Algebraic Closure Operators and Constructions in Ring Theory

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ABSTRACT

ALGEBRAIC CLOSURE OPERATORS AND CONSTRUCTIONS IN RING THEORY

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Suppose $R \longrightarrow S$ is a homomorphism of associative rings (with unit). For X a subset of a finitely generated free right R-module R^n, we define the S-closure of X, denoted $C_S(X) \subseteq R^n$, to be the set of elements of R^n whose images in S^n are S-linear combinations of images of elements of X. We study the inverse problem of constructing in a natural way a ring S and such a homomorphism, from an abstract algebraic closure operator C on finitely generated free R-modules. In fact, it turns out that what we want are such closure operators on <u>both</u> free right <u>and</u> left R-modules, satisfying a certain coherence condition between the two. Then we can construct a ring $\overline{R}$, which we call the <u>zigzag localization</u> of R with respect to the coherent pair of closure operators, and a homomorphism $R \longrightarrow \overline{R}$.

In fact, closure operators may be defined on ideals of any additive category (generalizing the category of finitely generated free R-modules), and the zigzag localization (a new additive category) can be constructed in the same way.

Given a closure operator on <u>right</u> ideals, we construct a new one (the "reflection") on <u>left</u> ideals which satisfies

the coherence condition; however, the construction is somewhat unnatural.

When both closure operators are S-closures derived from a homomorphism R⟶S as above, then the ring $\bar{R}$ is isomorphic to the dominion of R in S as defined by Isbell. When the right closure operator is an S-closure and the left one is the reflection, then our zigzag localization ring $\bar{R}$ may be bigger than the dominion.

When the right closure operator satisfies a certain "exchange" condition, then the constructed ring $\bar{R}$ is a division ring. This parallels P. M. Cohn's construction of a division ring from a "prime matrix ideal". A new direct form of the latter construction will also be given.

The zigzag localization may also be carried out in the additive category of all right R-modules. When the right closure operator is given by a torsion theory (as studied by Lambek and others), then the zigzag localization yields the ring of quotients with respect to that torsion theory.

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FOREWORD

In the presentation of this dissertation, I have made an effort to be self-contained and to assume no more background of the reader than a one-year graduate course in algebra. However, it seems that such efforts are generally doomed to substantial failure, especially in the area of providing motivation. The interested reader may wish to consult Zariski and Samuel ([10]) on commutative localization. Also, Cohn ([2]) and Lambek ([4]) may help to provide some of the motivation for considering kinds of noncommutative localizations.

During the preparation of the dissertation, my advisor George Bergman has been exceptionally generous with his time and judgment; I gratefully acknowledge the value of his guidance, especially his suggestions for how to generalize appropriately. I also wish to thank all of the people who tolerated my groaning and raving throughout these trying times.

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CHAPTER ONE

INTRODUCTION

Suppose we are given a fixed ring R (all our rings are associative with a unit element which is preserved by ring homomorphisms). Then given a ring homomorphism $R \longrightarrow S$, we can try to find information on R which is sufficient to determine the given homomorphism. More specifically, we may want to find enough such "R-information" to allow us to construct the ring S and the homomorphism.

For example, one kind of such R-information is a two-sided ideal I of R. This information is derived from a ring homomorphism $R \longrightarrow S$ by taking the kernel. It is "informative" enough to construct the factor ring R/I and the natural map $R \longrightarrow R/I$.

Another kind of R-information is a particular choice of a subsemigroup M of the multiplicative semigroup of R. Given $R \longrightarrow S$, we can get such an M by including all elements of R whose images in S are units. If R is commutative, then the subsemigroup M is enough to construct the localization $R \longrightarrow R_M$ of R with respect to M.

Now let F be a finitely generated free right R-module, and regard the elements of F as column vectors over R. Then given a ring homomorphism $R \longrightarrow S$ and a subset $X \subseteq F$, define $C_S(X)$ to be the set of elements of F whose images under the homomorphism (regarded as column vectors in the appropriate free S-module) are S-linear combinations of the images of the elements of X. Then this "S-closure operator" C_S gives another form of R-information.

In this paper we consider when this R-information determines the homomorphism $R \longrightarrow S$ and when such a homomorphism may be constructed from an abstract "closure operator". Ideas for this construction and for the axiomatization of the notion of closure operator come from some known constructions of division rings.

We find that construction of a ring $\bar{R}$ and homomorphism $R \longrightarrow \bar{R}$ may be made if we are given a closure operator on free left R-modules as well as one on free right R-modules, provided a certain coherence condition between the two is satisfied. In fact, the construction may be accomplished in the generality of any additive category, rather than just using the category of finitely generated free R-modules. If we have only one closure operator (say on free right modules), we show how to construct a closure operator on free left modules which satisfies the coherence conditions; however, the construction is somewhat unnatural.

If we start with right and left S-closure operators, defined from a homomorphism $R \longrightarrow S$, then the ring $\bar{R}$ that we construct is exactly the dominion of R in S, as defined by Isbell. Thus the closure operator information determines the homomorphism "as well as can be expected," in some sense. In particular, if $R \longrightarrow S$ is an epimorphism of rings, then $\overline{R}$ is just isomorphic to S and the homomorphism is thus completely determined.

We choose to designate this construction the "zigzag localization" of R, as suggested by the characterization of an element of Isbell's dominion as a "zigzag". To justify the "localization" part of this term, we give a condition on the closure operator which implies that the zigzag localization ring $\overline{R}$ is a division ring.

Another sort of localization in the literature is that given by the notion of a "torsion theory", as studied by Lambek, Gabriel and others. We show that closure operators give more general R-information than do torsion theories. Under certain conditions, the two kinds of R-information are equivalent, and in that case we show that the zigzag localization ring is the same as the ring obtained by localizing with respect to the given torsion theory.

CHAPTER TWO

MOTIVATING EXAMPLES

The choice of closure operators as the kind of R-information to be studied was made as a result of considering certain constructions of ring homomorphisms $R \longrightarrow K$, where K is a division ring. The intention was to generalize to a notion of localization of R which might yield rings K more general than just division rings. As the ideal model of this, the localization of a commutative ring with respect to a multiplicative subset is the appropriate generalization of the construction of the quotient field of an integral domain. In this chapter we point out some of these constructions of division rings and describe how they led to the definition of an abstract algebraic closure operator.

The first step in extending the construction of quotient fields of commutative integral domains to non-commutative rings was made by Ore (see [2], Chap. 0 for a discussion). He gave conditions on a ring without zero-divisors (namely, that any two non-zero right ideals should intersect in a non-zero ideal) under which the non-zero elements can all be inverted, to form a division ring. Various generalizations have been formulated based on Ore's method, involving the inversion of some of the non-zero elements of the ring.

<u>Matrix Localizations</u>. For rings which do not satisfy an Ore-like condition, P. M. Cohn supplied the fundamental idea of inverting <u>matrices</u> over the ring, rather than (just) elements of it. This method involves more than just Ore-like localization of a matrix ring, since square matrices of <u>all</u> sizes are available for inversion. (See [2], Chap. 7 for a full account.)

For R any ring, recall the notation $M_n(R)$ for the ring of all n-by-n matrices with entries in R. Then if R is a subring of a ring S, Cohn defined what he called the "rational closure" of R in S to be

$$\Sigma(R,S) = \left\{ s \in S \;\middle|\; \begin{array}{l} \text{there is a matrix } A \text{ in } M_n(R) \\ \text{such that } A^{-1} \text{ is in } M_n(S) \text{ and} \\ s \text{ occurs as an entry in } A^{-1} \end{array} \right\} .$$

For $H:R \longrightarrow S$ a ring homomorphism, he also put $\Sigma(R,S) = \Sigma(H(R),S)$. Then Cohn showed that $\Sigma(R,S)$ was in fact a subring of S which contains R (or H(R)). (For a discussion of why this should be called a "closure", see Appendix I.)

Given any set Σ of square matrices over any ring R, one can also construct a "universal Σ-inverting ring" R_Σ by adding to a system of generators and relations for R further generators, one for each entry of a matrix in Σ, and further relations, which make the matrices of new generators into inverses of the matrices in Σ. Then any homomorphism $R \longrightarrow S$ which takes each element of Σ to an invertible matrix must factor through the canonical homomorphism $R \longrightarrow R_\Sigma$. One still would like to know when $R \longrightarrow R_\Sigma$ is an embedding, and when R_Σ is (say) a division ring (or even when $R_\Sigma \neq 0$).

Suppose $R \longrightarrow K$ is a homomorphism of a ring R to a division ring K. Then if P is the set of all square matrices (of all sizes) over R whose images in K are singular matrices, we can see that P satisfies:

(1) If A and B are matrices over R, with A of size n-by-(n-1) and B of size (n-1)-by-n, and $X = AB$, then $X \in P$.

(2) If A, B, and C are square matrices over R of the same size which agree except in one row (or column), and if that row (or column) in C is the vector sum of the corresponding rows (or columns) in A and B, and if $A \in P$ and $B \in P$, then $C \in P$.

(3) If $A \in P$ and B is any square matrix over R, then the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in P$.

(4) The 1-by-1 matrix (1) is <u>not</u> in P.

(5) If $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in P$, then either $A \in P$ or $B \in P$.

Cohn calls any set P of square matrices over R (of arbitrary sizes) a <u>prime matrix ideal</u> of R if it satisfies the conditions (1) - (5). Then he showed that each such P comes (as above) from a homomorphism $R \longrightarrow K_P$ to a division ring, by constructing the division ring K_P from the "information" given by P. (See Appendix II for another proof.) This result also showed, for Σ the set of all square matrices over R which are <u>not</u> in P, that R_Σ is a local ring with factor ring K_P.

Dependence Relations. In the light of the fact that a matrix over a division ring is invertible if and only if its columns are linearly independent, George Bergman decided that a prime matrix ideal of R might be interpreted as prescribing which sets of vectors in finitely generated free R-modules become linearly dependent over the division ring which is to be constructed. This prescription gives R-information which may be axiomatized by the following definitions (from [3], p.252).

A collection $\mathfrak{D}$ is said to give a dependence relation on a set S if $\mathfrak{D}$ is a collection of subsets of S, called the "dependent" subsets, such that if $X \subseteq S$ then $X \in \mathfrak{D}$ if and only if some finite subset of X belongs to $\mathfrak{D}$. Given such a $\mathfrak{D}$, and $X \subseteq S$, we may form

$$\langle X\rangle = \left\{ y \in S \;\middle|\; \begin{array}{l} \text{there is } X_0 \subseteq X \text{ with } X_0 \notin \mathfrak{D}, \\ \text{but } X_0 \cup \{y\} \in \mathfrak{D} \end{array} \right\} \cup X\,,$$

the "span" of X. This abstract formulation describes both linear dependence over a field and algebraic dependence, as in van der Waerden ([9], sections 33 and 64) and Zariski and Samuel ([10], Ch. I, sec. 21 and Ch. II, sec. 12).

For R any ring, we define an algebraic dependence relation on finitely generated free right R-modules to be a dependence relation on each finitely generated free right R-module F, such that (for $X \subseteq F$):

(0) $\langle\langle X\rangle\rangle = \langle X\rangle$ ("transitivity").

(1) $\langle X\rangle$ is an R-submodule of F.

(2) If $f: F \longrightarrow F'$ is a homomorphism of finitely generated free R-modules, then $f(\langle X\rangle) \subseteq \langle f(X)\rangle$.

Recall that an R-module is <u>projective</u> if it is a direct summand of a free R-module. Then given a finitely generated projective module P, say a direct summand of R^n, and an algebraic dependence relation as above, we may extend the dependence relation to P by defining a subset to be dependent if its canonical image in R^n is dependent. Then using (2) it is not hard to show that this definition is independent of the choice of representation of P as a direct summand of a free module, and that this definition gives the unique extension of the original dependence relation to a new one on the class of finitely generated projective modules, with (0) - (2) still holding.

An algebraic dependence relation gives enough R-information to construct a division ring (as in Chapter Four). In considering how we might generalize, note the following immediate result of the definition of <u>any</u> dependence relation $\mathfrak{D}$:

("Exchange Condition") If $y \in \langle X \cup \{x\}\rangle$ but $y \notin \langle X\rangle$, then $x \in \langle X \cup \{y\}\rangle$.

(<u>Proof</u>: Suppose $X_0 \subseteq X \cup \{x\}$ and $X_0 \notin \mathfrak{D}$, but $\{y\} \cup X_0 \in \mathfrak{D}$. Then $x \in X_0$, or else $y \in \langle X\rangle$. Thus $\{x,y\} \cup (X_0 - \{x\}) \in \mathfrak{D}$. But we also have $\{y\} \cup (X_0 - \{x\}) \notin \mathfrak{D}$ (or again $y \in \langle X\rangle$). Hence $x \in \langle X \cup \{y\}\rangle$.)

Considering $\mathfrak{D}$ to indicate linear dependence, this condition comes from the invertibility of the coefficient of x in the linear combination which gives y. Thus, if we ever want to construct rings _other_ than division rings, we can't have this condition holding. Since it follows from the definition, we will have to change them in some fundamental way; in fact we decide to reformulate the notion of the "span".

Closure Operators. So we define a _closure operator_ on a set S to be an operator C, associating with each subset X of S another subset C(X) (or just CX), such that (for any subsets X, Y of S):

(1) $X \subseteq C(X)$.

(2) If $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

(3) $C(C(X)) = C(X)$.

The operator C is said to be _finitary_ if it satisfies the strengthened condition (for any $X \subseteq S$)

(2+) $C(X) = \bigcup_{X_o \text{ finite} \subseteq X} C(X_o)$.

Thus for a dependence relation on S as above, the "span" is a finitary closure operator.

Now for any ring R we define a (right) _algebraic closure operator_ on finitely generated free right R-modules to be a closure operator C on each such R-module, also satisfying (for $X \subseteq F$):

(4) C(X) is an R-submodule of F.

(5) If $f: F \to F'$ is a homomorphism of finitely generated free R-modules, then $f(CX) \subseteq C(f(X))$.

Such a closure operator easily extends uniquely to a corresponding one on finitely generated projective right R-modules, exactly as before.

As specific examples of algebraic closure operators we note the discrete operator, for which the closure of a subset X of F is just the R-submodule of F generated by X, and the indiscrete operator, for which the closure of any subset X of F is just F itself. Another example is the C_S defined in Chapter One, for any ring homomorphism $R \longrightarrow S$. In fact, the first two operators above are just C_S for $R \longrightarrow S$ being the identity map and the zero map, respectively. These closure operators are all finitary.

Closure Operators on Maps. Given a particular finitely generated projective right R-module P, consider the collection of all R-homomorphisms from any other such R-module to P. Then for any subset X of P, we can form the subcollection $\mathfrak{X}$ of all such maps whose images lie in the R-submodule of P generated by X. This subcollection has properties like those of a right ideal; namely, if $f:Q \longrightarrow P$ and $f':Q \longrightarrow P$ are both in $\mathfrak{X}$, and if $g:Q' \longrightarrow Q$ is any homomorphism, then both $f+f':Q \longrightarrow P$ and $fg:Q' \longrightarrow P$ are in $\mathfrak{X}$.

Then if an algebraic closure operator C is also given, we can form CX and the subcollection $\overline{\mathfrak{X}}$ of maps to P with images contained in CX. Then clearly $\mathfrak{X} \subseteq \overline{\mathfrak{X}}$ and we have a sort of closure operator on these "right ideals". This operator will satisfy properties similar to those of our original definition, and we can even recover the original C from it (by taking the union of the images).

With this kind of formulation we can speak about submodules and elements purely in terms of maps among the modules. Thus we can generalize the notion of an algebraic closure operator to the framework of a category; this generalization will be carried out in Chapter Three. The categorical formulation allows a little more generality in our construction of zigzag localizations and perhaps provides a more natural setting for some of the notions involved.

CHAPTER THREE

ZIGZAG LOCALIZATIONS OF ADDITIVE CATEGORIES

Recall that a <u>category</u> $\mathcal{C}$ consists of a collection of <u>objects</u>, together with a collection $\mathcal{C}(A,B)$ of <u>morphisms</u> for each (ordered) pair of objects A and B. The category also is given with an associative composition law and identity elements for the composition. We think of f in $\mathcal{C}(A,B)$ as a map from A (the <u>domain</u> of f) to B (the <u>codomain</u> of f), and we write $f:A \longrightarrow B$. We also have the notion of a <u>functor</u> F from $\mathcal{C}$ to another category $\mathcal{A}$; it assigns to each object A in $\mathcal{C}$ an object F(A) in $\mathcal{A}$, and assigns morphisms $f \in \mathcal{C}(A,B)$ to corresponding morphisms $F(f) \in \mathcal{A}(F(A),F(B))$ in such a way as to preserve the composition and identities.

As examples, there are the traditional categories of groups and group-homomorphisms, rings and ring-homomorphisms, etc. The "group ring" construction is a functor from the first example to the second.

Let us also recall the notions of fullness and faithfulness. A <u>subcategory</u> of a given category $\mathcal{C}$ is a choice of a subcollection of objects and a subcollection of $\mathcal{C}(A,B)$ for each pair of objects A and B in the chosen subcollection, nonetheless still containing the identity morphisms and closed under the composition law. Such a subcategory is said to be <u>full</u> if the chosen subcollection of $\mathcal{C}(A,B)$

is in fact all of $\mathcal{C}(A,B)$ in each case. (Note that the subcollection of objects may still be proper.)

A functor F from $\mathcal{C}$ to $\mathcal{A}$ is said to be <u>full</u> if its "image subcategory" is full; that is, if (for objects A,B of $\mathcal{C}$) every morphism in $\mathcal{A}(F(A),F(B))$ is F(f) for some morphism $f \in \mathcal{C}(A,B)$. We say F is <u>faithful</u> if any two morphisms $f \neq g$ in $\mathcal{C}(A,B)$ also satisfy $F(f) \neq F(g)$ in $\mathcal{A}(F(A),F(B))$. The "group ring" functor above is faithful but not full.

<u>A Remark on Foundations</u>. At times we may wish to speak of the "set" of objects of a category, or of a "set" of morphisms. In the case of the category of all sets and functions mapping between them (for example), this practice is inaccurate, since the collection of objects does not in fact form a set according to the axioms of set theory. We expect to involve no contradictions of these axioms through our abuse of these terms, but as a precaution (and to forestall objections), we will indicate an axiomatic foundation for our usage. The following is taken from Mac Lane ([5], pp. 21-24), where a more complete account exists.

We assume that there exists a set U, the <u>universe</u>, which satisfies:

(i) If $u \in U$ and $x \in u$, then $x \in U$.

(ii) If $u \in U$ and $v \in U$, then $u \times v \in U$.

(iii) If $x \in U$, then $\mathcal{P}x = \{y \mid y \subseteq x\} \in U$,
and $\cup x = \{y \mid y \in z \text{ for some } z \in x\} \in U$.

(iv) The set ω of all finite ordinals is an element of U.

(v) If $f: a \longrightarrow U$ is a function and $a \in U$, then the image $f(a) \in U$.

Then U is large enough so that most of "ordinary" mathematics may be done completely "inside" U.

Fixing such a universe U, we define a "U-set" to be a set which is an element of U. Then a function from one U-set to another is also an element of U (by the assumptions on U), and we get the category of U-sets, which has a _set_ of objects and a _set_ of morphisms. Likewise we define a "U-ring" to be an element of U with a prescribed ring structure, and we form the category of U-rings and (U-)ring homomorphisms. Doing this for every category we shall be interested in, we can make the convention that every occurrence of the term "category of sets" (or rings, etc.) will be an abbreviation for the "category of U-sets" (or U-rings, etc.). With this convention we are justified in the usage as described above.

Additive Categories. Define an _Ab-category_ to be a category $\underline{C}$ given with a structure of abelian group on each $\underline{C}(A,B)$ (for A,B objects of $\underline{C}$), under an operation + (called "addition"), with identity element 0. Further, the composition law of the category must distribute over addition (on both sides), just as multiplication distributes over addition in a ring. In fact, an Ab-category with only one object is exactly a ring. (See Mitchell ([6]) for the ring theory of Ab-categories.) We also define an _additive_ functor from one Ab-category to another to be a functor which preserves the abelian group structure on the morphisms.

Examples of Ab-categories include the category of (right, say) R-modules and R-homomorphisms, for R any ring. Furthermore, any full subcategory of an Ab-category inherits a structure of Ab-category, so that the category of finitely generated projective right R-modules and R-homomorphisms forms another example.

Let C be an Ab-category and P an object of C. Then following the motivating discussion in Chapter Two, we define a right ideal of C at P to be a set of morphisms of C, all having codomain P, such that if two morphisms f and f' in the ideal have the same domain as well, then f+f' is also in the ideal, as is fg whenever g is any morphism for which the composition is defined. Likewise and symmetrically we define a left ideal at P as an appropriate set of morphisms with domain P.

