FROM RING EPIMORPHISMS TO UNIVERSAL LOCALISATIONS

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ABSTRACT. For a fixed ring, different classes of ring epimorphisms and localisation maps are compared. In fact, we provide sufficient conditions for a ring epimorphism to be a universal localisation. Furthermore, we consider recollements induced by some homological ring epimorphisms and investigate whether they yield recollements of derived module categories.

Keywords: ring epimorphism; perfect localisation; universal localisation; recollement.

1. INTRODUCTION

It is well-known that Ore localisations yield ring epimorphisms with a flatness condition. Different generalisations of Ore localisation, notably localisation with respect to a Gabriel filter and universal localisation, usually lack this flatness property. Localisations with respect to Gabriel filters generalise Ore localisations from a torsion-theoretic point of view. From a homological perspective, however, these localisation maps are often difficult to deal with. In fact, they are not always ring epimorphisms. Still, this setting is large enough to include all flat ring epimorphisms and these localisations are called perfect (see [24] for details). Universal localisations, as developed by Cohn ([11]) and Schofield ([23]), provide a technique that largely differs from the one above. In particular, they yield ring epimorphisms satisfying some nice homological properties. Universal localisations have shown to be useful in algebraic K-theory ([19], [20]) and the study of tilting modules and derived module categories in representation theory ([1], [2], [6], [7], [8]).

In [1], both universal and perfect localisations were used to construct (large) tilting modules. Furthermore, [1] compares perfect and universal localisations for semihereditary rings and Prüfer domains. Also, in [17], it was proved that universal localisations are in bijection with homological ring epimorphisms for hereditary rings. These results motivate the study of universal localisations from a homological point of view, which we further in this paper, namely through our first theorem.

Theorem A (Theorem 3.3) *Let $f : A \to B$ be a ring epimorphism such that B is a finitely presented left A-module of projective dimension less or equal than one. Then f is homological if and only if it is a universal localisation.*

Recent work uses universal localisations to construct interesting examples of recollements of derived module categories ([2], [6], [7], [8]). In this setting, we prove the following theorem.

Theorem B (Theorem 4.1) *Let $f : A \to B$ be a homological ring epimorphism such that B is a finitely presented left A-module of projective dimension less or equal than one. If $Hom_A(coker(f), ker(f)) = 0$ holds then the derived restriction functor f_* induces a recollement of derived module categories*

$$\mathcal{D}(B) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(End_{\mathcal{D}(A)}(K_f)),$$

where K_f is the cone of f in $\mathcal{D}(A)$. Moreover, if B is a finitely presented projective left A-module then there is an isomorphism of rings $End_{\mathcal{D}(A)}(K_f) \cong A/\tau_B(A)$, where $\tau_B(A)$ is the trace of B in A.

The first named author is supported by DFG-SPP 1489 and the second named author by DFG-SPP 1388.

Note that theorem A cannot hold in full generality since universal localisations are not always homological ring epimorphisms (see example 2.14) just as homological ring epimorphisms are not necessarily universal localisations, notably through Keller's example in [15]. Using different methods, theorem A has also been proved independently by Chen and Xi in [8] (corollary 3.7).

Theorem B yields recollements in which both outer terms are derived module categories. These recollements are particularly relevant to recent results obtained in [3], [4] and [18], where a Jordan-Hölder-type theorem for derived module categories of some rings has been proved. Such a property cannot, however, hold for all rings and a counter example can be constructed using universal localisations ([6]).

This paper is organised as follows. In section 2 we recall some preliminaries and prove some easy facts. Remark 2.5 and lemma 2.9, in particular, give information on how to construct examples in our setting. Section 3 contains theorem A and consequences for the cases of finite, injective and surjective ring epimorphisms. Also, following subsection 2.4, we generalise the comparison between universal localisations, localisations with respect to Gabriel filters and flat ring epimorphisms initiated in [1]. In section 4 we prove theorem B, while examples illustrating this result are given in section 5. In particular, we use our methods to obtain a large class of algebras which are not derived simple.

2. Ring epimorphisms and localisations

Throughout, A will be a ring with unit and $\mathbb{K}$ a field. We will denote the category of left (respectively, right) A-modules by A-*Mod* (respectively, *Mod*-A), its subcategory of finitely generated modules by A-*mod* (respectively, *mod*-A) and its subcategory of finitely generated projective modules by A-$proj$. The derived category of left A-modules will be denoted by $\mathcal{D}(A)$.

2.1. **Ring epimorphisms.** We will be discussing some types of ring epimorphisms. Recall that a ring epimorphism is just an epimorphism in the category of rings with unit. Two ring epimorphisms $f : A \to B$ and $g : A \to C$ are said to be equivalent if there is a ring isomorphism $h : B \to C$ such that $g = hf$. We then say that B and C lie in the same epiclass of A.

Proposition 2.1 ([24], Proposition XI.1.2)**.** *For a ring homomorphism $f : A \to B$, the following statements are equivalent.*

(1) *f is a ring epimorphism;*
(2) *The restriction functor $f_* : B\text{-}Mod \to A\text{-}Mod$ (respectively, $f_\# : Mod\text{-}B \to Mod\text{-}A$) is fully faithful;*
(3) *$f \otimes_A B = B \otimes_A f : B \to B \otimes_A B$ is an isomorphism of B-B-bimodules;*
(4) *$B \otimes_A coker(f) = 0$.*

Moreover, the functor $B \otimes_A -$ (respectively, $- \otimes_A B$) is left adjoint to f_* (respectively, $f_\#$).

