ABSTRACT NILGROUPS OF FINITE ABELIAN GROUPS

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This thesis deals with the following problem: given π a finite abelian group, compute $NK_1(\underline{\underline{Z}}\pi)$. Here $NK_1(R) = \mathrm{Ker}(K_1(R[t]) \to K_1(R))$ where the map is that induced by augmentation. The group $NK_1(\underline{\underline{Z}}\pi)$ appears as a direct summand in the group $K_1(\underline{\underline{Z}}\pi')$ where π' is finitely generated abelian and π is the torsion part of π'.

These calculations consist of two parts. In the first part it is shown that $NK_1(\underline{\underline{Z}}\pi) = o$ for π of square free order. In the second we show that otherwise the group $NK_1(\underline{\underline{Z}}\pi)$ can be infinite. In particular we show that if $|\pi_{(p)}|>p^2$ p odd and $\pi_{(p)}$ cyclic then $NK_1(\underline{\underline{Z}}\pi)_{(p)}$ is infinite torsion and p-primary.

In addition several general facts about NK_1 and NK_2 are also proved and utilized in these computations. The following results are of independent interest.

i) A surjective map of Artin Rings $R \longrightarrow S$ induces a surjection $NK_2(R) \longrightarrow NK_2(S)$.

ii) A surjection of finite abelian groups $\Pi \longrightarrow \Pi'$ induces a surjection $NK_1(\underline{\underline{Z}}\pi) \longrightarrow NK_1(\underline{\underline{Z}}\pi')$.

Some other examples are given where the hypotheses of the theorems proved cannot be weakened and certain other examples for infinite NK_1's are produced.

CONTENTS

INTRODUCTION

The purpose of this thesis is the computation of nilgroups of finite cyclic groups. In order to put these computations in their proper perspective we first recall the definitions of the functors K_0, K_1 and K_2.

Let A be an associative ring with unity and denote by GL(A) the union over all n of $GL_n(A)$. Here we view $GL_n(A) \subset GL_{n+1}(A)$ by the map.

1) $$M \longmapsto \begin{pmatrix} M & O \\ O & I \end{pmatrix}$$

If e_{ij} is the standard matrix unit, (i.e. the $ij\underline{th}$ entry of e_{ij} is 1, all other entries 0) we consider the subgroup E(A) of GL(A) generated by matrices of the form $I_n + ae_{ij}$ $i \neq j$ $i,j \leq n$ $a \varepsilon A$. It can be shown that E(A) is a perfect group and moreover is the commutator subgroup of GL(A). Consequently GL(A)/E(A) is an abelian group denoted $K_1(A)$.

An, at first seemingly unrelated functor $K_0(A)$ can be defined as follows, Let $\underline{\underline{P}}(A)$ denote the category of finitely generated projective A-modules. For each $P \varepsilon \underline{\underline{P}}(A)$, let (P) denote the isomorphism class of P. Then M_A, the set of all such classes, is a monoid under the binary operation $(P) \oplus (Q) = (P \oplus Q)$, induced by direct sum. Since M_A is abelian there exists an abelian group $K_0(A)$ and an additive map $\Theta_A : M_A \to K_0(A)$, such that for all additive map $f : M_A \to G$, where G is an abelian group, there exists a homomorphism $\Phi : K_0(A) \to G$ such that $f = \Phi o \Theta_A$. A similar construction can be carried out for small categories with a product ([2], [12]). We shall see later there is an intimate relationship between K_0 and K_1 more particularly exact sequences relating these to yet a third functor K_2A.

To define K_2A we consider first some formal identities satisfied by $E(A)$. If G is any group put $[a,b] = aba^{-1}b^{-1}$ for all $a,b\varepsilon G$. Let $E^a_{ij} = I_n + ae_{ij}\varepsilon E(A)$. Then one can easily verify the following identities.

$$2) \quad \begin{array}{l} E^a_{ij}\, E^b_{ij} = E^{a+b}_{ij} \\ \\ [E^a_{ij}, E^b_{kl}] = \begin{cases} 1 & \text{if } i \neq k,\ l \neq j \\ E^{ab}_{i\ell} & \text{if } j=k \end{cases} \end{array} \qquad a,\ b\ \varepsilon\ A$$

We can then define the Steinberg group St(A) as the free group on the symbols X^a_{ij}, $a\varepsilon A$ subject to the relations 2) above with X^a_{ij} replacing E^a_{ij}. From this definition it is apparent that there is an epimorphism of groups $St(A)\to E(A)$. The kernel of this homomorphism is an abelian group denoted $K_2(A)$ ([7], [10]).

In the work of J.H.C. Whitehead a certain quotient group of $K_1(A)$ was found to contain valuable topological information when $A = \underline{\underline{Z}}\pi$ and π is the fundamental group of a C.W. complex. Since then in the work of C.T.C. Wall ([16]) and others (see [11]) the computation of $K_1(\underline{\underline{Z}}\pi)$ has become of considerable interest to topologists.

This computation for an arbitrary group, π, is difficult. Of the little that is known in general, the nontrivial theorems apply to essentially three situations.

1^0 π is finite.

2^0 π is finitely generated and abelian.

3^0 π is a generalized free product.

The results here apply to the first two situations. In particular the study of $K_1(\mathbb{Z}\pi)$ for π finitely generated and abelian is based upon two considerations, first a detailed study of related questions when π is finite, and second, considerations relating to the so called fundamental theorem of Algebraic K-Theory. Since the latter is essential for our purposes we recall briefly its statement.

Let F be any functor from rings to abelian groups. Let t denote an indeterminant. If $\varepsilon: A[t] \to A$ is the augmentation $t \mapsto 1$ we define,

3) $NF(A)=\mathrm{Ker}(F(A[t]) \xrightarrow{F(\varepsilon)} F(A))$

4) $LF(A)=\mathrm{coker}(F(A[t]) \oplus F(A[t^{-1}]) \to F(A[t,t^{-1}]))$

where the latter map is induced by the obvious inclusions. In this situation we have a natural decomposition 5) and a sequence 6)

5) $F(A[t])=F(A) \oplus NF(A)$

6) $0 \to F(A) \xrightarrow{\delta} F(A[t]) \oplus F(A[t^{-1}]) \longrightarrow F(A[t,t^{-1}]) \xrightarrow{\rho} LT(A) \longrightarrow 0$

In 6) δ is induced by the map $\delta: A \to A[t] \oplus A[t^{-1}], \delta(x)=(x,-x)$. It is obvious from these definitions that 6) is exact, except perhaps at $F(A[t] \oplus F([t^{-1}])$, and that the composition of any two morphisms is zero. We shall be concerned when 6) is a contractible complex of groups, that is, when it is exact and ρ has a natural section. We call F <u>contracted</u> if for all rings A this is the case. It follows that if F is contracted there is a natural decomposition 7).

7) $F(A[t,t^{-1}])=F(A) \oplus NF(A) \oplus NFA \oplus LF(A)$

We can now state the fundamental theorem ([2], [12]).

<u>Fundamental Theorem</u> K_0, K_1, K_2 are contracted functors. Moreover there is a natural isomorphism $LK_i \simeq K_{i-1}$ $i=1,2$.

This applies notably to the computation of $F(\underline{\underline{Z}}\pi)$ when $F=K_i$ $i=0,1,2$ and π is finitely generated and abelian. One proceeds inductively on the rank r of π. If $r=0$ then π is finite abelian and this situation must be treated directly ([2],[5],[13]). If $r>0$ write $\pi = \pi_0 \times T$, where T is an infinite cyclic group and π_0 has rank $r-1$.

Then putting $A=\underline{\underline{Z}}\pi_0$ we have $\underline{\underline{Z}}\pi=A[t,t^{-1}]$. By the fundamental theorem we find

8) $F(\underline{\underline{Z}}\pi)=F(\underline{\underline{Z}}\pi_0)\oplus 2NF(\underline{\underline{Z}}\pi_0)\oplus LF(\underline{\underline{Z}}\pi_0)$.

