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# CONSTRUCTION OF UNIVERSAL MATRIX LOCALIZATIONS

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Given a collection $\Sigma$ of square matrices over a ring R, the universal $\Sigma$-inverting homomorphism $\lambda: R \to R_\Sigma$ is the universal homomorphism carrying the elements of $\Sigma$ to invertible matrices. This has been considered by P.M. Cohn and others. It is generally constructed by generators and relations, which method gives little insight into (for example) the kernel of $\lambda$. In this article I propose another construction of $\lambda: R \to R_\Sigma$ under a mild closure condition on $\Sigma$. Some information about $\lambda$ may be derived, depending on how matrices in $\Sigma$ can be factored.

In the first part of the article we present the definitions and results, together with some explanatory material. The proofs are relegated to the second part.

## The Statements

Let R be an associative ring with unit (which is preserved by ring homomorphisms). An R-ring will mean a ring homomorphism from R to some other such ring. These objects form a category with morphisms being ring homomorphisms which make the obvious triangular diagrams commutative.

For $\Sigma$ a collection of square matrices over R, an R-ring $\phi: R \to S$ is said to be $\Sigma$-inverting if the image under $\phi$ of every element of $\Sigma$ is invertible over S. A $\Sigma$-inverting R-ring is universal if it factors uniquely through any $\Sigma$-inverting R-ring. Such an object is unique up to (unique) isomorphism of R-rings.

These definitions are from Cohn ([1], Chap. 7) in which the universal $\Sigma$-inverting ring is constructed using generators and relations. Cohn discusses the conditions under which $R_\Sigma$ is a local ring, leading to the definition of a "prime matrix ideal." The author has used so called "zigzag" methods to obtain similar results ([2]) and these methods will again be used in the present effort.

To describe this method, let us assume first that the collection $\Sigma$ of square matrices satisfies the following two conditions: 1) the $1 \times 1$ identity matrix is in $\Sigma$, and 2) if A and B are in $\Sigma$ and if C is of the appropriate size, then $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is in $\Sigma$. Under such conditions $\Sigma$ is called <u>multiplicative</u>.

When $\Sigma$ is multiplicative, Cohn has shown that every element of $R_\Sigma$ is an entry in the inverse of the image in $R_\Sigma$ of some element of $\Sigma$. Thus every element of $R_\Sigma$ is of the form $\lambda(f)\lambda(A)^{-1}\lambda(x)$, where $A \in \Sigma$ (say $n \times n$), f is $1 \times n$ and x is $n \times 1$, all over R. The basis of the zigzag method is to construct $R_\Sigma$ as a set of equivalence classes of such triples (f, A, x). The equivalence class of (f, A, x) is thus to be interpreted as the element $fA^{-1}x$ of $R_\Sigma$, with addition and multiplication defined according to that interpretation.

To this end, assume $\Sigma$ is multiplicative set of square matrices over R and let $T_\Sigma$ consist of all triples (f, A, x), where $A \in \Sigma$ and where (letting A be $n \times n$) f is $1 \times n$ and x is $n \times 1$, both over R. We will say that "f is a row the size of A" to describe this sort of shape, and similarly for the "column" x. Other elements of $T_\Sigma$ will be denoted by (g, B, y), (h, C, z), etc.

Define a relation ~ among elements of $T_\Sigma$ by (f, A, x) ~ (g, B, y) if there exist L, M, P, $Q \in \Sigma$, rows j and u the sizes of L and P, respectively, and columns w and v the sizes of M and Q, respectively, such that

$R_\Sigma/_R$ = { (M = coker(·), $\varphi \in M^\wedge \otimes_A M$) }/~

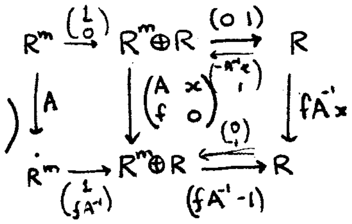
(M, φ) ~ 0 if φ = 0

(M, φ) ~ 0 if ∃ $0 \to L \xrightarrow{i} M \xrightarrow{j} N \to 0$

$\theta \in N^\wedge \otimes_A L \xmapsto{j^\wedge \otimes i} \varphi \in M^\wedge \otimes_A M$

$$\left(\begin{array}{c|c} m\times m & m\times 1 \\ \hline 1\times m & 1\times 1 \end{array}\right) = \begin{pmatrix} m\times m \\ 1\times m \end{pmatrix}\left(m\times m \mid m\times 1\right)$$






$$\left(\begin{array}{cccc|c} A & 0 & 0 & 0 & x \\ 0 & B & 0 & 0 & -y \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & M & w \\ \hline f & g & j & 0 & 0 \end{array}\right) = \left(\begin{array}{c} P \\ \hline u \end{array}\right)\left(Q \mid v\right)$$

Thus PQ is a block-diagonal matrix; we have written 0 for zero blocks, rows and columns as necessary.

