Project MUSE®
http://muse.jhu.edu

COCHAIN MULTIPLICATIONS

By MICHAEL A. MANDELL

Abstract. We describe a refinement of the Eilenberg-Steenrod axioms that provides a necessary and sufficient condition for functors from spaces to differential graded algebras or E_∞ differential graded algebras to be naturally quasi-isomorphic to the singular cochain functor.

Introduction. The introduction of the Eilenberg-Steenrod axioms in the 1940's revolutionized the understanding of cohomology. The axiomatic framework provided a powerful theorem for identifying ordinary cohomology and reorganized the study of cohomology theory by isolating key elements typically used for its basic calculations, maximizing flexibility. It was quickly seen that additional structures on cohomology such as multiplication and Steenrod operations also admit similarly elementary axiomatizations.

The singular cochain functor on spaces or, more generally, the normalized cochain functor on simplicial sets gives a particular model for ordinary cohomology where all known additional structure is visible. For example, the multiplication and Steenrod operations come from an "E_∞ algebra" structure [12, 6]. We now understand that for a finite type nilpotent space (or simplicial set), all p-adic homotopy information about the space is encoded in the quasi-isomorphism type of this E_∞ algebra structure [10]. In fact, we can recover the p-pro-finite completion (up to weak equivalence) of any connected space from the E_∞ algebra of its cochains (up to quasi-isomorphism) [10, App B]. This is nearly the theoretical maximum amount of homotopy information that can be preserved by the quasi-isomorphism class of any model for ordinary $\mathbf{Z}/p\mathbf{Z}$ cohomology.

While it is useful to have a refinement of ordinary cohomology theory carrying so much homotopy information, for calculations to be feasible, it would be preferable to have more flexibility in the model. While it is probable that the argument of [10] could be modified and extended to apply to any suitable model, this would be tedious to do on a case by case basis. A better alternative is to have axioms to identify the singular cochain functor up to quasi-isomorphism in the category of E_∞ algebras. In this paper, we provide such axioms. They turn out to be a cochain-level refinement of the Eilenberg-Steenrod axioms. We encode them in the definition of a cochain theory.

Manuscript received December 11, 2000; revised September 24, 2001.
Research supported in part by NSF Postdoctoral Research Fellowship DMS-9804421.
American Journal of Mathematics 124 (2002), 547–566.

Definition. Let T: **Top** $\to$ **M** be a contravariant functor from spaces to differential graded k-modules for some commutative ring k. We say that T is a *cochain theory* if it satisfies the following axioms.

A.1 (Homotopy). If $X \to Y$ is a weak equivalence, then $T(Y) \to T(X)$ is a quasi-isomorphism.

A.2 (Exactness/Excision). If X is a CW complex and $A \to X$ is an inclusion of a subcomplex, then the map from the homotopy fiber $F(T(X/A) \to T(*))$ to the homotopy fiber $F(T(X) \to T(A))$ is a quasi-isomorphism.

A.3 (Product). If $\{X_\alpha, \alpha \in S\}$ is a collection of CW complexes indexed on a set S, then the canonical map from $T(\coprod X_\alpha)$ to $\prod T(X_\alpha)$ is a quasi-isomorphism.

A.4 (Dimension). $H^0(T(*)) \cong k$ and $H^n(T(*)) = 0$ for $n \neq 0$.

We can give a homological interpretation of these axioms as follows. If we write $H^*_T(X, A)$ for $H^*F(T(X) \to T(A))$ and by convention $H^*_T(X)$ for $H^*_T(X, \emptyset)$, then for any $A \subset B \subset X$, we obtain a "connecting homomorphism"

$$\delta\colon\ H^*_T(B, A) \to H^{*+1}_T(X, B)$$

that associates to the map of pairs $(X, A) \to (X, B)$ a long exact sequence

$$\cdots \to H^{n-1}_T(B, A) \to H^n_T(X, B) \to H^n_T(X, A) \to H^n_T(B, A) \to \cdots.$$

When $H^*(T(\emptyset)) = 0$, the axioms above are equivalent to the requirement that H^*_T be naturally isomorphic to (singular) cohomology as a δ-functor, that is, by a natural isomorphism preserving the long exact sequences of pairs.

In the language of the previous definition, the following is the main result.

MAIN THEOREM. *Let T: **Top** $\to$ **E** be a contravariant functor from spaces to E_∞ k-algebras. Then T is naturally quasi-isomorphic in* **E** *to the singular cochain functor C^* if and only if the underlying functor T: **Top** $\to$ **M** is a cochain theory.*

We give a precise definition of the category **E** of E_∞ k-algebras in Section 4 as the category of algebras over a cofibrant operad $\mathcal{E}$. That section also explains the rationale behind this category. We review the construction of the cochain complex C^* as a functor to **E** in Section 1.

We have written the axioms in the most convenient but not the most general form. For the purposes of this paper, the reader is free to interpret the category **Top** of "spaces" to be the usual category of topological spaces or any one of the following: the category of simplicial sets, the category of finite simplicial sets, the category of cell complexes, or the category of finite cell complexes. When we consider the category of simplicial sets for example, we should understand all spaces to be CW for the purposes of the Exactness Axiom A.2 and the inclusions of subcomplexes to be the monomorphisms. Likewise, when we consider the

categories of finite objects, we should understand the Product Axiom A.3 to apply only to the disjoint unions that exist in the category, namely the finite ones. With the proper reinterpretation of CW complex and inclusion of subcomplex in A.2 and of the indexing set in A.3, the results of this paper apply as well to more general categories of spaces.

The axioms above resemble the modern rather than the classical Eilenberg-Steenrod axioms in that they include the Milnor Product Axiom A.3 and the strong form of the Homotopy Axiom in A.1. These modern axioms uniquely identify the singular cohomology theory, and this is reflected in the statement of the Main Theorem when we understand **Top** to denote the category of all topological spaces. When we understand **Top** to denote the category of finite cell complexes, both the axioms and the conclusions essentially reduce to those of the classical case.

To give a full analogue of the Eilenberg-Steenrod uniqueness theorem, we need to understand the uniqueness of the natural quasi-isomorphism whose existence the Main Theorem asserts. The most concise way to explain this requires the introduction of the localized functor category $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$. Let $\mathbf{E}^{\mathbf{Top}}$ denote the "category" of contravariant functors from spaces to E_∞ k-algebras with maps the natural transformations, and let $\mathcal{Q}$ denote the subcategory of natural quasi-isomorphisms. The "category" $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ is then the "category" obtained from $\mathbf{E}^{\mathbf{Top}}$ by formally inverting the maps in $\mathcal{Q}$, see for example [4, §I.1]. When **Top** denotes the category of all topological spaces or all simplicial sets, the set-theoretical difficulties in constructing this category are well-known, and are overcome by working in a set-theory with a hierarchy of universes; when **Top** denotes the category of finite cell complexes or finite simplicial sets, it has a small skeleton, and the construction of $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ has no set-theoretic difficulties [11, 2.1]. (The reader squeamish of set-theoretical difficulties should now restrict **Top** exclusively to this latter case.)

In this language, the Main Theorem states that a functor T: **Top** $\to$ **E** is isomorphic to C^* in $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ if and only if T is a cochain theory. To say that the isomorphism is unique is equivalent to saying that the only automorphism of C^* in $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ is the identity. We prove the following theorem.

