MASLOV INDEX, LAGRANGIANS, MAPPING CLASS GROUPS AND TQFT

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Abstract. Given a mapping class f of an oriented surface Σ and a lagrangian λ in the first homology of Σ, we define an integer $n_\lambda(f) \pmod 4$. We use $n_\lambda(f)$ to describe a universal central extension of the mapping class group of Σ as an index-four subgroup of the extension constructed from the Maslov index of triples of lagrangian subspaces in the homology of the surface. We give two formulas for $n_\lambda(f)$. One is topological using surgery, the other is homological and builds on work of Turaev and work of Walker. Some applications to TQFT are discussed. They are based on the fact that our construction allows one to precisely describe how the phase factors that arise in the skein theory approach to TQFT-representations of the mapping class group depend on the choice of a lagrangian for the surface.

Contents

1. Introduction

The mapping class group Γ_g of a surface of genus g has a long history in low-dimensional topology. In this paper, we are concerned with central extensions of Γ_g, which have proved to be important in TQFT. It follows from Harer's work [H] that Γ_g has a universal central extension by $\mathbb{Z}$, for $g \geq 5$ (later works improve this to $g \geq 4$). The cohomology class of such an extension is a generator of $H^2(\Gamma_g;\mathbb{Z})$ (this group is isomorphic to $\mathbb{Z}$ for $g \geq 3$). One way to obtain explicit 2-cocycles representing cohomology classes of central extensions of Γ_g is to pull back cocycles of the symplectic group $\mathrm{Sp}(g,\mathbb{Z})$ via the map $\Gamma_g \to \mathrm{Sp}(g,\mathbb{Z})$ which sends a mapping

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class $f \in \Gamma_g$ to the induced map $f_\star$ on the homology of the surface. The most prominent such 2-cocycle among topologists is probably the signature cocycle for $\mathrm{Sp}(g,\mathbb{Z})$, defined by Meyer [M] using signatures of certain 4-manifolds which fiber over a disk with two holes. We will use τ to denote the pull-back of Meyer's cocycle to the mapping class group Γ_g. Meyer's work implies that the cohomology class $[\tau]$ is divisible by four, and the class $[\tau]/4$ is a generator of $H^2(\Gamma_g;\mathbb{Z})$. However, τ itself is not divisible by 4, and Meyer did not give an explicit $\mathbb{Z}$-valued cocycle representing $[\tau]/4$. This was done by Turaev [T1, T2], who had independently studied the signature cocycle from a different point of view. Turaev showed how to modify τ by the coboundary of a certain explicit 1-cochain to find a cocycle which is divisible by four. Thus, Turaev's work gives an explicit cocycle for a universal central extension of Γ_g.

Renewed interest in these questions was sparked by Atiyah [A], who pointed out that the signature cocycle was closely related to the problem of resolving anomalies in TQFT. Anomalies are responsible for the fact that TQFT-representations of mapping class group are often only projective representations. Resolving the anomalies means replacing these projective representations by linear representations of appropriate central extensions of the mapping class group. In [A], Atiyah suggested the notion of 2-framings to resolve anomalies. Blanchet, Habegger, Masbaum and Vogel [BHMV2] used the notion of p_1-structures to resolve anomalies in their construction of TQFT's from the skein theory of the Kauffman bracket. The projective factors arising in the skein theoretical construction of TQFT were computed explicitly in Masbaum-Roberts [MR].

The central extensions of Γ_g considered in the present paper are constructed using yet another approach to resolving anomalies which was pioneered by Walker [W], and further developed by Turaev [T3]. As far as the mapping class group is concerned, this method depends on fixing a lagrangian subspace λ of the first rational homology of the surface. One then uses the Maslov index of triples of lagrangian subspaces to define a central extension of Γ_g. Let us denote this extension by $\widetilde{\Gamma}_g$. The group $\widetilde{\Gamma}_g$ is thus given explicitly as the set of pairs $\{(f,n)|f \in \Gamma_g, n \in \mathbb{Z}\}$, with multiplication defined by a certain cocycle m_λ which we call the Maslov cocycle. This cocycle is also known as the Shale-Weil cocycle, which is discussed for instance in [LV]. See formula (2) in Section 3 for an explicit formula for the multiplication in $\widetilde{\Gamma}_g$, and formula (15) in Section 6 for our definition of m_λ. (Note that in the body of the paper, we denote the extended mapping class group $\widetilde{\Gamma}_g$ by $\widetilde{\Gamma}(\Sigma)$, where Σ stands for the 'extended' surface consisting of a surface together with a fixed lagrangian; see the beginning of Section 3 for more details.)

