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THE TOPOLOGICAL SPHERICAL SPACE FORM PROBLEM—II EXISTENCE OF FREE ACTIONS

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RECENT advances in calculation of projective class groups and of surgery obstruction groups lead us to hope that it will shortly be possible to give a fairly complete account of the classification of free actions of finite groups on spheres. In the present paper, we determine which groups can so act, thus solving a problem of several years' standing. Further, we show that these actions can be taken to be smooth actions on smooth homotopy spheres.

Previously known results can be summarised as follows, where we say the finite group π satisfies the "pq-condition" (p, q primes not necessarily distinct) if all subgroups if π of order pq are cyclic.

0.1. (Cartan and Eilenberg[3]). If π acts freely on S^{n-1}, it has periodic cohomology with minimum period dividing n. Moreover, π has periodic cohomology if and only if it satisfies all p^2-conditions. And the p^2 condition is equivalent to the Sylow p-subgroup π_p of π being cyclic or perhaps (if $p = 2$) generalised quaternionic.

0.2. (Wolf[19]). If π acts freely and orthogonally on a sphere, it satisfies all pq-conditions. Conversely, if π is soluble and satisfies all pq-conditions, free orthogonal actions exist. However, for π non-soluble, the only non-cyclic composition factor allowed is the simple group of order 60.

0.3. (Milnor[9], see also Lee[8]). If π acts freely on any sphere, it satisfies all $2p$-conditions.

0.4. (Petrie[11]). Any extension of a cyclic group of odd order m by a cyclic group of odd prime order q prime to m can act freely on S^{2q-1}.

Petrie's result shows that pq-conditions are not all necessary for free topological actions. it is therefore not so surprising that

THEOREM 0.5. *A finite group π can act freely on a sphere if and only if it satisfies all $2p$- and p^2-conditions.*

We shall elaborate the group theory in the next section: the most interesting groups π included are perhaps the groups $SL_2(p)$ (p prime).

As to smooth actions, it will follow from a general result below that

THEOREM 0.6. *For each free action of π on S^{n-1} constructed in the proof of* (0.5), *S^{n-1} has a differential structure α such that π acts freely and smoothly on S_α^{n-1}.*

Clearly, in many cases one can deduce existence of free smooth actions on S^{n-1}, but in this paper we will confine ourselves to general arguments.

In principle, the proof of these theorems follows the pattern laid down in [15] and elaborated for this problem in a previous paper[14]. We construct first, a finite simplicial complex X; second, a normal invariant, and hence a normal cobordism class of normal maps $M \to X$; and thirdly we show that the corresponding surgery obstruction vanishes. This yields a manifold homotopy equivalent to X whose universal cover is homotopy equivalent, hence homeomorphic (but not necessarily diffeomorphic, when smooth) to a sphere.

In practice, the key idea of the proof is a careful choice of X, and of the normal invariant, so as to allow a simple proof of vanishing of the suergery obstruction.

The paper is set out in four sections. In the first, we summarise the group theory, and introduce notations for the groups involved. In the second, we choose the homotopy type of X: this involves circumventing the finiteness obstruction of Swan[13], and prepares the way for the surgery. In the third, we discuss normal invariants. General existence of normal invariants follows from the powerful techniques of modern homotopy theory. Topological normal invariants can then be studied using Sullivan's[10] analysis of the homotopy type of G/Top.

In the final section, we first state two general theorems about surgery, and show how to deduce Theorems 0.5 and 0.6 from these, using the preceding material to verify their hypotheses. The key underlying idea for these general theorems is reduction to subgroups. Induction theorems due to Andreas Dress[5] allow us to reduce to hyperelementary subgroups; then we evoke the main techniques of [17], [18] for calculation of surgery obstructions.

§1. GROUP THEORY

Although our arguments will not mainly proceed by lists of cases, it will be necessary to use general information about the structure of groups π with periodic cohomology.

If all Sylow subgroups of π are cyclic—e.g. if π has odd order—π is metacyclic. For the general case, there also exists a classification [19, pp. 179, 195–8]: the non-soluble cases are due to Suzuki[12]. Write $O(\pi)$ for the maximal normal subgroup of π of odd order. Then $\pi/O(\pi)$ has Sylow 2-subgroups isomorphic to those of π, and Sylow p-subgroups cyclic for p odd. This quotient may be of one of six types.