This proceedure effectively reduces the computation of $F(\underline{\underline{Z}}\pi)$ to the computation of F and related functors for $\underline{\underline{Z}}(\pi_0)$. We will be concerned particularly in the case π_0 is finite (i.e. rank $\pi=1$). To illustrate the kinds of questions we seek to answer about $K_1(\underline{\underline{Z}}\pi)$ we refer to the work of Bass and Murthy ([5]) see also ([2] pg 663). The investigation in ([5]) began as an attempt to answer the following question of Milnor ([3] pg 408). If π is finitely generated abelian is $Wh_1(\pi)$ finitely generated? Here $Wh_1(\pi)$ is the quotient of $K_1(\underline{\underline{Z}}\pi)$ by the subgroup of $GL_1(\underline{\underline{Z}}\pi)$ of elements of the form $\pm g$, $g\epsilon\pi$. Since π is finitely generated this is essentially the same as asking whether $K_1(\underline{\underline{Z}}\pi)$ is finitely generated. For π of rank 1, this is a question as to finite generation of $K_i(\underline{\underline{Z}}\pi_0)$ $i=0,1$ and $NK_1(\underline{\underline{Z}}\pi_0)$. Finite generation for $K_i(\underline{\underline{Z}}\pi_0)$ $i=0,1$ has been settled ([13], [3]). As for $NK_1(\underline{\underline{Z}}\pi_0)$ the question remained unsettled prior to this thesis. For rank $\pi>1$, $NK_0(\underline{\underline{Z}}\pi_0)$ is a subgroup of $K_1(\underline{\underline{Z}}\pi_0)$ and this question was completely settled by computing $NK_0(\underline{\underline{Z}}\pi_0)$. We explicitly describe these results below. The p-primary subgroup of an abelian group π will be denoted $\pi_{(p)}$.

Theorem A. Let π be finite abelian, then $NK_0(\underline{\underline{Z}}\pi)$ is a countable torsion group.

1) If $|\pi_{(p)}|\leq p$, then $(NK_0\underline{\underline{Z}}\pi))_{(p)}=0$

2) If $|\pi_{(p)}|\geqq p^2$, then $(NK_0(\underline{\underline{Z}}\pi))_{(p)}$ is infinite.

3) Consequently $NK_0(\underline{\underline{Z}}\pi)=0$ iff $|\pi|$ is square free.

Although we cannot prove the analogue of this theorem with K_1 replacing K_0, we have obtained partial results which we indicate as theorem B.

Theorem B Let π be as in the theorem A, then $NK_1(\underline{\underline{Z}}\pi)$ is a countable torsion group.

1) If $|\pi_{(p)}|\leq p$ then $(NK_1\underline{\underline{Z}}\pi)_{(p)}=0$

2) If $\pi_{(p)}$ is cyclic and

$$\left.\begin{array}{l} p \text{ odd and } |\pi_{(p)}|\geq p^2 \\ \qquad \text{or} \\ p=2 \text{ and } |\pi_{(p)}|\geqq 8 \end{array}\right\} \text{ then } NK_1(\underline{\underline{Z}}\pi)_{(p)} \text{ is infinite}$$

3) Consequently $NK_1(\underline{\underline{Z}}\pi)=0$ if $|\pi|$ is squarefree.

The approach we take in proving these results can be outlined as follows. We first prove 3) as theorem 2.1, this is simply an application of the functorialty of the NK_i $i=0,1$ developed in section 1. To prove 2) we first show that a surjection $\pi\to\pi'$ of finite abelian groups induces an epimorphism $NK_i(\underline{\underline{Z}}\pi)\longrightarrow NK_i(\underline{\underline{Z}}\pi')$ for $i=0,1$. This reduces 2) to the case where π itself is cyclic of prime power order. The proof of 1) when π is not squarefree can then be handled using the machinery set up in sections 2 and 3.

I would like to take this opportunity to thank my thesis advisor, Hyman Bass, without whose patience, this thesis would not have been written.

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.0 Preliminaries on Cartesian Squares

In this section we recall the representation of group rings as cartesian products. Most of the results here are well known. Recall that a commutative square 1) of additive groups and homomorphisms is called cartesian if $A=\{(a_1,a_2)\varepsilon A_1 x A_2 \mid f_1(a_1)=f_2(a_2)\}$.

1)
$$\begin{tikzcd} A \arrow[r, "p_1"] \arrow[d, "p_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & A' \end{tikzcd}$$

Example 0.1 In 1) above put $A_1=A_2$ and $A'=\{0\}$ then if 1) is cartesian A is nothing more than the direct product $A=A_1xA_1$.

If in 1) above the groups have additional structure (e.g. rings, k-modules, k-algebras) and the morphisms preserve this additional structure we speak of a cartesian square of rings, k-modules, k-algebras. Given a commutative square 1) we can define a homomorphism

2) $$h: \ A_1xA_2 \longrightarrow A' \quad h(a_1,a_2)=f_1(a_1)-f_2(a_2)$$

Then clearly ker h is the cartesian product of A_1 and A_2 over A. We have proved:

Prop 0.2 The commutative square 1) is cartesian iff the sequence 3) is exact.

3) $$0 \longrightarrow A \xrightarrow{p_1 x p_2} A_1 x A_2 \xrightarrow{h} A'$$

We now can construct the two principal types of cartesian squares which are important in what follows.

Remark: This proposition will be used repeatedly in the following situation A, A' are rings and $\mathscr{C}$ two sided ideal of A'. Then clearly 11) will be a cartesian square of rings.

Proof We must show that 12) is exact, that is ker h = $p_1 x p_2(A)$

$$12)\qquad 0 \to A \xrightarrow{p_1 x p_2} A/\mathscr{C} \xrightarrow{h} A'/\mathscr{C}$$

Let $(a+\mathscr{C}, a') \in \ker h$. Thus $a+ = a'+\mathscr{C}$ i.e. $a-a' \in A$. This implies $a' \in A$. Since clearly $(a+\mathscr{C}, a') = (a'+\mathscr{C}, a') = p_1 x p_2(a')$ we have $p_1 x p_2(a) \in \ker h$. Moreover $p_1 x p_2$ is clearly injective.

Example 0.6 In 0.5 we let A = $\underline{\underline{Z}}\pi$ where π is a finite group. We put $\mathscr{O}$ the integral closure of $\underline{\underline{Z}}\pi$ in $\underline{\underline{Q}}\pi$ and $\mathscr{C} = \{a \in \underline{\underline{Z}}\pi \mid a\mathscr{O} \subseteq \underline{\underline{Z}}\pi\}$ the conductor from $\mathscr{O}$ to $\underline{\underline{Z}}\pi$. Then $\mathscr{C}$ is non-zero ([2] pg 535) and the resulting square will be referred to as the <u>conductor situation.</u>

We can use proposition 0.2 to produce further examples of cartesian squares via

Prop 0.7 Let 1) be A cartesian square of k-modules, and assume B is flat k-module. Then the square 13) is cartesian.

$$13)\qquad \begin{array}{ccc} A \otimes_k B & \xrightarrow{p_1 \otimes 1_B} & A_1 \otimes_k B \\ {\scriptstyle P_2 \otimes 1_B}\downarrow & & \downarrow{\scriptstyle f_1 \otimes 1_B} \\ A_2 \otimes_k B & \xrightarrow{f_2 \otimes 1_B} & A' \otimes_k B \end{array}$$

Proof Exactness of 14) and k-flatness of B implies exactness of 15). Thus 13) is cartesian by proposition 0.2, and the natural isomorphism $(A_1 x A_2) \otimes_k B \simeq (A_1 \otimes_k B) \times (A_2 \otimes_k B)$

$$14)\qquad 0 \to A \to A_1 x A_2 \to A'$$

Prop. 0.3 Suppose that 1) is cartesian and f_1 and f_2 are surjective. Then there exist subgroups $\mathfrak{a}_i$, of A such that

1° $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$

2° $A/\mathfrak{a}_1 = A_1$, $A/\mathfrak{a}_2 = A_2$

3° $A/\mathfrak{a}_1+\mathfrak{a}_2 = A'$.

conversely given an additive group and two subgroups $\mathfrak{a}$ and $\mathfrak{b}$, the square 4) (all morphisms just quotient maps) is cartesian with f_1 and f_2 subjective.

4)
$$\begin{array}{ccc} A/\mathfrak{a}\cap\mathfrak{b} & \longrightarrow & A/\mathfrak{a} \\ \downarrow & & \downarrow \\ A/\mathfrak{b} & \longrightarrow & A/\mathfrak{a}+\mathfrak{b} \end{array}$$

Proof: Assume that 1) is cartesian and put $\mathfrak{a}_i = \ker p_i$. Then ker $(p_1 x p_2) = \mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$ by proposition 0.2. Moreover p_1 is surjective. In effect let $a_1 \varepsilon A_1$ and consider $f_1(a_1) \varepsilon A'$, surjectivity of f_2 implies $\exists$ an $a_2 \varepsilon A_2$ such that $f_2(a_2)=f_1(a_1)$ and therefore an $(a_1,a_1)\varepsilon A$ such that $p_1(a_1,a_2)=a_1$. Similarly p_2 is surjective and therefore $A_1 = A/\mathfrak{a}_1$ $A_2 = A/\mathfrak{a}_2$. Since f_1 and f_2 are surjective A' is a quotient of A(say $A/\mathfrak{h}$), we have a commutative diagram 5) with exact rows. By the "snake lemma" g is an ismorphism hence $\mathfrak{h} = \mathfrak{a}_1 + \mathfrak{a}_2$.

$$\begin{array}{lccccccccc} & 0 & \to & A & \to & A/\mathfrak{a}_1 \times A/\mathfrak{a}_2 & \to & A/\mathfrak{h} & \to & 0 \\ 5) & & & \| & & \| & & \downarrow g & & \\ 6) & 0 & \to & A & \to & A/\mathfrak{a}_1 \times A/\mathfrak{a}_2 & \to & A/\mathfrak{a}_1+\mathfrak{a}_2 & \to & 0 \end{array}$$

Remark: We note that $\ker f_1 = p_1(\mathfrak{a}_2) \simeq \mathfrak{a}_2$. And that by symmetry $\ker f_2 = p_2(\ker p_1) \simeq \ker p_1$. Conversely the exactness of 6) implies that 4) is cartesian by proposition 0.2.