Consider the following sequence of left A-modules given by a ring epimorphism $f : A \to B$

$$0 \longrightarrow ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow coker(f) \longrightarrow 0,$$

which we unfold into two short exact sequences, namely

$$0 \longrightarrow ker(f) \longrightarrow A \xrightarrow{\bar{f}} f(A) \longrightarrow 0, \tag{2.1}$$

$$0 \longrightarrow f(A) \longrightarrow B \longrightarrow coker(f) \longrightarrow 0. \tag{2.2}$$

The following easy observations follow from proposition 2.1.

Corollary 2.2. *Let $f : A \to B$ be a ring epimorphism. The following assertions hold.*

(1) *$B \otimes_A f(A) \cong B \otimes_A B \cong B$;*
(2) *$B \otimes_A ker(f) \cong Tor_1^A(B, f(A))$;*
(3) *If $Tor_1^A(B,B) = 0$ then $Tor_1^A(B, coker(f)) = 0$.*

Proof. To prove (1), consider the commutative diagram given by the epi-mono factorisation of f

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & B \\ & f(A) \arrow[ur] & \end{tikzcd}$$

and apply to it the functor $B \otimes_A -$. By proposition 2.1, $B \otimes_A f : B \otimes_A A \to B \otimes_A B$ is an isomorphism and, therefore, the induced epimorphism $B \otimes_A \bar{f}$ is also a monomorphism.

The statements (2) and (3) follow from (1) by considering the long exact sequences given by applying the functor $B \otimes_A -$ to the sequences (2.1) and (2.2), respectively.

□

Epiclasses of a ring A can be classified by suitable subcategories of A-Mod. For a ring epimorphism $f : A \to B$ we denote by $\mathcal{X}_B$ the essential image of the restriction functor.

Theorem 2.3 ([13], Theorem 1.2, [14], [22], Theorem 1.6.1)**.** *There is a bijection between:*

1. *ring epimorphisms $A \to B$ up to equivalence;*
2. *bireflective subcategories $\mathcal{X}_B$ of A-Mod (respectively, Mod-A), i.e., strict full subcategories of A-Mod (respectively, Mod-A) closed under products, coproducts, kernels and cokernels.*

If A is a finite dimensional $\mathbb{K}$-algebra, this bijection can be restricted between:

1. *ring epimorphisms $A \to B$ up to equivalence, where B is a finite dimensional $\mathbb{K}$-algebra;*
2. *bireflective subcategories $\mathcal{X}_B$ of A-mod (respectively, mod-A), i.e., strict full functorially finite subcategories of A-mod (respectively, mod-A) closed under kernels and cokernels.*

Given a ring epimorphism $f : A \to B$ consider the adjunction in proposition 2.1. For a left A-module M, let $\psi_M : M \to B \otimes_A M$ be the unit of this adjunction at M. Clearly, we have that

$$\psi_M(m) = 1_B \otimes m, \ \forall m \in M.$$

Note that ψ_M for a left B-module N is an isomorphism. The following easy lemma shows that the map ψ_M is the $\mathcal{X}_B$-reflection of the left A-module M.

Lemma 2.4. *Let $f : A \to B$ be a ring epimorphism and M a left A-module. For any left B-module N and for any A-homomorphism $g : M \to N$, g factors uniquely through ψ_M.*

Proof. Since the map ψ_N is an isomorphism, we can define a homomorphism of A-modules

$$\tilde{g} := \psi_N^{-1} \circ (B \otimes_A g).$$

It is clear that $g = \tilde{g} \circ \psi_M$ and, by construction, $\tilde{g}$ is the unique map satisfying this property. □

Remark 2.5. In particular, note that the ring epimorphism $f : A \to B$, regarded as a homomorphism of A-modules, is a $\mathcal{X}_B$-reflection. Moreover, if A is a finite dimensional $\mathbb{K}$-algebra, then f can be seen as the sum of the reflections of the indecomposable projective A-modules.

2.2. **Flat and finite ring epimorphisms.**

Definition 2.6. A ring epimorphism $f : A \to B$ is said to be

- flat, if f turns B into a flat left A-module;
- finite, if f turns B into a finitely generated projective left A-module;
- 1-finite, if f turns B into a finitely presented left A-module of projective dimension less or equal than one.

Clearly, every finite ring epimorphism is flat and 1-finite. Conversely, the following result holds.

Proposition 2.7 ([12], Corollary 1.4)**.** *If A is a perfect ring, then a ring epimorphism $f : A \to B$ is flat if and only if it is finite.*

Remark 2.8. For a perfect ring A, a ring epimorphism $A \to B$ is finite if and only if every finitely generated projective left B-module is finitely generated and projective as a left A-module. Equivalently, B is finitely generated as a left A-module and for all M in B-*mod* its projective cover in A-*mod* is also a left B-module.

Recall that finite dimensional $\mathbb{K}$-algebras are perfect rings. From remark 2.8 and theorem 2.3 we get the following immediate lemma.

Lemma 2.9. *Let A be a finite dimensional $\mathbb{K}$-algebra. There is a bijection between*

(1) *finite ring epimorphisms $A \to B$ up to equivalence;*
(2) *bireflective subcategories $\mathcal{X}_B$ of A-mod (respectively, mod-A) such that projective objects of $\mathcal{X}_B$ are projective A-modules.*

2.3. **Homological ring epimorphisms.** We are interested in ring epimorphisms with particularly nice homological properties. Following Geigle and Lenzing ([14]), a ring epimorphism $f : A \to B$ is said to be homological if $Tor_i^A(B,B) = 0$, for all $i > 0$.

For any ring epimorphism $f : A \to B$, we denote by K_f the object

$$A \xrightarrow{f} B$$

in the category of complexes of left A-modules, where A lies in position -1. Note that, regarded as an object of $\mathcal{D}(A)$, K_f is isomorphic to the cone of the map f, seen as a map of complexes concentrated in degree zero. The following well-known result is an analogue of proposition 2.1 for homological ring epimorphisms.