Following our interpretation of (f, A, x) as $fA^{-1}x$, we can see why this might be the correct definition (though not why it is so complicated) as follows: If all elements of $\Sigma$ are invertible, then

$$\begin{aligned} 0 = uv &= uQ(PQ)^{-1}Pv \\ &= fA^{-1}x - gB^{-1}y + jL^{-1}0 + 0M^{-1}w \\ &= fA^{-1}x - gB^{-1}y . \end{aligned}$$

Thus $fA^{-1}x$ should be the same as $gB^{-1}y$.

<u>LEMMA 1</u>. The relation ~ is an equivalence relation.

Let $R_\Sigma$ denote the set of equivalence classes $T_\Sigma/\sim$, and denote the equivalence class containing (f, A, x) by $(f/A\backslash x)$, reminding us of $fA^{-1}x$. Again following our interpretation, we are led to the appropriate definitions of operations in $R_\Sigma$ (as in [2]). For $(f/A\backslash x), (g/B\backslash y) \in R_\Sigma$, define

$$(f/A\backslash x) + (g/B\backslash y) = \left((f\ g)\Big/\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\Big\backslash\begin{pmatrix} x \\ y \end{pmatrix}\right),$$

$$(f/A\backslash x) \cdot (g/B\backslash y) = \left((f\ 0)\Big/\begin{pmatrix} A & -xg \\ 0 & B \end{pmatrix}\Big\backslash\begin{pmatrix} 0 \\ y \end{pmatrix}\right),$$

$$-(f/A\backslash x) = (f/A\backslash -x) .$$

Also define a map $\lambda : R \to R_\Sigma$ by $\lambda(r) = (1/1\backslash r)$. All these make sense because $\Sigma$ is multiplicative.

THEOREM. The above definitions give $R_\Sigma$ a well-defined structure of associative ring with unit. Further, the map $\lambda : R \to R_\Sigma$ is the universal $\Sigma$-inverting R-ring, and each element $(f/A\backslash x)$ of $R_\Sigma$ satisfies

$$(f/A\backslash x) = \lambda(f)\,\lambda(A)^{-1}\,\lambda(x) .$$

Corollary. An element $r \in R$ is in the kernel of $\lambda$ if and only if there exist $L, M, P, Q \in \Sigma$, rows $j$ and $u$ the sizes of $L$ and $P$, respectively, and columns $w$ and $v$ the sizes of $M$ and $Q$, respectively, such that

$$\left(\begin{array}{cc|c} L & 0 & 0 \\ 0 & M & w \\ \hline j & 0 & r \end{array}\right) = \left(\begin{array}{c} P \\ \hline u \end{array}\right) \left(\begin{array}{c|c} Q & v \end{array}\right).$$

The Proofs.

The proofs that follow will primarily be complicated factorizations of block matrices, as suggested by the definition of the equivalence relation. To make these easier to read, zeros will be replaced by dots and the matrices corresponding to $L$ and $M$ in the definition of $\sim$ will be outlined. Thus the factorization in the definition would be written:

$$(*) \qquad \left(\begin{array}{cccc|c} A & \cdot & \cdot & \cdot & x \\ \cdot & B & \cdot & \cdot & -y \\ \cdot & \cdot & \boxed{L} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \boxed{M} & w \\ \hline f & g & j & \cdot & \cdot \end{array}\right) = \left(\begin{array}{c} P \\ \hline u \end{array}\right) \left(\begin{array}{c|c} Q & v \end{array}\right).$$

We will also denote by $I$ the identity matrix and by $E_i$ the row (or column) block matrix which is zero in each block except for an identity matrix in the i-th block. The size and shape of these matrices will be indicated by context. For example, if $P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is a block matrix, then $E_2P = (0\ B)$ and $E_1E_2P = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ .



As an example of the techniques we will use, let us show that if $L$ or $M$ is a "null matrix" (i.e. does not appear) in a factorization (*), then there is a similar one in which they do appear.

Proposition 1. If $(f, A, x)$ , $(g, B, y)$ in $T_\Sigma$ are such that there is a factorization of any of these forms (with $L, M, P, Q \in \Sigma$):

(a) $$\left(\begin{array}{cc|c} A & \cdot & x \\ \cdot & B & -y \\ \hline f & g & \cdot \end{array}\right) = \left(\begin{array}{c} P \\ \hline u \end{array}\right) \left(\begin{array}{c|c} Q & v \end{array}\right) ;$$

(b) $$\left(\begin{array}{ccc|c} A & \cdot & \cdot & x \\ \cdot & B & \cdot & -y \\ \cdot & \cdot & L & \cdot \\ \hline f & g & j & \cdot \end{array}\right) = \left(\begin{array}{c} P \\ \hline u \end{array}\right) \left(\begin{array}{c|c} Q & v \end{array}\right) ;$$

(c) $$\left(\begin{array}{ccc|c} A & \cdot & \cdot & x \\ \cdot & B & \cdot & -y \\ \cdot & \cdot & M & w \\ \hline f & g & \cdot & \cdot \end{array}\right) = \left(\begin{array}{c} P \\ \hline u \end{array}\right) \left(\begin{array}{c|c} Q & v \end{array}\right) ;$$

then there is a factorization of the form (*) .

Proof:

(a) $$\left(\begin{array}{cccc|c} A & \cdot & \cdot & \cdot & x \\ \cdot & B & \cdot & \cdot & -y \\ \cdot & \cdot & \boxed{1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \boxed{1} & 1 \\ \hline f & g & 1 & \cdot & \cdot \end{array}\right) = \left(\begin{array}{ccc} P & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \hline u & 1 & \cdot \end{array}\right) \left(\begin{array}{ccc|c} Q & \cdot & \cdot & v \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \end{array}\right) ;$$

(b) $$\left(\begin{array}{cccc|c} A & \cdot & \cdot & \cdot & x \\ \cdot & B & \cdot & \cdot & -y \\ \cdot & \cdot & \boxed{L} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \boxed{1} & 1 \\ \hline f & g & j & \cdot & \cdot \end{array}\right) = \left(\begin{array}{cc} P & \cdot \\ \cdot & 1 \\ \hline u & \cdot \end{array}\right) \left(\begin{array}{cc|c} Q & \cdot & v \\ \cdot & 1 & 1 \end{array}\right) ;$$

(c) $$\left(\begin{array}{cccc|c} A & \cdot & \cdot & \cdot & x \\ \cdot & B & \cdot & \cdot & -y \\ \cdot & \cdot & \boxed{M} & \cdot & \cdot \\ \cdot & \cdot & \cdot & M & w \\ \hline f & g & \cdot & \cdot & \cdot \end{array}\right) = \left(\begin{array}{cc} P & -E_3 \\ \cdot & I \\ \hline u & \cdot \end{array}\right) \left(\begin{array}{cc|c} Q & QE_3 & v \\ \cdot & M & w \end{array}\right) ,$$

where the last equation follows because

$$PQE_3 = E_3M = \begin{pmatrix} 0 \\ 0 \\ M \end{pmatrix} ,$$

$$uQE_3 = (f\ g\ 0)E_3 = 0 \text{ , etc.}$$

We remark that many of the proofs to follow could be simplified if $\Sigma$ was assumed to be closed under multiplication by matrices invertible over $R$ . We proceed for the more general $\Sigma$ to improve the applicability of the results.