Before listing them, we introduce our notation for isomorphism classes of groups. For a cyclic resp. dihedral resp. generalised quaternion group of order 2^n, write $C(n)$ resp. $D(n)$ resp. $Q(n)$. For tetrahedral and octahedral groups, write $T = T_1$ and $O = O_1$: the corresponding binary groups are denoted $T^* = T_1^*$ and $O^* = O_1^*$. T resp. T^* is an extension of $D(2)$ resp. $Q(3)$ by a cyclic group of order 3; the analogous extension by a cyclic group of order 3^v is T_v resp. T_v^*, and O_v, O_v^* contain these as subgroups of index 2. Explicit generators and relations can be found in Wolf[19], see also [14]. We write $\mathbf{F}_p$ for the field of prime order p, $GL_2(p) = GL_2(\mathbf{F}_p)$ for the general linear group of 2×2 matrices over it, and $SL_2(p)$, $PGL_2(p)$, $PSL_2(p)$ for the corresponding special and projective groups. The map $SL_2(p)\to PSL_2(p)$ has kernel of order 2; we need also a 'double covering' $TL_2(p)\to PGL_2(p)$ defined as follows. Observe that (for p odd) $|PGL_2(p): PSL_2(p)| = 2$: the nontrivial coset is represented by the matrix $y = \begin{pmatrix} 0 & -1 \\ \omega & 0 \end{pmatrix}$ where ω generates $\mathbf{F}_p^\times$. Define

$$TL_2(p) = \langle SL_2(p), Y \,|\, Y^2 = -I,\ Y^{-1}gY = y^{-1}gy \text{ for } g \in SL_2(p)\rangle.$$

Now let π have periodic cohomology and satisfy all $2p$-conditions. Then $\pi/O(\pi)$ belongs to one of the following isomorphism classes: I. $C(n)$; II. $Q(n)$ $(n \geq 3)$; III. T^*; IV. O^*; V. $SL_2(p)$; VI. $TL_2(p)$. Here, p denotes a prime greater than 3: observe that $SL_2(3) = T^*$ and $TL_2(3) = O^*$. In cases. I, II, V and VI, the extension π of $O(\pi)$ splits. In cases III, IV, π is a split extension of a normal subgroup of order prime to 2 and 3 by some T_v^*, O_v^*. We denote by τ such a splitting subgroup (all cases).

Finally, we consider the subgroup structure in the non-soluble cases V and VI. According to Dickson[4], $PSL_2(p)$ has subgroups T. The Sylow 2-subgroup is dihedral, so has two conjugacy classes of four groups $D(2)$ in it, as has $PSL_2(p)$ (except when $p \equiv \pm 3 \pmod 8$, and $D(2)$ is the Sylow 2-subgroup); all these $D(2)$ become conjugate in $PGL_2(p)$.

Thus all subgroups $D(2)$ of $PSL_2(p)$ are contained in subgroups T. Now $PGL_2(p)$ also has dihedral Sylow 2-subgroups, and two conjugacy classes of subgroups $D(2)$. Those in one class are contained in the commutator subgroup $PSL_2(p)$, and hence in a subgroup T; those in the other class cannot be, for $D(2)$ is the commutator subgroup of T.

The kernel of $TL_2(p)\to PGL_2(p)$ is the unique element of order 2. Hence the subgroups $Q(3)$ resp. T^* of $TL_2(p)$ are just the preimages of the subgroups $D(2)$ resp. T of $PGL_2(p)$. Now as τ contains a Sylow 2-subgroup of π, it follows that

LEMMA 1.1. *For any π of type V, all subgroups $Q(3)$ are contained in subgroups T^*. For π of type* VI, *there are two conjugacy classes of subgroups $Q(3)$; those of one class are contained in subgroups T^*, and those of the other class are not.*

§2. CHOOSING A HOMOTOPY TYPE

We first recall [14, Theorem 2.2]. A CW-complex Y, dominated by a finite complex, is (π, n)-polarised if we are given an isomorphism $\pi_1(Y, y_0)\to \pi$ and a homotopy equivalence of the universal cover $\tilde{Y}\to S^{n-1}$ $(n \geq 3)$. The equivalence classes of Y, provided with such polarisations, correspond bijectively (via the first k-invariant) with generators g of $H^n(\pi; \mathbf{Z})$.