But an Ab-category is a little too general as a framework for defining our closure operators. We also want some kind of "direct sum" operation on objects. So we define an additive category to be an Ab-category such that, for any two (not necessarily distinct) objects A and B there is another object, denoted $A\oplus B$, and morphisms

$$\iota_A : A \longrightarrow A\oplus B \qquad \pi_A : A\oplus B \longrightarrow A$$
$$\iota_B : B \longrightarrow A\oplus B \qquad \pi_B : A\oplus B \longrightarrow B$$

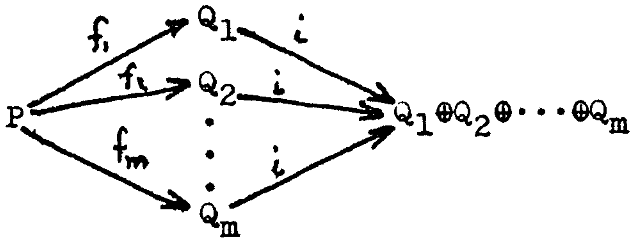
such that $\pi_A\iota_A = 1_A$, $\pi_B\iota_B = 1_B$, and $\iota_A\pi_A + \iota_B\pi_B = 1_{A\oplus B}$

(here we use the standard notation of 1_A for the identity morphism of A). Further, we assume the existence of a "null" object 0 satisfying $1_0 = 0_0$ as morphisms of 0. Then $\pi_A : A \oplus 0 \longrightarrow A$ is an isomorphism (i.e., has an inverse i_A), and we see that the set of isomorphism classes of objects becomes an abelian semigroup with identity, under the operation $\oplus$.

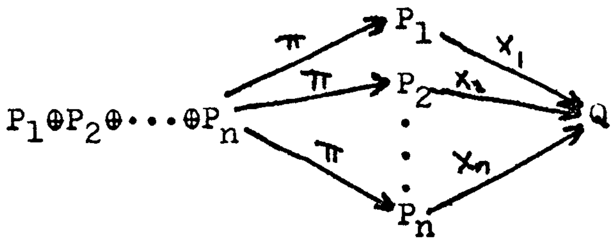
This operation $\oplus$ serves to "connect up" the category. For example, the category of all right or left R-modules, with only zero morphisms between a right and a left module, forms an Ab-category, but not an additive category.

Given an additive category M and a set X of morphisms, all having the same codomain P, we denote by XM the right ideal of M at P generated by X. For a single morphism f we can denote $\{f\}M$ by just fM. Then we note the following obvious properties of right ideals of M. The ideal $1_P M$ is the "unit" ideal of all morphisms with codomain P. A set I of morphisms is a right ideal of M at P if and only if $I \subseteq 1_P M$ and $IM = I$. For any subset $X \subseteq 1_P M$ and $f : P \longrightarrow Q$, define $fX = \{fx \mid x \in X\}$. Then if I is a right ideal at P, we have that fI is a right ideal at Q. Two right ideals I and J at the same P can be intersected to give an ideal $I \cap J$ or "added" to give the ideal $I + J = \{i+j \mid i \in I, j \in J$, having the same domain$\}$. Then I + J will contain both I and J, since each contains all zero maps into P.

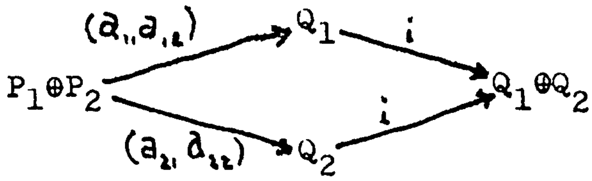
Suppose $f_1, f_2, \ldots, f_m$ are morphisms of M with $f_i : P \longrightarrow Q_i$. Then we will denote the sum of the compositions



by the "m-by-1 matrix" $\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}: P \longrightarrow Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m$. Likewise, for $x_1, x_2, \cdots, x_n$ morphisms with $x_j: P_j \longrightarrow Q$, we denote the sum of the compositions



by the matrix $(x_1 \cdots x_n): P_1 \oplus P_2 \oplus \cdots \oplus P_n \longrightarrow Q$. Finally, for $a_{ij}: P_j \longrightarrow Q_i$ (say for $i, j = 1,2$), we will denote the sum of the compositions



by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, instead of $\begin{pmatrix} (a_{11} \; a_{12}) \\ (a_{21} \; a_{22}) \end{pmatrix}$ or $\left(\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$. It can easily be seen that these "matrix-like" morphisms add like matrices and compose like matrix multiplication. In fact, when in the category of finitely generated free R-modules, the morphisms are exactly matrices over R.

Now with this notation, the ideal generated by the morphisms $x_1, x_2, \cdots, x_n$, previously denoted $x_1M + x_2M + \cdots + x_nM$, may be simplified to $(x_1 \; x_2 \; \cdots \; x_n)M$.

Thus in an additive category, "finitely generated" is equivalent to "principal" (or 1-generated).

Closure Operators on Right Ideals. Let M be an additive category. Then define an algebraic closure operator C on right ideals of M (abbreviated "right a.c.o.") to be a closure operator on $1_P M$ for each object P of M, satisfying (for any $X \subseteq 1_P M$):

(4) C(X) is a right ideal of M at P.

(5) For $a : P \longrightarrow Q$ in M, $aC(X) \subseteq C(aX) \subseteq 1_Q M$.

(Refer to conditions (1 - 3) in the definition of closure operator.) Note then that such a C satisfies $C(a) = C(\{a\}) = C(aM)$.

A right a.c.o. C is said to be finitary if it is finitary (condition (2+)) for each object P. The finitariness may also be written

$$(2+)\quad C(X) = \bigcup_{a \in XM} C(a),$$

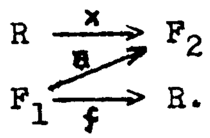
where we are now using strongly the existence of $\oplus$.

For C_1, C_2 two right a.c.o.'s on M, we will write $C_1 \le C_2$ if $C_1(X) \subseteq C_2(X)$ for every X. Also we define the discrete right a.c.o. C_0^R by $C_0^R(X) = XM$ for any P and any $X \subseteq 1_P M$. Then clearly C_0^R is the minimal right a.c.o., in the sense that $C_0^R \le C$ for any right a.c.o. C on M.

Let us now note that, as for any right ideal, the closure C(X) of any $X \subseteq 1_P M$ satisfies $(a\ b) \in C(X)$ if any only if $a \in C(X)$ and $b \in C(X)$. This follows from the equations $a = (a\ b)\binom{1}{0}$, $b = (a\ b)\binom{0}{1}$ and $(a\ b) = a(1\ 0) + b(0\ 1)$. Thus a right ideal is generated by just the columns of its generators. Hence we will think of $C(X)$ as giving the columns which can be obtained from the columns of X by "generalized column operations".

In a similar fashion we define an algebraic closure operator on left ideals of M. The discrete left a.c.o. is denoted C_0^L and defined by $C_0^L(X) = MX$ for any $X \subseteq Ml_Q$. It is minimal among all left a.c.o.'s under a similar partial ordering. We will think of left a.c.o.'s as giving "generalized row operations".

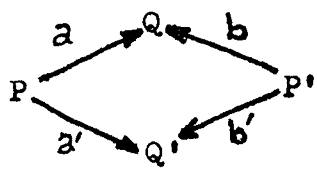
Coherent Pairs of Operators. To give some idea how the construction of the zigzag localization proceeds, consider the case of an inclusion of rings $R \subseteq S$, and the closure operator C_S (as in Chapter One) defined by the inclusion map. We expect the zigzag localization of R to contain some elements of S. Now if $a:F_1 \longrightarrow F_2$ is a map of free right R-modules with $\{e_1, e_2, \cdots, e_n\}$ a basis for F_1, and if $x \in C_S(a(F_1))$, then $x = \Sigma a(e_i)s_i$, for some coefficients $s_i \in S$. To get an element of S from x, we can think of $\Sigma e_i s_i$ as "$a^{-1}(x)$" and then apply some functional $f:F_1 \longrightarrow R$ to get "$f\ a^{-1}\ x$" $= \Sigma f(e_i)s_i$. If we think of $x \in F_2$ as a map $x:R \longrightarrow F_2$ (via $x(r) = xr$), then we can express this as a zigzag diagram



But if there is another expression $x = \Sigma a(e_i)t_i$, then we will want to get a well-defined element of S; that is, $\Sigma f(e_i)s_i = \Sigma f(e_i)t_i$. Thus we will have to allow only functionals f which "cancel" the uncertainty in the choice of $a^{-1}x$.

In constructing the zigzag localization, we will express the relationship between a and the allowable functionals f by the condition $f \in C^L(a)$, where C^L is a left a.c.o. on M. The following definition gives the condition on C^L and C^R which guarantees that "$f \in C^L a$ will indeed cancel the uncertainty in $a^{-1}x$."

Definition: Let M be an additive category, C^L a left a.c.o. on M and C^R a right a.c.o. on M. The pair (C^L, C^R) is said to be right coherent when, for any diagram



of morphisms of M which satisfies $b \in C^R(a) \subseteq 1_Q M$ and $(a'\ b') \in C^L(a\ b) \subseteq M 1_{P \oplus P'}$, the diagram also satisfies $b' \in C^R(a') \subseteq 1_{Q'} M$.

To support the contention that this is the condition we want, suppose we again have the zigzag $R \xrightarrow{x} F_2 \xleftarrow{a} F_1 \xrightarrow{f} R$ with $x \in C^R a$ and $f \in C^L a$, and assume that $x = ay = ay'$, so that there is uncertainty in $a^{-1}x$. Then using condition (5) for a.c.o.'s, we get $f(0\ y-y') \in C^L a(0\ y-y') = C^L(0\ 0)$. Since $0 \in C^R 0$, right coherence implies $fy - fy' \in C^R 0$. Thus f cancels the uncertainty in $a^{-1}x$, up to the small ideal $C^R 0$.

Regarding right coherence of pairs of a.c.o.'s, we get the following basic result.

Proposition 3.1:

(i) (C_o^L, C^R) is right coherent for any right a.c.o. C^R.

(ii) If (C_2^L, C^R) is right coherent and $C_1^L \leq C_2^L$, then (C_1^L, C^R) is right coherent.

Proof: (i) If $(a' \ b') \in C_o^L(a \ b) = M(a \ b)$, then $(a' \ b') = c(a \ b) = (ca \ cb)$ for some c. Hence if $b \in C^R(a)$, then $b' = cb \in C^R(ca) = C^R(a')$ by condition (5) for a.c.o.'s. Part (ii) is clear.

Note that no analogous form of 3.1(ii) holds for the right a.c.o.'s; that is, if (C^L, C_2^R) is right coherent, neither $C_1^R \leq C_2^R$ nor $C_2^R \leq C_1^R$ implies that (C^L, C_1^R) is right coherent.

We can also define the intersection of two (left or) right a.c.o.'s, by

$$C_1^R \cap C_2^R(X) = C_1^R(X) \cap C_2^R(X)$$

for any X. It is easily seen that this gives a (left or) right a.c.o. (which is finitary if C_1^R, C_2^R are). Then we also get the following result, whose proof is straightforward.

Proposition 3.2: If (C_1^L, C_1^R) and (C_2^L, C_2^R) are right coherent, then so is $(C_1^L \cap C_2^L, C_1^R \cap C_2^R)$.

The Construction of the Zigzag Localization. Suppose $Z = (C^L, C^R)$ is a right coherent pair of a.c.o.'s on the additive category M. We wish to construct an additive category M_Z, the right zigzag localization of M with respect to Z, and an additive functor E from M to M_Z. The new category M_Z will have the same objects as M, but new morphisms.

For S, T objects of M, let $A_Z(S,T)$ be the set of diagrams in M of the form

$$p = \begin{pmatrix} S & \xrightarrow{x} & Q \\ & \nearrow_{a} & \\ P & \xrightarrow[f]{} & T \end{pmatrix}$$

where $f \in C^L(a)$ and $x \in C^R(a)$. Define a relation $\approx$ on each $A_Z(S,T)$ by

$$p \approx p' = \begin{pmatrix} S & \xrightarrow{y} & Q' \\ & \nearrow_{b} & \\ P' & \xrightarrow[g]{} & T \end{pmatrix}$$

if and only if

$$\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} a & 0 \\ 0 & b \\ f & g \end{pmatrix} \subseteq 1_{Q\oplus Q'\oplus T}M.$$

<u>Theorem 3.3</u>: The relation $\approx$ is an equivalence relation.

<u>Proof</u>: For symmetry of $\approx$, let $t: Q\oplus Q' \longrightarrow Q'\oplus Q$ be the reversing isomorphism. Then if $p \approx p'$, we get

$$\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} 0 & b \\ a & 0 \\ f & g \end{pmatrix} = C^R\begin{pmatrix} b & 0 \\ 0 & a \\ g & f \end{pmatrix}$$

by applying the morphism $(t\ 1)$, using condition (5) for a.c.o.'s (and "generalized column operations"). Of course then $-\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$ is in this closure, since closures give right ideals. Thus $p' \approx p$.

For reflexivity of $\approx$, apply $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}: P \longrightarrow P\oplus P\oplus T$ to $x \in C^R(a)$, to obtain

$$\begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} a \\ -a \\ 0 \end{pmatrix},$$

where we use condition (5) again. But

$$\begin{pmatrix} a \\ -a \\ 0 \end{pmatrix} M \subseteq \begin{pmatrix} a \\ 0 \\ f \end{pmatrix} M + \begin{pmatrix} 0 \\ -a \\ -f \end{pmatrix} M = \begin{pmatrix} a & 0 \\ 0 & a \\ f & f \end{pmatrix} M, \text{ so}$$

$$\begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 \\ 0 & a \\ f & f \end{pmatrix} M = C^R \begin{pmatrix} a & 0 \\ 0 & a \\ f & f \end{pmatrix},$$

now using condition (2). Thus $p \approx p$.

For transitivity of $\approx$, suppose $p \approx p'$ and $p' \approx p''$, where p'' is the diagram $S \xrightarrow{z} Q'' \xleftarrow{c} P'' \xrightarrow{h} T$. Then $\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 \\ 0 & b \\ f & g \end{pmatrix}$ and $\begin{pmatrix} y \\ -z \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} b & 0 \\ 0 & c \\ g & h \end{pmatrix}$. By injecting both of these into $1_{Q \oplus Q' \oplus Q'' \oplus T} M$, we get

$$\begin{pmatrix} x \\ -y \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ -z \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \\ f & g & h \end{pmatrix}$$

and we can add these together to obtain

$$\begin{pmatrix} x \\ 0 \\ -z \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \\ f & g & h \end{pmatrix}.$$

Now $g \in C^L(b)$, so $(0\ g\ 0\ 0) \in C^L(0\ b\ 0\ 0)$, and we can use row operations to obtain

$$\begin{pmatrix} a & 0 & 0 & x \\ 0 & 0 & c & -z \\ f & 0 & h & 0 \end{pmatrix} \in C^L \begin{pmatrix} a & 0 & 0 & x \\ 0 & b & 0 & 0 \\ 0 & 0 & c & -z \\ f & g & h & 0 \end{pmatrix}.$$

Now using right coherence we obtain

$$\begin{pmatrix} x \\ -z \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & c \\ f & 0 & h \end{pmatrix} = C^R \begin{pmatrix} a & 0 \\ 0 & c \\ f & h \end{pmatrix},$$

or just $p \approx p''$.

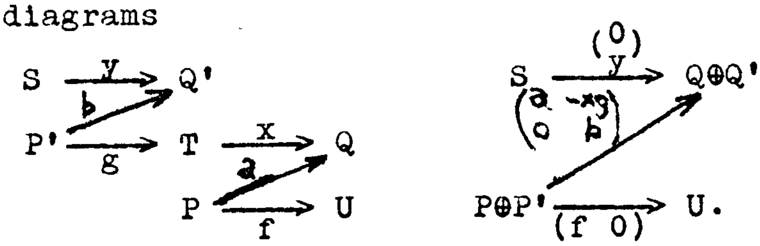
Now that $\approx$ has been shown to be an equivalence relation, we can denote the set of equivalence classes of

$A_Z(S,T)$ by $M_Z(S,T)$. We want to make this the set of morphisms: $S \longrightarrow T$ in the new category M_Z.

To define the appropriate composition law and identity morphisms, let us first denote the equivalence class of the diagram $(T \xrightarrow{x} Q \xleftarrow{a} P \xrightarrow{f} U) \in A_Z(T,U)$ by the symbol $(f/a\backslash x) \in M_Z(T,U)$. Then define

$$(f/a\backslash x)\cdot(g/b\backslash y) = ((f\ 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} 0 \\ y \end{pmatrix}),$$

from the diagrams



(That this is in $M_Z(S,U)$ will be proven below.) We also define the identity morphism in $M_Z(S,S)$ to be $(1_S/1_S\backslash 1_S)$.

To justify this definition of composition, suppose that a and b are actually invertible morphisms and we interpret $(f/a\backslash x)$ and $(g/b\backslash y)$ as the morphisms $fa^{-1}x$ and $gb^{-1}y$, respectively. Then the interpretation of $((f\ 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} 0 \\ y \end{pmatrix})$ is

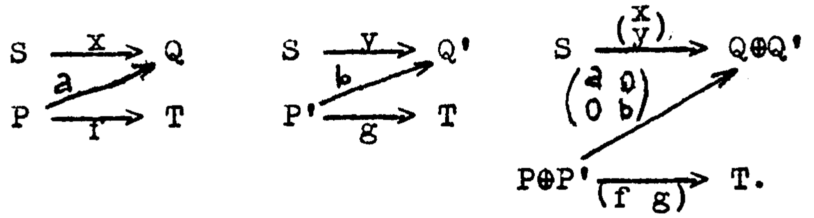
$$\begin{aligned}(f\ 0)\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ y \end{pmatrix} &= (f\ 0)\begin{pmatrix} a^{-1} & a^{-1}xgb^{-1} \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} \\ &= fa^{-1}xgb^{-1}y,\end{aligned}$$

exactly the composition.

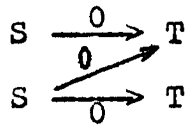
Following the same kind of justification, we define the addition on $M_Z(S,T)$ by

$$(f/a\backslash x) + (g/b\backslash y) = ((f\ g)/\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} x \\ y \end{pmatrix}),$$

where the diagrams are



Then also define $-(f/a\backslash x) = (-f/-a\backslash -x)$ and the zero map from S to T to be $(0/0\backslash 0)$, where the diagram is



There is also an additive functor E from M to M_Z, defined by $E(S) = S$ on objects S and $E(u:S \longrightarrow T) = (u/u\backslash u)$ (which we may wish to denote by $\bar{u}$) on morphisms u.

<u>Theorem 3.4</u>: For $Z = (C^L, C^R)$ a right coherent pair, the above definitions give a well-defined structure of additive category on M_Z and an additive functor $E:M \longrightarrow M_Z$.

<u>Proof</u>: (Long and computational; ends on page 33.) First we check that the diagrams defined actually lie in A_Z. We need:

(·) $\binom{0}{y} \in C^R\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}$, $(f\ 0) \in C^L\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}$.

(1) $1_S \in C^R 1_S$, $1_S \in C^L 1_S$.

(+) $\binom{x}{y} \in C^R\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $(f\ g) \in C^L\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

(-) $-x \in C^R(-a)$, $-f \in C^L(-a)$.

(0) $0 \in C^R 0$, $0 \in C^L 0$.

(E) $u \in C^R u$, $u \in C^L u$.

(Here the notations are as in the definitions.) Obviously (1), (-), (0), and (E) follow from the fact that the original diagrams were in A_Z and from the properties of a.c.o.'s. For (+), simply note that because $x \in C^R a$, we have $\binom{x}{0} \in C^R\binom{a}{0}$, which is contained in $C^R\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. The latter also contains $\binom{0}{y}$, so we get the first part of (+) by adding; the second part is similar.

To verify (·), note that $\binom{x}{0} \in C^R\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}$ just as above. Thus $\binom{xg}{0} \in C^R\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}$, because the closure is a right ideal. Adding this to $\binom{-xg}{b}$, we get $\binom{0}{b} \in C^R\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}$. But then $\binom{0}{y} \in C^R\binom{0}{b} \subseteq C^R\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}$. Again, the second part is similar.

Now we want to show that composition is well-defined. So suppose $(h/c\backslash z) = (g/b\backslash y)$ in $M_Z(S,T)$; then we need $(f/a\backslash x)\cdot(h/c\backslash z) = (f/a\backslash x)\cdot(g/b\backslash y)$ for $(f/a\backslash x) \in M_Z(T,U)$ and $(h/c\backslash z)\cdot(j/d\backslash w) = (g/b\backslash y)\cdot(j/d\backslash w)$ for $(j/d\backslash w) \in M_Z(Q,S)$, say. For the first statement, we need that $((f\ 0)/\begin{pmatrix} a & -xh \\ 0 & c \end{pmatrix}\backslash\binom{0}{z}) = ((f\ 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\backslash\binom{0}{y})$, or $\begin{pmatrix} 0 \\ z \\ 0 \\ -y \\ 0 \end{pmatrix} \in C^R u$, where u is the morphism $\begin{pmatrix} a & -xh & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & -xg \\ 0 & 0 & 0 & b \\ f & 0 & f & 0 \end{pmatrix}$. We have already seen in the reflexive part of the proof of 2.3 that $\begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} a & 0 \\ 0 & a \\ f & f \end{pmatrix}$, so here we also get $\begin{pmatrix} x \\ 0 \\ -x \\ 0 \\ 0 \end{pmatrix} \in C^R u$. Thus $\begin{pmatrix} xh \\ 0 \\ -xh \\ 0 \\ 0 \end{pmatrix} \in C^R u$, so adding to $\begin{pmatrix} -xh \\ c \\ 0 \\ 0 \\ 0 \end{pmatrix} \in uM$ we get that $\begin{pmatrix} 0 \\ c \\ -xh \\ 0 \\ 0 \end{pmatrix} \in C^R u$.