<u>Proof of Lemma 1</u>: For $(f, A, x) \in T_\Sigma$ , the factorization below (with null $L$, $M$) proves $\sim$ is reflexive:

$$\left(\begin{array}{cc|c} A & \cdot & x \\ \cdot & A & -x \\ \hline f & f & \cdot \end{array}\right) = \left(\begin{array}{cc} A & -I \\ \cdot & I \\ \hline f & \cdot \end{array}\right) \left(\begin{array}{cc|c} I & I & 0 \\ \cdot & A & -x \end{array}\right) .$$

Now assume $(f, A, x) \sim (g, B, y)$ via tha factorization (*). Symmetry for $\sim$ is given by the following factorization:

$$\left(\begin{array}{cccccc|c} B & \cdot & \cdot & \cdot & \cdot & \cdot & y \\ \cdot & A & \cdot & \cdot & \cdot & \cdot & -x \\ \cdot & \cdot & B & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & L & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & M & \cdot & -w \\ \cdot & \cdot & \cdot & \cdot & \cdot & B & y \\ \hline g & f & \cdot & j & \cdot & \cdot & \cdot \end{array}\right) = \left(\begin{array}{ccc} B & E_2P & \cdot \\ \cdot & P & -E_2 \\ \cdot & \cdot & I \\ \hline g & u & \cdot \end{array}\right) \left(\begin{array}{ccc|c} I & -E_2 & -I & \cdot \\ \cdot & Q & QE_2 & -v \\ \cdot & \cdot & B & y \end{array}\right) .$$



For transitivity, assume $(f, A, x) \sim (g, B, y)$ via (*) and also assume that $(g, B, y) \sim (h, C, z)$ via the following factorization (with $L'$, $M'$, $P'$, $Q' \in \Sigma$):

$$\left(\begin{array}{cccc|c} B & \cdot & \cdot & \cdot & y \\ \cdot & C & \cdot & \cdot & -z \\ \cdot & \cdot & \boxed{L'} & \cdot & \cdot \\ \cdot & \cdot & \cdot & M' & w' \\ \hline g & h & j' & \cdot & \cdot \end{array}\right) = \left(\frac{P'}{u'}\right) \left(Q' \mid v'\right) .$$

Then $(h, C, z) \sim (f, A, x)$ is justified by the factorization of the matrix

$$\left(\begin{array}{ccccccccccc|c} C & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z \\ \cdot & A & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x \\ \cdot & \cdot & B & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & L & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & M & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & L' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B & \cdot & \cdot & \cdot & \cdot & -y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C & \cdot & \cdot & \cdot & z \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & L' & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M' & \cdot & -w' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M & w \\ \hline h & f & g & j & \cdot & j' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

into the product

$$\left(\begin{array}{ccccc} C & \cdot & \cdot & E_2P' & \cdot \\ \cdot & P & \cdot & E_2E_1P' & E_4 \\ \cdot & \cdot & L' & E_3P' & \cdot \\ \cdot & \cdot & \cdot & P' & \cdot \\ \cdot & \cdot & \cdot & \cdot & I \\ \hline h & u & j' & u' & \cdot \end{array}\right) \qquad \left(\begin{array}{ccccc|c} I & \cdot & \cdot & -E_2 & \cdot & \cdot \\ \cdot & Q & \cdot & -QE_2E_1 & -QE_4 & -v \\ \cdot & \cdot & I & -E_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & Q' & \cdot & -v' \\ \cdot & \cdot & \cdot & \cdot & M & w \end{array}\right) .$$

Proof of Theorem. First we prove that the operations are well-defined. Suppose that $(f, A, x) \sim (g, B, y)$ via (*); we wish to show first that $(f/A\backslash x) + (h/C\backslash z) = (g/B\backslash y) + (h/C\backslash z)$ . According to the definition of addition above, this equation is justified by the factorization of

$$\left(\begin{array}{ccccccccc|c} A & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x \\ \cdot & C & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z \\ \cdot & \cdot & B & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -y \\ \cdot & \cdot & \cdot & C & \cdot & \cdot & \cdot & \cdot & \cdot & -z \\ \cdot & \cdot & \cdot & \cdot & \boxed{L} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & A & \cdot & \cdot & \cdot & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B & \cdot & \cdot & -y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & L & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M & w \\ \hline f & h & g & h & j & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

into the product

$$\left(\begin{array}{cccccc} A & \cdot & \cdot & \cdot & \cdot & E_1P \\ \cdot & C & \cdot & -I & \cdot & \cdot \\ \cdot & \cdot & B & \cdot & \cdot & E_2P \\ \cdot & \cdot & \cdot & I & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & L & E_3P \\ \cdot & \cdot & \cdot & \cdot & \cdot & P \\ \hline f & h & g & \cdot & j & u \end{array}\right) \qquad \left(\begin{array}{cccccc|c} I & \cdot & \cdot & \cdot & \cdot & -E_1 & \cdot \\ \cdot & I & \cdot & I & \cdot & \cdot & \cdot \\ \cdot & \cdot & I & \cdot & \cdot & -E_2 & \cdot \\ \cdot & \cdot & \cdot & C & \cdot & \cdot & -z \\ \cdot & \cdot & \cdot & \cdot & I & -E_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & Q & v \end{array}\right) .$$



This shows addition on the left is well-defined. For addition on the right the factorization is similar; or we may refer to commutativity, below. Under the same equivalence (*), we get $-(f/A\backslash x) = -(g/B\backslash y)$ by simply changing the sign of $v$ and $w$ in (*).