There is an obstruction $\theta(g)$ in the projective class group $\tilde{K}_0(\mathbf{Z}\pi)$ to the existence of a finite complex homotopy equivalent to Y. However, by the main theorem of Swan[13], it is possible to choose g so that $\theta(g)=0$ and hence Y is a finite CW-complex.

The following sharpening of this assertion will be crucial for our argument.

LEMMA 2.1. *There exists a finite (π, n)-polarised complex $Y = Y(\pi)$ such that for each $\rho \subset \pi$ which has a fixed-point-free orthogonal representation, the covering space $Y(\rho)$ of Y corresponding to ρ is homotopy equivalent to a manifold.*

Proof. By the result of Swan quoted above, there exists a finite (π, n_1)-polarised complex Y_1, say, corresponding to $g_1 \in H^{n_1}(\pi \cdot \mathbf{Z})$. There is a natural free action of π on the universal covering $\tilde{Y}_1$, and a homotopy equivalence $\tilde{Y}_1 \to S^{n_1-1}$. If we form the joint of a certain number t of copies of $\tilde{Y}_1$, this is homotopy equivalent to a sphere S^{tn_1-1}, and we inherit a free action of π on it. Observe also that the orbit space of this action has first k-invariant $g_1{}^t$.

Now let r denote the exponent of the multiplicative group of units of $\mathbf{Z}/N$ (N = order of π), and take $t = r$ above; we write $g = g_1{}^r$, $n = rn_1$ and Y for the corresponding orbit space. Since π acts cellularly on the join, Y is a finite CW-complex. Any other generator of $H^{n_1}(\pi;\mathbf{Z})$ is of the form ug_1 (u a unit of $\mathbf{Z}/N$), and $(ug_1)^r = u^r g_1{}^r = g_1{}^r = g$. Thus g is the only generator of $H^n(\pi;\mathbf{Z})$ which is an rth power. Also, if $\rho \subset \pi$ has order $M|N$, then $\mathbf{Z}/N$ maps onto $\mathbf{Z}/M$, so any unit of $\mathbf{Z}/M$ has order diving r. The same argument then shows that the restriction of g to ρ is the only generator of $H^n(\rho;\mathbf{Z})$ which is an rth power.

Now consider subgroups ρ of π which have fixed-point-free orthogonal actions on spheres. Since there are only a finite number of such subgroups ρ we may suppose (replacing g_1 by a power if necessary) that each one has a fixed-point-free orthogonal action on S^{n_1-1}, arising from a representation χ_ρ, and generator g_ρ, say, in $H^{n_1}(\rho;\mathbf{Z})$. The direct sum $r\chi_\rho$ of r copies of this representation corresponds to the join construction above. The corresponding action of ρ on S^{n-1} is smooth, so has orbit space a smooth manifold $Z(\rho)$. By the remarks in the preceding paragraph, the corresponding generator of $H^n(\rho;\mathbf{Z})$ is $g_\rho{}^r$, the restrictions of g to ρ. Hence $Z(\rho)$ is homotopy equivalent to $Y(\rho)$.

Remark. The above is vague as to the possible dimensions n. A better estimate can be obtained as follows. Let g_0 be a generator having the minimal possible period n_0, and corresponding to Y_0. Each ρ as above then has a generator of dimension n_0; checking the cases in Wolf's list, one can show (see our next paper) that it has a fixed-point-free representation of degree $2n_0$. Now take $g_1 = g_0{}^2$ in the above. All details are as stated, except that Y_0 need not be homotopy equivalent to a finite complex . However, (again see our next paper) θ is multiplicative on generators. Thus $\theta(g_1{}^r) = r\theta(g_1)$. But according to Swan[13], $\theta(g_1)$ has order dividing r. Thus $\theta(g)=0$ and we can take Y finite. This gives $n = 2rn_0$. It is not difficult to improve this in special cases: the best value in general is probably $2n_0$ (or n_0). Note for later reference that n is even: in fact this holds whenever π, or order greater than 2, has period n.