Now the assumption $(h/c\backslash z) = (g/b\backslash y)$ means $\begin{pmatrix} z \\ -y \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} c & 0 \\ 0 & b \\ h & g \end{pmatrix}$, so apply the morphism $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ to get $\begin{pmatrix} 0 \\ z \\ 0 \\ -y \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} 0 & 0 \\ c & 0 \\ -xh & -xg \\ 0 & b \\ 0 & 0 \end{pmatrix}$.

This last closure is contained in $C^R u$ by "generalized column operations", so we are done.

Now to prove the second statement we need that $((h\ 0)/\begin{pmatrix} c & -zj \\ 0 & d \end{pmatrix} \setminus \begin{pmatrix} 0 \\ w \end{pmatrix}) = ((g\ 0)/\begin{pmatrix} b & -yj \\ 0 & d \end{pmatrix} \setminus \begin{pmatrix} 0 \\ w \end{pmatrix})$, or

$\begin{pmatrix} 0 \\ w \\ 0 \\ -w \\ 0 \end{pmatrix} \in C^R u$, where now $u = \begin{pmatrix} c & -zj & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & b & -yj \\ 0 & 0 & 0 & d \\ h & 0 & g & 0 \end{pmatrix}$. Use $\begin{pmatrix} z \\ -y \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} c & 0 \\ 0 & b \\ h & g \end{pmatrix}$

to get $\begin{pmatrix} z \\ 0 \\ -y \\ 0 \\ 0 \end{pmatrix} \in C^R u$, so $\begin{pmatrix} zj \\ 0 \\ -yj \\ 0 \\ 0 \end{pmatrix} \in C^R u$. Then

$\begin{pmatrix} 0 \\ d \\ 0 \\ -d \\ 0 \end{pmatrix} = \begin{pmatrix} -zj \\ d \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} zj \\ 0 \\ -yj \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -yj \\ d \\ 0 \end{pmatrix} \in C^R u$. But $w \in C^R d$, so

$\begin{pmatrix} 0 \\ w \\ 0 \\ -w \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} 0 \\ d \\ 0 \\ -d \\ 0 \end{pmatrix} \subseteq C^R(C^R u) = C^R u$.

The next task is to show that + is well-defined; so take $(h/c\setminus z) = (g/b\setminus y)$ in $M_Z(S,T)$ again and try to show that $(f/a\setminus x) + (h/c\setminus z) = (f/a\setminus x) + (g/b\setminus y)$, for $(f/a\setminus x) \in M_Z(S,T)$. This means that we need to show that

$((f\ g)/\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \setminus \begin{pmatrix} x \\ y \end{pmatrix}) = ((f\ h)/\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \setminus \begin{pmatrix} x \\ z \end{pmatrix})$, or

$\begin{pmatrix} x \\ y \\ -x \\ z \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & c \\ f & g & f & h \end{pmatrix}$. This just follows from $\begin{pmatrix} x \\ 0 \\ -x \\ 0 \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 \\ 0 & 0 \\ 0 & a \\ 0 & 0 \\ f & f \end{pmatrix}$

(as before) and $\begin{pmatrix} 0 \\ y \\ 0 \\ -z \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} 0 & 0 \\ b & 0 \\ 0 & 0 \\ 0 & c \\ g & h \end{pmatrix}$ (assumption).

The equation $(g/b\backslash y) + (f/a\backslash x) = (h/c\backslash z) + (f/a\backslash x)$, necessary since + is not yet commutative, is shown by just rearranging the direct sums in the preceding proof.

The final verification of "well-definedness" is that if $(g/b\backslash y) = (h/c\backslash z)$, then $-(g/b\backslash y) = -(h/c\backslash z)$. For this we need that if $\begin{pmatrix} y \\ -z \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} b & 0 \\ 0 & c \\ g & h \end{pmatrix}$, then $\begin{pmatrix} -y \\ z \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} -b & 0 \\ 0 & -c \\ -g & -h \end{pmatrix}$; this is obvious since the closure is a right ideal.

Thus we have the operations defined; we now need to check the conditions for M_Z to be an additive category. These conditions include associativity of $\cdot$ and +, identity for $\cdot$ and +, commutativity of + and distributivity.

We will start with associativity for composition. So we want to show that $[(f/a\backslash x)\cdot(g/b\ y)]\cdot(h/c\backslash z) = (f/a\backslash x)\cdot[(g/b\backslash y)\cdot(h/c\backslash z)]$. The left-hand side is

$((f\ 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} 0 \\ y \end{pmatrix})\cdot(h/c\backslash z)$

$$= ((f\ 0\ 0)\Big/\begin{pmatrix} a & -xg & 0 \\ 0 & b & -yh \\ 0 & 0 & c \end{pmatrix}\Big\backslash\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}).$$

The right-hand side is $(f/a\backslash x)\cdot((g\ 0)/\begin{pmatrix} b & -yh \\ 0 & c \end{pmatrix}\backslash\begin{pmatrix} 0 \\ z \end{pmatrix})$

$$= ((f\ 0\ 0)\Big/\begin{pmatrix} a & -xg & 0 \\ 0 & b & -yh \\ 0 & 0 & c \end{pmatrix}\Big\backslash\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}).$$

Notationally we are lucky, for they are the same.

To check the identity property of $\cdot$ on the right, we need that $(f/a\backslash x) = (f/a\backslash x)\cdot(1/1\backslash 1)$

$$= ((f\ 0)/\begin{pmatrix} a & -x1 \\ 0 & 1 \end{pmatrix}\backslash\begin{pmatrix} 0 \\ 1 \end{pmatrix}),\ \text{or}$$

$\begin{pmatrix} x \\ 0 \\ -1 \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} a & 0 & 0 \\ 0 & a & -x \\ 0 & 0 & 1 \\ f & f & 0 \end{pmatrix}$. But $\begin{pmatrix} 0 \\ x \\ -1 \\ 0 \end{pmatrix} \in \begin{pmatrix} 0 \\ -x \\ 1 \\ 0 \end{pmatrix}M$ and $\begin{pmatrix} x \\ -x \\ 0 \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} a & 0 \\ 0 & a \\ 0 & 0 \\ f & f \end{pmatrix}$

(the familiar argument again), so adding these gives the result.

For the identity property on the left, we need that $(f/a\backslash x) = (1/1\backslash 1)\cdot(f/a\backslash x) = ((1\ 0)/(\begin{smallmatrix}1 & -1f\\ 0 & a\end{smallmatrix})\backslash(\begin{smallmatrix}0\\ x\end{smallmatrix}))$, or

$$\begin{pmatrix}x\\0\\-x\\0\end{pmatrix} \in C^R\begin{pmatrix}a&0&0\\0&1&-f\\0&0&a\\f&1&0\end{pmatrix}. \text{ But } \begin{pmatrix}a&0\\0&0\\0&a\\f&f\end{pmatrix} = \begin{pmatrix}a&0&0\\0&1&-f\\0&0&a\\f&1&0\end{pmatrix}\begin{pmatrix}1&0\\0&f\\0&1\end{pmatrix}, \text{ so } \begin{pmatrix}x\\0\\-x\\0\end{pmatrix}$$

must be in the required closure. At this point we have at least a category M_Z.

Now, for associativity of +, we need that

$$\left[(f/a\backslash x) + (g/b\backslash y)\right] + (h/c\backslash z) = (f/a\backslash x) + \left[(g/b\backslash y) + (h/c\backslash z)\right].$$

The left side is $((f\ g)/(\begin{smallmatrix}a&0\\0&b\end{smallmatrix})\backslash(\begin{smallmatrix}x\\y\end{smallmatrix})) + (h/c\backslash z)$

$$= \left((f\ g\ h)\Big/\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}\Big\backslash\begin{pmatrix}x\\y\\z\end{pmatrix}\right).$$

The right side will obviously be the same.

For commutativity of +, we need that $(f/a\backslash x) + (g/b\backslash y) = (g/b\backslash y) + (f/a\backslash x)$, or $((f\ g)/(\begin{smallmatrix}a&0\\0&b\end{smallmatrix})\backslash(\begin{smallmatrix}x\\y\end{smallmatrix})) = ((g\ f)/(\begin{smallmatrix}b&0\\0&a\end{smallmatrix})\backslash(\begin{smallmatrix}y\\x\end{smallmatrix}))$. Thus we need to show that $\begin{pmatrix}x\\y\\-y\\-x\\0\end{pmatrix} \in C^R\begin{pmatrix}a&0&0&0\\0&b&0&0\\0&0&b&0\\0&0&0&a\\f&g&g&f\end{pmatrix}$. This is done by the same argument as for proving + is well-defined.

To check the identity property for +, we need only (now that commutativity is done) show that $(f/a\backslash x) = (f/a\backslash x) + (0/0\backslash 0) = ((f\ 0)/(\begin{smallmatrix}a&0\\0&0\end{smallmatrix})\backslash(\begin{smallmatrix}x\\0\end{smallmatrix}))$, or

$$\begin{pmatrix}x\\-x\\0\\0\end{pmatrix} \in C^R\begin{pmatrix}a&0&0\\0&a&0\\0&0&0\\f&f&0\end{pmatrix}.$$ But this is obvious.

Next we check distributivity of $\cdot$ over +, first verifying it on the right. We want

$$\left[(f/a\backslash x) + (g/b\backslash y)\right]\cdot(h/c\backslash z)$$
$$= (f/a\backslash x)\cdot(h/c\backslash z) + (g/b\backslash y)\cdot(h/c\backslash z).$$

The left side is $((f\ g)/\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\backslash\begin{pmatrix} x \\ y\end{pmatrix})\cdot(h/c\backslash z)$

$$= \left((f\ g\ 0)\Big/\begin{pmatrix} a & 0 & -xh \\ 0 & b & -yh \\ 0 & 0 & c\end{pmatrix}\Big\backslash\begin{pmatrix} 0 \\ 0 \\ z\end{pmatrix}\right).$$

The right side is $((f\ 0)/\begin{pmatrix} a & -xh \\ 0 & c\end{pmatrix}\backslash\begin{pmatrix} 0 \\ z\end{pmatrix}) +$

$$((g\ 0)/\begin{pmatrix} b & -yh \\ 0 & c\end{pmatrix}\backslash\begin{pmatrix} 0 \\ z\end{pmatrix})$$

$$= \left((f\ 0\ g\ 0)\Big/\begin{pmatrix} a & -xh & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & b & -yh \\ 0 & 0 & 0 & c\end{pmatrix}\Big\backslash\begin{pmatrix} 0 \\ z \\ 0 \\ z\end{pmatrix}\right).$$

Thus we need to check that

$$\begin{pmatrix} 0 \\ 0 \\ z \\ 0 \\ -z \\ 0 \\ -z \\ 0\end{pmatrix} \in C^R u, \text{ where } u = \begin{pmatrix} a & 0 & -xh & 0 & 0 & 0 & 0 \\ 0 & b & -yh & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & -xh & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & -yh \\ 0 & 0 & 0 & 0 & 0 & 0 & c \\ f & g & 0 & f & 0 & g & 0\end{pmatrix}.$$

Now $\begin{pmatrix} x \\ 0 \\ 0 \\ -x \\ 0 \\ 0 \\ 0 \\ 0\end{pmatrix} \in C^R \begin{pmatrix} a & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & a \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ f & f\end{pmatrix} \subseteq C^R u$, and $\begin{pmatrix} 0 \\ y \\ 0 \\ 0 \\ 0 \\ -y \\ 0 \\ 0\end{pmatrix} \in C^R \begin{pmatrix} 0 & 0 \\ b & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b \\ 0 & 0 \\ g & g\end{pmatrix} \subseteq C^R u$.

So $\begin{pmatrix} 0 \\ 0 \\ c \\ 0 \\ -c \\ 0 \\ -c \\ 0\end{pmatrix} = \begin{pmatrix} -xh \\ -yh \\ c \\ 0 \\ 0 \\ 0 \\ 0 \\ 0\end{pmatrix} + \begin{pmatrix} x \\ 0 \\ 0 \\ -x \\ 0 \\ 0 \\ 0 \\ 0\end{pmatrix} h - \begin{pmatrix} 0 \\ 0 \\ 0 \\ -xh \\ c \\ 0 \\ 0 \\ 0\end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 0 \\ 0 \\ 0 \\ -y \\ 0 \\ 0\end{pmatrix} h - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -yh \\ c \\ 0\end{pmatrix}$

is in $C^R u + uM = C^R u$. But then

$$\begin{pmatrix} 0 \\ 0 \\ z \\ 0 \\ -z \\ 0 \\ -z \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \\ -c \\ 0 \\ -c \\ 0 \end{pmatrix} \subseteq C^R C^R u = C^R u.$$

To verify distributivity on the left, we need that

$(f/a\backslash x)\cdot\ (g/b\backslash y) + (h/c\backslash z)$

$$= (f/a\backslash x)\cdot(g/b\backslash y) + (f/a\backslash x)\cdot(h/c\backslash z).$$

The left side is $(f/a\backslash x)\cdot((g\ h)/\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}\backslash\begin{pmatrix} y \\ z \end{pmatrix})$

$$= \left((f\ 0\ 0)\Big/\begin{pmatrix} a & -xg & -xh \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\Big\backslash\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}\right).$$

The right side is $((f\ 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} 0 \\ y \end{pmatrix}) +$

$$((f\ 0)/\begin{pmatrix} a & -xh \\ 0 & c \end{pmatrix}\backslash\begin{pmatrix} 0 \\ z \end{pmatrix})$$

$$= \left((f\ 0\ f\ 0)\Big/\begin{pmatrix} a & -xg & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & -xh \\ 0 & 0 & 0 & c \end{pmatrix}\Big\backslash\begin{pmatrix} 0 \\ y \\ 0 \\ z \end{pmatrix}\right).$$

So we need that

$$\begin{pmatrix} 0 \\ y \\ z \\ 0 \\ -y \\ 0 \\ -z \\ 0 \end{pmatrix} \in C^R u, \text{ where we put } u = \begin{pmatrix} a & -xg & -xh & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & -xg & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & -xh \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & f & 0 & f & 0 \end{pmatrix}.$$

We use $\begin{pmatrix} x \\ 0 \\ 0 \\ -x \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & a \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ f & f \end{pmatrix} \subseteq C^R u$ to get

$$\begin{pmatrix}0\\b\\0\\0\\-b\\0\\0\\0\end{pmatrix} = \begin{pmatrix}-xg\\b\\0\\0\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}x\\0\\0\\-x\\0\\0\\0\\0\end{pmatrix} g - \begin{pmatrix}0\\0\\0\\-xg\\b\\0\\0\\0\end{pmatrix} \in C^R u + uM = C^R u, \text{ so that}$$

$$\begin{pmatrix}0\\y\\0\\0\\-y\\0\\0\\0\end{pmatrix} \in C^R \begin{pmatrix}0\\b\\0\\0\\-b\\0\\0\\0\end{pmatrix} \subseteq C^R C^R u = C^R u. \text{ Likewise, } \begin{pmatrix}0\\0\\z\\0\\0\\0\\-z\\0\end{pmatrix} \in C^R u,$$

so the sum is in $C^R u$, as needed.

So now M_Z is an Ab-category. Let us check that E is an additive functor. By definition it preserves 1, 0, and $-$.

To check functoriality, take two composable morphisms x, y in M. Then we want $E(xy) = E(x)E(y)$, or

$(xy/xy\backslash xy) = (x/x\backslash x)\cdot(y/y\backslash y) = ((x\ 0)/\begin{pmatrix}x & -xy\\0 & y\end{pmatrix}\backslash\begin{pmatrix}0\\y\end{pmatrix})$.

Thus we need $\begin{pmatrix}xy\\0\\-y\\0\end{pmatrix} \in C^R \begin{pmatrix}xy & 0 & 0\\0 & x & -xy\\0 & 0 & y\\xy & x & 0\end{pmatrix}$. But the morphism is even in the ideal itself, not just the closure, since

$$\begin{pmatrix}xy\\0\\-y\\0\end{pmatrix} = \begin{pmatrix}xy & 0 & 0\\0 & x & -xy\\0 & 0 & y\\xy & x & 0\end{pmatrix}\begin{pmatrix}1\\-y\\-1\end{pmatrix}.$$

For additivity of E, take addable morphisms x, z in M, and show $E(x+z) = E(x) + E(z)$, or

$(x+z/x+z\backslash x+z) = (x/x\backslash x) + (z/z\backslash z) = ((x\ z)/\begin{pmatrix}x & 0\\0 & z\end{pmatrix}\backslash\begin{pmatrix}x\\z\end{pmatrix})$.

Like the above, this follows from $\begin{pmatrix}x+z\\-x\\-z\\0\end{pmatrix} = \begin{pmatrix}x+z & 0 & 0\\0 & x & 0\\0 & 0 & z\\x+z & x & z\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$.

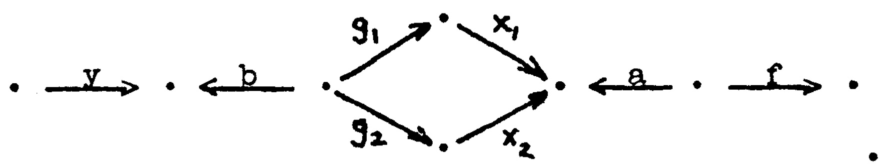
Finally, we can see that M_Z has ⊕'s and a zero object, since our additive functor E will preserve these. That is, their existence in M will guarantee their existence in M_Z, so that M_Z is additive. This completes the proof of 3.4.

Computational Tools on M_Z. We state and prove some direct results of the construction which serve both as tools for future computations and as support for the intuition of $fa^{-1}x$ for $(f/a\backslash x)$.

Proposition 3.5: $(f/a\backslash x) = 0$ if and only if $\binom{x}{0} \in C^R\binom{a}{f}$.

Proof: $(f/a\backslash x) = 0$ is equivalent to $(f/a\backslash x) = (0/0\backslash 0)$, or $\begin{pmatrix}x\\0\\0\end{pmatrix} \in C^R\begin{pmatrix}a&0\\0&0\\f&0\end{pmatrix}$, whence the result is clear.

Proposition 3.6: Suppose $(f/a\backslash x_i)$ and $(g_i/b\backslash y)$ (for i=1,2) all belong to M_Z, forming the diagram



If $x_1g_1 = x_2g_2$, then $(f/a\backslash x_1)\cdot(g_1/b\backslash y) = (f/a\backslash x_2)\cdot(g_2/b\backslash y)$.

Proof: Trivial from the definition of composition.

Proposition 3.7: If $(fu/au\backslash x)$ and $(f/va\backslash vx)$ are both in M_Z, then so is $(f/a\backslash x)$ and they are all equal.

Proof: We have $x \in C^R au$ by assumption, but $auM \subseteq aM$, so $x \in C^R a$. Likewise $g \in C^L va \subseteq C^L a$, so that $(f/a\backslash x)$ is in M_Z. Using $x \in C^R au$, we get

$$\begin{pmatrix}x\\-x\\0\end{pmatrix} \in C^R\begin{pmatrix}au&0\\0&au\\fu&fu\end{pmatrix} = C^R\begin{pmatrix}au&0\\0&a\\fu&f\end{pmatrix}\begin{pmatrix}1&0\\0&u\end{pmatrix} \subseteq C^R\begin{pmatrix}au&0\\0&a\\fu&f\end{pmatrix},$$ so that

$(fu/au\backslash x) = (f/a\backslash x)$. The other equality comes from

$$\begin{pmatrix} x \\ -vx \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} 1 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} au & 0 \\ 0 & a \\ fu & f \end{pmatrix} = C^R \begin{pmatrix} au & 0 \\ 0 & va \\ fu & f \end{pmatrix}, \text{ or}$$

$(fu/au\backslash x) = (f/va\backslash vx)$.

Corollary 3.8: (i) $E(u) = (u/u\backslash u) = (u/1\backslash 1) = (1/1\backslash u)$.

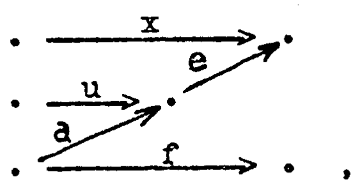
(ii) $E(v)\cdot(f/a\backslash x) = (vf/a\backslash x)$ and

$(f/a\backslash x)\cdot E(u) = (f/a\backslash xu)$.

(iii) $(va/a\backslash x) = E(vx)$ and $(f/a\backslash au) = E(fu)$.

Proof: Part (i) is immediate from 3.7. Part (ii) follows from 3.6 and (i), using $E(v) = (1/1\backslash v)$ and $E(u) = (u/1\backslash 1)$. Part (iii) follows from 3.7, (i) and (ii). (For example, $(va/a\backslash x) = E(v)(a/a\backslash x) = E(v)(1/1\backslash x) = E(vx)$.)

Proposition 3.9: If all three of the factors in the following equation are morphisms of M_Z, with diagram



then the equation $(f/a\backslash u)\cdot(1/eu\backslash x) = (f/ea\backslash x)$ is satisfied.