To show multiplication is well-defined, again assume $(f/A\backslash x) = (g/B\backslash y)$ via (*). To show $(f/A\backslash x) \cdot (h/C\backslash z) = (g/B\backslash y) \cdot (h/C\backslash z)$ we use the factorization of

$$\left(\begin{array}{cccccccccc|c} A & -xh & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & C & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z \\ \cdot & \cdot & B & -yh & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & C & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -z \\ \cdot & \cdot & \cdot & \cdot & \boxed{L} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & A & \cdot & \cdot & \cdot & -xh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B & \cdot & \cdot & yh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & L & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M & -wh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C & z \\ \hline f & \cdot & g & \cdot & j & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

into the product

$$\left(\begin{array}{ccccccc} A & -xh & \cdot & \cdot & \cdot & E_1P & \cdot \\ \cdot & C & \cdot & \cdot & \cdot & \cdot & I \\ \cdot & \cdot & B & -yh & \cdot & E_2P & \cdot \\ \cdot & \cdot & \cdot & C & \cdot & \cdot & -I \\ \cdot & \cdot & \cdot & \cdot & L & E_3P & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & P & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I \\ \hline f & \cdot & g & \cdot & j & u & \cdot \end{array}\right) \qquad \left(\begin{array}{ccccccc|c} I & \cdot & \cdot & \cdot & \cdot & -E_1 & \cdot & \cdot \\ \cdot & I & \cdot & \cdot & \cdot & \cdot & -I & \cdot \\ \cdot & \cdot & I & \cdot & \cdot & -E_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & I & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & \cdot & I & -E_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & Q & -vh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C & z \end{array}\right) .$$

To show $(h/C\backslash z) \cdot (f/A\backslash x) = (h/C\backslash z) \cdot (g/B\backslash y)$ we use the factorization of

$$\left(\begin{array}{ccccccccccc|c}
C & -zf & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & A & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x \\
\cdot & \cdot & C & -zg & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & B & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -y \\
\cdot & \cdot & \cdot & \cdot & C & -zj & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & L & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C & zf & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A & \cdot & \cdot & \cdot & x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B & \cdot & \cdot & -y \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & L & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & M & w \\
\hline
h & \cdot & h & \cdot & h & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

into the product

$$\left(\begin{array}{cccccccc}
C & -zf & \cdot & \cdot & \cdot & \cdot & -I & \cdot \\
\cdot & A & \cdot & \cdot & \cdot & \cdot & \cdot & E_1P \\
\cdot & \cdot & C & -zg & -I & \cdot & I & -zu \\
\cdot & \cdot & \cdot & B & \cdot & \cdot & \cdot & E_2P \\
\cdot & \cdot & \cdot & \cdot & I & -zj & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & L & \cdot & E_3P \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & P \\
\hline
h & \cdot & h & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
\left(\begin{array}{cccccccc|c}
I & \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot \\
\cdot & I & \cdot & \cdot & \cdot & \cdot & \cdot & -E_1 & \cdot \\
\cdot & \cdot & I & \cdot & I & \cdot & -I & \cdot & \cdot \\
\cdot & \cdot & \cdot & I & \cdot & \cdot & \cdot & -E_2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & C & \cdot & \cdot & -zjE_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot & -E_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C & zfE_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Q & v
\end{array}\right).$$