§3. NORMAL INVARIANTS

We will now establish the general existence of smooth normal invariants for our complexes, thus improving on the results of [14]. We then discuss in further detail the relation between topological normal invariants of $Y(\pi)$ and of $Y(\pi_2)$.

A (π, n)-polarised complex Y is a Poincaré complex in the sense of [15], hence has a 'Spivak normal bundle', classified by a map $Y \to BG$. A topological (resp. smooth) normal invariant is a homotopy class of liftings of this map to $B|\mathrm{Top}$ (resp. BO). If Y is a manifold, its tangent bundle provides such a lifting.

THEOREM 3.1. *Any finite (π, n)-polarised complex Y has a smooth normal invariant.*

Proof. Since the map $BO \to BG$ is a map of infinite loop spaces[2], the obstruction to existence of a smooth normal invariant is an element of $k^*(Y)$, where k is the cohomology theory represented by $B(G/O)$. Since this group is finitely generated, it will suffice to show that the obstruction vanishes when localised at any given prime, p.

We now compare the obstruction for $Y = Y(\pi)$ with that for the covering space $Y(\pi_p)$

corresponding to a Sylow p-subgroup π_p of π; using the generalised transfer due to Kahn and Priddy[7] (see also Becker and Gottlieb[1]).

Recall that for any finite covering $f: \tilde{X} \to X$ and cohomology theory h^*, there is a transfer $f_*: h^*(\tilde{X}) \to h^*(X)$ induced by an S-map $X^+ \to \tilde{X}^+$. If f is a k-fold cover and $k \in h^*$ (point) is invertible, then $f_* \circ f^*$: $h^*(X) \to h^*(X)$ is an isomorphism. For there is an induced endomorphism of the Atiyah–Hirzebruch spectral sequence $H^*(X; h^*(\text{point})) \Rightarrow h^*(X)$, given on the E^2-term by multiplication by k. In particular, f^* is then injective.

Now the inclusion $i: \pi_p \subset \pi$ induces a covering $Y(\pi_p) \to Y(\pi)$ which is compatible with Spivak tangent bundles. Thus $i^*: k^*(Y(\pi)) \to k^*(Y(\pi_p))$ maps the obstruction to existence of a smooth normal invariant for $Y(\pi)$ to that for $Y(\pi_p)$. Since the degree $|\pi: \pi_p|$ of the covering is prime to p, it follows from the above that i^* becomes injective when localised at p. Thus the proof will be concluded if we show that each $Y(\pi_p)$ has a smooth normal invariant.

If π_p is cyclic, it is well known that $Y(\pi_p)$ is homotopy equivalent to a (smooth) lens space: indeed, there is a unique one of the form $L(p^r; q, 1, \dots, 1)$. Otherwise, $p = 2$ and π_2 is generalised quaternionic, of order 2^r, say, $(r \geq 3)$. For $Y(\pi_2)$ of fixed formal dimension, the polarised homotopy types correspond to odd integers l mod 2^r. For $l = \pm 1$ (mod 8), we can again find corresponding smooth actions, coming from fixed-point-free orthogonal representations. For $l \equiv \pm 3$ (mod 8), on the other hand, $Y(\pi_2)$ (and hence $Y(\pi)$) is not homotopy equivalent to a finite complex: see, for example, [6].

It is probably the case that smooth normal invariants exist also in the case last-mentioned (certainly topological ones do), but the point is not material to our subsequent deductions.

In order to make calculations in the final section, we need to choose a normal invariant for $Y(\pi)$ so as to have some control over its restriction to $Y(\pi_2)$. We first recall the results of the previous paper[14]. We showed there that if π is soluble (i.e. in Cases I–IV) it contains a subgroup τ (the same as that mentioned above) such that

(A i). τ contains a Sylow 2-subgroup of π.

(A ii). The only prime divisors of $|\tau|$ are 2 and 3.

(A iii). The restriction homomorphism $H^2(\pi; \mathbf{Z}/2) \to H^2(\tau; \mathbf{Z}/2)$ is an isomorphism.

(A iv). There exist fixed point free orthogonal actions of π on spheres.

This formulation differs slightly from [14, 3.7], however it corresponds more closely to what was proved. We can also find such a τ for some non-soluble groups: if π contains $SL_2(p)$, we need $p \equiv \pm 3$ (mod 8) and then choose a subgroup T^* (type V) or O^* (type VI).