Example 0.4 Let A =$\underline{Z}[t]$ and f, g $\varepsilon \underline{Z}[t]$ satisfy 7), equivalently f and g have no common prime factor). Then the square 8) is cartesian.

7) $fA \cap gA = fgA$

8)
$$\begin{array}{ccc} A/fgA & \xrightarrow{p_1} & A/fA \\ p_2 \downarrow & & \downarrow f_1 \\ A/gA & \xrightarrow[f_2]{} & A/fA+gA \end{array}$$

Of particular importance is the situation where, for a fixed rational prime p, we write.

9)
$$f(t) = x-1$$
$$g(t) = x^{p-1} + x^{p-2} + \ldots + 1$$
$$x = t^{p^{n-1}}$$

Then;

10)
$$A/fgA = \underline{\underline{Z}}\pi_n \qquad \pi_n = \text{the cyclic group } \underline{\underline{Z}}/p^n\underline{\underline{Z}}$$
$$A/gA = \underline{\underline{Z}}[\zeta_n] \qquad \zeta_n\text{-a primitive } p^{n\,th} \text{ root of unity}$$
$$A/fA = \underline{\underline{Z}}\pi_{n-1}$$
$$A/fA+gA = \underline{\underline{F}}_p\pi_{n-1} \qquad (\underline{\underline{F}}_p = \text{the field with p elements}).$$

Under this identification the image of t will play the role of a generator for π_n respectively π_{n-1} in A/fgA, respectively in A/fA, A/fA+gA, and the role of ζ_n in A/gA. the maps in 8) then become reduction modulo the obvious ideals. In the special case that n=1 we notice that p_1 is the (split) augmentation $\underline{\underline{Z}}\pi_1 \to \underline{\underline{Z}}$.

Prop 0.5 Let $\mathscr{C} \subset A \subset A'$ be a additive groups. Then the square 11) is cartesian, where the vertical arrows are the quotient maps.

11)
$$\begin{array}{ccc} A & \subset & A' \\ \downarrow & & \downarrow \\ A/\mathscr{C} & \subset & A'/\mathscr{C} \end{array}$$

15) $$0 \to A \otimes_k B \to (A_1 \otimes_k B) \times (A_2 \otimes_k B) \to A' \otimes_k B$$

Cor 0.8 Let 1) be a cartesian square of rings and let T be a monoid. Then 16) is a cartesian square of rings.

$$\begin{tikzcd} A_2[T] \arrow[r, "P_1[T]"] \arrow[d, "P_2[T]"'] & A_1[T] \arrow[d, "f_1[T]"] \\ A_2[T] \arrow[r, "f_2[T]"'] & A'[T] \end{tikzcd}$$

Proof $\mathbb{Z}[T]$ is free hence flat over $\mathbb{Z}$.

Remark This applies notably when $T=\mathbb{N}$ or $\mathbb{Z}$ when we recover $A[T]=A[t]$ or $A[t,t^{-1}]$.

1. Cartesian Squares and Exact Sequences

In this section we present the important exact sequences of algebraic K-theory within the framework of cartesian squares. With the machinery devolped in the last section we show how to deduce the analogues of these results for the functors NK_i $i=0,1,2$. Our approach differs somewhat from that of Bass ([2] p 656) in being less axiomatic. The methods we use allow us to prove these results with less machinery. We begin this discussion with a definition.

Definition 1.1 Let $f:A \to B$ be a monomorphism of rings and assume B admits a decomposition as a finite product of rings say

1) $$B = \prod_{i=1}^{n} B_i.$$

If p_i denotes the projection $p_i : B \to B_i$ and all of the composites $p_i \circ f : A \to B_i$ are surjective we call f a subdirect monomorphism.

We give some examples of this phenomenon.

Example 1.2 If $\alpha_1,\dots,\alpha_n$ are two sided ideals in A then the monomorphism 2)

2) $$A/\alpha_1 \cap \dots \cap \alpha_n \to \prod_{i=1}^{n} A/\alpha_i$$

induced by the maps $A/\alpha_1 \cap \dots \cap \alpha_n \to A/\alpha_i$ is a subdirect monomorphism. In particular if 3) is cartesian and p_1 and p_2 are surjective then the map $p_1 \times p_2 : A \to A_1 \times A_2$ is a subdirect monomorphism.

3)
$$\begin{array}{ccc} A & \xrightarrow{p_1} & A_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow{f_2} & A' \end{array}$$

Example 1.3 If k is a flat $\underline{\underline{Z}}$ - algebra and f: A →B is a subdirect monomorphism, then so is 4)

4)
$$f \otimes_{\underline{\underline{Z}}} 1_k : \ A \otimes_{\underline{\underline{Z}}} k \to B \otimes_{\underline{\underline{Z}}} k.$$

The importance of this concept can be seen in the following theorem of Milnor see ([0],[6] App 2)

Theorem 1.4 Let 3)be a cartesian square of rings and assume either

1) f_1 and f_2 are surjective, or

2) f_1 is surjective and p_1 is a subdirect monomorphism.

Then there is an exact sequence 5) which is natural in the category of cartesian squares of rings.

5)
$$K_2(A)\to K_2(A_1 \times A_2)\to K_2(A') \longrightarrow K_1(A)\to K_1(A_1 \times A_2)\to K_1 A' \longrightarrow K_0 A\to K_0(A_1 \times A_2)\to K_0 A'$$

The importance of this result is that is allows us to "approximate" the groups $K_1(A)$ via the intervening groups, which are in many cases better understood. By virtue of corollary 0.8 we can extend this result as follows (see also [2] pg 674)

Theorem 1.5 Under either of the hypotheses of 1.4 there is an exact sequence 6) . Natural in the category of cartesian squares of rings.

6)
$$NK_2(A)\to NK_2(A_1 x A_2)\to NK_2(A')\to NK_1(A)\to NK_1(A_1 x A_2)\to NK_1(A')\to$$
$$NK_0(A)\to NK_0(A_1 x A_2)\to NK_0(A')$$

Proof By example 1.3 and corollary 0.8 the cartesian square 7) satisfies the hypotheses of 1.4 if 3) does. We can therefore apply 1.5 twice and deduce a homomorphism of exact sequences 8).

7)
$$\begin{array}{ccc} A_2[T] & \xrightarrow{p_1[T]} & A_1[T] \\ {\scriptstyle p_2[T]}\downarrow & & \downarrow{\scriptstyle f_1[T]} \\ A_2[T] & \xrightarrow[f_2[T]]{} & A'[T] \end{array}$$

8)
$$\begin{array}{ccccccccc} K_2(A[t]) & \to & K_2((A_1xA_2)[t]) & \to & K_2(A'[t]) & \to & K_1(A[t]) & \to\dots\to & K_0(A'[t]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_2(A) & \to & K_2(A_1xA_2) & \to & K_2(A') & \to & K_1(A) & \to\dots\to & K_0(A') \end{array}$$

This homorphism is induced by the augmentation ε: $A[t] \to A$ and therefore all the vertical maps split. From this we deduce exactness for the sequence of kernels 6).

From this result we can recover the exact sequence relating to a surjective homomorphism f: $A \to A/\alpha$. Define $A(\alpha)$ by the cartesian square 9).

9)
$$\begin{array}{ccc} A(\alpha) & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & A/\alpha \end{array}$$

Then there is a natural homomorphism $\Delta: A \to A(\alpha)$ given by $\Delta(a) = (\bar{a},\bar{a})$ which is split by both p_1 and p_2. If we apply 1.4 we get an exact sequence 10) putting $K_i(A,\alpha) = \ker(K_iA(\alpha) \xrightarrow{p_1} K_i(A))$ $i = 0, 1, 2$ we easily deduce the exact sequence 11).