UNIQUENESS THEOREM. *The only endomorphism of C^* in $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ is the identity.*

In the course of the proof of the Main Theorem and the Uniqueness Theorem, we prove the following more general theorem, reformulated as Theorem 3.1 below. In it, $\mathbf{E}[\mathcal{Q}^{-1}]$ denotes the category obtained from **E** by formally inverting the quasi-isomorphisms.

Definition. A functor T: **Top** $\to$ **M** is a *generalized cochain theory* if it satisfies Axioms A.1–A.3.

THEOREM. *Let* P: $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}] \to \mathbf{E}[\mathcal{Q}^{-1}]$ *denote the functor* $PT = T(*)$. *Then* P *induces an equivalence from the full subcategory of* $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ *of generalized cochain theories to the category* $\mathbf{E}[\mathcal{Q}^{-1}]$.

In particular, it follows that natural quasi-isomorphism classes of generalized cochain theories are in one-to-one correspondence with quasi-isomorphism classes of E_∞ algebras. Additionally, it also follows that P induces an equivalence of categories between the full subcategory of generalized cochain theories satisfying a dimension axiom and the category of (ungraded) commutative k-algebras: Standard obstruction theory arguments can be adapted to prove that the full subcategory of $\mathbf{E}[\mathcal{Q}^{-1}]$ of E_∞ k-algebras whose cohomology is concentrated in degree zero is equivalent to the category of commutative k-algebras.

For the reader unfamiliar with E_∞ k-algebras, we offer the following motivational corollary regarding the category **Com** of commutative differential graded k-algebras. The Thom-Sullivan PL De Rham complex Ω^* is a cochain theory that takes values in commutative **Q**-algebras. By tensoring over **Q** with k, we obtain a canonical cochain theory of commutative k-algebras for any commutative ring k that contains **Q**. On the other hand, when k contains **Q**, E_∞ k-algebras are not so different from commutative differential graded k-algebras: The forgetful functor $\mathbf{Com} \to \mathbf{E}$ induces an equivalence of the localized functor categories $\mathbf{Com}^{\mathbf{Top}}[\mathcal{Q}^{-1}] \to \mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$. (For convenience we include a proof as Theorem 4.9 below.) We obtain the following corollary of the Main Theorem.

COROLLARY. *Let* $k \supseteq \mathbf{Q}$, *and let* T: $\mathbf{Top} \to \mathbf{Com}$. *Then* T *is naturally isomorphic in* $\mathbf{Com}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ *to the De Rham functor* Ω^* *if and only if* T *is a cochain theory. When the isomorphism exists, it is unique.*

It is an easy consequence of [12, 3.3] that when k does not contain **Q** no functor to commutative differential graded k-algebras can be a cochain theory.

Finally, we also prove the following less structured versions of the Main Theorem.

THEOREM. *The Main Theorem and the Uniqueness Theorem hold with the category* **E** *replaced with the category of differential graded* k*-algebras.*

THEOREM. *The Main Theorem holds with the category* **E** *replaced with the category* **M** *of differential graded* k*-modules. The monoid of endomorphisms of* C^* *in* $\mathbf{M}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ *is isomorphic to the monoid of* k*-module endomorphisms of* k, *i.e., the multiplicative monoid of* k.

The latter theorem at least must already be known, but we have not been able to find it in the literature. It implies in particular that a functor T: $\mathbf{Top} \to \mathbf{E}$ is naturally quasi-isomorphic to C^* in **E** if and only if it is naturally quasi-isomorphic to C^* in **M**.

1. The Main Theorem. Although we are primarily concerned with the category of E_∞ algebras, we need relatively little about the specifics of the category for the proof of the Main Theorem. In fact, the properties of the category **E** that we need are largely unrelated to its concrete construction or its philosophical justification. For this reason, we postpone the detailed definition and explanation of the category **E** to Section 4. In this section, we prove a generalization of the Main Theorem that works in many categories. The proof, given in the next section, is a straightforward homotopy limit argument.

We use the following conventions. We fix the commutative ring k and work in categories of k-modules. Let **M** denote the category of differential graded k-modules, which we grade cohomologically (differential raising degree) until Section 4. In the abstract setting of this section and the next two, we have the choice of understanding differential graded k-modules to be **Z**-graded or nonnegatively graded; for the category **E**, we must use **Z**-graded differential modules. In our abstract setting, we consider a category **C** with a faithful functor to **M**, which we will call the "forgetful" or "underlying differential graded k-module" functor. We assume that **C** is complete (has categorical products and equalizers) and that the forgetful functor is continuous (preserves categorical products and equalizers). We also require **C** to have the following additional structure.

Definition 1.1. Let $N\colon \mathbf{\Delta C} \to \mathbf{C}$ be a functor from the category of cosimplicial objects on **C** to **C**. We say that N is a *cosimplicial normalization functor* if it is normalization on the underlying differential graded k-modules, and if, whenever $A^\bullet$ is a constant cosimplicial object (all face and degeneracy maps are the identity), the canonical isomorphism $A^0 \cong N(A^\bullet)$ in **M** is the underlying map of differential graded k-modules of a map in **C**.

We remind the reader that the normalization of a cosimplicial differential graded k-module is just the total complex of the double complex obtained by normalizing degreewise. When considering **Z**-graded differential graded k-modules, we should understand the total complex in the completed sense, that is, constructed with the cartesian product rather than the direct sum of the modules of a given total degree. In forming the total complex of a double complex such as this, where the two differentials commute, we must alter one of the differentials by a sign. We choose to alter the cosimplicial differential, attaching the sign $(-1)^{j+1}$ to the differential from cosimplicial degree j to $j+1$. In other words, for an element a in internal degree i and cosimplicial degree j, the total differential on a is given by

$$da = \partial a + (-1)^{j+1}(\delta^0 a - \delta^1 a + \cdots + (-1)^{j+1}\delta^{j+1}a),$$

where ∂ denotes the internal differential and the δ's are the coface maps. This choice is explained by Proposition 5.2 below. One consequence of this sign

convention is that when $A^\bullet$ is concentrated in internal degree zero, our differential differs from the usual alternating sum of the faces by a sign of $(-1)^{n+1}$. However, this sign is actually natural and desirable in considering cochain complexes: The normalized cochain complex of a simplicial set $X_\bullet$ is precisely the normalization of the cosimplicial differential graded k-module $k^{X_\bullet}$ (see below) when we take this sign convention. Another advantage of this sign convention is that when $A^\bullet$ is a constant cosimplicial differential graded k-module, the usual identification $A^0 = N(A^\bullet)$ introduces no signs in the differential.

For a simplicial set $X_\bullet$, we have the cosimplicial differential graded k-module $k^{X_\bullet}$, which in cosimplicial degree n is the product of copies of k indexed on the set X_n of n-simplices of $X_\bullet$; the simplicial face and degeneracy maps of $X_\bullet$ induce cosimplicial face and degeneracy maps for $k^{X_\bullet}$. With the sign convention above, the normalized cochain complex $C^*X_\bullet$ is $N(k^{X_\bullet})$. Likewise, for a space X, if we denote by X_n the set of continuous maps from the standard n-simplex $\Delta[n]$ into X, then $X_\bullet$ is a simplicial set, the singular simplicial set of X, and the singular cochain complex C^*X is $N(k^{X_\bullet})$. It follows that when $\mathbf{C}$ has a cosimplicial normalization functor and there is an object of $\mathbf{C}$ with underlying differential graded k-module isomorphic to k, we can lift C^* to a functor $\mathbf{Top} \to \mathbf{C}$. The object "$k$" in $\mathbf{C}$ may not be unique and choosing a different object chooses a different lift of C^*. For this reason, it is useful to generalize the normalized and singular cochain functors as follows.