Now if τ satisfies (Ai)–(Aiii), the map $f: Y(\tau) \to Y(\pi)$ satisfies

(B i). For all k, f^*: $H^k(Y(\pi); \pi_{k-1}(G/\mathrm{Top})) \to H^k(Y(\tau); \pi_{k-1}(G/\mathrm{Top}))$ is injective.

(B ii). For all k, f^*: $H^k(Y(\pi); \pi_k(G/\mathrm{Top})) \to H^k(Y(\tau); \pi_k(G/\mathrm{Top}))$ is surjective.

Hence, by an obstruction theory argument (given in [14]), follows

LEMMA 3.2. *Any (topological) normal invariant for $Y(\tau)$ extends to one for $Y(\pi)$.*

For the remaining (non-soluble) groups, we cannot manage with a single subgroup τ, but will need instead a small diagram of subgroups.

LEMMA 3.3. *A (topological) normal invariant c for $Y(\pi_2)$ extends to one of $Y(\pi)$ if and only if for each subgroup $Q(3)$ of π_2 contained in a subgroup T^* of π, the restriction $c|Y(Q(3))$ extends to $Y(T^*)$.*

Proof. Taking a fixed normal invariant for $Y(\pi)$, and its restrictions to $Y(\rho)$ for $\rho \subset \pi$, as basepoint, we can identify normal invariants of $Y(\rho)$ with homotopy classes of maps $Y(\rho) \to G/\mathrm{Top}$. The problem is thus to characterise the maps $Y(\pi_2) \to G/\mathrm{Top}$ which factor through $Y(\pi)$. The conditions stated are clearly necessary.

Write $G/\mathrm{Top} \simeq U \times V$, where U has nonvanishing homotopy groups only in dimensions $4k$; V only in dimensions $4k+2$. This is possible by [10]. Since $\pi_{4k}(U) \cong \mathbf{Z}$, and $H^{4k}(\pi; \mathbf{Z})$ maps onto $H^{4k}(\pi_2; \mathbf{Z})$ (to see this, it is enough to localise at 2: but then it follows since both π and π_2 have 2-period 4), any map $Y(\pi_2) \to U$ factors through $Y(\pi)$. It thus suffices to discuss $V = \Pi_k K(\mathbf{Z}/2, 4k+2)$.

The problem is thus reduced to characterising the image of $H^{4k+2}(\pi; \mathbf{Z}/2) \to H^{4k+2}(\pi_2; \mathbf{Z}_2)$. This is done in general in [3, 10.1]; here we look for something simpler. On account of periodicity, it suffices to consider the case $k = 0$. Since, for $\rho \subset \pi$, $H^3(\rho; \mathbf{Z}) = 0$, mod 2 reduction

$H^2(\rho;\mathbf{Z})\to H^2(\rho;\mathbf{Z}/2)$ is surjective. For quaternion 2-groups (e.g. π_2) it is bijective. Thus it suffices to consider $H^2(\rho;\mathbf{Z})$.

Now for any finite ρ we have isomorphisms $H^2(\rho;\mathbf{Z})\cong H^1(\rho;\mathbf{Q}/\mathbf{Z})\cong \mathrm{Hom}\,(\rho,\mathbf{Q}/\mathbf{Z})$, natural for restriction to subgroups.

In particular, $H^2(T^*;\mathbf{Z})\cong\mathbf{Z}/3$, with zero localisation at 2. Thus an element of $H^2(Q(3);\mathbf{Z})$ extends to T^* only if it vanishes.

Our subgroup π_2 is quaternionic: let

$$\pi_2=\langle x,y\,|\,x^{2^{t+1}}=1,\quad y^2=x^{2^t},\quad y^{-1}xy=x^{-1}\rangle=Q(t+2)$$

be a presentation. If $t\geq 2$, there are two conjugacy classes of subgroups $Q(3)$, respresented by $\langle x^{2^{t-1}},y\rangle$ and $\langle x^{2^{t-1}},xy\rangle$. The group $H^2(\pi_2;\mathbf{Z})\cong\mathbf{Z}/2\times\mathbf{Z}/2$, with generators corresponding to the homomorphisms h_1, h_2, where

$$h_1(x)=\frac{1}{2},\qquad h_1(y)=0,$$
$$h_2(x)=0,\qquad h_2(y)=\frac{1}{2}.$$

For $t\geq 2$, under the first inclusion of $Q(3)$, $h_1\to 0$ and $h_2\to\eta\neq 0$; under the second, both h_1 and h_2 restrict to η. Thus a class in $H^2(\pi_2;\mathbf{Z})$ vanishes if and only if it restricts to zero on both subgroups $Q(3)$.