10) $K_2(A\ \alpha) \to K_2(A \times A) \to K_2(A/\alpha) \to K_1(A(\alpha)) \to \dots \to K_0(A/\alpha)$

11) $0 \to K_2(A, \alpha) \to K_2(A) \to K_2(A/\alpha) \to K_1(A, \alpha) \to K_1(A) \to K_1(A/\alpha) \to \dots \to K_0(A/\alpha)$

By virtue of the fact that cartesian products commute with flat base change (Proposition 0.7), we have that

12) $$A[t]\ (\alpha[t]) = A(\alpha)[t],$$

and therefore can deduce an exact sequence 13), by the same method as in 1.5.

13) $0 \to NK_2(A, \alpha) \to NK_2(A) \to NK_2\ (A/\alpha) \to NK_1(A, \alpha) \to NK_1(A)$

$\to NK_1\ (A/\alpha) \to NK_0(A, \alpha) \to NK_0(A) \to NK_0(A/\alpha)$.

The only thing we need to show is that $NK_i(A,\alpha) = \ker\ (K_i(A[t], \alpha[t]) \to K_i(A,\alpha))$ is a direct summand. This follows from the commutative diagram 14).

14)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & NK_i(A, \alpha) \arrow[r] \arrow[d] & K_i(A[t], \alpha[t]) \arrow[r] \arrow[d] & K_i(A, \alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & NK_iA(\alpha) \arrow[r] \arrow[d] & K_i\ A(\alpha)[t] \arrow[r] \arrow[d] & K_iA(\alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & NK_iA \arrow[r] \arrow[d] & K_iA[t] \arrow[r] \arrow[d] & K_iA \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here all vertical sequences are split exact and induced by the projections and all rows except possibly the first row (of kernels) are split exact. Therefore the first row of 14) is also split exact and we have established 13).

Another result which we will require is the ability to compare $NK_i(A, \alpha)$ and $NK_i(A,\beta)$ whenever $\alpha \subset \beta$ are ideals in A. In this connection we have the following result ([10] pg 56).

<u>Theorem 1.6</u> Let $\alpha \subset \beta$ be ideals in A. Then there are exact sequences.

15) $$K_2(A/\alpha , \beta/\alpha) \to K_1(A,\alpha) \to K_1(A,\beta) \to K_1(A/\alpha , \beta/\alpha) \to K_0(A,\alpha) \to K_0(A,\beta) \to K_0(A/\alpha , \beta/\alpha)$$

16) $$NK_2(A/\alpha \ \ \beta/\alpha) \to NK_1(A,\alpha) \to NK_1(A,\beta) \to NK_1(A/\alpha, \beta/\alpha) \to NK_0 (A, \alpha) \to NK_0 (A,\beta) \to NK_0(A/\alpha , \beta/\alpha).$$

As was remarked above 15) is well known, 16) follows from 14) applied to $\alpha\subset\beta\subset A$ and $\alpha[t] \subset \beta [t] \subset A[t]$ and by the splitting argument immediately above.

2. Results on the Vanishing of Nilgroups

In this section we use the machinery so far developed to prove some results concerning the vanishing of the group $NK_1(A)$.
An associative ring A is called right regular in case A is right Noetherian and finitely generated right A-modules have finite projective dimension. The main result which we require for this discussion is due to Bass,Heller.Swan ([4]) and Quillen ([12]).

Theorem 2.1 If A is right regular then $NK_i(A) = 0$, $i = 0, 1, 2$.
We can now state the main result of this section. It is interesting to note that this theorem gives examples of rings, A, which are not regular but for which $NK_i(A) = 0$ $i = 0,1$. The case $i = 0$ was already known to Bass and Murthy but our method of proof will allow us to handle both cases at once.

Theorem 2.2 Let R_n denote the nth cyclotomic extension of the integers, and let π be an abelian group of order $|\pi|$. Then if $|\pi|$ is squarefree and either

1) $(|\pi|, n) = 1$ or

2) $(|\pi|, n) = 2$ and $4 \nmid n$

then $NK_i(R_n\pi) = 0$, $i=0,1$.

Proof The proof will be by induction. If m is a squarefree integer we define the length $\ell(m)$ to be the number of prime factors of m. Suppose first $\ell(|\pi|) = 1$, then $|\pi|$ is a prime p and we can obtain the cartesian square for $R_n\pi$ 2) by tensoring the square 1) for $\underline{\underline{Z}}\pi$

with R_n (see corollary 0.8 and example 0.4).

1)
$$\begin{tikzcd} \underline{\underline{Z}}\pi \arrow[r] \arrow[d] & R_p \arrow[d] \\ \underline{\underline{Z}} \arrow[r] & \underline{\underline{F}}_p \end{tikzcd}$$

2)
$$\begin{tikzcd} R_n\pi \arrow[r] \arrow[d] & R_n \otimes_{\underline{\underline{Z}}} R_p \arrow[d] \\ R_n \arrow[r] & R_n \otimes_{\underline{\underline{Z}}} \underline{\underline{F}}_p \end{tikzcd}$$

Regarding square 2) there are two cases to consider, if hypothesis 1) holds then R_n and $\underline{\underline{Z}}[\zeta_p]$ are linearly disjoint in $\underline{\underline{C}}$ (that is their quotient fields are). Therefore we can identify $R_n \otimes \underline{\underline{Z}}[\zeta_p]$ with R_{np}. Since $R_n \otimes \underline{\underline{F}}_p \simeq R_n/pR_n$ and $(p,n)=1$, $p\underline{\underline{Z}}$ does not ramify in R_n thus R_n/pR_n is reduced and therefore regular. In case hypothesis 2) above holds and $\ell(|\pi|)=1$ we have π cyclic of order 2 and $R_n = R_{2m}$ with $(m,2)=1$. In this situation we can obtain the square for $R_n\pi$ 4) by tensoring the square for $\underline{\underline{Z}}\pi$ 3) with R_n.

3)
$$\begin{tikzcd} \underline{\underline{Z}}\pi \arrow[r] \arrow[d] & \underline{\underline{Z}} \arrow[d] \\ \underline{\underline{Z}} \arrow[r] & \underline{\underline{F}}_2 \end{tikzcd}$$

4)
$$\begin{tikzcd} R_n\pi \arrow[r] \arrow[d] & R_n \arrow[d] \\ R_n \arrow[r] & R_n \otimes F_2 \end{tikzcd}$$

Again since the prime $2\underline{\underline{Z}}$ does not ramify in R_n unless $4 \nmid n$ $R_n \otimes \underline{\underline{F}}_2$ is a product of fields. In either case the proof of 2.2 for $\ell(|\pi|)=1$ follows from

<u>Lemma 2.3</u> Let 5) be a cartesian square and assume that $NK_i(A_1) = NK_i(A_2) = NK_{i+1}(A') = 0$ (e.g. A_1, A_2 and A' regular) for $i=0$ or $i=1$. Then $NK_i(A) = 0$.

<u>Proof</u> If we apply then 1.5 to 5) we obtain an exact sequence 6). The exactness of 6) together with the hypotheses implies the result.

5)
$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] & A' \end{tikzcd}$$

6) $NK_{i+1}(A') \to NK_i(A) \to NK_i(A_1 \times A_2)$

We next consider the general case when $\ell(|\pi|) = r$ Here we write $\pi = \pi_p \times \pi'$ π_p = a cyclic group of order p where $\ell(|\pi'|) = r-1$ $(p, |\pi|) = 1$. If we tensor 1) with $R_n\pi'$ we obtain the cartesian square 7)

7)
$$\begin{array}{ccc} R_n\pi & \longrightarrow & R_n\pi' \\ \downarrow & & \downarrow \\ R_n \otimes_{\underline{\underline{Z}}} R_p\pi' & \longrightarrow & R_n \otimes_{\underline{\underline{Z}}} \underline{\underline{F}}_p\pi' \end{array}$$

Under hypotheses 1) and 2) $R_n \otimes_{\underline{\underline{Z}}} R_p \simeq R_{np}$ and $\ell(|\pi'|) = \ell(|\pi|)-1$ therefore the rings adjacent to $R_n\pi$ in 7) have trivial NK_i by induction. On the other hand $R_n \otimes_{\underline{\underline{Z}}} \underline{\underline{F}}_p$ is a product of fields whose characteristic p does not divide $|\pi|$ so $R_n \otimes_{\underline{\underline{Z}}} \underline{\underline{F}}_p\pi'$ is semisimple hence regular. We are done by lemma 2.3. Note that if $2 = (n, |\pi|)$ the hypothesis $4 \nmid n$ cannot be relaxed (see 3.9).