Proposition 2.10. *The following are equivalent for a ring homomorphism $f : A \to B$.*

(1) *f is a homological ring epimorphism;*
(2) *The derived restriction functor $f_* : \mathcal{D}(B) \to \mathcal{D}(A)$ is fully faithful;*
(3) *$B \otimes_A^{\mathbb{L}} f : B \to B \otimes_A^{\mathbb{L}} B$ is an isomorphism in $\mathcal{D}(A)$;*
(4) *$B \otimes_A^{\mathbb{L}} K_f = 0$.*

Moreover, the functor $B \otimes_A^{\mathbb{L}} -$ is left adjoint to f_*.

Proof. The fact that (1) is equivalent to (2) can be found in [14] (theorem 4.4).

It is easy to see that (1) is equivalent to (3). Indeed, note that $H^0(B \otimes_A^{\mathbb{L}} f) = B \otimes_A f$ is an isomorphism if and only if f is a ring epimorphism. Also, for $i > 0$, $H^i(B \otimes_A^{\mathbb{L}} f) = Tor_i^A(B, f)$ is the zero map and it is an isomorphism if and only if $H^i(B \otimes_A^{\mathbb{L}} B) = Tor_i^A(B,B) = 0$.

Finally, we check that (3) is equivalent to (4). Consider the triangle in $\mathcal{D}(A)$

$$A \xrightarrow{f} B \longrightarrow K_f \longrightarrow A[1]$$

and apply to it the triangle functor $B \otimes_A^{\mathbb{L}} -$. Clearly, $B \otimes_A^{\mathbb{L}} f$ is an isomorphism if and only if $B \otimes_A^{\mathbb{L}} K_f = 0$, thus finishing the proof. $\square$

Homological ring epimorphisms of A play a role in understanding how to *decompose* the derived category $\mathcal{D}(A)$ into other triangulated categories. This *decomposition* is formalised by the notion of recollement.

Definition 2.11. Let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be triangulated categories. A recollement of $\mathcal{D}$ by $\mathcal{X}$ and $\mathcal{Y}$ is a diagram of six triangle functors, satisfying the properties below.

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{X} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} .$$

(1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, (j^*, j_*) are adjoint pairs;
(2) i_*, j_*, $j_!$ are full embeddings;
(3) $i^! \circ j_* = 0$ (and thus also $j^* \circ i_* = 0$ and $i^* \circ j_! = 0$);
(4) for each $Z \in \mathcal{D}$ there are triangles

$$i_* i^! Z \to Z \to j_* j^* Z \to i_* i^! Z[1]$$

$$j_! j^* Z \to Z \to i_* i^* Z \to j_! j^* Z[1].$$

We now recall the following result from [21], stating how homological ring epimorphisms give rise to recollements.

Theorem 2.12 ([21], §4). *Let $f : A \to B$ be a homological ring epimorphism. Then the derived restriction functor f_* induces a recollement*

$$\begin{tikzcd} \mathcal{D}(B) \arrow[r, "f_*"] & \mathcal{D}(A) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & Tria(K_f) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd} ,$$

where $Tria(K_f)$ denotes the smallest triangulated subcategory of $\mathcal{D}(A)$ containing K_f and closed under coproducts.

2.4. **Universal localisations.** The following theorem defines and shows existence of universal localisations.

Theorem 2.13 ([23], Theorem 4.1). *Let A be a ring and Σ a set of maps between finitely generated projective left A-modules. Then there is a ring A_Σ, unique up to isomorphism, and a ring homomorphism $f_\Sigma : A \to A_\Sigma$ such that*

(1) *$A_\Sigma \otimes_A \sigma$ is an isomorphism of left A-modules for all $\sigma \in \Sigma$;*
(2) *every ring homomorphism $g : A \to B$ such that $B \otimes_A \sigma$ is an isomorphism for all $\sigma \in \Sigma$ factors in a unique way through f_Σ, i.e., there is a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f_\Sigma"'] & & B \\ & A_\Sigma. \arrow[ur, "\exists ! \tilde{g}"'] & \end{tikzcd}$$

We say that the ring A_Σ in the theorem is the universal localisation of A at Σ. It is well-known that the homomorphism $f_\Sigma : A \to A_\Sigma$ is a ring epimorphism with $Tor_1^A(A_\Sigma, A_\Sigma) = 0$ ([23]). The functor $A_\Sigma \otimes_A -$ is called the localisation functor of the universal localisation and it is left adjoint to the restriction functor $f_{\Sigma *} : A_\Sigma\text{-}Mod \to A\text{-}Mod$ (see proposition 2.1). For a left A-module M we call the $\mathcal{X}_{A_\Sigma}$-reflection ψ_M the localisation map of M (see lemma 2.4).

We can also define universal localisations with respect to a certain set of A-modules. Indeed, let $\mathcal{U}$ be a set of finitely presented left A-modules of projective dimension less or equal than one. We denote by $A_\mathcal{U}$ the universal localistaion of A at $\Sigma = \{\sigma_U | U \in \mathcal{U}\}$, where $\sigma_U : P \to Q$ is a projective resolution of U in A-*mod*. Note that $A_\mathcal{U}$ is well-defined by [11] and we will call it the universal localisation of A at $\mathcal{U}$. The following easy example shows that universal localisations do not, in general, yield homological ring epimorphisms.