Definition 1.2. Let A be an object of $\mathbf{C}$, and let X be an object of **Top**. Let $A^{X_\bullet}$ be the cosimplicial object in $\mathbf{C}$ that in cosimplicial degree n is the product of copies of A indexed on the set X_n, with face and degeneracy maps induced by the face and degeneracy maps of $X_\bullet$. Define $C^*(X;A) = N(A^{X_\bullet})$, a functor of A and X. Let C^*_A: $\mathbf{Top} \to \mathbf{C}$ be the functor $C^*(-;A)$.

In the category of differential graded k-modules, we can also describe the functor C^*_A as the composite of the chain functor C_* and the function complex $\mathrm{Hom}(-,A)$,

$$C^*_A = \mathrm{Hom}(C_*(-), A). \tag{1.3}$$

It is easy to deduce from the properties of the chain functor that $H^*_{C^*_A}$ is a generalized cohomology theory. (In fact, $H^*_{C^*_A}$ always decomposes as a graded product of ordinary cohomology theories, but not in a canonical way: $\mathrm{Hom}_k(C_*(-;k),A)$ is naturally isomorphic to $\mathrm{Hom}_{\mathbf{Z}}(C_*(-;\mathbf{Z}),A)$ and A is (noncanonically) quasi-isomorphic as a differential graded $\mathbf{Z}$-module to the graded sum of its cohomology groups.) The following proposition is also an easy consequence of (1.3).

PROPOSITION 1.4. *For any differential graded k-module A, the functor C^*_A is a generalized cochain theory. A quasi-isomorphism $A \to B$ induces a natural quasi-isomorphism $C^*_A \to C^*_B$.*

Clearly, the three axioms of a generalized cochain theory are preserved by natural quasi-isomorphisms. Thus, for any functor T: **Top** $\to$ **M**, no zigzag of natural quasi-isomorphisms between T and $C^*_{T(*)}$ can exist unless T is a generalized cochain theory. Furthermore, it is easy to recognize when a natural transformation between generalized cochain theories is a natural quasi-isomorphism: exactly when the map on the one point space is a quasi-isomorphism. This is because when we define H^*_T as in the introduction, the axioms of a generalized cochain theory for T translate into the axioms of a generalized cohomology theory for H^*_T. A natural transformation $T \to U$ between generalized cochain theories induces a natural transformation of cohomology theories $H^*_T \to H^*_U$. Since we are using the strong form of the Homotopy Axiom A.1 and the Product Axiom A.3, the natural transformation of cohomology theories is an isomorphism on every object X if and only if it is an isomorphism on coefficients. We summarize these two observations in the following proposition.

PROPOSITION 1.5. *Let T, U: **Top** $\to$ **M** be contravariant functors, and let η: $T \to U$ be a natural transformation.*

(i) *Suppose η is a natural quasi-isomorphism. Then T is a generalized cochain theory if and only if U is.*

(ii) *Suppose T and U are generalized cochain theories. Then η is a natural quasi-isomorphism if and only if η_*: $T(*) \to U(*)$ is a quasi-isomorphism.*

As an immediate consequence, we get the last part of the following generalization of the Main Theorem. The remainder is proved in the next section.

THEOREM 1.6. *Let* **C** *be a complete category with a continuous faithful functor to* **M** *and a cosimplicial normalization functor, and let T: **Top** $\to$ **C** be a functor. There exists a functor E_T: **Top** $\to$ **C** and natural transformations*

$$T \to E_T \leftarrow C^*_{T(*)}$$

such that if T is a generalized cochain theory, then these natural transformations are natural quasi-isomorphisms. If T is not a generalized cochain theory, then T is not naturally quasi-isomorphic (by a zigzag of any length) to C^*_A for any A.

The deduction of the Main Theorem from the previous theorem is easy, once we know that the category **E** has a cosimplicial normalization functor; we prove this in Section 5. We need two additional facts that turn out to be obvious from the definition of the category **E** in Section 4. The first fact we need is that the category of commutative differential graded k-algebras has a natural embedding in **E** (induced by the augmentation of operads $\mathcal{E} \to \mathcal{C}om$), and so in particular there is a canonical object k in **E** whose underlying differential graded k-module is k. The lift of the cochain functor C^*: **Top** $\to$ **E** is defined to be C^*_k for this object k. The other fact we need is that **E** has an initial object ι (= $\mathcal{E}(0)$) that satisfies $H^*\iota = k$. For any object A in **E**, the cohomology H^*A is naturally a graded

commutative k-algebra, and the unique map $\iota \to A$ induces on cohomology the unit map $k \to H^*A$. It follows that when $H^*A = k$, the unique map $\iota \to A$ must be a quasi-isomorphism.

Now for any functor T: **Top** $\to$ **E**, we have the natural transformations

$$T \to E_T \leftarrow C^*_{T(*)} \leftarrow C^*_\iota \to C^*_k = C^*.$$

When T satisfies the Dimension Axiom A.4, the initial map $\iota \to T(*)$ is a quasi-isomorphism and so the zigzag $C^*_{T(*)} \leftarrow C^*_\iota \to C^*$ is a zigzag of natural quasi-isomorphisms. When T is a generalized cochain theory, the theorem above implies the zigzag $T \to E_T \leftarrow C^*_{T(*)}$ is a zigzag of quasi-isomorphisms. When T is not a generalized cochain theory, the theorem above also implies that no zigzag of natural quasi-isomorphisms between T and C^* can exist. This proves the Main Theorem.

For **C** = **A**, the category of differential graded k-algebras (either **Z**-graded or nonnegatively cohomologically graded), the proof of the analogue of the Main Theorem is even easier. The differential graded k-algebra k is the initial object in **A**, and we look at the zigzag

$$T \to E_T \leftarrow C^*_{T(*)} \leftarrow C^*.$$

For any T, the initial map $k \to T(*)$ induces the unit map $k \to H^*T(*)$ and so is a quasi-isomorphism if and only if T satisfies the Dimension Axiom 4. Thus, when T is a cochain theory, the zigzag above consists of natural quasi-isomorphisms by the theorem above. When T is not a cochain theory no zigzag of quasi-isomorphisms between T and C^* can exist.

Finally, we also consider the case **C** = **M**. Let $G = H^0T$, and let A be the differential graded k-module that in positive cohomological degrees is zero, in degree zero is the degree zero cocycles of $T(*)$, and in negative degrees is identical with $T(*)$. Then we have the inclusion map $A \to T(*)$ which is a quasi-isomorphism when H^*T is concentrated in nonpositive degrees. We also have the canonical map $A \to G$ that sends a degree zero cocycle of $T(*)$ to the cohomology class it represents; this is a quasi-isomorphism when H^*T is concentrated in nonnegative degrees. We consider the zigzag

$$T \to E_T \leftarrow C^*_{T(*)} \leftarrow C^*_A \to C^*_G = C^*(-; G).$$

Note that the functor $C^*(-; G)$ is the cochain functor with coefficients in the abelian group G in the classical sense. Suppose T is a generalized cochain theory and suppose furthermore that H^*T is concentrated in degree zero. Then the zigzag above consists of natural quasi-isomorphisms. On the other hand, when T is not a cochain theory or H^*T is not concentrated in degree zero, there can be no zigzag of natural quasi-isomorphisms between T and $C^*(-; G)$.