The various identities for an associative ring with unit will be verified below with null $L$ and $M$. We remark that the zero and unit are $\lambda(0)$ and $\lambda(1)$, respectively. For commutativity of addition, $(f/A\backslash x) + (g/B\backslash y) = (g/B\backslash y) + (f/A\backslash x)$ by the following factorization:



$$\left(\begin{array}{cccc|c}
A & \cdot & \cdot & \cdot & x \\
\cdot & B & \cdot & \cdot & y \\
\cdot & \cdot & B & \cdot & -y \\
\cdot & \cdot & \cdot & A & -x \\
\hline
f & g & g & f & \cdot
\end{array}\right) = \left(\begin{array}{cccc}
A & \cdot & \cdot & -I \\
\cdot & B & -I & \cdot \\
\cdot & \cdot & I & \cdot \\
\cdot & \cdot & \cdot & I \\
\hline
f & g & \cdot & \cdot
\end{array}\right) \left(\begin{array}{cccc|c}
I & \cdot & \cdot & I & \cdot \\
\cdot & I & I & \cdot & \cdot \\
\cdot & \cdot & B & \cdot & -y \\
\cdot & \cdot & \cdot & A & -x
\end{array}\right).$$

Associativity for both addition and multiplication follow from the reflexivity of $\sim$, since the two sides of the equation desired turn out to be identical.

To check that $\lambda(0)$ is an identity for addition requires $(f/A\backslash x) + (1/1\backslash 0) = (f/A\backslash x)$, which is verified by the following factorization:

$$\left(\begin{array}{ccc|c}
A & \cdot & \cdot & x \\
\cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & A & -x \\
\hline
f & 1 & f & \cdot
\end{array}\right) = \left(\begin{array}{ccc}
A & \cdot & -I \\
\cdot & 1 & \cdot \\
\cdot & \cdot & I \\
\hline
f & 1 & 0
\end{array}\right) \left(\begin{array}{ccc|c}
I & \cdot & I & \cdot \\
\cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & A & -x
\end{array}\right).$$

For $-(f/A\backslash x)$ to give an additive inverse requires $(f/A\backslash x) + (f/A\backslash -x) = (1/1\backslash 0)$, as verified by the following factorization:

$$\left(\begin{array}{ccc|c}
A & \cdot & \cdot & x \\
\cdot & A & \cdot & -x \\
\cdot & \cdot & 1 & \cdot \\
\hline
f & f & 1 & \cdot
\end{array}\right) = \left(\begin{array}{ccc}
A & -I & \cdot \\
\cdot & I & \cdot \\
\cdot & \cdot & 1 \\
\hline
f & \cdot & 1
\end{array}\right) \left(\begin{array}{ccc|c}
I & I & \cdot & \cdot \\
\cdot & A & \cdot & -x \\
\cdot & \cdot & 1 & \cdot
\end{array}\right).$$

Verification of distributivity requires larger matrices; to check $(f/A\backslash x) \cdot (h/C\backslash z) + (g/B\backslash y) \cdot (h/C\backslash z) = [(f/A\backslash x) + (g/B\backslash y)] \cdot (h/C\backslash z)$ requires the following factorization:

$$\left(\begin{array}{ccccccc|c} A & -xh & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & C & \cdot & \cdot & \cdot & \cdot & \cdot & z \\ \cdot & \cdot & B & -yh & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & C & \cdot & \cdot & \cdot & z \\ \cdot & \cdot & \cdot & \cdot & A & \cdot & -xh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & B & -yh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C & -z \\ \hline f & \cdot & g & \cdot & f & g & \cdot & \cdot \end{array}\right) =$$

$$\left(\begin{array}{ccccccc} A & -xh & \cdot & \cdot & -I & \cdot & \cdot \\ \cdot & C & \cdot & \cdot & \cdot & \cdot & -I \\ \cdot & \cdot & B & -yh & \cdot & -I & \cdot \\ \cdot & \cdot & \cdot & C & \cdot & \cdot & -I \\ \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I \\ \hline f & \cdot & g & \cdot & \cdot & \cdot & \cdot \end{array}\right) \left(\begin{array}{ccccccc|c} I & \cdot & \cdot & \cdot & I & \cdot & \cdot & \cdot \\ \cdot & I & \cdot & \cdot & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & I & \cdot & \cdot & I & \cdot & \cdot \\ \cdot & \cdot & \cdot & I & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & \cdot & A & \cdot & -xh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & B & -yh & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C & -z \end{array}\right) .$$