Example 2.14. Let A be the quotient of the path algebra over $\mathbb{K}$ of the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

by the ideal generated by $\beta\alpha$. Consider the universal localisation of A at $\mathcal{U} := \{P_2\}$. Note that $A_\mathcal{U}$ and A/Ae_2A lie in the same epiclass of A. It is easy to check that $Tor_2^A(A_\mathcal{U}, A_\mathcal{U}) \neq 0$ and, hence, the ring epimorphism $A \to A_\mathcal{U}$ is not homological.

2.5. **Localisations with respect to Gabriel filters.** These localisations generalise the torsion-theoretical properties of Ore localisations. In fact, right Gabriel filters in a ring A are in bijection with hereditary torsion classes in A-Mod. Also, in contrast with Ore or universal localisation, the localisation functor associated to a Gabriel filter is not necessarily the tensor product with the localised ring. For details and definitions we refer the reader to [24]. In what follows we discuss some properties of these localisations that motivate some of the questions answered in this paper. We start by discussing how flat ring epimorphisms relate to this notion of localisation.

Theorem 2.15 ([24], Theorem XI.2.1, Proposition XI.3.4)**.** *A localisation with respect to a Gabriel filter yields a flat ring epimorphism if and only if the localisation functor is naturally equivalent to the tensor product with the localised ring. Moreover, any flat ring epimorphism $f : A \to B$ lies in the same epiclass as the localisation of A with respect to a Gabriel filter of right ideals of A.*

A localisation with respect to a Gabriel filter is said to be perfect if it yields a flat ring epimorphism. The following corollary establishes a first connection between universal localisations, localisations with respect to Gabriel filters and flat ring epimorphisms.

Corollary 2.16. *If a universal localisation is a localisation with respect to a Gabriel filter then it is perfect, i.e., it yields a flat ring epimorphism.*

Proof. The localisation functor of a universal localisation is the tensor product with the localised ring. The result then follows from theorem 2.15. ☐

3. A SUFFICIENT CONDITION FOR UNIVERSAL LOCALISATION

In this section we provide sufficient conditions on a ring epimorphism for it to be a universal localisation. Recall that a quasi-isomorphism is a morphism of complexes inducing isomorphisms in the cohomologies.

Proposition 3.1. *Let $f : A \to B$ be a ring epimorphism. The following are equivalent.*

(1) *There is a quasi-isomorphism from P_f, a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of projective left A-modules, to K_f;*
(2) *B is a left A-module of projective dimension less or equal than one.*

Moreover, if these conditions hold, B is finitely presented if and only if P_f can be chosen as a complex of finitely generated projective left A-modules.

Proof. (1) ⇒ (2) Suppose we have a quasi-isomorphism as in the diagram

(3.1)
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & ker(g) & \xrightarrow{k_1} & P_f^{-1} & \xrightarrow{g} & P_f^0 & \xrightarrow{c_1} & coker(g) & \longrightarrow & 0 \\ & & \cong \downarrow k & & \downarrow \pi_2 & & \downarrow \pi_1 & & \cong \downarrow c & & \\ 0 & \longrightarrow & ker(f) & \xrightarrow{k_2} & A & \xrightarrow{f} & B & \xrightarrow{c_2} & coker(f) & \longrightarrow & 0. \end{array}$$

Define a complex as follows:

$$0 \longrightarrow P_f^{-1} \xrightarrow{p_1} A \oplus P_f^0 \xrightarrow{p_2} B \longrightarrow 0,$$

$$\begin{array}{ll} p_1 : P_f^{-1} \longrightarrow A \oplus P_f^0 & \qquad p_2 : A \oplus P_f^0 \longrightarrow B \\ x \mapsto (\pi_2(x), g(x)) & \qquad (y,z) \mapsto f(y) - \pi_1(z). \end{array}$$

It is easy to check, by diagram chasing in (3.1), that this is a short exact sequence. Hence, B has projective dimension less or equal than one.

(2) ⇒ (1): Choose a projective resolution of B of shortest length

$$0 \longrightarrow P_1^B \xrightarrow{h} P_0^B \xrightarrow{\pi} B \longrightarrow 0$$

and consider a Cartan-Eilenberg resolution of K_f given by

$$\begin{tikzcd}
 & P_1^B \arrow[d, "h"] \\
A \arrow[r, "\hat{f}"] \arrow[d, dotted, "id"'] & P_0^B \arrow[d, dotted, "\pi"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

It is well-known (see [25], §5.7) that there is a quasi-isomorphism from its total complex

$$A \oplus P_1^B \xrightarrow{\hat{f}+h} P_0^B$$

to K_f, thus finishing the proof. □

Remark 3.2. This proposition can be easily generalised to B of any finite projective dimension. Since our focus is on 1-finite ring epimorphisms, it is convenient to keep the statement and proof as above.

The following theorem shows that certain homological ring epimorphisms can be characterised as universal localisations.

Theorem 3.3. *Let $f : A \to B$ be a 1-finite ring epimorphism. Then f is homological if and only if it is a universal localisation.*

Proof. Suppose that f is a universal localisation. Then $Tor_1^A(B,B) = 0$ and, since B is a left A-module of projective dimension less or equal than one, f is homological.

