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TOPOLOGICAL CONTINUED FRACTIONS

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1. Introduction

The purpose of this paper is to prove that each integral solution of a unimodular Diophantine equation can not have an arbitrary number of terms in its negative continued fraction expansion, but in fact, the number of terms must satisfy a certain divisibility condition. The proof is based upon the introduction of quadratic forms into continued fractions in the spirit utilized by Milnor [4] for the construction of two (unexpected) distinct differential structures on the same topological manifold. The results may be further extended to Pontryagin type of divisibility conditions similar to the presentation of Adam [1] on the calculation of the co-kernel of J, and the formulation of K_2 by Bass [2] and Tate [7], if a more general setting is desired.

2. Statement of Results

Number theory enters topology in a number of ways. For instance, one of the most fundamental number theoretic concepts is that of the bilinear form. In topology, this notion makes itself known by considering the cup product on the middle dimensional cohomology of an even dimensional oriented manifold. This bilinear form is non-degenerate on the free part of the cohomology, and if the dimension of the manifold is divisible by four, this form is symmetric and one can study its associated quadratic form. The following theorem of Milnor [5],

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which is employed later, was used to give some algebraic perspective of this topological quadratic form.

A symmetric integral valued bilinear form (,) on a free abelian group A is called a type I form if there is an x in A such that $(x, x) \not\equiv 0 \mod 2$. Otherwise, the form is called a type II form. Two quadratic forms belong to the same genus if they are equivalent over the p-adic integers and the real numbers.

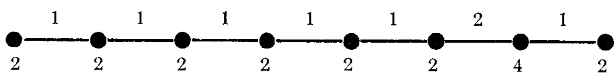
THEOREM 1. *The rank r, index s, and the type (I or II) form a complete system of invariants for the genus of an integral quadratic form with determinant* ± 1. *A given rank, index, and type actually occur if and only if the following conditions are satisfied:*

i) *r and s are integers with*

$$-r \leqslant s \leqslant r, \quad s \equiv r \mod 2$$

ii) *for forms of type II,* $s \equiv 0 \mod 8$
iii) *for forms of type I,* $r > 0$.

The quadratic form

1) 

shows that ii) in the above result is sharp i. e., q (x) is of type II, rank 8, determinant 1 and index 8. To give some idea of the effects of the topology, the next theorem [6] shows that the differential structure plays a non-trivial role in the related algebra of the manifold.

THEOREM 2. *If the quadratic form arising from the middle dimensional cohomology of a simply-connected smooth four dimensional manifold is of type II, then its index must be divisible by 16.*

A curious consequence of these two theorems is that if there is a topological manifold whose index is divisible by eight and not sixteen, then this would be a manifold which did not possess any type of differentiable structure. For further results concerning these ideas, see [3].

In this paper, the above procedure is reversed, that is a purely number theoretic result is proven which arose from topological considerations, namely, the plumbing of lens spaces.

If $[p_0, p_1, \ldots, p_n]$ are the terms in the expansion of c/d in a negative continued fraction of two coprime natural numbers, then it shall be assumed that $p_n > 1$. The result to be proved in this paper is:

THEOREM 3. *Let c, d be a positive integral solution of the Diophantine equation*

2) $$ax - b^2y = 1$$

where a and b are integers with $a \equiv 0 \bmod 2$. *Denote by* $[p_0, p_1, \ldots, p_n]$, *the terms in the expansion of* c/d *as a negative continued fraction. If every* p_i, $i = 0, \ldots, n$, *is even, then n must be of the form* $8k + 6$ *for some natural number k. Equivalently, if n is not congruent to six modulo eight, then some* p_i *is odd.*

This rather peculiar anomaly on the number of terms in a continued fraction expansion can be employed in addition to Pell's equation as follows.

COROLLARY 4. *Let c, d be a solution of*

3) $$2x^2 - Ny^2 = 1$$

where c, d, N are natural numbers. If the terms p_i, $i = 0, \ldots, n$, *in the expansion of* c/N *as a negative continued fraction are all even, then*

4) $$n \equiv 6 \text{ modulo } 8.$$

REMARK 5. It is clear from the proof of Theorem 3, that the right hand side of 2) and 3) may be replaced by -1 under appropriate assumptions.

EXAMPLE 6. Consider the equation

5) $$8x - 9y = 1.$$

A solution is given by the numbers 8 and 7. The terms in the negative continued fraction expansion of 8/7 are [2,2,2,2,2,2,2].

3. Proofs

Proof of COROLLARY 4. If c, d is a solution of 3), then c, N is also a solution of

$$6) \qquad 2\,cx - d^2\,y = 1.$$

However, this is the form of 2) and so Theorem 3 is applicable.

Proof of THEOREM 3. Let P_m/Q_m $m = 0, 1, \ldots, n$, denote the principal convergents of c/d with $P_n/Q_n = c/d$. Then P_m/Q_m can be written as the quotient of two continuants:

$$7) \quad P_m / Q_m = \frac{\begin{vmatrix} p_0 & 1 & & \\ 1 & p_1 & & \\ & & \ddots & 1 \\ & & 1 & p_m \end{vmatrix}_{m+1}}{\begin{vmatrix} p_1 & 1 & & \\ 1 & p_2 & & \\ & & \ddots & 1 \\ & & 1 & p_m \end{vmatrix}_{m}} = \frac{|\,p_1, p_2, \ldots, p_m\,|}{|\,p_0, p_1, \ldots, p_m\,|}.$$

It follows by induction that

$$8) \qquad P_m = |\,p_0, p_1, \ldots, p_m\,| > 1$$

for $m > 0$.

Consider the quadratic form defined by

$$9) \quad q\,(x) = \sum_{m=0}^{n} p_m\,x_m^2 + \sum_{m=0}^{n-1} 2\,x_m\,x_{m+1} + a x_{n+1}^2 + 2\,b x_0\,x_{n+1}.$$

This is an integral form of type II and by 2), its determinant is equal to one. Moreover, q (x) is equivalent (over the rationals) to the following form:

$$10) \qquad p(y) = \sum_{m=0}^{n+1} \frac{y_m^2}{P_{m-1} P_m}$$

where y_m is a linear form in the variables x_m, and P_{-1}, P_{n+1} are defined to be 1. By 8) above, the P_m are all positive and hence, the index of q (x) is n + 1. Therefore, by Theorem 1, n + 2 must be congruent to zero modulo eight. This concludes the proof.

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