Proof: We need that $(f/ea\backslash x) = ((f\ 0)/\begin{pmatrix} a & -u \\ 0 & eu \end{pmatrix}\backslash\begin{pmatrix} 0 \\ x \end{pmatrix})$,

or $\begin{pmatrix} x \\ 0 \\ -x \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} ea & 0 & 0 \\ 0 & a & -u \\ 0 & 0 & eu \\ f & f & 0 \end{pmatrix}$. Clearly $\begin{pmatrix} -u \\ u \\ 0 \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} a & 0 \\ 0 & a \\ 0 & 0 \\ f & f \end{pmatrix}$, so

$\begin{pmatrix} -eu \\ u \\ 0 \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} ea & 0 \\ 0 & a \\ 0 & 0 \\ f & f \end{pmatrix}$. But then $\begin{pmatrix} -eu \\ 0 \\ eu \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} ea & 0 & 0 \\ 0 & a & -u \\ 0 & 0 & eu \\ f & f & 0 \end{pmatrix}$, so we

use $\begin{pmatrix} x \\ 0 \\ -x \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} -eu \\ 0 \\ eu \\ 0 \end{pmatrix}$ to get the needed fact.

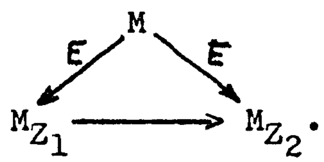
We can also give a description of the "kernel" of the functor E, an ideal of M at each object of M.

<u>Proposition 3.10</u>: For $a:P \longrightarrow Q$ a morphism of M, $E(a) = 0$ in M_Z if and only if $a \in C^R 0 \subseteq 1_Q M$.

<u>Proof</u>: By 3.5, $E(a) = 0$ if and only if $\binom{a}{0} \in C^R\binom{a}{a}$. If $a \in C^R 0$, then $\binom{a}{0} \in C^R\binom{0}{0} \subseteq C^R\binom{a}{a}$. Conversely, if $\binom{a}{0} \in C^R\binom{a}{a}$, then $a = (1\ -1)\binom{a}{0} \in C^R(1\ -1)\binom{a}{a} = C^R 0$.

<u>Functoriality on Pairs</u>. For two pairs $Z_1 = (C_1^L, C_1^R)$ and $Z_2 = (C_2^L, C_2^R)$ on M, define $Z_1 \leq Z_2$ if $C_1^L \leq C_2^L$ and $C_1^R \leq C_2^R$. Then we can get the following "functorial" relationship among the localizations M_Z.

<u>Proposition 3.11</u>: If $Z_1 \leq Z_2$ are right coherent pairs on M, then there is an additive functor from M_{Z_1} to M_{Z_2} which makes the following functorial diagram commute:



<u>Proof</u>: The functor just fixes the objects and takes morphisms $(f/a\backslash x)$ in M_{Z_1} to $(f/a\backslash x)$ in M_{Z_2}. The order assumption on the pairs makes all the appropriate properties (including well-definedness) obvious.

In some particular cases we can make statements about the fullness or faithfulness of the functor E.

<u>Proposition 3.12</u>: If $Z = (C^L, C^R)$ is right coherent and C^L is discrete (i.e., $C^L = C_o^L$), then every morphism $(f/a\backslash x)$ in M_Z is given by $E(a)$ for some morphism a in M (i.e., E is full).

<u>Proof</u>: For such a $(f/a\backslash x)$, $f \in C^L a = Ma$, since C^L is discrete. Thus $f = va$ and so $(f/a\backslash x) = E(vx)$ by 3.8(iii).

<u>Proposition 3.13</u>: If $Z = (C^L, C^R)$ is right coherent and C^R is discrete, then E gives an additive isomorphism of the additive categories M and M_Z.

<u>Proof</u>: As in 3.12 we can use 3.8(iii) to see that E is full. But now if E(a) = 0, we have by 3.10 that $a \in C^R 0 = 0M$, so a = 0 already. Thus E is faithful; since the objects of M and M_Z are the same, E is an isomorphism.

<u>Right and Left Coherent Pairs</u>. Of course, the dual of the entire construction can be completed, using a left coherent pair of a.c.o.'s on M, defining equivalence on A_Z in the dual fashion (namely that the two diagrams $S \xrightarrow{x} Q \xleftarrow{a} P \xrightarrow{f} T$ and $S \xrightarrow{y} Q' \xleftarrow{b} P' \xrightarrow{g} T$ are <u>left</u> equivalent if $(f\ -g\ 0) \in C^L\begin{pmatrix} a & 0 & x \\ 0 & b & y \end{pmatrix}$) and going through the dual proof to obtain the <u>left</u> zigzag localization. But what happens to the two constructions if the same pair is both right and left coherent? Then we can show them to be the same.

<u>Proposition 3.14</u>: If $Z = (C^L, C^R)$ is right coherent, and if the diagrams in the paragraphs above are left equivalent in A_Z, then they are right equivalent also, i.e. we have $(f/a\backslash x) = (g/b\backslash y)$.

<u>Proof</u>: We have assumed that $(f\ -g\ 0) \in C^L\begin{pmatrix} a & 0 & x \\ 0 & b & y \end{pmatrix}$, so also $(f\ g\ 0) \in C^L\begin{pmatrix} a & 0 & x \\ 0 & -b & y \end{pmatrix} = C^L\begin{pmatrix} a & 0 & x \\ 0 & b & -y \end{pmatrix}$, by "generalized row operations". Thus $\begin{pmatrix} a & 0 & x \\ 0 & b & -y \\ f & g & 0 \end{pmatrix} \in C^L\begin{pmatrix} a & 0 & x \\ 0 & b & -y \end{pmatrix}$. Now $x \in C^R a$ and $y \in C^R b$, so $\begin{pmatrix} x \\ -y \end{pmatrix} \in C^R\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Hence by right coherence we

have $\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} a & 0 \\ 0 & b \\ f & g \end{pmatrix}$, or $(f/a\backslash x) = (g/b\backslash y)$.

Corollary 3.15: If $Z = (C^L, C^R)$ is right and left coherent, then the right and left equivalence relations on A_Z are the same, so the right and left zigzag localizations M_Z are the same.

Proof: First note that the formation of A_Z does not depend on any coherence condition being assumed, and then apply 3.14 and its right-left dual.

The Reflected Left Closure Operator. Given a right a.c.o. C on M, we can try to form a left a.c.o. C^* (the "reflection" of C) which will give a right coherent pair. This is possible though the construction is perhaps not well motivated.

If Q, Q' are objects of M, note that $\begin{pmatrix} 0 \\ 1_{Q'} \end{pmatrix}M$ is a subideal of $1_{Q\oplus Q'}M$. We will say that an ideal $I \subseteq 1_{Q\oplus Q'}M$ is C-disjoint from Q' if for every ideal $J \subseteq 1_{Q'}M$ we have $C(I + (\begin{smallmatrix} 0 \\ J \end{smallmatrix})) \cap \begin{pmatrix} 0 \\ 1_{Q'} \end{pmatrix}M = C(\begin{smallmatrix} 0 \\ J \end{smallmatrix})$, where $(\begin{smallmatrix} 0 \\ J \end{smallmatrix}) = \begin{pmatrix} 0 \\ 1_{Q'} \end{pmatrix}J$. Clearly we have the inclusion $\supseteq$, so the point is that the Q'-part of I is "inessential". Thus of course $\begin{pmatrix} 1_Q \\ 0 \end{pmatrix}M$ is C-disjoint from Q'. Noting that $aC(I') = C(aI')$ for any ideal I' and invertible morphism a, we can see that the image $\begin{pmatrix} 1_Q \\ v \end{pmatrix}M$ of $\begin{pmatrix} 1_Q \\ 0 \end{pmatrix}M$ under the automorphism $\begin{pmatrix} 1_Q & 0 \\ v & 1_{Q'} \end{pmatrix}$ is also C-disjoint from Q'. Clearly any subideal of an ideal C-disjoint from Q' is also C-disjoint, so that $\begin{pmatrix} 1_Q \\ v \end{pmatrix}aM = (\begin{smallmatrix} a \\ va \end{smallmatrix})M$ is C-disjoint from Q'.

So now define $f:P \longrightarrow Q''$ to be <u>C-dominated</u> by $a:P \longrightarrow Q$ if $\binom{a}{vf}M$ is C-disjoint from Q' for all Q' and all $v:Q'' \longrightarrow Q'$. Then for such an a, denote by $C^*(a) \subseteq Ml_P$ the set of morphisms C-dominated by a.

<u>Lemma 3.16</u>: (1) $a \in C^*(a)$.

(2) If $v \in Ml_Q$, then $C^*(va) \subseteq C^*(a)$.

(3) If $f \in C^*(a)$ and $g \in C^*(f)$, then $g \in C^*(a)$.

(4') $vC^*(a) \subseteq C^*(a)$.

(4") Given $a:P \longrightarrow Q$, $b:P \longrightarrow Q'$ and $f \in C^*(a)$, $g \in C^*(b)$ with common codomain, then $f+g \in C^*(\binom{a}{b})$.

(5) $C^*(a)u \subseteq C^*(au)$.

(The numbers of the statements refer to the analogous parts of the definition of a.c.o.)

<u>Proof</u>: Part (1) is just the remark immediately preceding the definition of C-dominance.

Let us do (3) before (2). Take v with codomain Q' and try to show $\binom{a}{vg}M$ is C-disjoint from Q'. That is, pick $J \subseteq 1_{Q'}M$ and assume $\binom{0}{w} \in C(\binom{a}{vg}M + \binom{0}{J})$. Then

$$\begin{pmatrix}0\\0\\w\end{pmatrix} \in C(\begin{pmatrix}a\\0\\vg\end{pmatrix}M + \begin{pmatrix}0\\0\\J\end{pmatrix}) \subseteq C(\begin{pmatrix}a\\-f\\0\end{pmatrix}M + \begin{pmatrix}0\\f\\vg\end{pmatrix}M + \begin{pmatrix}0\\0\\J\end{pmatrix}).$$ Now put

$v' = \binom{-1_{Q'}}{0}$, $w' = \binom{0}{w}$ and $J' = \binom{f}{vg}M + \binom{0}{J}$ to get $\binom{0}{w'} \in C(\binom{a}{v'f}M + \binom{0}{J'})$. Now a C-dominates f by assumption, so we get $w' \in C(J')$, or $\binom{0}{w} \in C(\binom{f}{vg}M + \binom{0}{J})$. Similarly, using $g \in C^*(f)$, we may "cancel" the $\binom{f}{vg}$ part to get $\binom{0}{w} \in C\binom{0}{J}$. Thus $g \in C^*(a)$.

Now to prove (2), using (3) we need only show that $va \in C^*(a)$, or a C-dominates va. But this is obvious from the definition, as is (4').

Part (4"): Take v with codomain Q'' and $J \subseteq 1_{Q''}M$, and assume that $\begin{pmatrix}0\\0\\w\end{pmatrix} \in C(\begin{pmatrix}a\\b\\v(f+g)\end{pmatrix}M + \begin{pmatrix}0\\0\\J\end{pmatrix}) \subseteq C(\begin{pmatrix}a\\0\\vf\end{pmatrix}M + \begin{pmatrix}0\\b\\vg\end{pmatrix}M + \begin{pmatrix}0\\0\\J\end{pmatrix})$.

Then as before we may successively "cancel" the $\begin{pmatrix}a\\0\\vf\end{pmatrix}$ and $\begin{pmatrix}0\\b\\vg\end{pmatrix}$ parts to obtain $\begin{pmatrix}0\\0\\w\end{pmatrix} \in C\begin{pmatrix}0\\0\\J\end{pmatrix}$. Thus $\begin{pmatrix}a\\b\\v(f+g)\end{pmatrix}M$ is C-disjoint from Q'', so $f+g \in C^*(\binom{a}{b})$.

For part (5), take $f \in C^*(a)$. We want, for v with codomain Q', that $\binom{au}{vfu}M$ is C-disjoint from Q'. But this is a subideal of $\binom{a}{vf}M$, which is C-disjoint, so we are done.

Using this Lemma, we can define a left a.c.o. on M. For any subset X of $M1_P$, define

$$C^*(X) = \sum_{f \in MX} C^*(f).$$

<u>Theorem 3.17</u>: C^* is a finitary algebraic closure operator on left ideals of M.

<u>Proof</u>: Using $v = (0\ 1)$ and $v = (1\ 0)$ in 3.16(2), we see that $C^*\binom{a}{b} \supseteq C^*(a) \cup C^*(b)$. For $a = b$, we can use $v = \binom{1}{1}$ in 3.16(4') to get $C^*\binom{a}{a} = C^*(a)$, and then by 3.16(4',4"), $C^*(a)$ is a left ideal for all a. Thus $C^*\binom{a}{b} \supseteq C^*(a) + C^*(b)$, or "directedness" for the sum which gives $C^*(X)$. Further more, $C^*(a) = C^*(Ma)$, again using 3.16(2).

Now condition (1) for a.c.o.'s comes immediately from 3.16(1), and (2) for a.c.o.'s is now obvious. The "idempotency" condition (3) comes directly from 3.16(3). The algebraic condition (4) follows from $C^*(a)$ being an ideal, plus "directedness". The condition (5) follows directly from 3.16(5).

The a.c.o. C^* is finitary (condition (2+)) because anything in $C^*(X)$ is (by directedness) in $C^*(a)$ for some a in MX, and any element of MX is generated by a finite number of elements of X.

Coherence of the Reflected Operator. We would like to know if the pair (C^*,C) satisfies any coherence conditions (right or left). In some sense it seems to be the best possible.

Proposition 3.18: The pair (C^*,C) is right coherent.

Proof: Suppose $b \in C(a)$ and $(a'\ b') \in C^*(a\ b)$, as for defining right coherence. Now $\binom{b}{0} \in C\binom{a}{0} \subseteq C\binom{a}{a'}\binom{0}{a'}$ by column operations. Hence $\binom{0}{b'} = \binom{b}{b'} - \binom{b}{0} \in C\left(\binom{a}{a'}\ \binom{b}{b'}\ \binom{0}{a'}\right)$ as well. Since (a b) C-dominates $(a'\ b')$, we get $\binom{0}{b'} \in C\binom{0}{a'}$, or $b' \in C(a')$, as needed.

Proposition 3.19: The pair (C^*,C) is left coherent.

Proof: Translating the coherence condition to the left, we need to show that if $b \in C^*(a)$ and $\binom{a'}{b'} \in C\binom{a}{b}$, then $b' \in C^*(a')$. So we need a' to C-dominate b'. Take v with codomain Q' and $J \subseteq 1_{Q'}M$, and assume that $\binom{0}{w} \in C(\binom{a'}{vb'}M + \binom{0}{J})$. Now $\binom{a'}{b'} \in C\binom{a}{b}$ gives $\binom{a'}{vb'} \in C\binom{a}{vb}$. Thus $\binom{0}{w} \in C(\binom{a}{vb}M + \binom{0}{J})$ also. But a C-dominates b, so $w \in C(J)$. We did this for any Q', v and J, so we have shown $b' \in C^*(a')$.

Proposition 3.20: Suppose C^L and C^R are a left and right a.c.o. on M, respectively. If $C^L \leq C^{R*}$, then the pair (C^L, C^R) is right coherent; and if both C^L and C^R are finitary, then the converse also holds.

Proof: If $C^L \leq C^{R*}$, then right coherence follows from 3.18 and 3.1(ii). For the converse, we need only show (assuming coherence) that $C^L a \subseteq C^{R*}a$, since C^L is finitary. So take any $f \in C^L a$; we want $\binom{a}{vf}M$ to be C^R-disjoint from Q' for all v with codomain Q'. Thus take $J \subseteq 1_{Q'}M$ and assume $\binom{0}{w} \in C^R(\binom{a}{vf}M + \binom{0}{J})$. Because C^R is also finitary, we can assume that actually $\binom{0}{w} \in C^R(\binom{a}{vf}M + \binom{0}{x}M)$ $= C^R\begin{pmatrix} a & 0 \\ vf & x \end{pmatrix}$, for some $x \in J$. Since $f \in C^L a$, we get $(vf\ 0\ 0) \in C^L(a\ 0\ 0)$ and so $(0\ x\ w) \in C^L\begin{pmatrix} a & 0 & 0 \\ vf & x & w \end{pmatrix}$, using row operations. Now right coherence gives $w \in C^R(0\ x)$ $= C^R x \subseteq C^R J$, just as needed.

We are perhaps fortunate that the reflected a.c.o. satisfies such nice properties. However, this construction is not functorial, in the sense that if $C_1 \leq C_2$, then we have no information about whether $C_1^* \leq C_2^*$. This failure of functoriality will be demonstrated in Chapter Four.

Inducing Closure Operators. Suppose we have an additive functor H from an additive category M to another, say N. Suppose also that there is a right a.c.o. C on N. Then we easily get an induced a.c.o. C_H on M, by putting $C_H(X) = \{y \in 1_P M \mid H(y) \in C(H(X))\}$ for $X \subseteq 1_P M$. We can say that C_H is obtained by "pulling back C along H." The induced a.c.o. is finitary if the original one is.

In particular, we may pull back the discrete operator C_o^R on N. Our motivating case (Chapter One) of defining C_S from $R \longrightarrow S$ is just an example of this, using the categories of finitely generated free (or projective) modules.

We may also induce a.c.o.'s from left a.c.o.'s on N. We will distinguish these by our usual "L" and "R" superscripts.

<u>Proposition 3.21</u>: Suppose (C^L, C^R) is right coherent on N. Then (C_H^L, C_H^R) is right coherent on M.

<u>Proof</u>: Assume $b \in C_H^R a$ and $(a' \; b') \in C_H^L(a \; b)$. This means $H(b) \in C^R(H(a))$ and $(H(a') \; H(b')) = H(a' \; b')$ belongs to $C^L(H(a \; b))$. By right coherence of (C^L, C^R) we get $H(b') \in C^R(H(a'))$, or $b' \in C_H^R(a')$.

<u>Proposition 3.22</u>: Suppose $Z = (C^L, C^R)$ is right coherent on N, and put $Z_H = (C_H^L, C_H^R)$. Then there is a faithful additive functor from M_{Z_H} to N_Z, making the following functorial diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{H} & N \\ {\scriptstyle E_M}\downarrow & & \downarrow{\scriptstyle E_N} \\ M_{Z_H} & \longrightarrow & N_Z \end{array} .$$

<u>Proof</u>: M_{Z_H} exists by 3.21. Suppose $(f/a\backslash x)$ is a morphism of M_{Z_H} and P an object. Then the functor associates to these the morphism $(H(f)/H(a)\backslash H(x))$ and the object H(P) (recall that the objects of M_{Z_H} are the same as those of M). Clearly this is well-defined and the diagram commutes. For faithfulness, suppose $(H(f)/H(a)\backslash H(x))$ is zero; then $\begin{pmatrix} H(x) \\ 0 \end{pmatrix} \in C^R \begin{pmatrix} H(a) \\ H(f) \end{pmatrix}$ by 3.5. But then $\binom{x}{0} \in C_H^R\binom{a}{f}$ by definition, so $(f/a\backslash x) = 0$ already.

Regarding the reflected a.c.o., we get:

Proposition 3.23: If C is a right a.c.o. on N, then $(C_H)^* \geq (C^*)_H$.

Proof: This just follows from the obvious fact that if H(I) is C-disjoint from H(Q'), then I is C_H-disjoint from Q'.

Now given a right coherent pair $Z = (C^L, C^R)$ on M, we can form the localization M_Z and then pull back the discrete a.c.o.'s on M_Z along the functor E to get a new right coherent pair $Z_E = (C^L_{oE}, C^R_{oE})$ on M.

Proposition 3.24: $Z_E \leq Z$.

Proof: Let $f \in C^R_{oE}(X)$ and try to show $f \in C^R(X)$. The discrete a.c.o. is finitary, so C^R_{oE} is finitary; hence we may assume that $f \in C^R_{oE}(a)$ for a single $a \in XM$. But then $E(f) \in C^R_o(E(a)) = E(a)M$ by definition, so $(f/f\backslash f) = (a/a\backslash a)\cdot(g/b\backslash y) = ((a\ 0)/\begin{pmatrix} a & -ag \\ 0 & b\end{pmatrix}\backslash\begin{pmatrix}0\\ y\end{pmatrix})$, for some $(g/b\backslash y)$. Thus we get

$$\begin{pmatrix} f \\ 0 \\ -y \\ 0 \end{pmatrix} \in C^R\begin{pmatrix} f & 0 & 0 \\ 0 & a & -ag \\ 0 & 0 & b \\ f & a & 0 \end{pmatrix}.$$

Now apply (1 1 0 -1) on the left to get $f \in C^R(0\ 0\ -ag) = C^R(ag) \subseteq C^R(a) \subseteq C^R(X)$.

The inclusion $C^L_{oE} \leq C^L$ is similar.

One might expect that there should be equality of the coherent pairs here, or at least that M_{Z_E} should be isomorphic to M_Z. That this may not be true will be shown in Chapter Four, where we develop some specific examples.

CHAPTER FOUR

ZIGZAG LOCALIZATIONS OF PROJECTIVE MODULE CATEGORIES

In this chapter we will consider the case when the additive category M is in fact the category of finitely generated (say right) projective modules over a ring R, as in Chapter Two. This category (with just R-module homomorphisms as morphisms) will be denoted P-Mod, or P_R-Mod if necessary to specify the ring or the "sidedness".

