Madsen, C. B. Thomas, and C. T. C. Wall, 'The topological spherical space form problem', *Topology*, 15 (1978), 375–382.

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14. R. J. MILGRAM, 'Evaluating the Swan finiteness obstruction for periodic groups', preprint, Stanford University, 1979.
15. R. J. MILGRAM, 'Odd index subgroups of units in cyclotomic fields and applications', preprint, Stanford University, 1980.
16. R. J. MILGRAM, 'Patching techniques in surgery and the solution of the compact space form problem', preprint, Stanford University, 1981.
17. J. MILNOR, 'Whitehead torsion', *Bull. Amer. Math. Soc.*, 72 (1966), 358–426.
18. J. MILNOR, 'Groups which act on S^n without fixed points', *Amer. J. Math.*, 79 (1957), 623–630.
19. A. S. MISHCHENKO, 'Homotopy invariants of non-simply connected manifolds III. Higher signatures', *Izv. Akad. Nauk SSSR Ser. Mat.*, 35 (1971), 1316–1355.
20. R. OLIVER, 'SK_1 for finite group rings: III', *Algebraic K-theory: Evanston 1980*, Lecture Notes in Mathematics 854 (Springer, Berlin, 1981).
21. A. A. RANICKI, 'The algebraic theory of surgery: I, II', *Proc. London Math. Soc.* (3), 40 (1980), 87–192, 193–283.
22. A. A. RANICKI, *Exact sequences in the algebraic theory of surgery* (Princeton University Press, 1981).
23. J.-P. SERRE, *Local fields*, Graduate Text in Mathematics (Springer, Berlin, 1979).
24. R. SWAN, 'Periodic resolutions for finite groups', *Ann. of Math.*, 72 (1960), 267–291.
25. R. SWAN, 'Induced representations and projective modules', *Ann. of Math.*, 71 (1960), 552–578.
26. C. B. THOMAS and C. T. C. WALL, 'The topological spherical space form problem I', *Compositio Math.*, 23 (1971), 101–114.
27. C. T. C. WALL, 'Foundations of algebraic L-theory', *Algebraic K-theory III, Battelle Institute Conference 1972*, Lecture Notes in Mathematics 343 (Springer, Berlin, 1973), pp. 266–300.
28. C. T. C. WALL, 'Periodic projective resolutions', *Proc. London Math. Soc.* (3), 39 (1979), 509–553.
29. C. T. C. WALL, 'On the classification of hermitian forms VI: Group rings', *Ann. of Math.*, 103 (1976), 1–80.
30. C. T. C. WALL, 'On the classification of hermitian forms III', *Invent. Math.*, 19 (1973), 59–71.
31. C. T. C. WALL, 'Norms of units in group rings', *Proc. London Math. Soc.* (3), 29 (1974), 593–632.
32. C. T. C. WALL, 'On the classification of hermitian forms IV: Adele rings', *Invent. Math.*, 23 (1974), 241–260.

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