Conversely, let P_f be a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of finitely generated projective left A-modules quasi-isomorphic to K_f, which exists by proposition 3.1. Since f is homological, by proposition 2.10, we have

$$0 = B \otimes_A^{\mathbb{L}} K_f \cong B \otimes_A^{\mathbb{L}} P_f = B \otimes_A P_f$$

in $\mathcal{D}(A)$, showing that $B \otimes_A g$ is an isomorphism of left A-modules. Therefore, by theorem 2.13, there is a commutative diagram of ring epimorphisms

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f_g"'] & & B \\
 & A_{\{g\}} \arrow[ur, "h"'] &
\end{tikzcd}$$

showing that, in particular, the essential images of the corresponding restriction functors for right modules satisfy, by proposition 2.1,

$$\mathcal{X}_B \subseteq \mathcal{X}_{A_{\{g\}}} \subseteq Mod\text{-}A.$$

In order to prove the reverse inclusion, we will see that $A_{\{g\}} \otimes_A f$ is an isomorphism of left (and right) A-modules. To do so, consider the short exact sequence

$$0 \longrightarrow ker(g) \longrightarrow P_f^{-1} \xrightarrow{\bar{g}} g(P_f^{-1}) \longrightarrow 0 \tag{3.2}$$

induced by the map g. Observe that a similar argument to the one in the proof of corollary 2.2(1) shows that $A_{\{g\}} \otimes_A \bar{g}$ is an isomorphism. Using the commutative diagram (3.1) given by the quasi-isomorphism from P_f

to K_f and applying the functor $A_{\{g\}} \otimes_A -$ to the short exact sequences (3.2) and (2.1) we get the following diagram of left A-modules

$$\begin{tikzcd} A_{\{g\}} \otimes_A ker(g) \arrow[r, "0"] \arrow[d, "A_{\{g\}} \otimes_A k"] & A_{\{g\}} \otimes_A P_f^{-1} \arrow[r, "\cong"] \arrow[d] & A_{\{g\}} \otimes_A g(P_f^{-1}) \arrow[r] \arrow[d] & 0 \\ A_{\{g\}} \otimes_A ker(f) \arrow[r, "A_{\{g\}} \otimes_A k_1"] & A_{\{g\}} \otimes_A A \arrow[r, "A_{\{g\}} \otimes_A \bar{f}"] & A_{\{g\}} \otimes_A f(A) \arrow[r] & 0. \end{tikzcd}$$

It shows that, since $A_{\{g\}} \otimes_A k$ is an isomorphism, $A_{\{g\}} \otimes_A k_1 = 0$ and thus $A_{\{g\}} \otimes_A \bar{f}$ is an isomorphism. Now, applying the functor $A_{\{g\}} \otimes_A -$ to the sequence (2.2), we get

$$Tor_1^A(A_{\{g\}}, coker(f)) \longrightarrow A_{\{g\}} \otimes_A f(A) \longrightarrow A_{\{g\}} \otimes_A B \longrightarrow 0.$$

In order to compute $Tor_1^A(A_{\{g\}}, coker(f))$, consider a projective resolution of $coker(f)$ of the form

$$\dots \longrightarrow P^{-2} \xrightarrow{d} P_f^{-1} \xrightarrow{g} P_f^0 \longrightarrow coker(f) \longrightarrow 0$$

and apply to it the functor $A_{\{g\}} \otimes_A -$. By definition, $A_{\{g\}} \otimes_A g$ is an isomorphism and, therefore, the first cohomology of the new complex is zero. This shows precisely that $Tor_1^A(A_{\{g\}}, coker(f)) = 0$ and, thus, using the epi-mono factorisation of f, we can conclude that

$$A_{\{g\}} \otimes_A f : A_{\{g\}} \otimes_A A \to A_{\{g\}} \otimes_A B$$

is an isomorphism of left A-modules. It is, however, easy to check that this is also an isomorphism of right A-modules. Hence, $A_{\{g\}}$ has a natural right B-module structure, i.e, it lies in $\mathcal{X}_B$. Since $A_{\{g\}}$ is a generator of $\mathcal{X}_{A_{\{g\}}}$, this shows that $\mathcal{X}_{A_{\{g\}}} \subseteq \mathcal{X}_B$ and, thus, $\mathcal{X}_{A_{\{g\}}} = \mathcal{X}_B$. By proposition 2.1, this means that $A_{\{g\}}$ and B lie in the same epiclass of A and, therefore, are isomorphic. □

Remark 3.4. As mentioned in the introduction, theorem 3.3 can be derived from independent current work of Chen and Xi by observing that, under our assumptions, the generalised localisation in [8] (corollary 3.7) is a universal localisation.

Remark 3.5. Note that, for a homological 1-finite ring epimorphism $f : A \to B$, the above proof together with the proof of proposition 3.1 explicitly constructs a map g in A-$proj$ with $B \cong A_{\{g\}}$. Indeed, g depends only on the choice of a projective resolution of B of shortest length in A-mod.

In particular, for finite ring epimorphisms, we have the following result.

Corollary 3.6. *Let $f : A \to B$ be a finite ring epimorphism. Then B lies in the same epiclass of A as the universal localisation $A_{\{f\}}$, where f is seen as an element of A-$proj$.*

With further assumptions on the ring epimorphism f, the universal localisation in theorem 3.3 takes a particularly nice form.

Corollary 3.7. *Let $f : A \to B$ be a homological 1-finite ring epimorphism. The following holds.*

(1) *If f is injective then $coker(f) = B/A$ is a finitely presented A-module of projective dimension less or equal than one and B and $A_{\{B/A\}}$ lie in the same epiclass of A.*

(2) *If f is surjective then $ker(f)$ is a finitely presented projective A-module and B and $A_{\{ker(f)\}}$ lie in the same epiclass of A.*

Moreover, if A is a finite dimensional $\mathbb{K}$-algebra and f is surjective then B and A/AeA lie in the same epiclass of A, for some idempotent e in A.

Proof. Let P_f be a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of finitely generated projective left A-modules quasi-isomorphic to K_f, which exists by proposition 3.1.

(1) Since f is injective, g is injective and $coker(f) \cong coker(g)$ is a finitely presented A-module of projective dimension less or equal than one. By theorem 3.3, it follows that B lies in the same epiclass of A as $A_{\{g\}} = A_{\{coker(f)\}}$.