If now π has type V, its commutator subgroup π' has odd index, so $H^2(\pi;\mathbf{Z})$ has odd order, and zero image in $H^2(\pi_2;\mathbf{Z})$. But we have just seen that an element of $H^2(\pi_2;\mathbf{Z})$ is 0 precisely when all its restrictions to subgroups $Q(3)$ are 0, i.e. extend to T^*.

If π has type VI, $\pi'\cap\pi_2$ has index 2 in π_2; we can choose x and y so that it is generated by x^2 and y. Then our first subgroup $Q(3)$ lies in a group $T^*\subset\pi$; the second does not. The image of $H^2(\pi;\mathbf{Z})$ in $H^2(\pi_2;\mathbf{Z})$ is the subgroup $\{0,h_1\}$, kernel of the first restriction to $Q(3)$. Hence the result follows in this case also.

§4. SURGERY

The arguments to be used for proving our theorems have more general validity. We will begin by stating two general results, deduce Theorems 0.5 and 0.6 from these, and finally give the proofs of the general theorems.

THEOREM 4.1. *Let $\phi: M\to Y$ be a normal map of degree* 1; *M a closed manifold, Y a finite Poincaré complex of formal dimension at least 5 with finite fundamental group π. Then surgery on ϕ to obtain a homotopy equivalence is possible if and only if*

(a) *For each 2-hyperelementary subgroup $\rho\subset\pi$, the covering space $Y(\rho)$ is homotopy equivalent to a manifold,*

(b) *Surgery is possible for the covering normal map $\tilde{\phi}: X(\pi_2)\to Y(\pi_2)$, π_2 the Sylow 2-subgroup of π,*

and, if dim *M is even.*

(c) *The equivariant signature of M is a multiple of the regular representation of π.*

For the smooth case, we have

THEOREM 4.2. *Let X be a closed topological manifold of odd dimension with finite fundamental group π. Suppose*

(i) *X has a smooth normal invariant, and*

(ii) *the covering space $X(\pi_2)$ is smoothable.*

Then X is homotopy equivalent to a smooth manifold.

Proof. of Theorem 0.5. First, suppose π soluble. Choose Y as in Lemma 2.1. The subgroup τ of Lemma 3.2 has (by A iv)(a fixed-point-free orthogonal representation, and by (2.1) one such yields $Z(\tau)\simeq Y(\tau)$. By Lemma 3.2, the normal invariant defined by $Z(\tau)$ extends to a normal invariant for $Y=Y(\pi)$. We will now apply Theorem 4.1.

To verify (a) observe that any $\rho\subset\pi$ which is 2-hyperelementary is soluble. Every subgroup of ρ of odd order is cyclic, and (by our main hypothesis on π) every subgroup of order $2p$ is cyclic. By (0.2), ρ is homotopy equivalent to a manifold.

As to (b), by construction surgery on the covering map corresponding to τ yields a homotopy equivalence $Z(\tau)\to Y(\tau)$. Since—by (A i)—τ contains a Sylow 2-subgroup π_2, the same is true for the covering corresponding to π_2.

Now suppose π insoluble. Again choose Y as in (2.1). Then $Y(\pi_2)\simeq X(\pi_2)$, coming from a fixed point free orthogonal representation χ of π_2. We claim that the corresponding normal invariant of $Y(\pi_2)$ extends to $Y(\pi)$. For by Lemma 3.3, it suffices to check extensibility when a subgroup $Q(3)$ of π_2 lies in a subgroup T^* of π. The restriction of χ to $Q(3)$ must be a multiple of the unique irreducible fixed-point-free representation. But this extends to a fixed-point-free representation χ^* of T^* and by the choice of Y in (2.1), the covering $Y(T^*)$ is homotopy equivalent to the quotient space of χ^*, which thus defines an extension of the normal invariant. Thus our claim is justified.