This theorem which I proved in 1972 produced the first known examples of rings R which although not regular have $NK_1(R)=0$. This type of vanishing also occurs in the following context. Let A be a Noetherian ring of Krull dimension $= 1$. Assume that the integral closure B of A, in the ring of fractions of A is finite over A. In this situation $\mathscr{C} = \{b \in B \mid bB \subset A\}$ is a nonzero ideal of B contained in A. We call $\mathscr{C}$ the conductor from B to A, and if $\sqrt[B]{\mathscr{C}} = \mathscr{C}$ we call such A seminormal. By a similar technique we can deduce the next result.

Theorem 2.5 Let A be seminormal, then $NK_i(A) = 0$, $i = 0,1$.

Proof Using the notation above and proposition 0.4 we have a cartesian square 8). Since B is the integral closure of A, Krull dim. $B = 1$. Consequently $B/\mathscr{C}$ is finite, and since $\sqrt[B]{\mathscr{C}} = \mathscr{C}$, $B/\mathscr{C}$ is reduced. Therefore $A/\mathscr{C}$ is also finite and reduced. Consequently $A/\mathscr{C}$ and $B/\mathscr{C}$ are products of regular local rings thus regular. Since B is integrally closed it is also regular. We finish by applying Lemma 2.3.

8)
$$\begin{array}{ccc} A & \subset & B \\ \downarrow & & \downarrow \\ A/\mathscr{C} & \subset & B/\mathscr{C} \end{array}$$

<u>Example 2.6</u> In 2.5 the hypothesis that $\sqrt[B]{\mathscr{C}} = \mathscr{C}$ is essential to the theorem. Consider $\mathbb{Z}[2i]$ (Gaussian integers with even imaginary part), then $\mathbb{Z}[2i]$ has $\mathbb{Z}[i]$ as its integral closure, but the conductor $2\mathbb{Z}[i] = \mathscr{C}$ is not its own radical, e.g. $(1+i)^2 = 2i \in \mathscr{C}$. With this in mind 8) becomes 9).

9)
$$\begin{array}{ccc} \mathbb{Z}[2i] & \subset & \mathbb{Z}[i] \\ \downarrow & & \downarrow \\ \mathbb{F}_2 = \mathbb{Z}[2i]/2\mathbb{Z}[i] & \subset & \mathbb{Z}[i]/2\mathbb{Z}[i] = \mathbb{F}_2[\varepsilon] \quad (\varepsilon^2 = 0) \end{array}$$

By applying theorem 1.5 to 9) we get $N\mathbb{K}_1(\mathbb{Z}[2i]) = N\mathbb{K}_2(\mathbb{F}_2[\varepsilon])$
We prove, in Chapter 3, that $NK_2(\mathbb{F}_2[\varepsilon]) = \mathbb{F}_2[t]$.

<u>Remark</u>: We can produce examples of this phenomenon for all primes p namely the ring $\mathbb{Z} + p\mathbb{Z}[\zeta_p]$. It also can be shown that if $|\pi| = p$ the ring $\mathbb{Z}[p\pi]$ will also have large NK_1 (Here if t = a generator of π, or ζ_p the rings described above are the subrings consisting of elements of the form $x = Z_0 + pZ_1 t + pZ_2 t^2 + \ldots + pZ_{p-1} t^{p-1}$, $Z_i \in \mathbb{Z}$.

3. Nonvanishing for $NK_1(Z\pi)$.

In this rather long section we prove the results 1) and 3) alluded to in the introduction. We begin by recalling some fairly well known results about the functor K_1.

When R is a commutative ring the determinant homomorphism $\det_n : GL_n(R) \to U(R)$ (units of R) induces a homomorphism $\det : GL(R) \to U(R)$ which upon abelianization induces a homomorphism $\mathrm{Det}: K_1(R) \to U(R)$ which is easily seen to be split by the inclusion $U(R) \subset K_1(R)$. This results in a direct product decomposition 1).

1) $\quad K_1(R) = SK_1(R) \oplus U(R)$

Here $SK_1(R)$ denotes the kernel of Det. Applying this decomposition to $K_1(R[t])$ we obtain a similar decomposition for $NK_1(R)$ 2).

2) $\quad NK_1(R) = NSK_1(R) \oplus NU(R)$

To understand $NK_1(R)$ we study each summand separately. The less exotic piece $NU(R)$ is completely understood.

Proposition 3.1.([2] pg 671) When R is commutative there is an isomorphism. $1 + \mathrm{Nil}(R)[t].t \simeq NU(R)$. Consequently if R is reduced then $NU(R)=0$.

Proof: If $g(t) \in \mathrm{Nil}(R)[t].t$ then the binomial theorem shows that $g(t)$ is nilpotent, therefore $1+g(t)$ is a unit congruent to 1 modulo $tR[t]$. Conversely if $f(t) \in NU(R)$ then $f(t) \equiv 1 \pmod{tR[t]}$ and being a unit this forces the coefficients of t^i $i>0$ to all be nilpotent([1] Chapt 1).

The next proposition shows that when R is an integral group ring that $NU(R)=0$.

<u>Proposition 3.2</u> Let R be an integral domain with quotient field k, and π a finitely generated abelian group. Let π_0 denote the torsion part of π and assume;

1) k has characteristic p and $(|\pi_0|,p)=1$,or

2) k has characteristic 0.

Then $R\pi$ is reduced.

<u>Proof</u>.In either situation above the Maschke theorem assures us that $k\pi_0$ is semisimple hence reduced. Consequently $R\pi_0 \subset k\pi_0$ is also reduced. The theorem now follows by viewing $R\pi$ as a localization of the reduced polynomial ring $R\pi_0[t_0,\ldots,t_n]$ at the multiplicative set generated by $t_o,\ldots,t_n$.

With these results we can now concentrate our attention on the groups $NSK_1(R)$. The cornerstone of this investigation is the following theorem due to Bass ([2] pg 685).

<u>Theorem 3.3</u> Let R be a commutative Artin ring. Then $NSK_1(R)=0$. Consequently, if $S \to R$ is a homomorphism with S cummutative and reduced then $NK_1(S) \to NK_1(R)$ is zero.

Using this result we can prove an interesting result concerning NK_2 for Artin rings (compare [8], pg 12 thru 27).

<u>Theorem 3.4</u> Let f: R S be a surjective homomorphism of commutative Artin rings, then the induced homomorphism $NK_2(f)$: $NK_2(R) \to NK_2(S)$ is surjective.

Proof. Let I denote the kernel of f. Then by (page 9, 11).) we have an exact sequence 3).

3) $$NK_2(R) \longrightarrow NK_2(S) \xrightarrow{\delta} NK_1(R,I)$$

It is clear ([10] pg 54) that the image of the map δ is contained in the group $NSK_1(R,I)$. Hence the result will follow if we show that this latter group is trivial. Recall that $NSK_1(R,I)$ is the kernel of the map $NSK_1(R(I)) \longrightarrow NSK_1(R)$ induced by p_1 in the cartesian square 4).

4) $$\begin{tikzcd} R(I) \arrow[r] \arrow[d, "p_1"'] & R \arrow[d] \\ R \arrow[r, dash] & R/I \end{tikzcd}$$

This square gives rise to an exact sequence of R-modules 5).

5) $$0 \to R(I) \to R \oplus R \to R/I \to 0$$

Here R acts on R(I) via the diagonal $\Delta: R \to R(I)$. Thus when R is Noetherian, or more generally when I is finitely generated, R(I) is a finitely generated R-module. R Artin implies that R(I) has finite length as an R-module and is therefore also Artin. The theorem now follows from 3.3.

As an immediate consequence of this we obtain

Theorem 3.5. Let $\alpha \subset \beta$ be ideals of a commutative ring R and assume that R/α is Artin. Then the natural homomorphism 6) is surjective.

6) $$NSK_1(R,\alpha) \to NSK_1(R,\beta)$$

Proof. By 1, 13) there is an exact sequence, part of which is 7).

7) $$NK_1(R,\alpha) \to NK_1(R,\beta) \to NK_1(R/\alpha,\beta/\alpha)$$

Using the naturality of the decomposition 2) above we obtain the exact sequence 8).

8) $$NSK_1(R,\alpha) \to NSK_1(R,\beta) \to NSK_1(R/\alpha,\beta/\alpha)$$

Since R/α is Artin then by 3.4 $R/\alpha(\beta/\alpha)$ is also Artin, therefore we have that $NSK_1(R/\alpha,\beta/\alpha)=0$.

In order to show that $NSK_1(\mathbb{Z}\pi)$ is nonzero in many interesting cases it is convenient to reduce this question to one about various special cases. The next result accomplishes this.

__Theorem 3.6__ Let $\pi_0 \to \pi_1$ be a surjective homomorphism of finite abelian groups. Then the induced map $NK_1(\mathbb{Z}\pi_0)\to NK_1(\mathbb{Z}\pi_1)$ is surjective.