In this case we get a nice correspondence between ideals of P-Mod and submodules of objects of P-Mod. This is because projectives have the "lifting" property; that is, if a map $f:P \longrightarrow Q$ from a projective P has its image $Im(f) \subseteq Im(g)$ for some $g:P' \longrightarrow Q$, then f "lifts" to a map $F:P \longrightarrow P'$ such that $gF = f$. Thus (letting M = P-Mod) if f and g are any morphisms in M with the same codomain, then $Im(f) \subseteq Im(g) \iff fM \subseteq gM \iff f \in gM$.

Also in this case we may correspond an algebraic closure operator on right ideals of M to an algebraic closure operator on the submodules of objects of M, as defined in Chapter Two. To describe this correspondence, define the <u>image</u> Im(I) of a set $I \subseteq 1_P M$ of morphisms with codomain P to be the union of the images of elements of I. Then Im(IM) is the image of the right ideal IM and forms a submodule of P (here again we use that M is additive). Given a right a.c.o. on M, P an object of M, and $X \subseteq P$, let I be the ideal of morphisms with codomain P and image contained in the submodule of P generated by X. Then we define the corresponding closure $C(X,P)$ of X in P to be $Im(C(I))$. This is easily seen to satisfy the conditions in Chapter Two for an algebraic closure operator. If, for $y \in P$ we define $m_y: R \longrightarrow P$ by $m_y(r) = yr$, then note that an equivalent definition is $C(X,P) = \{y \mid m_y \in C(I)\}$.

Another convenient fact is that the category of finitely generated left projective modules ${}_R P\text{-Mod}$ is anti-isomorphic to $P_R\text{-Mod}$, via the "dual" functor which associates to the module P its dual $P^* = Hom(P,R)$. For $f: P \longrightarrow Q$, we get $f^*: Q^* \longrightarrow P^*$, and P^{**} is naturally isomorphic to P. Thus for f and g morphisms in $P_R\text{-Mod}$ with the same domain, we get

$$\begin{aligned} Im(f^*) \subseteq Im(g^*) &\iff f^*({}_R P\text{-Mod}) \subseteq g^*({}_R P\text{-Mod}) \\ &\iff f^* \in g^*({}_R P\text{-Mod}) \\ &\iff (P_R\text{-Mod})f \subseteq (P_R\text{-Mod})g \\ &\iff f \in (P_R\text{-Mod})g. \end{aligned}$$

(Here $(P_R\text{-Mod})g$ is the left ideal generated by g.) Thus a left a.c.o. on $P_R\text{-Mod}$ may also be considered as a right a.c.o. on ${}_R P\text{-Mod}$.

Zigzag Localizations of Rings. Suppose $Z = (C^L, C^R)$ is a right coherent pair of a.c.o.'s on $M = P_R\text{-Mod}$, and we construct the zigzag localization M_Z. Let R_Z be the endomorphism ring $M_Z(R,R)$ of the object R in the new category M_Z. Note that the elements of R_Z are of the form $(f/a\backslash x)$ with diagram $R \xrightarrow{x} Q \xleftarrow{a} P \xrightarrow{f} R$, so x "is" an element of Q and f a functional on P, as in Chapter Two. Then since E is an additive functor from M to M_Z, and R $\cong$ M(R,R), we get a ring homomorphism $E: R \longrightarrow R_Z$, so we can call R_Z the zigzag localization of R with respect to Z.

In general, given any additive functor $H: P_R\text{-Mod} \longrightarrow N$ we can form the ring S = N(H(R),H(R)). Any object P of P_R-Mod is a direct summand of a direct sum of copies of R, and H will preserve this fact. Hence N(H(R),H(P)) is a projective S-module, and in fact N(H(R),H(P)) = $P \otimes_R S$.

In the specific case described above, the functor E is bijective on objects, so we can see that M_Z is equivalent to the full subcategory of P_{R_Z}-Mod whose objects are of the form $P \otimes_R R_Z$, for P in P_R-Mod.

Isbell's Dominion. In our discussion of specific examples of zigzag localizations of P-Mod categories, we will need the construction of a subring somewhat like the rational closure $\Sigma(R,S)$ of Chapter Two. For these purposes let us assume R is a subring of a ring S. Then there is the following known theorem, whose proof we include for completeness.

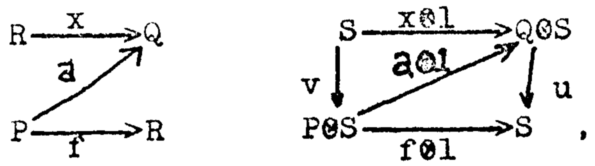
Theorem 4.1: For $w \in S$, the following conditions are equivalent:

(i) For any ring T and any pair of ring homomorphisms from S to T, if the homomorphisms agree on every element of R, then they agree on w.

(ii) In the tensor product $S \otimes_R S$, we have $1 \otimes w = w \otimes 1$.

(iii) There are positive integers n and m, an m-by-n matrix A over R, a 1-by-m matrix u over S, and an n-by-1 matrix v over S, such that both uA and Av are matrices over R, and such that w = uAv.

(iv) There are finitely generated right projective R-modules P and Q and maps as in the diagrams



where x, a, f are R-homomorphisms and u,v are S-homomorphisms, satisfying u(a⊗1) = f⊗1, (a⊗1)v = x⊗1, and u(a⊗1)v(1) = w.

(The so-called "zigzags" in (iii) and (iv) are a further justification for the term "zigzag localization".)

Proof: (i) $\Longrightarrow$ (ii): (Taken from Bergman ([1], p. 2) or Silver ([8], p. 46).) Make $S \oplus (S \otimes_R S)$ into a ring by defining the product of two elements $(s, \Sigma a_i \otimes b_i)$ and $(t, \Sigma c_i \otimes d_i)$ to be $(st, \Sigma(sc_i)\otimes d_i + \Sigma a_i \otimes (b_i t))$. Then note that $(1, 1\otimes 1)$ is a unit, with inverse $(1, -1\otimes 1)$. There is an obvious homomorphism H_1 from S into this ring, given by $H_1(s) = (s,0)$. Another homomorphism H_2 may be obtained by conjugating H_1 with $(1, 1\otimes 1)$; that is, $H_2(s) = (1,1\otimes 1)\cdot(s,0)\cdot(1,-1\otimes 1) = (s, 1\otimes s - s\otimes 1)$. The two homomorphisms agree on R ($1\otimes r = r\otimes 1$ for r in R), so they agree on w; that is, $1\otimes w - w\otimes 1 = 0$.

(ii) $\Longrightarrow$ (iii): (Taken partially from Mazet ([7], pp. 2-1, 2, 3).) Let us take generators $s_0, s_1, \cdots$ for S as a left R-module, with $s_0 = 1$ and $s_1 = w$. Then we have an exact sequence of left R-modules $0 \longrightarrow K \xrightarrow{i} F \xrightarrow{p} S \longrightarrow 0$, where F is a free R-module on generators $\{e_0, e_1, \cdots\}$, $p(e_j) = s_j$ for each j, and K is the kernel. Tensoring with S, the sequence $S\otimes_R K \xrightarrow{1\otimes i} S\otimes_R F \xrightarrow{1\otimes p} S\otimes_R S \longrightarrow 0$ is still exact, and $1\otimes p(w\otimes e_0 - 1\otimes e_1) = 0$ by the assumption (ii). Thus

$w\otimes e_0 - 1\otimes e_1 \in 1\otimes i(S\otimes K)$, say $w\otimes e_0 - 1\otimes e_1 = 1\otimes i(\sum_{j=1}^{m} t_j\otimes d_j)$

$= \sum_{j=0}^{m} t_j\otimes i(d_j)$, with each t_j in S and d_j in K. Now write

$i(d_j) = \sum_{k=0}^{n} a_{jk}e_k$ in F, with each a_{jk} in R. Then

$$\begin{aligned} w\otimes e_0 - 1\otimes e_1 &= \sum_{j=1}^{m}(t_j\otimes \sum_{k=0}^{n} a_{jk}e_k) \\ &= \sum_{j=1}^{m}\sum_{k=0}^{n} t_j\otimes a_{jk}e_k \\ &= \sum_{k=0}^{n}(\sum_{j=1}^{m} t_j a_{jk})\otimes e_k. \end{aligned}$$

Since F is free, we get $w = \sum_{j=1}^{m} t_j a_{j0}$, $-1 = \sum_{j=1}^{m} t_j a_{j1}$, and

$0 = \sum_{j=1}^{m} t_j a_{jk}$ for $k=2,3,\cdots,n$. But also $0 = p(i(d_j))$

$= \sum_{k=0}^{n} a_{jk}s_k$ for each $j=1,2,\cdots,m$. Now let A be the m-by-n matrix (a_{jk}), for $1\leq j\leq m$ and $1\leq k\leq n$, which has entries in R (note that we leave out the k=0 terms). Take u to be the 1-by-m matrix (t_j) for $1\leq j\leq m$ and v to be the n-by-1 matrix $(-s_k)$, $1\leq k\leq n$. Then $uA = (\sum_{j=1}^{m} t_j a_{jk})$, which is the R-matrix with -1 as the first entry and zeroes elsewhere. Also $Av = (-\sum_{k=1}^{n} a_{jk}s_k) = (a_{j0}s_0) = (a_{j0})$, which is a matrix over R. Finally, $uAv = (uA)v = s_1 = w$.

(iii) $\Longrightarrow$ (iv): Given u, A, v of (iii), the diagrams are

$$\begin{array}{ccc} R & \xrightarrow{Av} & R^m \\ & \nearrow A & \\ R^n & \xrightarrow[uA]{} & R \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{Av} & S^m \\ v\downarrow & \nearrow A & \downarrow u \\ S^n & \xrightarrow[uA]{} & S, \end{array}$$

the modules R^m and R^n being free modules on m and n generators, respectively.

(iv) $\Longrightarrow$ (i): Suppose we have the diagrams of (iv) and we are given two homomorphisms $H_1, H_2 : S \longrightarrow T$ which agree on R. Then we get tensor product functors from P_S-Mod to P_T-Mod, which we also call H_1 and H_2, and which agree on modules and morphisms (like $P \otimes S$, $a \otimes 1$) which come from P_R-Mod. Then

$$\begin{aligned} H_1(w) &= H_1(u(a\otimes 1)v) = H_1(u)H_1(x\otimes 1) \\ &= H_1(u)H_2(x\otimes 1) = H_1(u)H_2(a\otimes 1)H_2(v) \\ &= H_1(u)H_1(a\otimes 1)H_2(v) = H_1(f\otimes 1)H_2(v) \\ &= H_2(f\otimes 1)H_2(v) = H_2(w). \end{aligned}$$

This completes the proof of 4.1.

The set of such elements w of S is called the <u>dominion</u> of R in S, denoted D(R,S). By 4.1(i), it clearly forms a subring of S containing R. Since inverses (even of matrices) are unique, each pair of homomorphisms from S to T which agree on R will also agree on each entry of the inverse of a matrix over R (if the inverse has all entries in S). This says that the rational closure of R in S is contained in the dominion.

The dominion D(R,S) tells how much of S can be "determined" by R. How much of S can D(R,S) itself determine? If $w \in S$ is <u>not</u> in D(R,S), then there are two homomorphisms from S to some T which agree on R but not on w. But these two homomorphisms also must agree on D(R,S) by definition, so w is not determined by D(R,S). Thus D(D(R,S),S) = D(R,S). Then how much of D(R,S) is determined by R (or what is D(R,D(R,S)))? Surprisingly, the "second dominion" $D^2(R,S) = D(R,D(R,S))$ may be strictly contained in D(R,S).

As an example, take k a field and $R = k[a,ax]$, $S = k[a,x]$ (commutative polynomial rings). Then $R \subseteq S$ and using 4.1(iii) we can see that $x(a)x = ax^2 \in D(R,S)$. But x is not in D(R,S), as we can see by the two k-homomorphisms $H, H': S \longrightarrow k$ given by

$$H(a) = 0, \; H(x) = 1$$

$$H'(a) = 0, \; H'(x) = 0.$$

In fact, $D(R,S) = k[a,ax,ax^2,ax^3,\cdots]$. Now well-defined k-homomorphisms from D(R,S) to $k[e]/(e^2)$ may be given by

$$H_1(ax^i) = 0 \quad \text{for } i = 0,1,2,\cdots$$

$$H_2(ax^i) = \begin{cases} 0 & \text{for } i \neq 2 \\ e & \text{for } i = 2. \end{cases}$$

This shows that ax^2 is not in $D^2(R,S)$. But using 4.1(iii), clearly $a^3x^4 = ax^2(a)ax^2$ is in $D^2(R,S)$ but not in R. In this case $D^i(R,S) = D(R,D^{i-1}(R,S))$ forms a strictly decreasing sequence of rings containing R.

In general, there is a "stabilized" dominion $D^\infty(R,S)$, satisfying $D(R,D^\infty(R,S)) = D^\infty(R,S)$. To do this the iterated dominion $D^u(R,S)$ must be defined for all ordinal numbers u (for u a limit ordinal, D^u is given by an intersection).

Then $D^{\infty}(R,S) = D^{u}(R,S)$ for any ordinal u of the same cardinality as the set of subsets of S. In the specific example above, $D^{\infty}(k[a,ax],k[a,x]) = k[a,ax]$.

<u>The Dominion as a Zigzag Localization</u>. Given any homomorphism of rings $H: R \longrightarrow S$, we get an additive functor $H: P_R\text{-Mod} \longrightarrow P_S\text{-Mod}$ by extending scalars to S in the familiar fashion. Let $Z_H = (C^L_{oH}, C^R_{oH})$ be the (right and left coherent) pair on P_R-Mod obtained by pulling back the discrete pair $Z_o = (C^L_o, C^R_o)$ on P_S-Mod. Then the closure operator on submodules of objects of P_R-Mod corresponding to C^R_{oH} is exactly the C_S of Chapter One.

Now we get a diagram of additive functors and zigzag localizations from Proposition 3.22 as follows:

$$\begin{array}{ccc} P_R\text{-Mod} & \xrightarrow{\ H\ } & P_S\text{-Mod} \\ {\scriptstyle E_R}\downarrow & & \downarrow{\scriptstyle E_S} \\ (P_R\text{-Mod})_{Z_H} & \longrightarrow & (P_S\text{-Mod})_{Z_o} \end{array},$$

where the bottom functor is faithful. But E_S is an isomorphism by Proposition 3.13, since Z_o consists of discrete a.c.o.'s. Thus we may as well rewrite the diagram above as

$$\begin{array}{ccc} & P_R\text{-Mod} & \\ {\scriptstyle E_R}\swarrow & & \searrow{\scriptstyle H} \\ (P_R\text{-Mod})_{Z_H} & \xrightarrow[\ \overline{H}\ ]{} & P_S\text{-Mod} \end{array},$$

with $\overline{H}$ faithful. We also get a corresponding commutative diagram of rings

$$\begin{array}{ccc} & R & \\ {\scriptstyle E}\swarrow & & \searrow{\scriptstyle H} \\ \overline{R} & \xrightarrow[\ \overline{H}\ ]{} & S \end{array},$$

where $\overline{R}$ is the zigzag localization ring R_{Z_H} and $\overline{H}$ is one-to-one.

Theorem 4.2: The image $\overline{H}(\overline{R})$ in S is just the dominion of H(R) in S. (This D(H(R),S) is also called the dominion D(R,S) of R in S.)

Proof: An arbitrary element of the ring $\overline{R}$ is a zigzag (f/a\x) with $f \in C^L_{oH}(a)$ and $x \in C^R_{oH}(a)$. Hence $H(f) \in C^L_o(H(a))$ and $H(x) \in C^R_o(H(a))$. The a.c.o.'s C^L_o and C^R_o are discrete, so H(f) = uH(a) and H(x) = H(a)v with u and v morphisms of P_S-Mod. But H is just the functor extending scalars to S, so we get exactly the diagrams of 4.1(iv). Also the functor $\overline{H}$ takes the zigzag (f/a\x) to the interpretation of (H(f)/H(a)\H(x)) as an element of S — namely, to u•H(a)•v(1). Thus we see that elements of $\overline{H}(\overline{R})$ are exactly elements of the dominion D(H(R),S) as characterized by 4.1(iv).

A homomorphism of rings H:R⟶S is called an epimorphism if D(R,S) = S (i.e., if every element of S is "determined" by H(R)). Note that as a result of 4.2 every such epimorphism can be constructed as a zigzag localization with respect to an appropriate pair of a.c.o.'s on P_R-Mod. In particular, when S is a division ring which is generated by the image of R using the operations of +, -, •, and taking the inverse of non-zero elements (what P. M. Cohn calls an "R-field" ([2], p. 253)), then R⟶S is an epimorphism and hence can be constructed as a zigzag localization.

We can also give at this point the example referred to in Chapter Three.

Example 4.3: A right coherent pair $Z = (C^L, C^R)$ on a category M such that, if we let $Z_E = (C^L_{oE}, C^R_{oE})$ be the a.c.o.'s obtained by pulling back the discrete a.c.o.'s C^L_o, C^R_o on M_Z along $E: M \longrightarrow M_Z$, then the natural faithful additive functor $M_{Z_E} \longrightarrow M_Z$ is not an isomorphism.

Description and Proof: The faithful additive functor comes from Proposition 3.22 and the fact (3.13) that the zigzag localization of M_Z with respect to a pair of discrete a.c.o.'s is isomorphic to M_Z. As an example in which it is not an isomorphism, take the case where $M = P_R$-Mod and Z is induced from the discrete a.c.o.'s on P_S-Mod with $R \subseteq S$ any inclusion of rings for which $D^2(R,S) \neq D(R,S)$. Clearly the endomorphism ring of R in M_Z is $D(R,S)$ by 4.2, while in M_{Z_E} the endomorphism ring is taken to $D(R,D(R,S)) = D^2(R,S)$ in M_Z, a smaller set.

Right and Left Superdominions. We have been pulling back discrete a.c.o.'s on P_S-Mod along the ring homomorphism $H:R \longrightarrow S$ to get C^L_{oH} and C^R_{oH}. Now consider the left a.c.o. C^{R*}_o on P_S-Mod which is the reflection of the right discrete a.c.o. C^R_o, as in Chapter Three. Then we can pull this back along H also, to get a left a.c.o. $(C^{R*}_o)_H$ on P_R-Mod. Clearly we have $(C^{R*}_o)_H \geq C^L_{oH}$, since $C^{R*}_o \geq C^L_o$ in P_S-Mod already.

Now (C^{R*}_o, C^R_o) is right and left coherent by Propositions 3.18 and 3.19. Hence so is $((C^{R*}_o)_H, C^R_{oH})$, and we can use Proposition 3.23 to get $(C^R_{oH})^* \geq (C^{R*}_o)_H$. Thus we have the inclusions

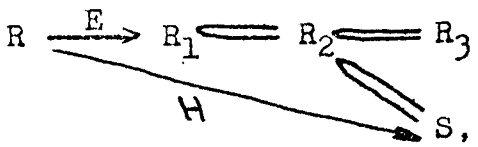
$$(C^L_{oH}, C^R_{oH}) \leq ((C^{R*}_o)_H, C^R_{oH}) \leq ((C^R_{oH})^*, C^R_{oH}),$$

and so by Proposition 3.11 we get functors between the respective zigzag localizations, forming a diagram

$$M \xrightarrow{E} M_1 \longrightarrow M_2 \longrightarrow M_3 .$$

Also, the functors of 3.11 just take the equivalence class of $(f/a\backslash x)$ to the bigger equivalence class of this zigzag which is determined by the bigger right a.c.o. Here the right a.c.o.'s determining the equivalence relations are all the same, so the functors among the M_i's above are all faithful. Also, since $((C^{R*}_o)_H, C^R_{oH})$ is induced from P_S-Mod, we get by Proposition 3.22 a faithful functor from M_2 to the zigzag localization of P_S-Mod with respect to (C_o^{R*}, C_o^R), which by Proposition 3.13 is just P_S-Mod itself.

Now pass to the corresponding ring homomorphisms. Then we get rings R_1, R_2, R_3, and a diagram



where we may write inclusions because of the faithfulness. We know from 4.2 that $R_1 = D(R,S)$. Let us call R_2 the (single) right (interior) superdominion of R in S, denoted $D^R(R,S)$, and R_3 the (single) right exterior superdominion $D^R_{ex}(R,S)$. Of course, all this exists on the left, too.

As a quick example for which $D^R(R,S)$ is bigger than $D(R,S)$, take our previous case $R = k[a,ax] \subseteq k[a,x] = S$. Then $x \notin D(R,S)$ as before, but x is given by the zigzag $(1/a \backslash ax)$ in $D^R(R,S)$. This zigzag is in $D^R(R,S)$ because $1 \in (C_o^{R*})_H(a)$ (while $1 \notin C^L_{oH}(a)$); to prove this, we want $1 \in C_o^{R*}(H(a))$, or that a C_o^R-dominates 1. (Here we will drop the notation for the functor H since it is just an inclusion map.) Equivalently, we need $\binom{a}{v}P_S$-Mod to be C_o^R-disjoint from the codomain V of v. So assume

$$\binom{0}{w} \in C_o^R\left(\binom{a}{v}P_S\text{-Mod} + \binom{0}{J}\right) \cap \binom{0}{1_V}P_S\text{-Mod}$$

for some ideal $J \subseteq 1_V(P_S\text{-Mod})$. Then $\binom{0}{w} = \binom{a}{v}s + \binom{0}{j}$ (with s in P_S-Mod and $j \in J$) since the a.c.o. is discrete. Hence $a \cdot s = 0$; since a has no annihilator in S, s must be the zero morphism. Thus $w = j$, and $\binom{0}{w} \in C_o^R\binom{0}{J}$, as needed. Hence in this case $D(R,S) = k[a,ax,ax^2,\cdots]$ and $D^R(R,S) = S$.