2. The proof of Theorem 1.6. In this section, we prove Theorem 1.6. The majority of the argument consists of constructing the functor E_T and the natural transformations $T \to E_T$ and $C^*_{T(*)} \to E_T$. The functor E_T is a homotopy limit version of a more basic construction, which we denote as $F_T(X)$.

The object $F_T(X)$ will essentially be the differential graded k-module of simplicial maps from the simplicial set $X_\bullet$ to the simplicial differential graded k-module $T(\Delta[\bullet])$; we make this an object of $\mathbf{C}$ as follows. Consider the differential graded k-modules $F_T^{m,n}(X)$ defined by

$$F_T^{m,n}(X) = \prod_{X_m} T(\Delta[n]),$$

the cartesian product of copies of $T(\Delta[n])$ indexed on the set X_m; here as in Definition 1.2, X_m denotes the set of maps from $\Delta[m]$ to X in **Top**. The simplicial structure maps of $X_\bullet$ make $F_T^{m,n}(X)$ cosimplicial in m and the cosimplicial structure maps of the cosimplicial space $\Delta[\bullet]$ make $F_T^{m,n}(X)$ simplicial in n. In other words, $F_T^{\bullet\bullet}(X)$ is a cosimplicial simplicial differential graded k-module, or equivalently, a covariant-contravariant bifunctor from the category $\mathbf{\Delta}$ of the finite sets $0, 1, \ldots$ and monotonic maps to $\mathbf{C}$.

In categorical terms, the differential graded k-module of simplicial maps from $X_\bullet$ to $T(\Delta[\bullet])$ is the *end* [9, IX.5] of $F_T^{\bullet\bullet}(X)$: This is by definition the differential graded k-module that makes the following diagram an equalizer

$$F_T(X) \longrightarrow \prod_n F_T^{n,n}(X) \rightrightarrows \prod_{\mathbf{n}\to\mathbf{m}} F_T^{m,n}(X). \tag{2.1}$$

In other words, this is the differential graded k-module consisting of the subset of $\prod F_T^{n,n}$ on which both arrows agree. The latter product above is taken over the maps in the category $\mathbf{\Delta}$; the top arrow in the right-hand pair is induced by the cosimplicial structure, the bottom by the simplicial structure. Since we have assumed that the forgetful functor from $\mathbf{C}$ to differential graded k-modules preserves products and equalizers, we get the end as an object of $\mathbf{C}$.

A map $\Delta[n] \to X$ induces a map $T(X) \to T(\Delta[n])$. The collection of all these maps defines a map $T(X) \to \prod F_T^{n,n}(X)$. Since this map is equalized by the pair of arrows in (2.1) above, we obtain a map $T(X) \to F_T(X)$ in $\mathbf{C}$.

If we think of $T(\Delta[n])$ as the T-valued "forms" on a simplex, we can think of $F_T(X)$ as being the T-valued (simplicial or singular) forms on X. When T satisfies the "extension property" (cf. [1, 1.2]) that a form on the boundary of a simplex extends over the whole simplex (in modern terms, when $T(\Delta[\bullet])$ is "Reedy fibrant" [7]), F_T preserves weak equivalences in X and quasi-isomorphisms of such T. For general T, $F_T(X)$ is harder to analyze and may behave less well. To fix this, in place of the end above we use the *homotopy end*. Like all homotopy limits, the homotopy end is constructed by noticing that the reflexive equalizer defining the end above is the first stage in a cobar construction. Namely, define

$E^\bullet_T(X)$ to be the cosimplicial differential graded k-module with

$$E^n_T(X) = \coprod_{\mathbf{m_n}\to\cdots\to\mathbf{m_0}} F_T^{m_0,m_n}(X),$$

the product over collections of n composable arrows in $\mathbf{\Delta}$. The zeroth coface map is induced by the cosimplicial structure on $F^{\bullet\bullet}_T(X)$; the last coface map is induced by the simplicial structure on $F^{\bullet\bullet}_T(X)$; the middle coface maps are induced by composition. The codegeneracy maps are induced by inserting identity maps. We have that $F_T(X)$ is "π^0" of $E^\bullet_T(X)$, i.e. $F_T(X)$ is the equalizer of the two face maps in cosimplicial degree zero. We therefore obtain a map of cosimplicial differential graded k-modules from the constant cosimplicial differential graded k-module on $F_T(X)$ to $E^\bullet_T(X)$.

Definition 2.2. Let $E_T(X)$ be the normalization $N(E^\bullet_T(X))$ of $E^\bullet_T(X)$.

We obtain a map $F_T(X) \to E_T(X)$ in $\mathbf{C}$. We define $\tau_X\colon\ T(X) \to E_T(X)$ to be the composite $T(X) \to F_T(X) \to E_T(X)$. The following proposition is clear from the construction. (We will need the statements regarding naturality in T in the next section.)

PROPOSITION 2.3. *$E_T(X)$ is a functor contravariant in the space X and covariant in the functor T. The map $\tau_X\colon\ T(X) \to E_T(X)$ is natural in X and T.*

We construct the natural transformation $C^*_{T(*)} \to E_T$ for Theorem 1.6 as follows. Let S denote the constant functor from **Top** to $\mathbf{C}$ that takes every space to the object $T(*)$ and every map to the identity. Naturality then gives us a natural transformation $E_S \to E_T$. Since S is a constant functor, we have that $F_S^{m,n}(X) = \prod_{X_m} T(*)$, and E_S admits a simpler description. Namely, we have a natural isomorphism

$$E_S(X) \cong C^*_{T(*)}(\operatorname{Hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet),$$

where $\operatorname{Hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet$ is the simplicial set

$$(\operatorname{Hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet)_n = \coprod_{\mathbf{m_n}\to\cdots\to\mathbf{m_0}} X_{m_0}.$$

A natural weak equivalence $\operatorname{Hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet \to X_\bullet$ is described in [2, XII.3.4], and we obtain a natural transformation $\eta\colon\ C^*_{T(*)} \to E_T$ by defining η_X to be the composite

$$C^*_{T(*)}(X) = C^*_{T(*)}(X_\bullet) \to C^*_{T(*)}(\operatorname{Hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet) \cong E_S(X) \to E_T(X).$$

Since the map $\operatorname{Hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet \to X_\bullet$ is a weak equivalence, the map $C^*_{T(*)}X_\bullet \to C^*_{T(*)}\operatorname{Hocolim} X_\bullet$ is a quasi-isomorphism. When T satisfies the Homotopy Ax-

iom A.1, the map $E_S(X) \to E_T(X)$ is the normalization of a degreewise quasi-isomorphism and is therefore a quasi-isomorphism. (The normalization of a cosimplicial differential graded module is complete with respect to the filtration by cosimplicial degree; a degreewise quasi-isomorphism induces an isomorphism on the spectral sequence associated to this filtration.) We have proven the following proposition.

PROPOSITION 2.4. *The map η_X: $C_{T(*)}(X) \to E_T(X)$ is natural in X and T. When T satisfies the Homotopy Axiom A.1, η is a natural quasi-isomorphism.*

Now we return to the natural transformation τ. To complete the proof of Theorem 1.6, we need to see that the map τ_X: $T(X) \to E_T(X)$ is a quasi-isomorphism when X is the one point space $*$. For this space, we have that X_n is a single point for all n, and $C^*_{T(*)} = T(*)$. The map τ: $T(*) \to E_T(*)$ coincides with the map η: $T(*) \to E_T(*)$, since both are the composite of the diagonal map $T(*) \to \prod_{\mathbf{m}} T(*)$ and the map $\prod_{\mathbf{m}} T(*) \to \prod_{\mathbf{m}} T(\Delta[m])$ induced by the unique maps $\Delta[m] \to \Delta[0]$. The following proposition therefore follows from the previous proposition.