__Proof.__ We consider the embeddings $\mathbb{Z}\pi \subset \mathcal{O}_i$ of the integral group rings into their maximal orders ([14] pg 63). The unique extension of the surjection $\mathbb{Z}\pi_0 \to \mathbb{Z}\pi_1$ to the surjection $\mathbb{Q}\pi_0 \to \mathbb{Q}\pi_1$ induces a surjective homomorphism $\mathcal{O}_0 \to \mathcal{O}_1$ of the integral closures. Letting $\mathscr{C}_i$ denote the respective conductors and f the surjection of the maximal orders we clearly have that $f(\mathscr{C}_1)\subset\mathscr{C}_2$. Thus the diagram 9) commutes.

9)
$$\begin{tikzcd}
\mathcal{O}_0 \arrow[r] \arrow[d] & \mathcal{O}_1 \arrow[d] \\
\mathcal{O}_0/\mathscr{C}_0 \arrow[r] & \mathcal{O}_1/\mathscr{C}_1
\end{tikzcd}$$

Here the verticals denote the canonical quotient homomorphisms. From these considerations it is clear that the cube 10) commutes. This diagram is by definition a homomorphism of the cartesian squares which comprise the front and back faces of the cube. Functoriality of the exact sequences of 1.5 yields the commutative diagram with exact rows 11).

10)
$$\begin{tikzcd}
& \mathbb{Z}\pi_1 \arrow[rr] \arrow[dd] & & \mathcal{O}_1 \arrow[dd] \\
\mathbb{Z}\pi_0 \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{O}_0 \arrow[ur] \arrow[dd] & \\
& \mathbb{Z}\pi_1/\mathcal{C}_1 \arrow[rr] & & \mathcal{O}_1/\mathcal{C}_1 \\
\mathbb{Z}\pi_0/\mathcal{C}_0 \arrow[ur] \arrow[rr] & & \mathcal{O}_0/\mathcal{C}_0 \arrow[ur] &
\end{tikzcd}$$

11)
$$\begin{tikzcd}
NK_2(\mathcal{O}_0/\mathcal{C}_0) \arrow[r] \arrow[d] & NK_1(\mathbb{Z}\pi_0) \arrow[r] \arrow[d] & NK_1(\mathbb{Z}\pi_0/\mathcal{C}_0) \arrow[d] \\
NK_2(\mathcal{O}_1/\mathcal{C}_1) \arrow[r] & NK_1(\mathbb{Z}\pi_1) \arrow[r] & NK_1(\mathbb{Z}\pi_1/\mathcal{C}_1)
\end{tikzcd}$$

Now in 11) both maps $NK_1\mathbb{Z}\pi_i \to NK_1\mathbb{Z}\pi_i/\mathcal{C}_i$ are trivial by virtue of 3.2 and 3.3. Moreover $NK_2\,\mathcal{O}_0/\mathcal{C}_0 \to NK_2\,\mathcal{O}_1/\mathcal{C}_1$ is surjective by theorem 3.4. By exactness, the result follows by considering the diagram 12),

12)
$$\begin{tikzcd}
NK_2\,\mathcal{O}_0/\mathcal{C}_0 \arrow[r] \arrow[d] & NK_1\mathbb{Z}\pi_0 \arrow[r] \arrow[d] & 0 \\
NK_2\,\mathcal{O}_1/\mathcal{C}_1 \arrow[r] & NK_1\mathbb{Z}\pi_1 \arrow[r] & 0
\end{tikzcd}$$

With the following few results we will be set to prove the main result of this section.

Theorem 3.7 (NK_1 - Excision). Let $f: A \to B$ be a homomorphism of rings and assume either

1) f is surjective or

2) f is a subdirect monomorphism, (1,1)

Let α and β be ideals s.t. $f(\alpha)=\beta$. Then $NK_1(A,\alpha) \simeq NK_1(B,\beta)$

Proof Under either of the hypotheses above it is known ([10] pg 55, [2] pg 484) that there is an isomorphism $K_1(A,\alpha) \simeq K_1(B,\beta)$. As flat base change preserves 1 and 2 we also have in this situation $K_1(A[t], \alpha[t]) \simeq K_1(B[t], \beta[t])$. Commutativity of 13) plus exactness of the columns and all except possibly the first row yields the conclusion.

13)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & NK_1(A,\alpha) \arrow[r] \arrow[d] & NK_1(B,\beta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1(A[t],\alpha[t]) \arrow[r] \arrow[d] & K_1(B[t],\beta[t]) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1(A,\alpha) \arrow[r] \arrow[d] & K_1(B,\beta) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Prop. 3.8 Let 14) be a cartesian square of rings and assume f_1 and f_2 surjective and A_2 regular. Then $NK_1(A) \simeq NK_1(A_1, \ker f_1)$

14)
$$\begin{tikzcd}
A \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\
A_2 \arrow[r, "f_2"'] & A'
\end{tikzcd}$$

Proof Since 14) is cartesian f_1 surjective implies g_2 surjective. By inspection of the exact sequence of the surjection g_2 15) one deduces $NK_1(A) \simeq NK_1(A, \ker g_2)$ from the regularity of A_2.

15) $$NK_2(A_2) \to NK_1(A, \ker g_2) \to NK_1(A) \to NK_1(A_2)$$

As g_1 is surjective 3.7 implies $NK_1(A, \ker g_2) \simeq NK_1(A_1, \ker f_1)$ and the result.

We can now give an example of a nonzero NK_1 for an integral group ring and at the same time show that (see theorem 2.2) in proving that $NK_1(R_n\pi) = 0$ when $|\pi|$ is square free and $(n,|\pi|)=2$, the hypothesis $4 \nmid n$ cannot be deleted. Below we let π_i be a cyclic group of order p^i

Example 3.9 $NK_1(\underline{\underline{Z}}\pi_2 x \pi_1)$ is infinite two-torsion when p=2.

To see this we consider the cartesian square 16).

16) $$\begin{array}{ccc} \underline{\underline{Z}}\pi_1 \times \pi_2 & \longrightarrow & \underline{\underline{Z}}[i]\pi_1 \\ \downarrow & & \downarrow \\ \underline{\underline{Z}}\pi_1 \times \pi_1 & \longrightarrow & \underline{\underline{F}}\pi_1 \times \pi_1 \end{array}$$

The exact sequence of 16) reads in part

17) $$NK_1(\underline{\underline{Z}}\pi_1 x \pi_2) \to NK_1(\underline{\underline{Z}}\pi_1 x \pi_1) \oplus NK_1(\underline{\underline{Z}}[i]\pi_1) \to NK_1(\underline{\underline{F}}_2\pi_1 \times \pi_1)$$

Since $\underline{\underline{F}}_2(\pi_1 \times \pi_1)$ is Artin and all other intervening rings are reduced we have a surjection.

18) $$NK_1(\underline{\underline{Z}}\pi_1 x \pi_2) \to NK_1(\underline{\underline{Z}}[i]\pi_1).$$

The result will follow from an explicit computation of the latter group. Notice that $Z[i]\pi_1$ is the simplest example of failure for the above mentioned hypothesis.

To see that $NK_1 Z[i]\pi_1$ is infinite we apply 3.8 to the cartesian square 19) obtained by tensoring $\underline{\underline{Z}}\pi_1$ with Z[i].

19)
$$\begin{tikzcd}
\mathbb{Z}[i]\pi_1 \arrow[r] \arrow[d] & \mathbb{Z}[i] \arrow[d, "g"] \\
\mathbb{Z}[i] \arrow[r] & \mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[i]
\end{tikzcd}$$

From 3.8 we have $NK_1(Z[i]\pi_1)=NK_1(Z[i], \ker g)$. Since $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{Z}[i] \simeq \mathbb{F}_2[\varepsilon]$ ("Dual numbers" over $\mathbb{F}_2$) and g is just reduction modulo the ideal $2\mathbb{Z}[i]$, regularity of $\mathbb{Z}[i]$ implies that $NK_1(\mathbb{Z}[i], \ker g) \simeq NK_2(\mathbb{F}[\varepsilon])$ To complete this example we have only to show that $NK_2\mathbb{F}[\varepsilon]$ is infinite (a later quoted result will show that it is torsion. To see this we first quote a highly non-trivial theorem of Van der Källen ([15]).

Theorem 3.10 Let R be a commutative ring. Then $K_2(R[\varepsilon]) \simeq K_2R \oplus V(R)$ where;

V(R) is an abelian group with the following presentation ;

generators: $d(r)$, $r \varepsilon R$

relations:
$$d(r+r') = d(r)+d(r') + F(rr') \quad \text{where } (F(r)=d(r+1)-d(r)$$
$$d(rr') = rd(r') + r\,d(r)$$
$$F(r+r') = F(r) + F(r'),\ F(r)$$

There is a natural surjection $V(R \to \Omega_{R/\mathbb{Z}}(R)$ (Kähler differentials). It is bijective if $2 \varepsilon R^{\cdot}$ or R is a perfect field.