For other superdominions, reflect C_o^{R*} again, to get a new right a.c.o. C_o^{R**} on P_S-Mod. Then pull back the pair (C_o^{R*}, C_o^{R**}) along $H:R \longrightarrow S$ to get $((C_o^{R*})_H, (C_o^{R**})_H)$, which is $\geq ((C_o^{R*})_H, C_{oH}^R)$. Also, Proposition 3.23 (in dualized form) says that $((C_o^{R*})_H, ((C_o^{R*})_H)^*) \geq ((C_o^{R*})_H, (C_o^{R**})_H)$. These two pairs will yield second superdominions, which it appears might be bigger than the previous superdominions. We may also use the pairs like (C_o^{R***}, C_o^{R**}), etc.

Non-Functoriality of the Reflection. Not much is known about these large superdominions, because the reflected a.c.o. C^* is hard to calculate. In fact, the construction is unnatural enough that $C_1 \leq C_2$ need not imply $C_1^* \leq C_2^*$ (or the other inclusion either). In this section we will give quick examples for which the inclusions fail.

First take $R = k[a]$ for k a field, and consider the embedding $R \subseteq S = k[a,x]/(a^2x - a)$. We have the discrete right a.c.o. C_o^R on P_R-Mod and also the a.c.o. (call it C_S^R, say) induced on P_R-Mod from the discrete a.c.o. on P_S-Mod. Clearly $C_o^R \leq C_S^R$. But here we have $C_o^{R*} \not\leq C_S^{R*}$; in particular $1 \in C_o^{R*}(a)$ but $1 \notin C_S^{R*}(a)$. To prove the latter statement, just note that $\binom{0}{1} = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}\binom{1-ax}{x}$, or $\binom{0}{1} \in C_S^R\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$; but $\binom{0}{1} \notin C_S^R\binom{0}{a}$ (there is no inverse to a in S). Thus a does not C_S^R-dominate 1. But a does C_o^R-dominate 1, by the same argument as in paragraph showing $D^R(R,S) \neq D(R,S)$. This proves the former statement and completes this example.

To disprove the other possible inclusion, take $R' = k[a,x]/(ax)$ and $H:R' \longrightarrow R = k[a]$ by sending x to zero. Again we have the discrete a.c.o. C_o^R on $P_{R'}$-Mod and again $C_o^R \leq C_{oH}^R$, the induced a.c.o. on $P_{R'}$-Mod. Again $1 \in (C_{oH}^R)^*(a)$, by the same old argument. But now $1 \notin C_o^{R*}(a)$, since $\binom{0}{x} = \binom{a}{1}x$, but $\binom{0}{x} \notin (0)$. Thus $C_o^{R*} \not\leq (C_{oH}^R)^*$ either.

Exchange Properties. Suppose temporarily that we have rings $R \subseteq S$, with S a division ring. Keep the notation C_S^R for the right a.c.o. on $M = P_R$-Mod induced from the discrete a.c.o. on P_S-Mod. Suppose also that $y \in C_S^R(J + xM)$, for x and y morphisms from R to Q in M, and for J an ideal of M at Q. Then $y = js' + xs$ (with $j \in J$) in P_S-Mod. Here s is a morphism from S to S, so if it isn't zero (i.e., if $y \notin C_S^R(J)$), then we can invert it and write $x = ys^{-1} - j(s's^{-1})$, or $x \in C_S^R(J + yM)$.

Thus the object R of P_R-Mod and the right a.c.o. C_S^R satisfy the following condition, which we state in general.

Definition: Let C be a right a.c.o. on an additive category M, P an object of M, and suppose that, for every object Q of M, every ideal $J \subseteq 1_Q M$, and every $x:P \longrightarrow Q$ and $y:P \longrightarrow Q$ in M, we have that whenever $y \in C(J + xM)$ but $y \notin C(J)$, then $x \in C(J + yM)$. Then C is said to have the exchange property with respect to P.

If we think of $M = P_R$-Mod, $P = R$, and regard x, y, and J as giving elements and subsets of Q, then this is just the exchange condition on the corresponding closure operator on submodules of Q, as defined in Chapter Two.

Thus this C is the same as the <u>span</u> for an algebraic dependence relation on projective R-modules.

As suggested in the discussion of dependence relations in Chapter Two, we should be able to construct a division ring from an a.c.o. with an exchange property. Let us first give a general lemma on inverses.

<u>Lemma 4.4</u>: Let $Z = (C^L, C^R)$ be a right coherent pair on M. Given a zigzag $(f/a\backslash x)$ in M_Z, suppose that $\binom{0}{1} \in C^R\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}$ and $(0\ 1) \in C^L\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}$. Then $-((0\ 1)/\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}\backslash\binom{0}{1})$ is the inverse of $(f/a\backslash x)$ in M_Z.

<u>Proof</u>: First check it to be a right inverse, or

$$0 = (1/1\backslash 1) + (f/a\backslash x)((0\ 1)/\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}\backslash\binom{0}{1})$$

$$= \left((1\ f\ 0\ 0) \Big/ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & -x \\ 0 & 0 & a & x \\ 0 & 0 & f & 0 \end{pmatrix} \Big\backslash \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right).$$ By 3.5 we need

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in C^R u$ for $u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & -x \\ 0 & 0 & a & x \\ 0 & 0 & f & 0 \\ 1 & f & 0 & 0 \end{pmatrix}$. Since $\binom{0}{1} \in C^R\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}$, we get

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \in C^R\begin{pmatrix} 0 & 0 \\ -a & -x \\ a & x \\ f & 0 \\ -f & 0 \end{pmatrix} \subseteq C^R u, \text{ so } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \in C^R u.$$

Also check $0 = (1/1\backslash 1) + ((0\ 1)/\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}\backslash\binom{0}{1})(f/a\backslash x)$

$$= \left((1\ 0\ 1\ 0) \Big/ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & x & 0 \\ 0 & f & 0 & -f \\ 0 & 0 & 0 & a \end{pmatrix} \Big\backslash \begin{pmatrix} 1 \\ 0 \\ 0 \\ x \end{pmatrix}\right),$$ or

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ x \\ 0 \end{pmatrix} \in C^R u$, where now $u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & x & 0 \\ 0 & f & 0 & -f \\ 0 & 0 & 0 & a \\ 1 & 0 & 1 & 0 \end{pmatrix}$. Clearly we have

$$\begin{pmatrix}0\\x\\0\\x\\0\end{pmatrix} \in C^R\begin{pmatrix}0\\a\\0\\a\\0\end{pmatrix} \subseteq C^R\begin{pmatrix}0&0\\a&0\\f&-f\\0&a\\0&0\end{pmatrix} \subseteq C^R u, \text{ so } \begin{pmatrix}1\\0\\0\\x\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix} - \begin{pmatrix}0\\x\\0\\0\\1\end{pmatrix} + \begin{pmatrix}0\\x\\0\\x\\0\end{pmatrix} \in C^R u.$$

Theorem 4.5: For C a right a.c.o., let the zigzag localization of P_R-Mod with respect to $Z = (C^*, C)$ give a ring homomorphism $R \longrightarrow R_Z$. If C has the exchange property with respect to the object R, then R_Z is a division ring.

Proof: Take any $(f/a\backslash x)$ in R_Z with $R \xrightarrow{x} Q \xleftarrow{a} P \xrightarrow{f} R$, and assume $(f/a\backslash x) \neq 0$. Then equivalently by Proposition 3.5, $\binom{x}{0} \notin C\binom{a}{f}$. Now $\begin{pmatrix}a&0\\f&1\end{pmatrix}M = \begin{pmatrix}a&0\\0&1\end{pmatrix}M$, so we get $\binom{x}{0} \in C\begin{pmatrix}a&0\\0&1\end{pmatrix} = C\begin{pmatrix}a&0\\f&1\end{pmatrix}$. Thus using the exchange property, $\binom{0}{1} \in C\begin{pmatrix}a&x\\f&0\end{pmatrix}$.

We also claim that $(0\ 1) \in C^*\begin{pmatrix}a&x\\f&0\end{pmatrix}$. To see this, take any $v: R \longrightarrow Q'$, $J \subseteq 1_{Q'}M$ and $w \in 1_{Q'}M$ and assume that

$$\begin{pmatrix}0\\0\\w\end{pmatrix} \in C(\begin{pmatrix}a&x\\f&0\\0&v\end{pmatrix}M + \begin{pmatrix}0\\0\\J\end{pmatrix}).$$ We want to show that $\begin{pmatrix}0\\0\\w\end{pmatrix} \in C\begin{pmatrix}0\\0\\J\end{pmatrix}$.

The domain of w is finitely generated projective over R, so w is in the right ideal generated by $\{we \mid e \text{ has domain } R\}$. Thus it will be enough to show that each such we satisfies $\begin{pmatrix}0\\0\\we\end{pmatrix} \in C\begin{pmatrix}0\\0\\J\end{pmatrix}$. But if $\begin{pmatrix}0\\0\\we\end{pmatrix} \notin C\begin{pmatrix}0\\0\\J\end{pmatrix}$, then clearly also $\begin{pmatrix}0\\0\\we\end{pmatrix} \notin C(\begin{pmatrix}a\\f\\0\end{pmatrix}M + \begin{pmatrix}0\\0\\J\end{pmatrix})$. Then we can use the exchange property to get $\begin{pmatrix}x\\0\\v\end{pmatrix} \in C(\begin{pmatrix}a&0\\f&0\\0&we\end{pmatrix}M + \begin{pmatrix}0\\0\\J\end{pmatrix})$. Composing with $\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$ on the left, we get $\binom{x}{0} \in C\binom{a}{f}$, contradicting our assumption.

Hence we have $\begin{pmatrix}0\\0\\w\end{pmatrix} \in C\begin{pmatrix}0\\0\\J\end{pmatrix}$, so we have shown that $(0\ 1) \in C^*\begin{pmatrix}a & x\\ f & 0\end{pmatrix}$.

Now by the Lemma (f/a\x) is invertible in R_Z, so we are done.

Of course the object R is not very special in this situation. Any projective generator P will behave the same, and in fact $P_{End(P)}$-Mod is equivalent to P_R-Mod. So if C has the exchange property with respect to P, zigzag localization would give a homomorphism from End(P) to a division ring.

Note that then P cannot be a direct sum Q^n for some projective Q and $n > 1$, for then Q would become a vector space of dimension $1/n$ over the division ring. Or, rather, if P is such a Q^n, then C is indiscrete and the zigzag localization gives the zero ring (which is a division ring, in the universal sense).

We can also see this directly; assume C has the exchange property with respect to P and $P = Q \oplus Q'$, even. A map from P to P is a 2-by-2 matrix of maps among Q and Q'. Clearly we have $\begin{pmatrix}1_Q & 0\\ 0 & 0\end{pmatrix} \in C\begin{pmatrix}1_Q & 0\\ 0 & 1_{Q'}\end{pmatrix} = C(1_P)$. Now if $\begin{pmatrix}1_Q & 0\\ 0 & 0\end{pmatrix} \in C\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, then we get $1_Q \in C(0)$, and C is indiscrete on Q. If $\begin{pmatrix}1_Q & 0\\ 0 & 0\end{pmatrix} \notin C\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, then $\begin{pmatrix}1_Q & 0\\ 0 & 1_{Q'}\end{pmatrix} \in C\begin{pmatrix}1_Q & 0\\ 0 & 0\end{pmatrix}$ by the exchange property with respect to P. So we get $\begin{pmatrix}0\\ 1_{Q'}\end{pmatrix} \in C\begin{pmatrix}1_Q\\ 0\end{pmatrix}$ and $1_{Q'} \in C(0)$, and C is indiscrete on Q'.

When $Q' = Q^{n-1}$, then C is indiscrete on Q. But Q is a projective generator (since P is), so C indiscrete on Q implies C indiscrete on all of P_R-Mod.

CHAPTER FIVE

ZIGZAG LOCALIZATIONS OF MODULE CATEGORIES AND TORSION THEORIES

In this chapter we consider the case when the additive category M is the category of all right modules over a fixed ring R. We denote this category by Mod-R. We show that an algebraic closure operator on right ideals of Mod-R can be induced from one such on P_R-Mod, and we compare closure operators on Mod-R with pre-torsion and torsion theories on Mod-R. In particular, rings and modules of quotients, as given by the notion of a torsion theory, can be constructed using the zigzag localization.

Comparison with the P-Mod Case. For M = Mod-R, we do not have the nice correspondence between right ideals of M at an object N and submodules of N. Here we have only that $Im(f) \subseteq Im(g) \Longleftarrow f \in gM$, for f and g having codomain N. If we define the image Im(I) of a set I of morphisms with codomain N as before (the union of the images), then a morphism f in $1_N M$ with image contained in Im(IM) may not be in IM.

We can also denote the right ideal of morphisms in $1_N M$ whose images are contained in a submodule $N' \subseteq N$ by ${}_N i_{N'} M$, where ${}_N i_{N'}$ is the inclusion map of N' into N. Then we get some kind of correspondence, namely that if f has codomain N and $I \subseteq 1_N M$, then

$$f \in {}_N 1_{Im(IM)}{}^M \iff Im(f) \subseteq Im({}_N 1_{Im(IM)}{}^M) = Im(IM).$$

Now given an algebraic closure operator C on right ideals of P_R-Mod, we have a corresponding closure operator C on submodules of each object P of P_R-Mod, as in Chapter Four. Then we want to extend this to a closure operator $\overline{C}$ on submodules of objects N of M = Mod-R (just as defined in Chapter Two, but we now allow all modules instead of just finitely generated projective ones), and thence to a right a.c.o. on M. If $p:P \longrightarrow N$, we will at least want $\overline{C}(p(X),N) \supseteq p(C(X,P))$ for $X \subseteq P$. This turns out to be just enough to extend, provided that the original C is finitary. So assume C is finitary, and, for object N and submodule N' of N, define

$$\overline{C}(N',N) = \left\{ n \in N \;\middle|\; \begin{array}{l} n \in p(C(X,P)) \text{ for some object } P \\ \text{of } P_R\text{-Mod, some } X \subseteq P, \text{ and some} \\ p:P \longrightarrow N \text{ satisfying } p(X) \subseteq N'. \end{array} \right\}.$$

Then for $I \subseteq 1_N M$, we can define the closure $\overline{C}I$ to be the ideal ${}_N 1_{\overline{C}(Im(IM),N)}{}^M$.

<u>Proposition 5.1</u>: Suppose C is a finitary right a.c.o. on P_R-Mod. Then defining $\overline{C}$ on M as above, $\overline{C}$ is a right a.c.o. on M, and if $f:P \longrightarrow Q$ is in P_R-Mod and $I \subseteq 1_Q(P_R\text{-Mod})$, then $f \in CI \iff f \in \overline{C}I \subseteq 1_Q M$.

<u>Proof</u>: We have the five conditions to verify.

(1): $I \subseteq \overline{C}I$. Given $z \in I$, we want to show that $z \in {}_N 1_{\overline{C}(Im(IM),N)}{}^M$, or that $Im(z) \subseteq \overline{C}(Im(IM),N)$. So take $n \in Im(z)$, and let $p:R \longrightarrow N$ be given by $p(r) = nr$. For

$X = \{1\} \subseteq R$, $p(X) = \{n\} \subseteq Im(IM)$, since $z \in I$. Thus $n \in \overline{C}(Im(IM),N)$, as needed.

(2): $I \subseteq J \Longrightarrow \overline{C}I \subseteq \overline{C}J$. We can see from the definition of $\overline{C}$ that $N' \subseteq N'' \subseteq N \Longrightarrow \overline{C}(N',N) \subseteq \overline{C}(N'',N)$. Then the needed result follows immediately.

(3): $\overline{C}\overline{C}I = \overline{C}I$. The inclusion $\supseteq$ is by (1) and (2), so assume $z \in \overline{C}\overline{C}I$, or $z \in {}_{N}i_{\overline{C}(Im(\overline{C}I),N)}{}^{M}$. Thus

$$Im(z) \subseteq \overline{C}(Im(\overline{C}I),N) = \overline{C}(Im({}_{N}i_{\overline{C}(Im(IM),N)}{}^{M})) = \overline{C}(\overline{C}(Im(IM),N),N).$$

Hence it is sufficient to show that $\overline{C}(\overline{C}(N',N),N) = \overline{C}(N',N)$. So take any $n \in \overline{C}(\overline{C}(N',N),N)$, or $n \in p(C(X,P))$ with $X \subseteq P$ and $p:P \longrightarrow N$, $p(X) \subseteq C(N',N)$. Here we may take X to be a finite set, since C was originally finitary. So for each $x \in X$, $p(x) \in \overline{C}(N',N)$; hence there is $Y_x \subseteq P_x$ and $p_x:P_x \longrightarrow N$ with $p_x(Y_x) \subseteq N'$ and $p(x) \in p_x(C(Y_x,P_x))$. Since X is finite we may take the direct sum of all the P_x (i.e., for all x in X), and call this P'. We then get a big direct sum map $p':P' \longrightarrow N$ and a big set $X' \subseteq P'$ satisfying $p'(X') \subseteq N'$ and $p(x) \in p'(C(X',P'))$ for each $x \in X$. So now we get $(p\ p'):P \oplus P' \longrightarrow N$ and a set

$$X_o = \left\{ \binom{0}{x'} \,\middle|\, x' \in X' \right\} \cup \left\{ \binom{x}{-y'} \,\middle|\, \begin{array}{l} x \in X,\ y' \in C(X',P') \\ \text{with } p(x) = p'(y') \end{array} \right\}.$$

Clearly $X_o \subseteq P \oplus P'$ and $(p\ p')(X_o) \subseteq N'$. Furthermore $C(X_o,P \oplus P')$ includes $\binom{0}{y'}$ for each $y' \in C(X',P')$, so $C(X_o,P \oplus P')$ also includes $\binom{y}{0}$ for each $y \in C(X,P)$. Hence $(p\ p')C(X_o,P \oplus P')$ includes n, and so $n \in \overline{C}(N',N)$.

(4): $\overline{C}I$ is a right ideal of M. This is clear; the ideal is generated by ${}_N1_{\overline{C}(\mathrm{Im}(IM),N)}$.

(5): For $f:N \longrightarrow L$, $f(\overline{C}I) \subseteq \overline{C}(fI)$. Take $z \in \overline{C}I$, so $\mathrm{Im}(z) \subseteq \overline{C}(\mathrm{Im}(IM),N)$, and try to show $fz \in \overline{C}(fI)$, for which we need $\mathrm{Im}(fz) \subseteq \overline{C}(\mathrm{Im}(fIM),N)$. But $\mathrm{Im}(fz) = f(\mathrm{Im}(z))$ and $\mathrm{Im}(fIM) = f(\mathrm{Im}(IM))$, so we need only show that $f(\overline{C}(N',N)) \subseteq \overline{C}(f(N'),L)$. This is clear by just using fp instead of p in the definition of $\overline{C}(N',N)$.

To prove the last statement, first note that we are regarding the object Q of P_R-Mod as also an object of Mod-R. The two closures C, $\overline{C}$ on submodules of Q are the same, since we may take $p = 1_Q$ in the definition. Thus $\overline{C}(\mathrm{Im}(IM),Q) = \mathrm{Im}(CI)$, because the correspondence between submodules and ideals is exact in P_R-Mod. So

$$f \in \overline{C}I = {}_Q1_{C(\mathrm{Im}(IM),Q)}M \Longleftrightarrow \mathrm{Im}(f) \subseteq C(\mathrm{Im}(IM),Q) = \mathrm{Im}(CI)$$
$$\Longleftrightarrow f \in CI.$$

Regarding the finitariness of $\overline{C}$, each $\overline{C}$ on submodules of N may be seen to be finitary, but $\overline{C}$ is not necessarily finitary on right ideals of M. This occurs because M = Mod-R includes modules which are not finitely generated.

Comparison with Pre-torsion Theories. In this section we recall the notion of a (right) pre-torsion theory, noting that it is slightly less general than that of a right a.c.o. on M = Mod-R. This discussion, as well as that for torsion theories, is taken largely from Lambek ([4], Chapter 0).

A pre-torsion theory aims to describe abstractly what it means for a module to be a "torsion" module. Specifically, one may give a collection B of right modules, called a pre-torsion class, such that B is closed under taking isomorphic images, factor modules, module extensions, and direct sums of elements of B. (We say B is closed under taking module extensions if whenever a submodule N' of N and the factor module N/N' both belong to B, then N does also. (Note Lambek says "group extensions".)) To motivate this notion, suppose we are given a ring homomorphism H:R⟶S. If we take B to be the set of all objects N of Mod-R such that $N \otimes_R S = 0$, then B is a pre-torsion class on Mod-R.

Likewise, one may give a pre-torsion-free class: a collection C of modules which is closed under taking isomorphic images, submodules, module extensions, and direct products.

Suppose we define a radical to be a function T which assigns to each module N a submodule T(N) of N, and which satisfies the conditions: T(N/T(N)) = 0 for all N; and if f:N⟶L is in Mod-R, then $f(T(N)) \subseteq T(L)$. This T is said to be idempotent if T(T(N)) = T(N) for all N. Such an idempotent radical abstractly gives the "torsion part" of each module.