PROPOSITION 2.5. *If T satisfies the Homotopy Axiom A.1, then for the one point space, the map τ: $T(*) \to E_T(*)$ is a quasi-isomorphism.*

Theorem 1.6 is now an immediate consequence of Proposition 1.5 and Propositions 2.4 and 2.5.

3. The Uniqueness Theorem. In this section we prove the Uniqueness Theorem of the introduction. Again, $\mathbf{C}$ denotes a complete category with a continuous faithful functor to differential graded k-modules and a cosimplicial normalization functor. We denote by $\mathbf{C}[\mathcal{Q}^{-1}]$ the category obtained from $\mathbf{C}$ by formally inverting the quasi-isomorphisms. We denote by $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ the category obtained from the category of contravariant functors $\mathbf{Top} \to \mathbf{C}$ by formally inverting the natural quasi-isomorphisms. For objects A in $\mathbf{C}$, we can consider C^*_A as a functor of A. Since by Proposition 1.4, C^*_A converts quasi-isomorphisms in A to natural quasi-isomorphisms, any map $A \to B$ in $\mathbf{C}[\mathcal{Q}^{-1}]$ induces a map $C^*_A \to C^*_B$ in $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$. We can therefore regard $C^*_{(-)}$ as a functor $\mathbf{C}[\mathcal{Q}^{-1}] \to \mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$. Likewise, we have a functor P: $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}] \to \mathbf{C}[\mathcal{Q}^{-1}]$ that takes an object of T of $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ to the object $T(*)$ of $\mathbf{C}[\mathcal{Q}^{-1}]$. We prove the following generalization of the Uniqueness Theorem.

THEOREM 3.1. *Let $\mathbf{C}$ be a complete category with a continuous faithful functor to differential graded k-modules and a cosimplicial normalization functor, and let A be an object of $\mathbf{C}$. Then the functors*

$$C^*_{(-)}\colon \mathbf{C}[\mathcal{Q}^{-1}] \to \mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}], \qquad \textit{and} \qquad P\colon \mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}] \to \mathbf{C}[\mathcal{Q}^{-1}]$$

induce inverse equivalences of $\mathbf{C}[\mathcal{Q}^{-1}]$ *with the full subcategory of* $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ *consisting of the generalized cochain theories.*

When $\mathbf{C} = \mathbf{E}$, the initial map $\iota \to k$ is a quasi-isomorphism, and so the only endomorphism of k in $\mathbf{E}[\mathcal{Q}^{-1}]$ is the identity. This gives the Uniqueness Theorem of the introduction. Likewise, when $\mathbf{C} = \mathbf{A}$, k is the initial object, and the only endomorphism of k in $\mathbf{A}[\mathcal{Q}^{-1}]$ is the identity. When $\mathbf{C} = \mathbf{M}$, the monoids of endomorphisms and automorphisms of k in $\mathbf{M}[\mathcal{Q}^{-1}]$ are isomorphic to the monoids of endomorphisms and automorphisms of k in the category of k-modules, and are given by the multiplicative monoid of k and the units of k respectively.

We now prove Theorem 3.1. Clearly the composite $P \circ C^*_{(-)}$ is the identity functor on $\mathbf{C}[\mathcal{Q}^{-1}]$. It therefore suffices to show that the composite $C^*_{(-)} \circ P$ is naturally isomorphic to the identity in the full subcategory of $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ of generalized cochain theories. When T is a generalized cochain theory, the zigzag of Theorem 1.6

$$T \xrightarrow{\tau} E_T \xleftarrow{\eta} C^*_{PT}$$

provides a well-defined isomorphism $\eta^{-1} \circ \tau$ in $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ from T to C^*_{PT}; we show that this isomorphism is natural.

Let $\phi\colon S \to T$ be a map in $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ from a generalized cochain theory S to a generalized cochain theory T. By the definition of the category $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$, we can decompose ϕ as an iterated composition of natural transformations with the formal inverses of natural quasi-isomorphisms. In other words, there are functors $U_i, V_i\colon \mathbf{Top} \to \mathbf{C}$ and natural transformations

$$S \xrightarrow{\phi_0} U_1 \xleftarrow{\theta_1} V_1 \to U_2 \cdots \to V_n \xleftarrow{\theta_n} \iota_{\mathbf{h}} \xrightarrow{\phi_n} T$$

where the leftward arrows θ_i are natural quasi-isomorphisms and

$$\phi = \phi_n \circ \theta_n^{-1} \circ \cdots \circ \theta_1^{-1} \circ \phi_0$$

in $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$. Applying the functors and natural transformations of the previous section, we obtain a commutative diagram

$$\begin{array}{ccccccccccccc}
S & \longrightarrow & U_1 & \longleftarrow & V_1 & \longrightarrow & \cdots & \longrightarrow & U_n & \longleftarrow & V_n & \longrightarrow & C^* \\
{\scriptstyle\tau}\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\tau} \\
E_S & \longrightarrow & E_{U_1} & \longleftarrow & E_{V_1} & \longrightarrow & \cdots & \longrightarrow & E_{U_n} & \longleftarrow & E_{V_n} & \longrightarrow & E_T \\
{\scriptstyle\eta}\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\eta} \\
C^*_{PS} & \longrightarrow & C^*_{PU_1} & \longleftarrow & C^*_{PV_1} & \longrightarrow & \cdots & \longrightarrow & C^*_{PU_n} & \longleftarrow & C^*_{PV_n} & \longrightarrow & C^*_{PT}.
\end{array}$$

The natural transformations in the bottom row are $C^*_{P\phi_i}$ and $C^*_{P\theta_i}$. Note that since we are not assuming that the T_i's and U_i's are generalized cochain theories, we do not know that the inner vertical natural transformations are natural quasi-isomorphisms. We should read this diagram as saying that in $\mathbf{C}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$,

$$\tau \circ \phi = E_{\phi_n} \circ E_{\theta_n}^{-1} \circ \cdots \circ E_{\theta_1}^{-1} \circ E_{\phi_0} \circ \tau,$$
$$\eta \circ C^*_{P\phi} = E_{\phi_n} \circ E_{\theta_n}^{-1} \circ \cdots \circ E_{\theta_1}^{-1} \circ E_{\phi_0} \circ \eta.$$

It follows that

$$\eta^{-1} \circ \tau \circ \phi = C^*_{(P\phi)} \circ \eta^{-1} \circ \tau.$$

Thus, $\eta^{-1} \circ \tau$ is natural. This proves Theorem 3.1.

4. The category of E_∞ algebras. In this section, we define the category **E** of E_∞ k-algebras. Since this definition involves a choice, the bulk of this section justifies the generality of this category and argues that any functor to any category of E_∞ k-algebras can be regarded as landing in **E** in an essentially unique way. We construct the cosimplicial normalization functor for **E** in the next section.