Notice that from this theorem $K_2(\mathbb{F}[\varepsilon,t]) = K_2(\mathbb{F}_2[t]) \oplus V(\mathbb{F}_2[t])$ and $K_2(\mathbb{F}_2[\varepsilon]) = K_2\mathbb{F}_2, V(\pi) = \Omega\, \mathbb{F}_2/\mathbb{Z}$ (Since $\mathbb{F}_2$ is perfect and $K_2\mathbb{F}[t] \simeq K_2\mathbb{F}_2 = 0$). As $\Omega\pi_{2/\mathbb{Z}} = 0$ ([9] pg 71)we have $NK_2\mathbb{F}[\varepsilon] = V(\mathbb{F}_2[t])$. We show that this latter group is nonzero by considering the fundamental exact sequence for Ω __ ([9] Theorem 57)

Theorem 3.11 Let $K \to A \to B$ be homormorphisms of rings. Then the sequence 20) is exact.

$$\text{20)} \qquad \Omega_{A/K} \otimes_A B \longrightarrow \Omega_{B/K} \longrightarrow \Omega_{B/A} \longrightarrow 0$$

For a definition of the maps the reader is referred to (Loc. Cit.). Putting $K = \underline{\underline{Z}}$, $A = \underline{\underline{F}}_2$, $B = \underline{\underline{F}}[t]$ and using the fact that $\Omega_{\underline{\underline{F}}_2/\underline{\underline{Z}}} = 0$, we have $\Omega_{\underline{\underline{F}}_2[t]/\underline{\underline{Z}}} = \Omega_{\underline{\underline{F}}_2[t]/\underline{\underline{F}}_2} \cong \underline{\underline{F}}_2[t]$. ([9] pg 184). Since $V(\underline{\underline{F}}_2[t]) \longrightarrow \Omega_{\underline{\underline{F}}_2[t]/\underline{\underline{Z}}}$ is surjective this completes 3.9.

We now turn to the proof of the main result of this section namely

Theorem 3.12 Assume that either p is odd and $n \geq 2$ or p is even and $n \geq 3$ then $NK_1(Z\pi_n)$ is infinite torsion.

In the course of the proof we shall isolate certain other results which are of independent interest. By the next result due to Bass we know $NK_1(Z\pi)$ is torsion for finite abelian π. ([2] p 648)

Theorem 3.13 Let $A \subset B$ be a subdirect monomorphism of rings with B a regular ring and assume $mB \subset A$ for some $m \in \underline{\underline{Z}}$. Then if T denotes a finitely generated free commutative monoid then any element of $L = \ker(K_1(A[T]) \to K_1(A))$ has order dividing some power of m.

By theorem 3.6 we can assume $n = 2$ if p is odd and $n = 3$ if $p = 2$. We have however the following result valid for all n.

Prop. 3.14 If $\left\{\begin{array}{l}(m,p) = 2,\ 4 \nmid n \\ \text{or} \\ (m,p) = 1\end{array}\right.$ then there is a natural isomorphism

$$NK_1(R_m \pi_n) \cong NK_1(R_m \pi_{n-1},\ p R_m \pi_{n-1})$$

Under either of the above hypotheses the ring $R_{mp}n \simeq R_m \otimes R_p n$. Therefore cartesian square 21) obtained by tensoring the square for $\underline{\underline{Z}}\pi_n$ (0.7) with R_m satisfies the hypothesis of 3.8.

Moreover as f is just reduction modulo p $R_m \pi_{n-1}$ the result follows

21)
$$\begin{array}{ccc} R_m \pi_n & \longrightarrow & R_m \pi_{n-1} \\ \downarrow & & \downarrow f \\ R_{mp}n & \longrightarrow & \underline{\underline{F}}_p \pi_{n-1} \otimes R_m \end{array}$$

By considering this situation with $m = 2$ we get

22)
$$NK_1(\underline{\underline{Z}}\pi_n) \simeq NK_1(\underline{\underline{Z}}\pi_{n-1}, p\underline{\underline{Z}}\pi_{n-1})$$

Thus we can prove 3.12 if we can show that the latter group is infinite under the hypothesis p odd, $n \geq 2$ or $p = 2, n \geq 3$.

<u>Theorom 3.15</u> If p odd and $n \geq 2$ or $p = 2$ $n \geq 3$ then $NK_1(\underline{\underline{Z}}\pi_{n-1}, p\underline{\underline{Z}}\pi_{n-1})$ is infinite.

We can assume that if p is odd $n = 2$ and if $p = 2$ $n = 3$ since by 22) and Theorem 3.6 the natural map

23)
$$NK_1(\underline{\underline{Z}}\pi_i, p\underline{\underline{Z}}\pi_i K) \twoheadrightarrow NK_1(\underline{\underline{Z}}\pi_{i-1}\ p\underline{\underline{Z}}\pi_{i-1})$$

is surjective. For the rest of this proof let π_1 denote a cyclic group of odd prime order and π_2 a cyclic group of order 4. Now $\underline{\underline{Z}}\pi_i$ is a $\underline{\underline{Z}}$-order in the semisimple $\underline{\underline{Q}}$ algebra $\underline{\underline{Q}}\pi_i$. It is well known ([14] pg 63) that $\underline{\underline{Z}}\pi_i$ can be embedded (subdirectly) in a maximal $\underline{\underline{Z}}$-order $\mathcal{O}_i$. Since $\mathcal{O}_i$ is maximal $\mathcal{O}_i$ is hereditary hence regular ([14] p 94). In this situation we have a cartesian square 23) where $\mathcal{S}_i$ denotes any $\mathcal{O}_i$ ideal $\mathcal{S}_i \subset \underline{\underline{Z}}\pi_i$ (0.6).

23)
$$\begin{array}{ccc} \underline{\underline{Z}}\pi_i & \subset & \mathcal{O}_i \\ \downarrow & & \downarrow \\ \underline{\underline{Z}}\pi_i/\mathcal{S}_i & \subset & \mathcal{O}_i/\mathcal{S}_i \end{array}$$

24) $NK_2\,\mathcal{O}_i \longrightarrow NK_2\,\mathcal{O}_i/\mathcal{C}_i \longrightarrow NK_1(\mathcal{O}_i,\mathcal{C}_i) \longrightarrow NK_1(\mathcal{O}_i)$.

Since $|\pi_i|\,\mathcal{O}_i \subset \underline{\underline{Z}}\pi_i$ ([14] pg. 63) we can take $\mathcal{C}_i = |\pi_i|\,\mathcal{O}_i$. Since $\mathcal{O}_i$ is regular the exact sequence of the surjection f_2 24) yields $NK_2\,\mathcal{O}_i/\mathcal{C}_i \simeq NK_1(\mathcal{O}_i,\mathcal{C}_i)$. Moreover a direct computation yields.

25)
$$\begin{aligned} \mathcal{O}_1 &= \underline{\underline{Z}} \times \underline{\underline{Z}}[\zeta] \\ \mathcal{O}_2 &= \underline{\underline{Z}} \times \underline{\underline{Z}} \times \underline{\underline{Z}}[i] \end{aligned} \qquad \zeta \text{ a primitive } p^{th} \text{ root of } 1$$

26)
$$\mathcal{O}_1/\mathcal{C}_1 = \underline{\underline{F}}_p \times \underline{\underline{F}}_p[\tau] \qquad \tau^{p-1} = 0$$
$$\mathcal{O}_2/\mathcal{C}_2 = \underline{\underline{Z}}/4\underline{\underline{Z}} \times \underline{\underline{Z}}/4\underline{\underline{Z}} \times \underline{\underline{Z}}[i]/4\underline{\underline{Z}}[i]$$

It is clear that $NK_2(\mathcal{O}_1/\mathcal{C}_1)$ is infinite since $\mathcal{O}_1/\mathcal{C}_1$ maps surjectively onto $\underline{\underline{F}}_p[\varepsilon]$ (apply 3.4). Moreover Dennis and Stein ([8] pg 14) have shown that $NK_2(\underline{\underline{Z}}/4\underline{\underline{Z}})$ is an infinite elementary two group of countable rank, hence $NK_2(\mathcal{O}_2/\mathcal{C}_2)$ is likewise infinite.
To complete the proof we notice that $NK_1(\underline{\underline{Z}}\pi_i,\mathcal{C}_i) = NK_1(\mathcal{O}_i,\mathcal{C}_i)$ by 3.7 and furthermore that $NK_1(\underline{\underline{Z}}\pi_i, p\underline{\underline{Z}}\pi_i)$ maps surjectively to $NK_1(\underline{\underline{Z}}\pi_i,\mathcal{C}_i)$ by applying 3.5 to $p\underline{\underline{Z}}\pi_i \subset \mathcal{C}_i$. Thus we have constructed a chain of maps according to the scheme 27).