Finally, we may give an algebraic closure operator C on submodules of objects of Mod-R. This closure operator will be said to be pre-modular if (besides the conditions

(1-5)) it satisfies (for any N of Mod-R with submodule N'):

(6) C(N',C(N',N)) = C(N',N).

(7) C(N',N)/N' = C(0,N/N').

Then we get the following equivalence:

Proposition 5.2: For any ring R, the following data on M = Mod-R are equivalent:

(a) a pre-torsion class B.

(b) a pre-torsion-free class C.

(c) an idempotent radical T.

(d) a pre-modular closure operator C.

Proof: Given a pre-torsion class B, take C to be the set of modules L such that M(N,L) = 0 for all N $\in$ B. The verification that C is then a pre-torsion-free class is straightforward. Conversely, given C we may take B to be the set of modules N such that M(N,L) = 0 for all L $\in$ C. The verification of the equivalence in this case is again straightforward.

For the other parts, all the verifications continue to be straightforward. We will indicate only how to construct one piece of data from another.

Given C, we can T(N) to be the intersection of all submodules X of N such that N/X $\in$ C. Conversely, given T we take C to include just the modules N with T(N) = 0.

Given T, we take C(N',N) to be the (full) inverse image of T(N/N') under the canonical map N$\longrightarrow$N/N'. From C we construct the radical T by just putting T(N) = C(0,N) for each module N.

Torsion Theories. The pre-torsion theories that have been of most interest and have been studied by Lambek and others are assumed to satisfy an additional "hereditariness" condition. We state the various forms of this condition in the following proposition, whose proof we omit, since it is straightforward.

Proposition 5.3: Suppose we are given a pre-torsion theory, with all of its equivalent forms: $\underline{B}$, $\underline{C}$, an idempotent T, and a pre-modular C. Then the following conditions are equivalent:

(a) $\underline{B}$ is closed under taking submodules.

(b) $\underline{C}$ is closed under injective hulls.

(c) For N' a submodule of N, $T(N') = T(N) \cap N'$.

(d) For N' and N" submodules of N, C also satisfies the strengthened condition

(6+) $C(N'' \cap N', N') = C(N'', N) \cap N'$.

Definition: If the equivalent conditions of 5.3 are satisfied, then ($\underline{B}$,$\underline{C}$) is called a torsion theory on Mod-R, T is called a torsion radical, and the closure operator C is called modular.

In this case there is still another formulation of a torsion theory. We will call a set F of right ideals of the ring R an idempotent filter if the following conditions hold:

(0) The unit ideal R is in F.

(1) If $D \in F$ and $D' \supseteq D$, then $D' \in F$.

(2) If $D \in F$ and $r \in R$, then

$$r^{-1}D = \{x \in R \mid rx \in D\} \in F.$$

(3) If $D \in F$ and $d^{-1}D' \in F$ for all $d \in D$, then $D' \cap D \in F$.

(Note that these conditions also imply that if $D, D' \in F$, then $D \cap D' \in F$.) To construct such an F from a modular closure operator C, just take F to consist of those right ideals D of R such that $C(D,R) = R$ (the <u>dense</u> right ideals). Conversely, given F, define C by putting

$$C(N',N) = \{n \in N \mid \{r \in R \mid nr \in N'\} \in F\}.$$

<u>Rings and Modules of Quotients</u>. Given a torsion theory and any module in M = Mod-R, one can construct a module of quotients with respect to that theory. The presentation of this construction which we give here is apparently somewhat obsolete (Lambek ([4], p.22)), but it is convenient for our purposes.

Suppose we are given a torsion theory, in the form of an idempotent filter F of dense right ideals of R. For a right R-module N, put

$$L(N) = \{m:D \longrightarrow N \text{ in Mod-}R \mid D \in F\}/\sim,$$

where $(m:D \longrightarrow N) \sim (m':D' \longrightarrow N)$ if m and m' agree on the intersection ideal $D \cap D' \in F$. Let us denote the $\sim$-equivalence class of $m:D \longrightarrow N$ by $[m]$. Then $L(N)$ is easily made into an abelian group by $[m:D \longrightarrow N] + [m':D' \longrightarrow N] = [m+m':D \cap D' \longrightarrow N]$. To make it an R-module, put $[m:D \longrightarrow N]r = [mr:r^{-1}D \longrightarrow N]$, where $mr(d) = m(rd)$ for $d \in r^{-1}D$. We also get a natural map $N \longrightarrow L(N)$ by taking an element n of N to $[m_n:R \longrightarrow N]$, for $m_n(r) = nr$.

Recall that for a torsion theory in the form of a filter F, the radical T is given by

$$T(N) = C(0,N) = \left\{ n \in N \,\middle|\, \{ r \in R \mid nr = 0 \} \in F \right\}.$$

Then we define the module of quotients Q(N) to be L(N/T(N)) for any N. This is easily shown to be a functor on M and we get a natural map $N \longrightarrow N/T(N) \longrightarrow L(N/T(N)) = Q(N)$. Lambek ([4], pp.16,25) shows that applying the functor Q to this map gives a natural isomorphism $Q(N) \longrightarrow Q(Q(N))$, and that the homomorphism $M(Q(R),Q(N)) \longrightarrow Q(N)$ given by composing with $R \longrightarrow Q(R)$ is actually an isomorphism.

Thus $Q(R) \cong M(Q(R),Q(R))$ becomes a ring, the _ring of quotients_ of R with respect to the torsion theory originally chosen. This construction has been studied by Lambek, Findlay, Popescu, and others.

The Ring of Quotients as a Zigzag Localization. In fact, the ring (and modules) of quotients can be obtained by a zigzag localization of M = Mod-R with respect to $Z = (C^*,C)$, where the right a.c.o. C arises from a modular closure operator.

Theorem 5.4: Given a torsion theory in the form of a modular closure operator C, define a right a.c.o. C on M = Mod-R by $C(I) = {}_N1_{C(Im(IM),N)}M$, for $I \subseteq 1_N M$. Also put $Z = (C^*,C)$, and take Q to be the functor from M to M giving the module of quotients with respect to the torsion theory. Then there is a full and faithful additive functor G from the localization M_Z to M, such that GE = Q as functors.

Proof: (comprising essentially the remainder of the chapter). Our definition of Q uses the idempotent filter F of right ideals, so recall that the closure operator C is given from such a filter by

$$C(N',N) = \left\{ n \in N \,\middle|\, \{ r \in R \mid nr \in N' \} \in F \right\}.$$

Also, we need to determine the reflected left a.c.o. C^* more specifically.

Lemma 5.5: Let $a{:}N \longrightarrow P$ and $f{:}N \longrightarrow V$ be morphisms in M. Then $f \in C^*(a)$ if and only if for every $n \in N$ with $a(n) = 0$ we have $f(n) \in C(0,V)$.

Proof of Lemma: If $f \in C^*(a)$, then in particular $\begin{pmatrix} a \\ f \end{pmatrix}M$ is C-disjoint from V, and so (taking J = 0)

$$C\left(\begin{pmatrix} a \\ f \end{pmatrix}\right) \cap \begin{pmatrix} 0 \\ 1_V \end{pmatrix}M = \begin{pmatrix} 0 \\ C(0) \end{pmatrix}.$$

This translates to $C\left(\begin{pmatrix} a \\ f \end{pmatrix}(N), P \oplus V\right) \cap \begin{pmatrix} 0 \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ C(0,V) \end{pmatrix}$. If $a(n) = 0$ then $\begin{pmatrix} 0 \\ f(n) \end{pmatrix}$ is in the left side of the latter equation, hence in the right side; so $f(n) \in C(0,V)$.

Conversely, suppose the condition on f. We want to show that $\begin{pmatrix} a \\ vf \end{pmatrix}M$ is C-disjoint from U for $v{:}V \longrightarrow U$, or $C\left(\begin{pmatrix} a \\ vf \end{pmatrix}M + \begin{pmatrix} 0 \\ J \end{pmatrix}\right) \cap \begin{pmatrix} 0 \\ 1_U \end{pmatrix}M = C\begin{pmatrix} 0 \\ J \end{pmatrix}$. This translates to

$C\left(\begin{pmatrix} a \\ vf \end{pmatrix}(N) + \begin{pmatrix} 0 \\ Im(J) \end{pmatrix}, P \oplus U\right) \cap \begin{pmatrix} 0 \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ C(Im(J),U) \end{pmatrix}$. Because of condition (6+) on C, the left side is

$$C\left(\begin{pmatrix} a \\ vf \end{pmatrix}(Ker(a)) + \begin{pmatrix} 0 \\ Im(J) \end{pmatrix}, \begin{pmatrix} 0 \\ U \end{pmatrix}\right)$$

$$= \begin{pmatrix} 0 \\ C(vf(Ker(a)) + Im(J),U) \end{pmatrix}$$

$$\subseteq \begin{pmatrix} 0 \\ C(v(0,V) + Im(J),U) \end{pmatrix}$$

$$\subseteq (C(C(0,U)^0 + \mathrm{Im}(J),U))\ ,$$

where we have used the assumption on f. But clearly this last closure is just $(C(\mathrm{Im}^0(J),U))$, since the C(0,U) part cannot add to the closure. Thus the left side of the desired equation is contained in the right; but we know that the other inclusion is always true.

This concludes the proof of the lemma.

To continue the proof of the theorem, we want to define the functor G. On objects E(N) of M_Z (for N an object of M), just put G(E(N)) = Q(N) in M. Now for a morphism (f/a\x) in $M_Z(U,V)$, we need to get an R-homomorphism $G(f/a\backslash x): L(U/T(U)) \longrightarrow L(V/T(V))$. Let us suppose the diagram is $U \xrightarrow{x} P \xleftarrow{a} N \xrightarrow{f} V$, with $f \in C^*(a)$ and $x \in C(a)$. Then f is as in the Lemma, while x satisfies $x(U) \subseteq C(a(N),P)$. This means that for each $u \in U$, $\{r \in R \mid x(u)r \in a(N)\} \in F$.

Now go to the diagram $U/T(U) \longrightarrow P/T(P) \longleftarrow N/T(N) \longrightarrow V/T(V)$ obtained by factoring out the "torsion part" of each module. The new R-homomorphisms are induced by the old ones, because of the functorial property of the radical T. We will generally denote this new diagram by $\overline{U} \xrightarrow{\overline{x}} \overline{P} \xleftarrow{\overline{a}} \overline{N} \xrightarrow{\overline{f}} \overline{V}$.

In this new situation, we claim that if $\overline{n} \in \overline{N}$ and $\overline{a}(\overline{n}) = 0$, then $\overline{f}(\overline{n}) = 0$. Indeed, if $\overline{a}(\overline{n}) = 0$, then (taking a representative n in N for $\overline{n}$) $a(n) \in T(P)$, so $\{r \in R \mid a(n)r = 0\} \in F$. Hence the larger ideal $\{r \in R \mid f(n)r \in C(0,V)\}$ is also dense, so that

$f(n) \in C(C(0,V),V) = C(0,V) = T(V)$ (and this is independent of the choice of n). Thus $\bar{f}(\bar{n}) = 0$. Also in this new situation $\bar{x} \in C(\bar{a})$. To see this, pick any $\bar{u} \in \bar{U}$. Then $K_{\bar{u}} = \{r \in R \mid \bar{x}(\bar{u})r \in \bar{a}(\bar{N})\} \supseteq \{r \in R \mid x(u)r \in a(N)\}$ (for some representative u of $\bar{u}$). Since the latter ideal is dense, so is $K_{\bar{u}}$, as needed. Now notice that for $r \in K_{\bar{u}}$, we get a well-defined element of $\bar{V}$ by taking $\overline{fa}^{-1}\bar{x}(\bar{u}r)$, because of the condition on $\bar{f}$.

Suppose now that we are given an element of $L(U/T(U))$, and we pick a representative $m:D \longrightarrow U/T(U) = \bar{U}$, for D dense. If $K = \{s \in D \mid \bar{x}(m(s)) \in \bar{a}(\bar{N})\}$ is dense, then we can get a well-defined $\overline{fa}^{-1}\bar{x}m:K \longrightarrow \bar{V}$, which we can use to give the element $G(f/a\ x)([m])$ of $L(\bar{V})$. But for any $r \in D$, $r^{-1}K = \{s \in D \mid \bar{x}(m(rs)) \in \bar{a}(\bar{N})\} \supseteq K_{m(r)} \in F$, so that $r^{-1}K \in F$ for all $r \in D$; this implies that K _is_ dense, so we get $G(f/a\backslash x)[m]$. This construction of the class $[\overline{fa}^{-1}\bar{x}m] \in L(\bar{V})$ is independent of the choice of m, because checking equivalence just involves a restriction of the domain. Hence we have a function $G(f/a\backslash x):L(\bar{U}) \longrightarrow L(\bar{V})$.

This function is clearly additive by its construction, so let us check it is an R-homomorphism. Recall that for $[m:D \longrightarrow \bar{U}] \in L(\bar{U})$, scalar multiplication by r gives the class $[mr:r^{-1}D \longrightarrow \bar{U}]$, with $mr(s) = m(rs)$. For $K = \{s \in D \mid \bar{x}(m(s)) \in \bar{a}(\bar{N})\}$ as above, $r^{-1}K$ is suitable on which to define $\overline{fa}^{-1}\bar{x}mr$, since $\bar{x}mr(r^{-1}K) \subseteq \bar{a}(\bar{N})$. Then clearly $(\overline{fa}^{-1}\bar{x}m)r:r^{-1}K \longrightarrow \bar{V}$ is the same map as $\overline{fa}^{-1}\bar{x}(mr)$, or $(G(f/a\backslash x)[m])r = G(f/a\backslash x)[mr]$, as needed.

Now we need to show that the definition of $G(f/a\backslash x)$ depends only on the equivalence class $(f/a\backslash x)$. So assume $(g/b\backslash y)$, $U \xrightarrow{y} P' \xleftarrow{b} N' \xrightarrow{g} V$, is a zigzag equivalent to $(f/a\backslash x)$, so that $\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \in C\begin{pmatrix} a & 0 \\ 0 & b \\ f & g \end{pmatrix}$. Factoring out the torsion part as before, we get $\bar{U} \xrightarrow{\bar{y}} \bar{P}' \xleftarrow{\bar{b}} \bar{N}' \xrightarrow{\bar{g}} \bar{V}$ and $\begin{pmatrix} \bar{x} \\ -\bar{y} \\ 0 \end{pmatrix} \in C\begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \\ \bar{f} & \bar{g} \end{pmatrix}$ (same argument as before). For $m:D \longrightarrow \bar{U}$ with $D \in F$, put

$$K' = \left\{ s \in D \;\middle|\; \begin{pmatrix} \bar{x} \\ -\bar{y} \\ 0 \end{pmatrix}(m(s)) \in \begin{pmatrix} \bar{a} & 0 \\ 0 & \bar{b} \\ \bar{f} & \bar{g} \end{pmatrix}(\bar{N} \oplus \bar{N}') \right\}.$$

By an argument as for the previous K, we get $K' \in F$ and so we may define both $\overline{fa}^{-1}\bar{x}m$ and $\overline{gb}^{-1}\bar{y}m$ on K'. But for $s \in K'$, there are $\bar{n} \in \bar{N}$ and $\bar{n}' \in \bar{N}'$ such that $\bar{x}(m(s)) = \bar{a}(\bar{n})$ and $\bar{y}(m(s)) = \bar{b}(\bar{n}')$, with $\bar{f}(\bar{n}) - \bar{g}(\bar{n}') = 0$. Thus $\overline{fa}^{-1}\bar{x}m = \overline{gb}^{-1}\bar{y}m$ on K', and so $G(f/a\backslash x)[m] = G(g/b\backslash y)[m]$.

Examination of the above part of the proof shows that it works precisely because equivalence in the zigzag localization M_Z is defined in just the way that makes "$fa^{-1}x$" well-defined. But composition and addition of morphisms and the zero and identity morphisms are also defined just so that the "$fa^{-1}x$" interpretation is composed, added, etc. Thus we can easily get proofs along the same lines as the above paragraph which will show that G respects composition, addition, and zero and identity morphisms.

Hence G is an additive functor, and $GE = Q$ on objects of M. For a morphism $c:U \longrightarrow V$ in M, we get $\bar{c}:\bar{U} \longrightarrow \bar{V}$ and thence $Q(U) = L(\bar{U}) \xrightarrow{Q\bar{c}} L(\bar{V}) = Q(V)$; this map $Q\bar{c}$ is given by composing a representative $m:D \longrightarrow \overline{U}$ with $\overline{c}$. On the other hand $E(c) = (c/c\backslash c)$, so $G(c/c\backslash c)[m] = [\overline{cc}^{-1}\overline{cm}] = [\overline{c}m]$, and we see that $GE(c) = Q(c)$.

Since G is additive, to show faithfulness we need only assume $G(f/a\backslash x) = 0$ and show $(f/a\backslash x) = 0$. Assume the diagram is $U \xrightarrow{x} P \xleftarrow{a} N \xrightarrow{f} V$, and take any $u \in U$ (and its image $\overline{u}$ in $\overline{U}$). Then $K = \{s \in R \mid x(u)s \in a(N)\}$ is dense, since $x \in C(a)$. If we define $m_u:R \longrightarrow \overline{U}$ by $m_u(r) = \overline{u}r$, then $G(f/a\backslash x)[m_u] = 0$ means that $[\overline{fa}^{-1}\overline{x}m_u] = 0$. Hence $\overline{fa}^{-1}\overline{x}m_u = 0$ on some dense ideal D, say. Now for $s \in K \cap D$, we get $x(u)s = a(n)$ for some $n \in N$, and also $\overline{f}(\overline{n}) = 0$ (where $\overline{n}$ is the image of n in $\overline{N}$); this means that $f(n) \in T(V)$. Now put $K' = \{r \in R \mid \binom{x(u)}{0}r \in \binom{a}{f}(N)\}$; we want K' to be dense. But for $s \in K \cap D$ and $x(u)s = a(n)$, we get

$$\begin{aligned} s^{-1}K' &= \left\{r \in R \;\middle|\; \binom{x(u)}{0}sr \in \binom{a}{f}(N)\right\} \\ &= \left\{r \in R \;\middle|\; \binom{a(n)}{0}r \in \binom{a}{f}(N)\right\} \\ &\supseteq \{r \in R \mid f(n)r = 0\}. \end{aligned}$$

This last ideal is dense because $f(n) \in T(V)$, so that $s^{-1}K'$ is also dense for each $s \in K \cap D$. Hence $K' \cap K \cap D$ is dense and so K' is also dense; thus $\mathrm{Im}\binom{x}{0} \subseteq C(\mathrm{Im}\binom{a}{f}, P \oplus V)$, or $(f/a\backslash x) = 0$.

Finally, we need G to be full. So take any Mod-R morphism $c:Q(U) \longrightarrow Q(V)$. There are natural maps $i:U \longrightarrow Q(U)$, $j:V \longrightarrow Q(V)$, so form the diagram $U \xrightarrow{ci} Q(V) \xleftarrow{j} V \xrightarrow{1} V$. For this to give an M_Z-morphism, we need $1 \in C^*(j)$ and $ci \in C(j)$. For the former, we use Lemma 5.5. Assume $j(v) = 0$ (for $v \in V$ and $\bar{v}$ its image in $\bar{V}$). Then $m_v : R \longrightarrow \bar{V}$ given by $m_v(r) = \bar{v}r$ is the zero element of $L(\bar{V})$, by the defintion of j. Thus m_v is the zero map on some dense ideal D. But then $D \subseteq \{r \in R \mid vr \in T(V)\}$, so $v = 1v \in C(T(V),V) = C(0,V)$. Thus $1 \in C^*(j)$. To get $ci \in C(j)$, we show that in fact $C(j(V),Q(V))$ is all of $Q(V)$. So pick any $m : D \longrightarrow \bar{V}$ a representative in $L(\bar{V}) = Q(V)$. Recall that multiplication of m by $r \in R$ gives $mr : r^{-1}D \longrightarrow \bar{V}$ with $mr(s) = m(rs)$. For $[m]$ to be in $C(j(V),Q(V))$, we need that

$$\{r \in R \mid \text{there is } v \in V \text{ with } mr = j(v) \text{ on some } K \in F\}$$
$$= \{r \in R \mid \text{there is } \bar{v} \in \bar{V} \text{ with } mr(s) = \bar{v}s \text{ for all } s \in K \in F\}$$
$$= \{r \in R \mid \text{there is } \bar{v} \in \bar{V} \text{ with } m(rs) = \bar{v}s \text{ for all } s \in K \in F\}$$

is dense. But for $r \in D$, $m(rs) = m(r)s$, so the dense ideal D is contained in the above ideal; hence we are done.