We continue to work exclusively in the category of k-modules for a fixed commutative ring k, and we continue to use **M** to denote the category of differential graded k-modules. However now we must specify that differential graded k-modules be **Z**-graded since E_∞ operads are concentrated in nonpositive cohomological degrees (nonnegative homological degrees). Working with operads, the standard convention is to grade homologically so that the differential lowers degrees. For the remainder of this paper, we follow this convention, and use lower indexes to denote homologically graded degrees. No sign is associated to this regrading: We should understand $M_n = M^{-n}$, $d_n = d^{-n}$, and $H_nM = H^{-n}M$ for a differential graded module M.

We assume familiarity with the basic definition of an operad in **M**, and we refer the reader to [8, §I] (and its references) for a good introduction. We generally follow the conventions and terminology of [8, §I]; the only exception is the following definition (cf. [8, I.1.3]).

Definition 4.1. Let $\mathcal{C}om$ denote the operad of commutative algebras, the operad with $\mathcal{C}om(n) = k$ for each $n \geq 0$ with Σ_n acting by the identity, and multiplication induced by the ring multiplication.

(i) An *augmented* operad is a map of operads $\mathcal{O} \to \mathcal{C}om$.

(ii) An *acyclic* operad is an augmented operad $\mathcal{O} \to \mathcal{C}om$ where $\mathcal{O}(n) \to \mathcal{C}om(n)$ is a quasi-isomorphism for each n.

(iii) A Σ-*free* operad is an operad $\mathcal{O}$ such that for each n, the underlying graded $k[\Sigma_n]$-module of $O(n)$ is free in each degree.

(iv) An E_∞ *operad* is a Σ-free acyclic operad $\mathcal{O} \to \mathcal{C}om$ such that each

module $\mathcal{O}(n)$ is concentrated in nonnegative (homologically graded) degrees. A map of augmented, acyclic, or E_∞ operads is a map of operads over $\mathcal{C}om$.

The previous definition differs from [8, I.1.3], which requires the zeroth space of an E_∞ operad to be k. The extra generality afforded by the previous definition allows us to consider appropriate "cofibrant" operads to be E_∞ operads. All we need of the theory of cofibrant operads is the following definition and proposition.

Definition 4.2. An operad $\mathcal{C}$ is said to be *cofibrant* if it has the following lifting property: Whenever $\mathcal{P} \to \mathcal{Q}$ is a map of operads that is a surjection and quasi-isomorphism for each $\mathcal{P}(n) \to \mathcal{Q}(n)$, then any map of operads $\mathcal{C} \to \mathcal{Q}$ lifts to a map of operads $\mathcal{C} \to \mathcal{P}$.

The following proposition is an easy consequence of Quillen's small object argument and the formulation of operads as algebras over a monad in [5, 1.10–1.12].

PROPOSITION 4.3. *Given any operad $\mathcal{O}$, there exists a cofibrant operad $\mathcal{C}$ and a map of operads $f\colon \mathcal{C} \to \mathcal{O}$ that is both a quasi-isomorphism and a surjection $\mathcal{C}(n) \to \mathcal{O}(n)$ for each n. The operad $\mathcal{C}$ can be chosen to be Σ-free. In addition, if each $\mathcal{O}(n)$ is concentrated in nonnegative degrees, then $\mathcal{C}$ can be chosen so that each $\mathcal{C}(n)$ is concentrated in nonnegative degrees.*

In particular, there exists a cofibrant E_∞ operad. We choose and fix such an operad $\mathcal{E}$, and we make the following definition.

Definition 4.4. Let $\mathbf{E}$ be the category of algebras over the cofibrant E_∞ operad $\mathcal{E}$.

Since the category of $\mathcal{E}$-algebras is the category of algebras over a monad [8, I.3.4] of differential graded k-modules, it follows that $\mathbf{E}$ is complete and that the forgetful functor is continuous. As with any category of algebras over an operad, $\mathbf{E}$ has as an initial object, the $\mathcal{E}$-algebra $\mathcal{E}(0)$, and by construction, the map $\mathcal{E}(0) \to \mathcal{C}om(0) = k$ is a quasi-isomorphism.

To justify our choice of the category $\mathbf{E}$ as "the" category of E_∞ k-algebras, we offer the following three lemmas and Theorem 4.8 below.

LEMMA 4.5. *Let $\mathcal{O}$ be an acyclic operad. Then there is a map of acyclic operads $\mathcal{E} \to \mathcal{O}$*

Proof. Since the maps $\mathcal{O}(n) \to \mathcal{C}om(n) = k$ are quasi-isomorphisms they are necessarily surjections, and the lemma follows from the definition of cofibrant. □

Having chosen a map of acyclic operads $\mathcal{E} \to \mathcal{O}$, we can "pull back" any $\mathcal{O}$ structure to an $\mathcal{E}$ structure. In particular, we can regard any functor to the category $\mathcal{O}$-algebras as a functor to the category of $\mathcal{E}$-algebras. Of course the pulled back

structure depends on the map of operads chosen. The following lemma implies that any two choices of maps give naturally quasi-isomorphic choices of functors. Thus, for the purposes of the Main Theorem, the precise map chosen is irrelevant.

LEMMA 4.6. *Let $\mathcal{O}$ be an acyclic operad and let f and g be two maps of acyclic operads $\mathcal{E} \to \mathcal{O}$. Then the pullback functors f^* and g^* from $\mathcal{O}$-algebras to $\mathcal{E}$-algebras are naturally quasi-isomorphic.*

The proof of Lemma 4.6 appears below. It is closely related to the following lemma proved along with [8, II.1.1] in [8, §II.4–5].

LEMMA 4.7. *Let $f\colon \mathcal{O} \to \mathcal{P}$ be a map of E_∞ operads. Then there is a functor B_f from the category of $\mathcal{O}$-algebras to the category of $\mathcal{P}$-algebras that preserves all quasi-isomorphisms and such that the composite functors $f^* \circ B_f$ and $B_f \circ f^*$ are naturally quasi-isomorphic to the identity functors.*

As a consequence of these three lemmas, we obtain the following theorem.

THEOREM 4.8. *Let $\mathcal{E}'$ be an E_∞ operad and let $\mathbf{E}'$ be the category of $\mathcal{E}'$-algebras. The localized functor categories $\mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ and $\mathbf{E}'^{\mathbf{Top}}[\mathcal{Q}^{-1}]$ are equivalent. The equivalence is canonical up to natural isomorphism.*

The previous theorem easily generalizes to categories where the E_∞ operad is allowed to vary with the object; we leave the details to the interested reader.

In the case when k contains the rational numbers, the commutative operad is not quite an E_∞ operad by the definition above, but only by a technicality: It is "Σ-projective" instead of Σ-free. This distinction is irrelevant in the proof of Lemma 4.7 in [8]. We therefore obtain the following theorem mentioned in the introduction.

THEOREM 4.9. *Let $k \supseteq \mathbf{Q}$ and let* **Com** *denote the category of commutative differential graded k-algebras. The augmentation $\mathcal{E} \to \mathcal{C}om$ induces an equivalence of localized functor categories $\mathbf{Com}^{\mathbf{Top}}[\mathcal{Q}^{-1}] \to \mathbf{E}^{\mathbf{Top}}[\mathcal{Q}^{-1}]$.*

Finally, we close this section with the proof of Lemma 4.6.