27)
$$\begin{tikzcd}
NK_1(\underline{\underline{Z}}\pi_{i+1}) & NK_1(\underline{\underline{Z}}\pi_i, p\underline{\underline{Z}}\pi_i) \arrow[l, "3.14"', "\simeq"] \arrow[d, "3.5"'] & \\
& NK_1(\underline{\underline{Z}}\pi_i,\mathcal{C}_i) \arrow[r, "3.7", "\simeq"'] & NK_2(\mathcal{O}_i;\mathcal{C}_i) \arrow[d, "{26),\ \text{with } i = 1}"] \\
& NK_2(\underline{\underline{F}}_p[\varepsilon]) & NK_2(F_p[\tau]) \arrow[l, two heads, "3.4"']
\end{tikzcd}$$

Combining 27) with theorem 3.13 we see a surjection

$NK_1(Z\pi_1) \longrightarrow NK_2(F_p[\tau])$ which gives the following corolary

Corollary 3.16 Every element of $NK_2(\mathbb{F}_p[\tau])$ is p-torsion

This does not appear to follow easily from the presentation for $K_2\, F_p[\tau,t]$ of Dennis & Stein. ([8]).

We can use theorems 2.5, 3.6 and 3.15 to show if π is cylic and $4 \nmid |\pi|$ then $NK_1\mathbb{Z}\pi = 0$ iff $|\pi|$ is square free. The troublesome restriction $4 \nmid |\pi|$ is due to the fact that theorem 3.15 does not apply to the cyclic group of order 4. At the moment there is no indication as to $NK_1(Z\pi)$, π cyclic $|\pi| = 4$ is nonzero or not. Also we have no indication as to the behavior of $NK_1\mathbb{Z}\pi$ for π and elementary p- group of rank ≥ 2. To examine this case it suffices by 3.6 to first look at the rank 2 case. We show next that the comportment of $NK_1(\mathbb{Z}\pi)$ for π elementary of rank 2 can be reduced to the study of a partial converse of 2.5.

Proposition 3.17 Let p be a rational prime and π a cyclic group of order p. Then there is an isomorphism

28) $$NK_1(\mathbb{Z}\pi\times\pi) \longrightarrow NK_1(\mathbb{Z}[\zeta]\pi,\ (1-\zeta)\ \mathbb{Z}[\zeta]\pi)$$

(here ζ is a primitive p^{th} root of unity)

Proof We consider the cartesian square 29) for $\mathbb{Z}\pi \times \pi$ and note that f_1 is a split epimorphism.

29) $$\begin{array}{ccc} \mathbb{Z}\pi \times \pi & \xrightarrow{f_2} & \mathbb{Z}[\zeta]\pi \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ \mathbb{Z}\pi & \xrightarrow[g_1]{} & \mathbb{F}_p\pi \end{array}$$.

It follows that the Nil exact sequence for f_1 reads

30) $$0 \to NK_1(\underline{\underline{Z}}\pi\times\pi, \ker f_1) \to NK_1(\underline{\underline{Z}}\pi\times\pi) \to NK_1(\underline{\underline{Z}}\pi)$$

by theorem 2.9 $NK_1(Z\pi) = 0$ hence,

31) $$NK_1(\underline{\underline{Z}}\pi\times\pi, \ker f_1) \simeq NK_1(\underline{\underline{Z}}\pi\times\pi).$$

By excision since f_2 is surjective we deduce

32) $$NK_1(\underline{\underline{Z}}\pi\times\pi, \ker f_1) \simeq NK_1(\underline{\underline{Z}}[\zeta]\pi, \ker g_2).$$

Since the kernel of g_2 is $(\zeta-1)\underline{\underline{Z}}[\zeta]\pi$ the result is clear. Notice that $NK_1 f_2$ is a surjective map, this is by considering the Mayer Vietoris sequence of 29) and using the fact that the far right hand map is trivial 33).

33) $$NK_1(\underline{\underline{Z}}\pi\times\pi) \to NK_1(\underline{\underline{Z}}\pi) \to NK_1(\underline{\underline{Z}}[\zeta]\pi) \to NK_1\underline{\underline{F}}_p\pi$$

Now the ring $Z[\zeta] = R_p\pi$ is the simplest example of the failure of the hypothesis in 2.5 that the order of the extension and that of the group be relatively prime. (Compare with example 3.9). We ask is $NK_1(R_p\pi)\neq 0$? We now turn to the proof of 1) in theorem B Above.

Theorem 3.18) Let π be finite abelian and assume that $|\pi_{(p)}| = p$ then $NK_1(\underline{\underline{Z}}\pi)_{(p)} = 0$.

Proof We can write $\underline{\underline{Z}}\pi = \underline{\underline{Z}}\pi^1 \times \pi_p$ where π_p is cyclic of order p and $(|\pi^1|, p) = 1$. This allows us to express $\underline{\underline{Z}}\pi$ as a cartesian product 34).

34) $$\begin{array}{ccc} \underline{\underline{Z}}\pi & \longrightarrow & \underline{\underline{Z}}\pi^1 \\ \downarrow & & \downarrow \\ \underline{\underline{Z}}[\zeta]\pi^1 & \longrightarrow & \underline{\underline{F}}_p\pi^1 \end{array}$$ ζ-a primitive pth root of unity.

From the fact that $(p, |\pi^1|) = 1$ we deduce semisimplicity for the ring $\underline{\underline{F}}_p\pi^1$, hence its regularity. The Nil exact sequence for 34), is therefore 35).

35) $0 \to NK_1(\mathbb{Z}\pi) \to NK_1(\mathbb{Z}\pi^1) \to NK_1(\mathbb{Z}[\zeta]\pi^1) \to 0$

From the fact that passing to p - torsion is exact we recover 36)

36) $0 \to NK_1(\mathbb{Z}\pi)_{(p)} \to NK_1(\mathbb{Z}\pi^1)_{(p)} \to NK_1(\mathbb{Z}[\zeta]\pi^1)_{(p)} \to 0$

Now for any Dedekind domain R with quotient field of characteristic 0 we have

37) $$|\pi|\mathcal{O} \subset R\pi$$

for any R order $\mathcal{O} \supset R\pi$. Hence it follows that $\mathbb{Z}\pi^1$, $Z[\zeta]\pi^1$ satisfy the hypotheses of 3.13 with B a maximal order and $m = |\pi^1|$. It follows that the groups $NK_1(\mathbb{Z}\pi^1)_{(p)}$ and $NK_1(Z[\zeta]\pi^1)_{(p)}$ are trivial $(p, |\pi^1|) = 1)$.

Concluding Remarks

As was remarked above the techniques developed here do not apply to the situation where π is cyclic of order 4 or elementary abelian of rank ≥ 2. If we could prove results analogous to those treated in theorem B part 2) we could prove theorem A with K_1 replacing K_0. To extend these results more knowledge of the functor K_2 must become available.

We also remark that the techniques employed here will not extend to analogous results for NK_2. This is because the Mayer-Vietoris sequence (1.4) does not extend to K_3. (For a discussion of this we refer the reader to R.W. Swan. Excision in Algebraic K-Theory. Journal of Pure and Applied Algebra 1, 1971)

Some interesting topics for further consideration are the following questions:

1) Does a surjection $\pi \rightarrow \pi^1$ of finite groups induce a surjection for $NK_1(\underline{\underline{Z}}\pi) \rightarrow NK_1(\underline{\underline{Z}}\pi^1)$?

2) Same question for NK_2. In particular what can be said in case π, π^1 are abelian?

It should be remarked here that an affirmative response to 2) or 3) below in the abelian case would allow us to deduce the results alluded to in the first paragraph.

3) Compute $NK_2(\underline{\underline{Z}}\pi)$ for cyclic groups . Is $NK_2(\underline{\underline{Z}}\pi)$ trivial for $|\pi|$ squarefree?

4) Extend the results of this thesis to arbitrary finite groups. If π is finite does $NK_1(\underline{\underline{Z}}\pi)=0$ for $|\pi|$ squarefree?

5) Find reasonable nessessary and sufficient conditions on a ring A so that $NK_1(A)=0$.

We remark here that the techniques of sections 1) and 2) along with the Hilbert Basis and Syzygy theorems allow us to assert

$$K_i(A[T]) = K_i(A) \qquad i = 0, 1$$

for any finitely generated free commutative monoid and any ring A satisbying the hypotheses of theorems 2.2 or 2.5.

6) To what extent does this hold in general i.e. does $NK_1(A)=0$ imply $NK_1(A[t])=0$?

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