(2) Since f is surjective, g is surjective and thus a split map. It follows that $ker(f) \cong ker(g)$ is a finitely presented projective A-module. Again, by theorem 3.3, we get that B lies in the same epiclass of A as $A_{\{g\}}$, which is easily seen to be the universal localisation $A_{\{0\to ker(f)\}} = A_{\{ker(f)\}}$.

Note that, if f is surjective then $ker(f)$ is an idempotent ideal of A, since we have

$$0 = Tor_1^A(B,B) = Tor_A^1(A/ker(f), A/ker(f)) = ker(f)/ker(f)^2.$$

Thus, if A is a finite dimensional $\mathbb{K}$-algebra then $ker(f)$ is generated by an idempotent e in A. □

As a consequence of theorem 3.3 we can also establish a comparison between universal localisations and localisations with respect to Gabriel filters, motivated by the results in [1].

Corollary 3.8. *Let A be a perfect ring and $f : A \to B$ a ring epimorphism. Then f is a universal localisation and a localisation with respect to a Gabriel filter if and only if f is flat.*

Proof. If f is both a universal localisation and a localisation with respect to a Gabriel filter, it is flat by corollary 2.16. Conversely, if f is flat, it is a localisation with respect to a Gabriel filter by theorem 2.15. By proposition 2.7, since A is perfect, f is finite and, thus, a universal localisation by corollary 3.6. □

4. Recollements of derived module categories

We will now use homological 1-finite ring epimorphisms to construct recollements of derived module categories. For two left A-modules M, N we denote by $\tau_M(N)$ the trace of M in N, i.e., the submodule of N given by the sum of the images of all A-homomorphisms from M to N.

Theorem 4.1. *Let $f : A \to B$ be a homological 1-finite ring epimorphism with $Hom_A(coker(f), ker(f)) = 0$. Then the derived restriction functor f_* induces a recollement of derived module categories*

$$\mathcal{D}(B) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(End_{\mathcal{D}(A)}(K_f)).$$

Moreover, if f is finite then there is an isomorphism of rings $End_{\mathcal{D}(A)}(K_f) \cong A/\tau_B(A)$.

Proof. By theorem 2.12, we have the following recollement of triangulated categories induced by the derived restriction functor f_*

$$\mathcal{D}(B) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} Tria(K_f).$$

Since B is 1-finite, by proposition 3.1, K_f is quasi-isomorphic to P_f, a complex $P_f^{-1} \xrightarrow{g} P_f^0$ of finitely generated projective left A-modules, and therefore it is compact in $\mathcal{D}(A)$. We will prove that it is exceptional. Recall that (see, for example, [25], corollary 10.4.7), for all X in $\mathcal{D}(A)$, $Hom_{\mathcal{D}(A)}(K_f, X) \cong Hom_{\mathcal{K}(A)}(P_f, X)$, where $\mathcal{K}(A)$ denotes the homotopy category of complexes of left A-modules. Clearly, for all $i \geq 2$ and $i \leq -2$, we have

$$Hom_{\mathcal{D}(A)}(K_f, K_f[i]) \cong Hom_{\mathcal{K}(A)}(P_f, K_f[i]) = 0.$$

Since, by assumption, we know that

$$Hom_A(coker(g), ker(f)) \cong Hom_A(coker(f), ker(f)) = 0,$$

we also get

$$Hom_{\mathcal{D}(A)}(K_f, K_f[-1]) \cong Hom_{\mathcal{K}(A)}(P_f, K_f[-1]) = 0.$$

It remains to show that

$$Hom_{\mathcal{D}(A)}(K_f, K_f[1]) \cong Hom_{\mathcal{K}(A)}(P_f, K_f[1]) = 0.$$

Note that every element Φ in $Hom_{\mathcal{K}(A)}(P_f, K_f[1])$ is uniquely determined by a morphism ϕ in $Hom_A(P_f^{-1}, B)$ which, by lemma 2.4, factors through the $\mathcal{X}_B$-reflection $\psi_{P_f^{-1}}$. This shows that Φ factors through $B \otimes_A P_f$, which is zero in $\mathcal{D}(A)$ (see argument in the proof of theorem 3.3). Since $B \otimes_A P_f$ is a two term complex, it is also zero in $\mathcal{K}(A)$. Thus, we have $\Phi = 0$ and

$$Hom_{\mathcal{D}(A)}(K_f, K_f[i]) = 0, \ \forall i \neq 0.$$

We conclude that K_f is a compact exceptional object in $\mathcal{D}(A)$. Therefore, by a result of Keller ([16], theorem 8.5), we get a recollement of derived module categories

$$\begin{tikzcd} \mathcal{D}(B) \arrow[r] & \mathcal{D}(A) \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & D(End_{\mathcal{D}(A)}(K_f)). \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd}$$

Suppose now that f is finite and $P_f = K_f$. We will describe $End_{\mathcal{D}(A)}(K_f) \cong End_{\mathcal{K}(A)}(K_f)$. Note that, for any element a in A, there is a unique morphism in $End_{\mathcal{K}(A)}(K_f)$ defined by $k_A(1_A) = a$ and $k_B(1_B) = f(a)$ as in the following commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "k_A"'] & B \arrow[r] \arrow[d, "k_B"] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

It is easy to see that we get a surjective ring homomorphism $\Omega : A \to End_{\mathcal{K}(A)}(K_f)$, whose kernel can be described by homotopy. It turns out that an element a in A lies in the kernel of Ω if and only if it exists h in $Hom_A(B, A)$ with $h(1_B) = a$ making the diagram

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] \arrow[dl, "h"'] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

commute. It remains to show that $ker(\Omega) = \tau_B(A)$. It is clear that $ker(\Omega) \subseteq \tau_B(A)$. Conversely, let a be an element in $\tau_B(A)$. Let h be a map in $Hom_A(B, A)$ such that $a = h(b)$ for some $b \in B$. We define a morphism $\tilde{h} \in Hom_A(B, B) \cong End_B(B)$ by mapping 1_B to b. Therefore, $h \circ \tilde{h}$ lies in $Hom_A(B, A)$ and it satisfies $h \circ \tilde{h}(1_B) = a$. Hence, a lies in $ker(\Omega)$, finishing the proof. $\square$

Following [26], we say that a ring A is derived simple if it does not admit a non-trivial recollement of derived module categories.