Proof of Lemma 4.6. Let $\tilde{\mathcal{O}}$ be the connective cover of $\mathcal{O}$, the operad with $\tilde{\mathcal{O}}(n)$ zero in negative degrees, equal to the degree zero cycles of $\mathcal{O}$ in degree zero, and identical with $\mathcal{O}$ in positive degrees. The operadic multiplication on $\tilde{\mathcal{O}}$ is the restriction of the operadic multiplication of $\mathcal{O}$. The inclusion $\tilde{\mathcal{O}} \to \mathcal{O}$ is therefore a map of operads, and since $\mathcal{E}$ is concentrated in nonnegative degrees, the map $\mathcal{E} \to \mathcal{O}$ factors through $\tilde{\mathcal{O}}$. Thus, it suffices to consider the case when $\mathcal{O}$ is concentrated in nonnegative degrees.

By applying Proposition 4.3 to $\mathcal{O}$ and using the lifting property of $\mathcal{E}$, we can assume without loss of generality that $\mathcal{O}$ is a cofibrant E_∞ operad. The remainder of the argument is to construct a Quillen right homotopy between the maps f and

g and to apply Lemma 4.7 to produce the natural quasi-isomorphisms. Since we do not assume familiarity with this theory, we write out this argument in detail.

Let $\mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ be the pullback of

$$\mathcal{O} \to \mathcal{C}om \leftarrow \mathcal{O}$$

in the category of operads. In other words the nth space of $\mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ is the pullback in the category of differential graded $k[\Sigma_n]$-modules

$$(\mathcal{O} \times_{\mathcal{C}om} \mathcal{O})(n) = \mathcal{O}(n) \times_{\mathcal{C}om(n)} \mathcal{O}(n)$$

and the operadic multiplication is the unique map that the projection maps send to the operadic multiplication on each factor. The operad $\mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ comes with an augmentation to $\mathcal{C}om$, and it is clear from the description of nth space above that $\mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ is an acyclic operad.

We have a diagonal map of acyclic operads $\mathcal{O} \to \mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ that projects to the identity map of $\mathcal{O}$ on each factor, and we have a map of acyclic operads $\mathcal{E} \to \mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ that projects to f on one factor and g on the other. Applying Proposition 4.3 to $\mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$, we can find a cofibrant E_∞ operad $\mathcal{C}$ and a map of acyclic operads $\mathcal{C} \to \mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ that is a surjective quasi-isomorphism on the nth space for all n. The lifting property of $\mathcal{E}$ gives a map $\mathcal{E} \to \mathcal{C}$ lifting the map $\mathcal{E} \to \mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$. Since we have reduced to the case when $\mathcal{O}$ is cofibrant, we can also lift the diagonal map $\mathcal{O} \to \mathcal{O} \times_{\mathcal{C}om} \mathcal{O}$ to a map $s\colon\ \mathcal{O} \to \mathcal{C}$.

Write a and b for the maps $\mathcal{C} \to \mathcal{O}$ obtained by projection. Since pulling back operadic algebra structures is transitive and preserves quasi-isomorphisms, it suffices to produce natural quasi-isomorphisms between a^* and b^*. Since $a \circ s = b \circ s = \mathrm{id}_{\mathcal{O}}$, we have that $s^* \circ a^* = s^* \circ b^* = \mathrm{Id}$ on the category of $\mathcal{O}$-algebras. Lemma 4.7 gives a functor B_s from the category of $\mathcal{O}$-algebras to the category of $\mathcal{C}$-algebras and zigzag of natural quasi-isomorphisms between $B_s \circ s^*$ and the identity functor in the category of $\mathcal{C}$-algebras. Composing with the functors a^* and b^* we get zigzags of natural quasi-isomorphisms

$$a^* \longleftrightarrow B_s \circ s^* \circ a^* = B_s = B_s \circ s^* \circ b^* \longleftrightarrow b^*.$$

This completes the proof.

5. Cosimplicial $\mathcal{E}$-algebras. This section is devoted to the construction of the cosimplicial normalization functor in the category **E**. For this we use the theory of homotopy limits of operadic algebras developed in [6], which we review. The main idea goes back to the work of Dold [3], and it is to generalize the Alexander-Whitney map and parametrize it and all similar maps by an operad. We begin by recalling the definition of the Dold operad $\mathcal{Z}$.

Definition 5.1. Let $\Delta_*[m]$ denote the standard algebraic m-simplex differential graded k-module (the normalized chain complex of the standard m-simplex simplicial set), and let $\Delta_*^{(n)}[m]$ be its nth tensor power, $\Delta_*[m]\otimes\cdots\otimes\Delta_*[m]$. Let $\mathcal{Z}(n)$ be the normalization of the cosimplicial differential graded k-module $\Delta_*^{(n)}[\bullet]$.

To explain the operadic multiplication, we need the following observation.

PROPOSITION 5.2. *For a cosimplicial differential graded k-module $A^\bullet$, the normalization $N(A^\bullet)$ is the end of mapping complex* $\mathrm{Hom}\,(\Delta_*[\bullet], A^\bullet)$, *regarded as a contravariant-covariant functor from $\mathbf{\Delta}$ to differential graded k-modules.*

In other words, $N(A^\bullet)$ can be identified with the differential graded k-module formed by the subset of $\prod \mathrm{Hom}\,(\Delta_*[n], A^n)$ of elements on which the maps

$$\prod \mathrm{Hom}\,(\Delta_*[n], A^n) \rightrightarrows \prod_{\mathbf{m}\to\mathbf{n}} \mathrm{Hom}\,(\Delta_*[m], A^n)$$

induced by the cosimplicial operations on $A^\bullet$ and by the cosimplicial operations on $\Delta_*[\bullet]$ coincide.

For differential graded k-modules M, N, the mapping complex $\mathrm{Hom}\,(M, N)$ is the differential graded k-module formed by the graded maps of graded k-modules from M to N. We can therefore think of the end above as the differential graded k-module of graded cosimplicial maps from $\Delta_*[\bullet]$ to $A^\bullet$. In particular, an element of $N(A^\bullet)$ in degree n is a cosimplicial degree n map of graded k-modules $\Delta_*[\bullet]\to A^\bullet$. We use this latter interpretation in the following definition.

Definition 5.3. For cosimplicial differential graded k-modules $A_1^\bullet,\ldots, A_n^\bullet$, we denote by α_n the natural map $\mathcal{Z}(n)\otimes(N(A_1^\bullet)\otimes\cdots\otimes N(A_n^\bullet))\to N(\,\mathrm{diag}\, A_1^\bullet\otimes\cdots\otimes A_n^\bullet)$ defined by

$$\alpha_n(f\otimes(a_1\otimes\cdots\otimes a_n)) = (-1)^{ab}(a_1\otimes\cdots\otimes a_n)\circ f,$$

where $b = \deg f$ and $a = \deg a_1 + \cdots + \deg a_n$, and where "$\circ$" is induced by function complex composition, i.e., composition of cosimplicial graded maps of graded k-modules. We call the maps α *generalized Alexander-Whitney* maps. When $A_i^\bullet = \Delta_*^{(j_i)}[\bullet]$, we write

$$\gamma_{n;j_1,\ldots,j_n}\colon\ \mathcal{Z}(n)\otimes(\mathcal{Z}(j_1)\otimes\cdots\otimes\mathcal{Z}(j_n))\to\mathcal{Z}(j_1+\cdots+j_n)$$

for the map α_n.