The proof is now concluded exactly as in the soluble case, but using π_2 in place of τ.

Proof of Theorem 0.6. We have constructed a free action of π on S^{n-1} with quotient space X, say; it will suffice to prove X homotopy equivalent to a smooth manifold. We seek to apply Theorem 4.2. But the existence of a smooth normal invariant is guaranteed by Theorem 3.1. Since the normal invariant of X was obtained by extending the normal invariant of $Z(\tau)$ (using (3.2)) or of $Z(\pi_2)$ (using (3.3)) which came from a fixed-point-free representation, its restriction to $X(\pi_2)$ is a smooth normal invariant, so $X(\pi_2)$ is smoothable.

We now come to the proofs of (4.1) and (4.2). Although motivated by the rest of the paper, these depend on techniques drawn from elsewhere: in particular, we quote three results from earlier papers. For ease of reference, we use the intermediate L-groups of those papers: other possibilities are discussed in a concluding remark.

LEMMA 4.3. [17, 2.4]. *Let ρ be p-hyperelementary with p odd. Then $\rho=\rho_1\times\sigma$, where ρ_1 has odd order and σ is a cyclic 2-group. We have $L_i(\rho_1\times\sigma)=L_i(\sigma)\oplus\tilde{L}_i(\mathbf{R}(\rho_1\times\sigma))$, where the first summand is mapped in by the inclusion. The second summand vanishes for i odd: for i even it is free abelian, and detected by signature.*

LEMMA 4.4. [16, 7.3 and 7.4]. *For π finite, $L_i(\pi)$ is finitely generated. The torsion subgroup has exponent dividing* 8. *The free part is detected by signatures (and vanishes for i odd).*

Proposition 4.5. [18, Theorem 12]. *Let $\phi: M\to V$ be a normal map between closed manifolds with finite fundamental group π. Surgery is possible on ϕ if and only if it is possible for the covering space with fundamental group the Sylow* 2-*subgroup π_2.*

Proof of Theorem 4.1. The stated conditions are clearly necessary for surgery to be possible: for (c) this follows from the extension [15, 14B2] of the Atiyah–Singer theorem to topological manifolds. For the converse, we first note that by [15, 3.2] there is a single obstruction $\Sigma(\pi)$ in $L_i(\pi)$ to performing the surgery. According to Dress [5, Theorem 1] the natural restriction map $L_i(\pi)\to\Sigma\{L_i(\rho):\rho\subset\pi \text{ hyperelementary}\}$ is injective. Now the restriction $\Sigma(\pi)$ to ρ is precisely the obstruction to surgery for the covering space with fundamental group ρ. So it suffices to show that each $\Sigma(\rho)$ vanishes.

First suppose ρ p-hyperelementary with p odd. Then we apply Lemma 4.3. By hypothesis (c), the free part of $\Sigma(\pi)$ vanishes if and only if the ordinary signature does, which is the case (if applicable) by (b). So we can ignore the second summand. The first component of $\Sigma(\rho)$ is, however, $\Sigma(\sigma)$. For, since σ has odd index in ρ and the torsion subgroup of $L_i(\sigma)$ has exponent 2 (or 1), this follows by a simple calculation, or by using the transfer as in the proof of the next theorem.

Now suppose ρ 2-hyperelementary, with Sylow 2-subgroup σ. By hypothesis (a), the surgery obstruction $\Sigma(\rho)$ is that for a map between two closed manifolds. Then by Proposition (4.5), $\Sigma(\rho)$ vanishes if and only if $\Sigma(\sigma)$ does.

By hypothesis (b), the surgery obstruction $\Sigma(\sigma)$ vanishes if σ is a Sylow 2-subgroup of π, and hence for any 2-subgroup (contained in one of these). The result now follows by combining the last four paragraphs.

Proof of Theorem 4.2. This argument will be based on the transfer techniques already used in the proof of Theorem 3.1. The stated conditions are clearly necessary; suppose them satisfied. If $\dim X=1$ or 3, X is smoothable, so we may assume $\dim X\geqslant 5$ and use surgical techniques.