Corollary 4.2. *If A admits a non-trivial homological 1-finite ring epimorphism $f : A \to B$ which is either injective or surjective, then A is not derived simple.*

Let $f : A \to B$ be a finite ring epimorphism. It is well-known that, as the trace of a projective A-module in A, $\tau_B(A)$ is a two-sided idempotent ideal. In particular, if A is a finite dimensional $\mathbb{K}$-algebra, then $\tau_B(A)$ is generated by an idempotent e, i.e., $\tau_B(A) = AeA$. More precisely, we have the following easy lemma.

Lemma 4.3. *If A is a finite dimensional $\mathbb{K}$-algebra, B a finitely generated projective left A-module and $I := \{e_1, ..., e_n\}$ a complete set of primitive orthogonal idempotents in A, then we have*

$$\tau_B(A) = \sum_{\substack{e_i \in I \\ Ae_i | B}} Ae_iA.$$

Following ([10], §2.1), for a finite dimensional $\mathbb{K}$-algebra A, we call an idempotent ideal AeA of A stratifying if the associated ring epimorphism $A \to A/AeA$ is homological.

5. Examples

In this section we will discuss recollements arising from theorem 4.1 for three classes of homological 1-finite ring epimorphisms. Examples 5.1 and 5.2 consider the cases of injective and surjective ring epimorphisms, while proposition 5.3 and example 5.5 focus on finite ring epimorphisms which are neither injective nor surjective.

Example 5.1. Let $f : A \to B$ be a 1-finite, homological and injective ring epimorphism. Then, by corollary 3.7, B lies in the same epiclass of A as the universal localisation $A_{\{B/A\}}$ and, by [5] (theorem 3.5), the finitely generated left A-module $T := A \oplus B/A$ is tilting. Using theorem 4.1, we get the following recollement of derived module categories

$$\mathcal{D}(B) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(End_A(B/A)).$$

Note that B/A is isomorphic to K_f in $\mathcal{D}(A)$. If B/A is a left A-module of projective dimension one, this recollement is precisely the one induced by the universal localisation $A_{\{B/A\}}$ and by the tilting module T in [2] (theorem 4.8).

Indeed, take A to be the quotient of the path algebra over $\mathbb{K}$ of the quiver

$$\begin{tikzcd} 1 \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] & & 2 \\ & 3 \arrow[ur, "\beta"'] & \end{tikzcd}$$

by the ideal generated by $\beta\alpha$. Consider the map $\gamma^* : P_2 \to P_1$ in A-$proj$ given by multiplication with γ. Using remark 2.5, it is not difficult to see that $A \to A_{\{\gamma^*\}}$ is a 1-finite, homological and injective ring epimorphism and, thus, it yields the recollement

$$\mathcal{D}(A_{\{\gamma^*\}}) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(End_{\mathcal{D}(A)}(A_{\{\gamma^*\}}/A)).$$

In fact, we can describe explicitly the outer terms of the recollement. On one hand, the universal localisation $A_{\{\gamma^*\}}$ is Morita equivalent to the $\mathbb{K}$-algebra C given by the quotient of the path algebra over $\mathbb{K}$ of the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

by the ideal generated by $\beta\alpha$. On the other hand, since $A_{\{\gamma^*\}}/A$ is isomorphic to $coker(\gamma^*)^{\oplus 2}$ as a left A-module, it follows that $End_{\mathcal{D}(A)}(A_{\{\gamma^*\}}/A)$ is isomorphic to $\mathbb{K} \oplus \mathbb{K}$. Moreover, it is easy to check, on a case by case analysis, that this recollement is not induced by a stratifying ideal of A.

Example 5.2. Let $f : A \to B$ be a 1-finite, homological and surjective ring epimorphism. Then, by corollary 3.7, $ker(f)$ is a finitely generated projective left A-module and $B \cong A/ker(f)$ lies in the same epiclass of A as the universal localisation $A_{\{ker(f)\}}$. Using theorem 4.1, we get the following recollement of derived module categories

$$\mathcal{D}(A/ker(f)) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(End_A(ker(f))).$$

Note that we have $K_f \cong ker(f)[1]$ in $\mathcal{D}(A)$.

Moreover, if A is a finite dimensional $\mathbb{K}$-algebra then, again by corollary 3.7, B and A/AeA lie in the same epiclass of A, for some idempotent e in A. The above recollement is then the one induced by the stratifying ideal AeA of A, namely

$$\mathcal{D}(A/AeA) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \mathcal{D}(eAe).$$

We now give sufficient conditions for universal localisations to yield finite ring epimorphisms. In what follows, an element $w \neq 0$ of an admissible ideal I of the path algebra of a quiver is called a relation if it is a linear combination of paths with the same source and target such that for any non-trivial factorisation $w = uv$ neither u nor v lie in I. Note that I is generated by its relations.