We have a canonical isomorphism $\mathcal{Z}(0) = k$ and so we can interpret the multiplication $\gamma_{n;0,\ldots,0}$ as a map $\mathcal{Z}(n)\to k$. This map coincides with the map on normalization induced by the canonical collapse map $\Delta_*^{(n)}[\bullet]\to k$, and is therefore a quasi-isomorphism since it is the normalization of a degreewise quasi-isomorphism. The remainder of the following proposition from [6] is an easy check of the definition.

PROPOSITION 5.4. *The maps γ make $\mathcal{Z}$ an acyclic operad.*

The following theorem, essentially from [6], is the main result on $\mathcal{Z}$ we need. For it, recall that the tensor product of operads is an operad with the tensor product of the multiplications.

THEOREM 5.5. *Let $\mathcal{O}$ be an operad and let $A^\bullet$ be a cosimplicial $\mathcal{O}$-algebra. Then $N(A^\bullet)$ is an $(\mathcal{O}\otimes\mathcal{Z})$-algebra, naturally in $\mathcal{O}$ and $A^\bullet$.*

Proof. Let $\zeta_n\colon\ \mathcal{O}(n)\otimes N(\operatorname{diag}(A^\bullet\otimes\cdots\otimes A^\bullet))\to N(A^\bullet)$ be the composite of the natural map $\mathcal{O}(n)\otimes N(\operatorname{diag}(A^\bullet)^{(n)})\to N(\mathcal{O}(n)\otimes\operatorname{diag}(A^\bullet)^{(n)})$ and the normalization of the cosimplicial action map $\mathcal{O}(n)\otimes(A^\bullet)^{(n)}\to A^\bullet$. Let

$$\xi_n\colon\ (\mathcal{O}(n)\otimes\mathcal{Z}(n))\otimes N(A^\bullet)\otimes\cdots\otimes N(A^\bullet)\to N(A^\bullet)$$

be the composite $\zeta_n\circ(\mathrm{id}_{\mathcal{O}(n)}\otimes\alpha_n)$. It is straightforward to check that the maps ξ construct an $(\mathcal{O}\otimes\mathcal{Z})$-algebra structure on $N(A^\bullet)$ that is natural in maps of the operad $\mathcal{O}$ and of the cosimplicial $\mathcal{O}$-algebra $A^\bullet$. □

We can apply the previous theorem in particular to the constant cosimplicial object on an $\mathcal{O}$-algebra A. This then gives A an $(\mathcal{O}\otimes\mathcal{Z})$-algebra structure. On the other hand the augmentation of $\mathcal{Z}$ induces a map of operads $\mathcal{O}\otimes\mathcal{Z}\to\mathcal{O}$, also giving A an $(\mathcal{O}\otimes\mathcal{Z})$-algebra structure. The following consistency observation is an easy check of the definitions.

PROPOSITION 5.6. *The two $(\mathcal{O}\otimes\mathcal{Z})$-algebra structures on the normalization of a constant cosimplicial object coincide.*

We also need the following construction.

LEMMA 5.7. *There is a map of operads $\mathcal{E}\to\mathcal{E}\otimes\mathcal{E}$ such that both composites*

$$\mathcal{E}\to\mathcal{E}\otimes\mathcal{E}\to\mathcal{E}\otimes\mathcal{C}om=\mathcal{E},\qquad \mathcal{E}\to\mathcal{E}\otimes\mathcal{E}\to\mathcal{C}om\otimes\mathcal{E}=\mathcal{E}$$

are the identity.

Proof. Let $\mathcal{E}\times_{\mathcal{C}om}\mathcal{E}$ be the pullback in the category of operads of two copies of the augmentation $\mathcal{E}\to\mathcal{C}om$. In other words the nth space of $\mathcal{E}\times_{\mathcal{C}om}\mathcal{E}$ is the pullback in the category of differential graded $k[\Sigma_n]$-modules

$$(\mathcal{E}\times_{\mathcal{C}om}\mathcal{E})(n)=\mathcal{E}(n)\times_{\mathcal{C}om(n)}\mathcal{E}(n)$$

and the operadic multiplication is induced by the projection. We have the diagonal map $\mathcal{E}\to\mathcal{E}\times_{\mathcal{C}om}\mathcal{E}$, and a map

$$\mathcal{E}\otimes\mathcal{E}\to\mathcal{E}\times_{\mathcal{C}om}\mathcal{E}$$

induced by the two maps $\mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$. It is easy to see that this latter map is a surjective quasi-isomorphism $\mathcal{E}(n) \otimes \mathcal{E}(n) \to \mathcal{E}(n) \times_{\mathcal{C}om(n)} \mathcal{E}(n)$ for each n. Since $\mathcal{E}$ is cofibrant, we can find a map $\mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ that lifts the diagonal. □

Finally, we prove the main result of this section.

THEOREM 5.8. *The category* **E** *has a cosimplicial normalization functor.*

Proof. Choose and fix a map of acyclic operads $\mathcal{E} \to \mathcal{Z}$. Then we have a composite map of operads

$$\mathcal{E} \to \mathcal{E} \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{Z}.$$

This map together with Theorem 5.5 gives the normalization of a cosimplicial $\mathcal{E}$-algebra a natural $\mathcal{E}$-algebra structure. Since the map $\mathcal{E} \to \mathcal{Z}$ preserves the augmentation, the composite

$$\mathcal{E} \to \mathcal{E} \otimes \mathcal{E} \to \mathcal{E} \otimes \mathcal{Z} \to \mathcal{E} \otimes \mathcal{C}om = \mathcal{E}$$

is the identity map on $\mathcal{E}$. For a constant cosimplicial $\mathcal{E}$-algebra $A^\bullet$, the isomorphism of differential graded k-modules $A^0 \cong N(A^\bullet)$ is therefore a map in **E**. □

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637
E-mail: mandell@math.uchicago.edu

REFERENCES

[1] A. K. Bousfield and V. K. A. M. Gugenheim, On PL De Rham theory and rational homotopy type, *Mem. Amer. Math. Soc.* **179** (1976).
[2] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, *Lecture Notes in Math.*, vol. 304, Springer-Verlag, New York, 1972.
[3] A. Dold, Über die Steenrodschen Kohomologieoperationen, *Ann. of Math.* (2) **73** (1961), 258–294.
[4] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, *Ergebn. Math. Grenzgeb.*, vol. 35, Springer-Verlag, Berlin, 1967.
[5] E. Getzler and J. D. S. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, preprint, hep-th/9403055.
[6] V. A. Hinich and V. V. Schechtman, On homotopy limit of homotopy algebras, *K-Theory, Arithmetic, and Geometry*, *Lecture Notes in Math.*, vol. 1289, Springer-Verlag, New York, 1987, pp. 240–264.
[7] C. L. Reedy, Homotopy theory of model categories, preprint, http://www-math.mit.edu/~psh/reedy.dvi.
[8] I. Kriz and J. P. May, *Operads, algebras, modules, and motives*, *Asterisque* **233** (1995).
[9] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1971.
[10] M. A. Mandell, E_∞ *algebras and p-adic homotopy theory*, *Topology* (to appear); preprint, http://www.math.uchicago.edu/~mandell/papers/einffinal.dvi.

[11] ———, Equivariant p-adic homotopy theory, preprint, 2000, http://www.math.uchicago.edu/~mandell/papers/finite.dvi.

[12] J. P. May, A general algebraic approach to Steenrod operations, *Steenrod Algebra and its Applications: A Conference to Celebrate N. E. Steenrod's Sixtieth Birthday*, *Lecture Notes in Math.*, vol. 168, Springer-Verlag, New York, 1970, pp. 153–231.