Using the given smooth normal invariant of X as base-point, we can identify the set $[X: G/O]$

of homotopy classes of maps $X \to G/O$ with the set of all smooth normal invariants, and have compatible identifications of $[X: G/\mathrm{Top}]$ with all normal invariants; likewise for the covering space $X(\pi_2)$. The inclusion $i: \pi_2 \subset \pi$ induces a covering $X(\pi_2) \to X(\pi)$ and hence a map i^* and transfer i_* of cohomology theories. The map $f: G/O \to G/\mathrm{Top}$ which forgets the smooth structure is a map of infinite loop spaces[2], so the induced map of cohomology theories commutes with the transfer:

$$\begin{array}{ccccc} [X(\pi_2): G/O] & \xrightarrow{i_*} & [X: G/O] & \xrightarrow{i^*} & [X(\pi_2): G/O] \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ [X(\pi_2): G/\mathrm{Top}] & \xrightarrow{i_*} & [X: G/\mathrm{Top}] & \xrightarrow{i^*} & [X(\pi_2): G/\mathrm{Top}]. \end{array}$$

Now the given manifold X has normal invariant $\alpha \in [X: G/\mathrm{Top}]$. Since its covering space $X(\pi_2)$ is smoothable, this has a normal invariant say $\beta \in [X(\pi_2): G/O]$ with $f_*\beta = i^*\alpha$. For any integer m, $mi_*\beta$ determines a smooth normal invariant for X. We seek to choose m so that the corresponding surgery obstruction vanishes. By Proposition 4.5, this is the case if and only if the surgery obstruction for $i^*(mi_*\beta)$ vanishes.

Since smoothness is irrelevant to surgery obstructions, this is the same as the surgery obstruction for $f_*i^*(mi_*\beta) = mi^*i_*f_*\beta = mi^*i_*i^*\alpha$. Now by [10, §8], G/Top, localised at 2, is a product of Eulenberg-MacLane spaces. Thus i_*i^*, as for cohomology, coincides with multiplication by the degree $|\pi: \pi_2|$ of the covering. We now re-choose our base point for normal invariants of $X(\pi_2)$ so as to start from $f_*\beta$, representing the given smooth manifold. Thus we have $(m|\pi: \pi_2| - 1)f_*\beta v$ (perhaps modulo odd torsion).

Now by Lemma 4.4, $L_{2k+1}(\pi_2)$ has exponent 8. Thus we can ignore odd torsion, and indeed work modulo 8. But [18, Theorem 3] gives a formula for the surgery obstruction mod 8 whose dependence on $[X(\pi_2): G/\mathrm{Top}]$ is linear in the characteristic classes induced from $k \in H^{4*+2}(G/\mathrm{Top}; \mathbf{Z}/2)$ and $l \in H^{4*}(G/\mathrm{Top}; \mathbf{Z}/8)$. Both these classes are primitive for l by [10, Theorem 8.8]; the result for k is well-known, e.g. [15, 13B5]. Thus the obstruction depends additively on the normal invariant. It thus suffices to choose m so that $m|\pi: \pi_2| - 1$ is divisible by 8. A suitable choice is $m = |\pi; \pi_2|$.

Remarks. In the even-dimensional case, we know again that the torsion-free part of the obstruction boils down to the usual signature, so if M is nonorientable, or if the dimension is $\equiv 2$ (mod 4), the argument remains valid. In the remaining case, the conclusion seems unlikely to hold in general.

The crucial point of this argument—linearity of the surgery obstruction—depends essentially on the torsion of $L_*(\pi_2)$ having exponent 8. This holds, in fact, for all the intermediate L-groups (including L^s and L^h) since in this case torsion in the intermediate group of [17] has exponent 4. This follows from [17, 5.2.2] and the following argument. $L_i^K(\hat{\mathbf{Z}}_2\pi_2)$ has order 2, so that nonzero element here is the only one which can produce an element of $L_i^Y(\mathbf{Z}_2\pi_2)$ or order 4. But for i odd, this lifts to the 'surgery' element in $L_i^Y(\hat{\mathbf{Z}}\pi_2)$ which maps to 0 in $L_i'(\pi_2)$; and for $i \equiv 2$ (mod 4), it corresponds to the Kervaire–Arf invariant: a direct summand.

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