Proposition 5.3. *Let $A = \mathbb{K}Q/I$ be a finite dimensional $\mathbb{K}$-algebra given by a connected quiver Q and an admissible ideal I in $\mathbb{K}Q$. Assume that there are vertices i and j and an arrow $\alpha : i \to j$ in Q such that:*

(1) *α is the unique arrow in Q starting at vertex i;*
(2) *α is the unique arrow in Q ending at vertex j;*
(3) *there is no relation in I ending at vertex j.*

Then the ring epimorphism $f : A \to A_{\{\alpha^*\}}$ is finite, where $\alpha^* : P_j \to P_i$ is the map in A-$proj$ given by multiplication with α. Moreover, f_* induces a recollement of derived module categories

$$\mathcal{D}(A_{\{\alpha^*\}}) \rightleftarrows \mathcal{D}(A) \rightleftarrows \mathcal{D}(\mathbb{K}).$$

Proof. By our combinatorial assumptions and lemma 2.4, it is easy to check the following isomorphism of left A-modules for each indecomposable projective A-module P_k

$$A_{\{\alpha^*\}} \otimes_A P_k \cong \begin{cases} P_k, & k \neq j \\ P_i, & k = j. \end{cases}$$

Using remark 2.5, we conclude that $f : A \to A_{\{\alpha^*\}}$ is a finite ring epimorphism and, when regarded as an A-module homomorphism,

$$f : \bigoplus_k P_k \longrightarrow \bigoplus_k (A_{\{\alpha^*\}} \otimes_A P_k)$$

is given by right multiplication with the square matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \cdots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \cdots & \\ & & & & & & 1 \end{pmatrix},$$

where α lies in position (j, j).

We now show that $Hom_A(coker(f), ker(f)) = 0$. Clearly, we have

$$coker(f) = coker(\alpha^*) = S_i,$$
$$ker(f) = ker(\alpha^*).$$

Note that f is injective if and only if there is no relation in I starting at vertex i. Now assume that $Hom_A(coker(f), ker(f)) = Hom_A(S_i, ker(\alpha^*)) \neq 0$. Consequently, there is a non-trivial element u in e_iAe_j such that αu is zero in A, a contradiction to condition (3) in the assumptions. Therefore, by theorem 4.1, we get the following recollement of derived module categories

$$\mathcal{D}(A_{\{\alpha^*\}}) \rightleftarrows \mathcal{D}(A) \rightleftarrows \mathcal{D}(A/\tau_{A_{\{\alpha^*\}}}(A)),$$

where, by lemma 4.3, $\tau_{A_{\{\alpha^*\}}}(A)$ is isomorphic to AeA for $e := \sum_{k \neq j} e_k$. Hence, we have

$$A/\tau_{A_{\{\alpha^*\}}}(A) \cong A/AeA \cong \mathbb{K}.$$

□

Remark 5.4. Note that similar conditions to the ones above are considered in [9] (example 3.6.2), in the setting of expansions of abelian categories. Indeed, they prove that the inclusion functor $\mathcal{X}_{A_{\{\alpha^*\}}} \hookrightarrow A$-$mod$ is a right expansion. It is also a left expansion if the map α^* is injective.

We provide an application for the proposition.

Example 5.5. Let $n \in \mathbb{N}_{>1}$ and A be the quotient of the path algebra over $\mathbb{K}$ of the quiver Q below

$$\begin{tikzcd}
 & 1 \arrow[r] & 2 \arrow[dr] & \\
n \arrow[ur] & & & 3 \arrow[dl] \\
 & \cdots \arrow[ul] & 4 \arrow[l] &
\end{tikzcd}$$

by an admissible ideal I which is not a power of the ideal generated by the arrows of Q. Consequently, there are vertices i and j and an arrow $\alpha : i \to j$ in Q such that there is no relation in I ending at vertex j. We can now apply proposition 5.3, yielding the recollement

$$\mathcal{D}(A_{\{\alpha^*\}}) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(A) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(\mathbb{K}).$$

In particular, A is not derived simple. This conclusion can also be obtained by observing that A admits a stratifying ideal AeA, for some idempotent e in A. Again by assumption, there are vertices r and s and an arrow $\beta : r \to s$ in Q such that there is no relation in I starting at vertex r. Hence, by multiplication with β we get an injective morphism $\beta^* : P_s \to P_r$ and $coker(\beta^*) = S_r$ is of projective dimension 1. Now consider the universal localisation of A at $\mathcal{U} := \{\bigoplus_{k \neq r} P_k\}$, where $A_{\{\mathcal{U}\}}$ lies in the same epiclass of A as A/AeA for $e := \sum_{k \neq r} e_k$. Since $\mathcal{X}_{A_{\{\mathcal{U}\}}}$ is equivalent to $add\{S_r\}$, the ring epimorphism $A \to A_{\{\mathcal{U}\}}$ is 1-finite and, hence, homological. We conclude that the idempotent ideal AeA is stratifying and it yields the following recollement of derived module categories

$$\mathcal{D}(\mathbb{K}) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(A) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(eAe).$$

Note that in many cases the algebra eAe in the above recollement can be chosen to be Morita equivalent to $A_{\{\alpha^*\}}$. For example, let B be the quotient of the path algebra over $\mathbb{K}$ of the quiver

$$\begin{tikzcd}
1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

by the ideal generated by $\beta\alpha\beta$. On one hand, the finite ring epimorphism $A \to A_{\{\alpha^*\}}$, where $A_{\{\alpha^*\}}$ is Morita equivalent to $\mathbb{K}[x]/x^2$, yields the recollement

$$\mathcal{D}(\mathbb{K}[x]/x^2) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(A) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(\mathbb{K}).$$

On the other hand, the stratifying ideal Ae_2A induces the recollement

$$\mathcal{D}(\mathbb{K}) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(A) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \mathcal{D}(e_2Ae_2),$$

where e_2Ae_2 and $\mathbb{K}[x]/x^2$ are isomorphic as rings.

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