To complete the proof we need only show that $G(1/j\backslash ci) = c$. So take $m : D \longrightarrow \bar{U}$ any representative in $L(\bar{U}) = Q(U)$ and show $G(1/j\backslash ci)[m] = c([m])$. Now to define $G(1/j\backslash ci)[m]$, put $K = \{r \in D \mid \bar{c}\bar{1}m(r) \in \bar{j}(\bar{V})\}$, using the diagram $\bar{U} \xrightarrow{\bar{1}} Q(U) \xrightarrow{c} Q(V) \xleftarrow{\bar{1}} \bar{V} \xrightarrow{1} \bar{V}$. (It is easy to show that $T(Q(V)) = 0$, so $c = \bar{c}$; or see Lambek.) We have seen that K is dense, and $G(1/j\backslash ci)[m] = [\bar{j}^{-1}\bar{c}\bar{1}m]$ is defined at least on K. Now take D' to be a choice of dense ideal which is a domain for $c[m]$, so write $c[m] = [m']$, for $m' : D' \longrightarrow \bar{V}$. We intend to show that

m' and $\bar{j}^{-1}c\bar{i}m$ agree on $D' \cap D \cap K$. Recall that the action of $r \in R$ on m gives $mr: r^{-1}D \longrightarrow \bar{U}$, where $mr(s) = m(rs)$. Then if $r \in D$, clearly $r^{-1}D = R$ and so $[m]r = [mr] = \bar{i}(m(r))$ in $Q(U)$. If in fact $r \in D' \cap D \cap K$, then $c[mr] = c\bar{i}m(r)$ is in $\bar{j}(\bar{V})$. But c is an R-homomorphism, so $c[mr] = c([m]r) = c([m])r = [m']r = [m'r]$, where $m'r: r^{-1}D' \longrightarrow \bar{V}$ is given by $m'r(s) = m'(rs)$. Since $r \in D'$, $r^{-1}D' = R$, so $\bar{j}^{-1}c\bar{i}m(r) = \bar{j}^{-1}c[mr] = \bar{j}^{-1}[m'r] = m'r(1) = m'(r)$. Since $D' \cap D \cap K$ is dense, $c[m] = [m'] = [\bar{j}^{-1}c\bar{i}m]$, which is $G(1/j\backslash ci)[m]$. Since m was arbitrary, $G(1/j\backslash ci) = c$.

This completes the proof of 5.4.

Lambek has called the modules which are isomorphic to $Q(N)$ for some N the "torsion-free divisible" modules. (He actually gives a more constructive definition.) Since G has the property that every $Q(N)$ is isomorphic to some $G(N)$, we have shown that G is an equivalence between M_Z and the full subcategory of Mod-R = M of all torsion-free divisible modules.

<u>Rings not Rings of Quotients</u>. We can quickly show also that not all rings constructible as zigzag localizations are constructible as rings of quotients with respect to some torsion theory. To do this, put $R = k\langle x,y\rangle$, the free associative algebra in variables x and y over a field k. Also put $S = k\langle x,y,x^{-1},y^{-1}\rangle$, obtained by simply adjoining inverses of x and y to R. Then $R \subseteq S$ is an epimorphism of rings, so every element of S is expressible as a zigzag in $(P_R\text{-Mod})_Z$, for Z some coherent pair of

a.c.o.'s. Each such zigzag can be considered as a zigzag on Mod-R as well, so S is constructible as a zigzag localization of R.

If we assume that S is also a ring of quotients with respect to some torsion theory, then a result of Lambek ([4], Prop. 2.6) says that the right R-module S is an essential extension of R; that is, each R-submodule of S intersects R nontrivially. But the R-submodule $(x^{-1} + y^{-1})R$ of S can easily be seen to intersect R only in $\{0\}$. (This can be seen, for example, by noting that S is exactly the group algebra of the free group on two generators.) Thus S is not a ring of quotients of R.

APPENDIX I

THE RATIONAL CLOSURE

Suppose R and S are rings with $R \subseteq S$. Recall the definition of the rational closure $\Sigma(R,S)$ as the set of all elements of S which occur as an entry in a square matrix over S which is the inverse of a matrix over R. Cohn ([2], pp. 249-251) has shown that $\Sigma(R,S)$ is a subring of S containing R.

Cohn's term "rational closure" derives from the fact that, if S is a division ring, then $\Sigma(R,S)$ is the sub-division ring of S generated by R. Hence $\Sigma(R,S)$ is closed under taking inverses of elements of $\Sigma(R,S)$ which are invertible in S.

We are interested in further justifying the term "closure" for this subring, by showing that, for fixed S, $\Sigma(-,S)$ satisfies the three conditions for a closure operator, namely:

(1) For a subring R of S, $R \subseteq \Sigma(R,S)$.

(2) For subrings $R_1 \subseteq R_2$ of S, $\Sigma(R_1,S) \subseteq \Sigma(R_2,S)$.

(3) For a subring R of S, $\Sigma(\Sigma(R,S),S) = \Sigma(R,S)$.

Part (1) is by Cohn's result, while (2) is obvious. The main purpose of this Appendix is to prove (3).

Let us put $\Sigma = \Sigma(R,S)$ and take A to be a Σ-matrix which is invertible over S. Our task is to show that all the entries of A^{-1} are in fact in Σ already. We will do this by producing a (large) R-matrix whose inverse in S contains the matrix A^{-1} as a submatrix.

Let A be n-by-n with entry a_{ij} in the row-i, column-j position. Since $a_{ij} \in \Sigma$, we have that a_{ij} is an entry in A_{ij}^{-1}, for A_{ij} an R-matrix invertible over S. We can get a large R-matrix B (say m-by-m) whose inverse over S contains every a_{ij} as an entry, by putting

$$B = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{pmatrix}.$$

ACTUALLY WE NEED ROW AND COLUMN DECOMPOSITION. So $f,g: \{1..n\} \times \{1,..n\} \longrightarrow \{1,..m\}$ AND EVENTUALLY $U = \sum_{i,j} E_{i f(i,j)}$ $V = \sum_{i,j} E_{g(i,j) j}$

Let us give functions $f,g: \{1,2,\cdots,n\} \longrightarrow \{1,2,\cdots,m\}$ by requiring that a_{ij} occurs as the row-f(i), column-g(j) entry of B^{-1}. Then take n-by-m matrices $E_{if(i)}$ (for each i) which have only zero entries, except for a 1 in the (i,f(i)) position. Likewise take m-by-n matrices $E_{g(j)j}$ (for each j), and we see that $E_{if(i)}B^{-1}E_{g(j)j}$ is just an n-by-n matrix of zeroes, except for a_{ij} in the (i,j) position.

Hence $A = \sum_{i,j} E_{if(i)}B^{-1}E_{g(j)j} = UB^{-1}V$, where $U = \sum_i E_{if(i)}$ and $V = \sum_j E_{g(j)j}$. Recall that A is assumed to be invertible over S. Then the R-matrix $\begin{pmatrix} -B & -V \\ -U & 0 \end{pmatrix}$ has an inverse over S, namely

$$\begin{pmatrix} -B & -V \\ -U & 0 \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1}VA^{-1}UB^{-1} - B^{-1} & -B^{-1}VA^{-1} \\ -A^{-1}UB^{-1} & A^{-1} \end{pmatrix}.$$

Hence A^{-1} is a Σ-matrix, completing the proof.

More generally, we can let $\Sigma_k(R,S)$ be the set of elements of S which occur as an entry in an m-by-n matrix over S which is the inverse of a matrix over R, where m and n are any positive integers satisfying $m \equiv n \pmod{k}$. (An n-by-m matrix is invertible if it has an m-by-n inverse matrix, such that the product of the matrix by its inverse on each side is the identity matrix of the appropriate size.) Cohn's original $\Sigma(R,S)$ definition is the $k = 0$ case. Then a slight variation of Cohn's proof shows that $\Sigma_k(R,S)$ is a subring of S containing R for each k. Furthermore, the above proof of condition (3) can be slightly modified to demonstrate that $\Sigma_k(-,S)$ is a closure operator on subrings of S, for each k.

Note also that these rational closures satisfy the conditions for $\Sigma_k(R,-)$ to be a "coclosure operator" on overrings S, for fixed R. These conditions are:

(1') For an overring $S \supseteq R$, $\Sigma_k(R,S) \subseteq S$.

(2') For overrings $S_1 \subseteq S_2$ of R,

$$\Sigma_k(R,S_1) \subseteq \Sigma_k(R,S_2).$$

(3') For an overring S of R,

$$\Sigma_k(R,\Sigma_k(R,S)) = \Sigma_k(R,S).$$

Parts (1') and (2') are obvious, while (3') follows because the inverse of an appropriate R-matrix over S has <u>all</u> its entries in $\Sigma_k(R,S)$.

APPENDIX II

A PRIME MATRIX IDEAL YIELDS A DIVISION RING

Given a ring R, recall Cohn's definition of a prime matrix ideal P over R as a collection of square R-matrices (of any sizes) satisfying:

(1) If A is n-by-(n-1) and B is (n-1)-by-n, then $AB \in P$.

(2) If matrices A, B, and C agree except on one row (or column), and if the corresponding row (or column) of C is the vector sum of the corresponding rows (or columns) of A and B, and if $A \in P$ and $B \in P$, then $C \in P$.

(3) If $A \in P$ and B is square, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in P$.

(4) The 1-by-1 matrix (1) is not in P.

(5) If $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in P$, then either $A \in P$ or $B \in P$.

Then Cohn ([2], pp. 268-279) showed that such a P gives rise to a division ring K_P and a ring homomorphism $R \longrightarrow K_P$, such that the image of a square R-matrix is invertible over K_P if and only if the matrix is not in P.

In this Appendix we intend to construct such a division ring K_P in a manner similar both to Cohn's method and to the zigzag localization construction. The author expects eventually to be able to do this by zigzag localization alone, using Theorem 4.5, but as yet the correct choice of right a.c.o. is not confirmed.

Before we describe the construction, let us recall a few of Cohn's lemmas which will make it easier.

Lemma II.1: If A is n-by-n and B is m-by-m, and if C is n-by-m, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in P$ if and only if $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in P$.

Lemma II.2: Given any matrix in P, each of the following operations on it results in an element of P:

(i) exchanging rows (or columns);

(ii) adding one row to another (or one column to another);

(iii) multiplying a row by a scalar on the left (or multiplying a column by a scalar on the right).

These lemmas allow what we have called "generalized row and column operations", so that the proof that this construction works will be much like that for the zigzag localization. Hence we will omit much of it.

To construct K_P, first take L_P to be the set of all triples (f,a,x) such that a is an n-by-n R-matrix (for some n) which is not in P, f is 1-by-n and x is n-by-1. Put a relation $\approx$ on L_P by putting $(f,a,x) \approx (g,b,y)$ exactly when the matrix $\begin{pmatrix} a & 0 & x \\ 0 & b & -y \\ f & g & 0 \end{pmatrix} \in P$.

Theorem II.3: The relation $\approx$ is an equivalence relation.

Proof: Symmetry is obvious by changing signs and rearranging rows and columns. Reflexivity follows since $\begin{pmatrix} a & -x \\ 0 & 0 \end{pmatrix} \in P$ by (1), and so $\begin{pmatrix} a & 0 & 0 \\ 0 & a & -x \\ f & 0 & 0 \end{pmatrix} \in P$ by (3) and II.1;

then we add and subtract rows and columns to get

$\begin{pmatrix} a & -a & x \\ 0 & a & -x \\ f & 0 & 0 \end{pmatrix} \in P$, and finally $\begin{pmatrix} a & 0 & x \\ 0 & a & -x \\ f & f & 0 \end{pmatrix} \in P$, as needed. To do transitivity, assume both $(f,a,x) \approx (g,b,y)$ and $(g,b,y) \approx (h,c,z)$. Then $\begin{pmatrix} a & 0 & x \\ 0 & b & -y \\ f & g & 0 \end{pmatrix} \in P$, $\begin{pmatrix} b & 0 & y \\ 0 & c & -z \\ g & h & 0 \end{pmatrix} \in P$ imply $\begin{pmatrix} a & 0 & 0 & x \\ 0 & b & 0 & -y \\ 0 & 0 & c & 0 \\ f & g & h & 0 \end{pmatrix} \in P$ and $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & y \\ 0 & 0 & c & -z \\ f & g & h & 0 \end{pmatrix} \in P$, so use condition (2) to get $\begin{pmatrix} a & 0 & 0 & x \\ 0 & b & 0 & 0 \\ 0 & 0 & c & -z \\ f & g & h & 0 \end{pmatrix} \in P$. Then (5) says the "b" part may be dropped, or $\begin{pmatrix} a & 0 & x \\ 0 & c & -z \\ f & h & 0 \end{pmatrix} \in P$, completing the proof.

Now we put $K_P = L_P/\approx$ and denote the equivalence class of (f,a,x) by the familiar $(f/a\backslash x)$. Then we make the definitions of $+$, $-$, $\cdot$, 0, 1, E in a manner almost the same as for the zigzag localization:

(+) $(f/a\backslash x) + (g/b\backslash y) = ((f\ g)/\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} x \\ y \end{pmatrix})$.

(−) $-(f/a\backslash x) = (f/a\backslash -x)$.

(•) $(f/a\backslash x)\cdot(g/b\backslash y) = ((f\ 0)/\begin{pmatrix} a & -xg \\ 0 & b \end{pmatrix}\backslash\begin{pmatrix} 0 \\ y \end{pmatrix})$.

(0) $0 = (1/1\backslash 0)$.

(1) $1 = (1/1\backslash 1)$.

(E) For $r \in R$, $E(r) = (1/1\backslash r)$.

The only changes are in (−), (0), (E), since $(f/a\backslash x)$ is only defined for $a \notin P$.

<u>Theorem II.4</u>: The above definitions give a well-defined ring structure on K_P, under which K_P is a division ring.

Furthermore, $E: R \longrightarrow K_P$ is a ring homomorphism, and E takes a square matrix A to an invertible matrix over K_P if and only if $A \notin P$.

Proof: Each of the definitions above clearly yields an element of L_P. The proof that K_P is a ring involves checking that +, - and · are well-defined, + and · are associative and have identity elements 0 and 1, + is commutative, and · distributes over +. These all follow the proof of Theorem 3.4, except perhaps the proof of well-definedness for ·; let us prove that only.

Assume $(f/a\backslash x) = (g/b\backslash y)$ and $(h/c\backslash z) = (j/d\backslash w)$, and show $(f/a\backslash x)\cdot(h/c\backslash z) = (g/b\backslash y)\cdot(j/d\backslash w)$. Thus we want

$$\begin{pmatrix} a & -xh & 0 & 0 & 0 \\ 0 & c & 0 & 0 & z \\ 0 & 0 & b & -yj & 0 \\ 0 & 0 & 0 & d & -w \\ f & 0 & g & 0 & 0 \end{pmatrix} \in P. \quad \text{From} \begin{pmatrix} c & 0 & z \\ 0 & d & -w \\ h & j & 0 \end{pmatrix} \in P \text{ we get}$$

$$\begin{pmatrix} c & 0 & z & 0 \\ 0 & d & -w & 0 \\ h & j & 0 & g \\ 0 & 0 & 0 & b+yg \end{pmatrix} \in P, \text{ whence } p = \begin{pmatrix} c & 0 & z & 0 \\ 0 & d & -w & 0 \\ h & j & 0 & g \\ -yh & -yj & 0 & b \end{pmatrix} \in P. \quad \text{Also}$$

$$\begin{pmatrix} c & 0 & z & 0 \\ 0 & d & -w & 0 \\ h & j & 0 & 0 \\ -yh & -yj & 0 & b \end{pmatrix} \in P, \text{ whence } q = \begin{pmatrix} c & 0 & z & 0 \\ 0 & d & -w & 0 \\ -h & -j & 0 & 0 \\ -yh & -yj & 0 & b \end{pmatrix} \in P.$$

Applying condition (2) to p and q we get $\begin{pmatrix} c & 0 & z & 0 \\ 0 & d & -w & 0 \\ 0 & 0 & 0 & g \\ -yh & -yj & 0 & b \end{pmatrix} \in P,$

so $\begin{pmatrix} c & 0 & z & 0 & 0 \\ 0 & d & -w & 0 & 0 \\ 0 & 0 & 0 & g & f \\ -yh & -yj & 0 & b & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \in P.$ Rearrange to obtain

$$t = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & z \\ 0 & -yh & b & -yj & 0 \\ 0 & 0 & 0 & d & -w \\ f & 0 & g & 0 & 0 \end{pmatrix} \in P.$$ To finish, we need only show that the entry $-yh$ in the third row of t can be changed to a $-xh$ in the first row (same column). This must be done by changing one column of $-yh$ at a time; we show the technique for the first column only.

Let $h = (h_1 \; h')$ and $c = (c_1 \; c')$, where h_1 is a scalar, h' a row vector, and c_1 a column vector. Then

$$\begin{pmatrix} a & 0 & x \\ 0 & b & -y \\ f & g & 0 \end{pmatrix} \in P \text{ gives } \begin{pmatrix} a & 0 & -xh_1 & 0 & 0 & 0 \\ 0 & b & yh_1 & -yh' & -yj & 0 \\ f & g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c' & 0 & z \\ 0 & 0 & 0 & 0 & d & -w \end{pmatrix} \in P.$$ Rearrange this to obtain

$$\left(\begin{array}{c|cc|ccc} a & -xh_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c' & 0 & 0 & z \\ 0 & yh_1 & -yh' & b & -yj & 0 \\ 0 & 0 & 0 & 0 & d & -w \\ f & 0 & 0 & g & 0 & 0 \end{array}\right) \in P,$$ so we can use condition (2) with t to get

$$\left(\begin{array}{c|cc|ccc} a & -xh_1 & 0 & 0 & 0 & 0 \\ 0 & c_1 & c' & 0 & 0 & z \\ 0 & 0 & -yh' & b & -yj & 0 \\ 0 & 0 & 0 & 0 & d & -w \\ f & 0 & 0 & g & 0 & 0 \end{array}\right) \in P.$$

Continuing similarly, we obtain the needed result.

The map E is easily shown to be a homomorphism, as in Theorem 3.4; note that its kernel in this case consists of of those $r \in R$ such that $(r) \in P$. Further, K_P is a division ring because a non-zero $(f/a\backslash x)$ has $\begin{pmatrix} a & x \\ f & 0 \end{pmatrix} \notin P$, so we can use $((0\;1)/\begin{pmatrix} a & x \\ f & 0 \end{pmatrix}\backslash\begin{pmatrix} 0 \\ 1 \end{pmatrix})$ as the $(f/a\backslash x)^{-1}$, just as in Lemma 4.4.

Now for a square matrix $a \notin P$, we want to invert $E(a)$ over K_P. Let e_1 be the column vector of zeroes, except for a 1 in the i-th place, and let e_i^t be its transpose, a row vector. Then it is easy to see that the (i,j)-entry of $E(a)^{-1}$ is just $(e_i^t/a\backslash e_j) \in K_P$. We can write this as $E(a)^{-1} = (I/I\backslash a)^{-1} = (I/a\backslash I)$, where I is the identity matrix of the appropriate size.

Conversely, suppose $a \in P$; we wish to show E(a) is not invertible in K_P — equivalently, that the columns of E(a) are linearly dependent over K_P. By rearranging rows and columns of a (this does not change non-invertibility), let us assume that the upper left corner of a is the maximal square submatrix of a which does not belong to P (if there is none, then E(a) = 0, not invertible). So let us write $a = \begin{pmatrix} A_1 & a_2 & * \\ A_3 & a_4 & * \end{pmatrix}$, where $A_1 \notin P$ is this maximal submatrix (say m-by-m), a_2 and a_4 are column vectors, and the *'s represent the remainder of the matrix a. Then we can show that the column $\binom{a_2}{a_4}$ is a right linear combination over K_P of the preceding columns in a. If we recall our slightly generalized notation of $(I/A_1\backslash a_2)$ for a column vector with i-th entry $(e_i^t/A_1\backslash a_2)$, then in fact $E\begin{pmatrix} a_2 \\ a_4 \end{pmatrix} = E\begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \cdot (I/A_1\backslash a_2)$.

To check this, we check each row has equality; or that $E(e_i^t \cdot \begin{pmatrix} a_2 \\ a_4 \end{pmatrix}) = E(e_i^t \cdot \begin{pmatrix} A_1 \\ A_3 \end{pmatrix}) \cdot (I/A_1\backslash a_2)$ for each i. It is easily checked that what we need is the equalities

$(1/1\backslash e_i^t \cdot \begin{pmatrix} a_2 \\ a_4 \end{pmatrix}) = (e_i^t \cdot \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} \Big/ A_1\backslash a_2)$, or in matrix form

$$\begin{pmatrix} A_1 & 0 & a_2 \\ 0 & 1 & -e_i^t \cdot \begin{pmatrix} a_2 \\ a_4 \end{pmatrix} \\ e_i^t \cdot \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} & 1 & 0 \end{pmatrix} \in P.$$

Equivalently by a row operation, we need

$$\begin{pmatrix} A_1 & 0 & a_2 \\ 0 & 1 & -e_i^t \cdot \begin{pmatrix} a_2 \\ a_4 \end{pmatrix} \\ e_i^t \cdot \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} & 0 & e_i^t \cdot \begin{pmatrix} a_2 \\ a_4 \end{pmatrix} \end{pmatrix} \in P,$$

or $\begin{pmatrix} A_1 & a_2 \\ e_i^t \cdot \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} & e_i^t \cdot \begin{pmatrix} a_2 \\ a_4 \end{pmatrix} \end{pmatrix} \in P$. But this is now clear; if $i \leq m$, then we have a repeated row; if $i > m$, then the matrix is in P because it is a submatrix of a and A_1 is maximal. Hence $E(a)$ is not invertible over K_P if and only if $a \in P$, completing the proof.

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