For the reverse, $(h/C\backslash z) \cdot [(f/A\backslash x) + (g/B\backslash y)] = (h/C\backslash z) \cdot (f/A\backslash x) + (h/C\backslash z) \cdot (g/B\backslash y)$ requires the factorization:

$$\left(\begin{array}{ccccccc|c} C & -zf & -zg & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & A & \cdot & \cdot & \cdot & \cdot & \cdot & x \\ \cdot & \cdot & B & \cdot & \cdot & \cdot & \cdot & y \\ \cdot & \cdot & \cdot & C & -zf & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & A & \cdot & \cdot & -x \\ \cdot & \cdot & \cdot & \cdot & \cdot & C & -zg & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B & -y \\ \hline h & \cdot & \cdot & h & \cdot & h & \cdot & \cdot \end{array}\right)$$



$$= \left(\begin{array}{ccccccc} C & -zf & -zg & -I & \cdot & -I & \cdot \\ \cdot & A & \cdot & \cdot & -I & \cdot & \cdot \\ \cdot & \cdot & B & \cdot & \cdot & \cdot & -I \\ \cdot & \cdot & \cdot & I & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & I & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I \\ \hline h & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right) \left(\begin{array}{ccccccc|c} I & \cdot & \cdot & I & \cdot & I & \cdot & \cdot \\ \cdot & I & \cdot & \cdot & I & \cdot & \cdot & \cdot \\ \cdot & \cdot & I & \cdot & \cdot & \cdot & I & \cdot \\ \cdot & \cdot & \cdot & C & -zf & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & A & \cdot & \cdot & -x \\ \cdot & \cdot & \cdot & \cdot & \cdot & C & -zg & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & B & -y \end{array}\right) .$$

The proofs that $\lambda$ is a homomorphism and that $\lambda(1)$ acts as a unit element are subsumed in the following:

Lemma 2. The following equations hold in $R_\Sigma$ :

(i) $(f_1/A\backslash x) + (f_2/A\backslash x) = (f_1 + f_2/A\backslash x)$ ;

(i') $(f/A\backslash x_1) + (f/A\backslash x_2) = (f/A\backslash x_1 + x_2)$ ;

(ii) $\lambda(r) \cdot (f/A\backslash x) = (rf/A\backslash x)$ ;

(ii') $(f/A\backslash x) \cdot \lambda(r) = (f/A\backslash xr)$ .

Proof: The statements are successively justified by the following factorizations:

(i) $$\left(\begin{array}{ccc|c} A & \cdot & \cdot & x \\ \cdot & A & \cdot & x \\ \cdot & \cdot & A & -x \\ \hline f_1 & f_2 & f_1+f_2 & \cdot \end{array}\right) = \left(\begin{array}{ccc} A & \cdot & -I \\ \cdot & A & -I \\ \cdot & \cdot & I \\ \hline f_1 & f_2 & \cdot \end{array}\right) \left(\begin{array}{ccc|c} I & \cdot & I & \cdot \\ \cdot & I & I & \cdot \\ \cdot & \cdot & A & -x \end{array}\right) ;$$

(i') $$\left(\begin{array}{ccc|c} A & \cdot & \cdot & x_1 \\ \cdot & A & \cdot & x_2 \\ \cdot & \cdot & A & -x_1-x_2 \\ \hline f & f & f & \cdot \end{array}\right) = \left(\begin{array}{ccc} A & -I & -I \\ \cdot & I & \cdot \\ \cdot & \cdot & I \\ \hline f & \cdot & \cdot \end{array}\right) \left(\begin{array}{ccc|c} I & I & I & \cdot \\ \cdot & A & \cdot & x_2 \\ \cdot & \cdot & A & -x_1-x_2 \end{array}\right) ;$$

(ii) $$\left(\begin{array}{ccc|c} 1 & -rf & \cdot & \cdot \\ \cdot & A & \cdot & x \\ \cdot & \cdot & A & -x \\ \hline 1 & \cdot & rf & \cdot \end{array}\right) = \left(\begin{array}{ccc} 1 & -rf & \cdot \\ \cdot & A & -I \\ \cdot & \cdot & I \\ \hline 1 & \cdot & \cdot \end{array}\right) \left(\begin{array}{ccc|c} 1 & \cdot & rf & \cdot \\ \cdot & I & I & \cdot \\ \cdot & \cdot & A & -x \end{array}\right) ;$$

(ii')
$$\left(\begin{array}{ccc|c} A & -x & \cdot & \cdot \\ \cdot & 1 & \cdot & r \\ \cdot & \cdot & A & -xr \\ \hline f & \cdot & f & \cdot \end{array}\right) = \left(\begin{array}{ccc} A & -x & -I \\ \cdot & 1 & \cdot \\ \cdot & \cdot & I \\ \hline f & \cdot & \cdot \end{array}\right) \left(\begin{array}{ccc|c} I & \cdot & I & \cdot \\ \cdot & 1 & \cdot & r \\ \cdot & \cdot & A & -xr \end{array}\right) .$$

To show that the homomorphism $\lambda : R \to R_\Sigma$ is $\Sigma$-inverting, let $A$ be an arbitrary matrix in $\Sigma$. We claim that the $(i,j)$-entry of $\lambda(A)^{-1}$ is $(E_i/A\backslash E_j)$, where here $E_i$ and $E_j$ denote a row and column respectively. To verify the claim on one side, we will show that $\sum_i \lambda(E_k A E_i)(E_i/A\backslash E_j) = \delta_{kj}$, the Kronecker delta. Using Lemma 2 successively, what we need to show is $(E_k A/A\backslash E_j) = \lambda(\delta_{kj})$. This is proved by the factorization

$$\left(\begin{array}{cc|c} A & \cdot & E_j \\ \cdot & 1 & -\delta_{kj} \\ \hline E_k A & 1 & \cdot \end{array}\right) = \left(\begin{array}{cc} I & \cdot \\ \cdot & 1 \\ \hline E_k & 1 \end{array}\right) \left(\begin{array}{cc|c} A & \cdot & E_j \\ \cdot & 1 & -\delta_{kj} \end{array}\right) .$$

A similar factorization proves that $(E_j/A\backslash E_j)$ works as a left inverse for $\lambda(A)$.

Further applications of Lemma 2 show that $(f/A\backslash x) = \lambda(f)\lambda(A)^{-1}\lambda(x)$. Now given a $\Sigma$-inverting $R$-ring $\varphi : R \to S$, we may define $\varphi^+ : R_\Sigma \to S$ by $\varphi^+(f/A\backslash x) = \varphi(f)\,\varphi(A)^{-1}\,\varphi(x)$. This is well-defined by the computation preceding the statement of Lemma 1, and it is easy to check that $\varphi^+$ is a homomorphism satisfying $\varphi^+\lambda = \varphi$. Furthermore, $\varphi^+$ is the unique such homomorphism, since the inverse of $\varphi(A)$ is uniquely determined by $\varphi(A)$. Thus $\lambda : R \to R_\Sigma$ is the universal $\Sigma$-inverting $R$-ring.

Proof of Corollary: If $\lambda(r) = 0$, then there is a factorization as follows:



$$\left(\begin{array}{cccc|c} 1 & \cdot & \cdot & \cdot & r \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & L' & \cdot & \cdot \\ \cdot & \cdot & \cdot & M' & w' \\ \hline 1 & 1 & j' & \cdot & \cdot \end{array}\right) = \left(\frac{P'}{u'}\right) \left(Q' \mid v'\right) ,$$

where $L', M', P', Q' \in \Sigma$, etc. Then the following factorization

$$\left(\begin{array}{ccccc|c} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & L' & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & M' & \cdot & -w' \\ \cdot & \cdot & \cdot & \cdot & 1 & r \\ \hline 1 & 1 & j' & \cdot & \cdot & r \end{array}\right) = \left(\begin{array}{cc} P' & E_1 \\ \cdot & 1 \\ \hline u' & 1 \end{array}\right) \left(\begin{array}{cc|c} Q' & -Q'E_1 & -v' \\ \cdot & 1 & r \end{array}\right)$$

allows us to put $j = (1\ 1\ j')$, etc. Conversely if there is a factorization as in the Corollary then $\lambda(r) = 0$ follows from the factorization:

$$\left(\begin{array}{cccc|c} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & -r \\ \cdot & \cdot & L & \cdot & \cdot \\ \cdot & \cdot & \cdot & M & w \\ \hline 1 & 1 & j & \cdot & \cdot \end{array}\right) = \left(\begin{array}{ccc} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & P \\ \hline 1 & 1 & u \end{array}\right) \left(\begin{array}{ccc|c} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & -r \\ \cdot & \cdot & Q & v \end{array}\right) .$$

## References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