In contrast with the signature cocycle τ, the Maslov cocycle depends on the chosen lagrangian λ. But it turns out that in cohomology, one has $[m_\lambda] = -[\tau]$. Thus the class $[m_\lambda]/4$ corresponds to an index-four subgroup of $\widetilde{\Gamma}_g$ which we denote by $\widetilde{\Gamma}_g^{++}$. If $g \geq 4$, $\widetilde{\Gamma}_g^{++}$ is a universal central extension Γ_g. The main aim of the present paper is to explain how one can get one's hands on explicit elements of this group $\widetilde{\Gamma}_g^{++}$, and to understand the role played by the chosen lagrangian λ in this description.

Our first result in this direction is Theorem 4.2. It associates to every word $\mathfrak{w}$ in Dehn twists representing a given mapping class f an element $(f, n_\lambda(\mathfrak{w})) \in \widetilde{\Gamma}_g^{++}$. The integer $n_\lambda(\mathfrak{w})$ is defined in terms of the signature of the linking matrix of a framed link constructed from the word $\mathfrak{w}$ and the lagrangian λ. This framed link is really a surgery description of a related 3-manifold. So we call this the surgery description of $\widetilde{\Gamma}_g^{++}$. Our construction here is somewhat similar to the work of Roberts and one of us in [MR], but the context is different, as there were no lagrangians in [MR]. Also, the framed link we are using is different from the one used in [MR]. The framed link used in [MR] would be appropriate for our purposes only for words $\mathfrak{w}$ representing the identity mapping class, but not in general.

Our working definition of $\widetilde{\Gamma}_g^{++}$ (see Definition 3.8) is as the subgroup of $\widetilde{\Gamma}_g$ generated by certain specific lifts of Dehn twists to $\widetilde{\Gamma}_g$. The fact that $\widetilde{\Gamma}_g^{++}$ has index four in $\widetilde{\Gamma}_g$ is not obvious from this definition. This fact is stated in Theorem 3.9 and proved in Section 4. A similar statement was proved in [MR], and we use [MR] crucially in our proof here as well. Once we know this fact, it is clear that the possible values of $n_\lambda(\mathfrak{w})$, for words $\mathfrak{w}$ representing a fixed mapping class f, form a coset of $4\mathbb{Z}$ in $\mathbb{Z}$. This means that there is a function n_λ on Γ_g with values in $\mathbb{Z}/4$ such that $(f,n) \in \widetilde{\Gamma}_g^{++}$ if and only if $n \equiv n_\lambda(f) \pmod 4$, see Definition 3.10.

In our second description of $\widetilde{\Gamma}_g^{++}$, we then give a formula for $n_\lambda(f) \in \mathbb{Z}/4$ which does not require writing f as a product of Dehn twists. It is computed directly from the lagrangian λ and the action of f on the homology of the surface. See Theorem 5.6 for the precise statement. Our formula uses Turaev's 1-cochain from [T1, T2], but adds to it a term which explicitly depends on the lagrangian λ. It is remarkable that Turaev's cochain is defined using a certain non-symmetric bilinear form depending only on f, while our additional term is the signature of this same form restricted to a subspace on which the form is symmetric (but the subspace depends on the lagrangian). The proof of our formula uses the surgery description and a formula of Walker [W, p. 124] relating the signature cocycle to the Maslov cocycle. We remark that Walker's formula is in an unfinished manuscript, which does not claim to get the signs right. We state a version of his formula, in terms of our definitions and conventions, as Theorem 6.10, and give a detailed version of the proof Walker outlines. Also, Turaev defined his version of the signature cocycle in a purely algebraic fashion, and he did not give the precise relationship with Meyer's definition. In fact, Turaev's cocycle turns out to be equal to $-\tau$, see Prop. 6.5. Since we are proving a congruence modulo four (and not just modulo two), getting the signs right is important for us, so we have tried to deal with these sign issues in some detail.

The preceding results all extend to the mapping class group of a surface with boundary. The (small) modifications required to do so are explained in Section 7. We also explain briefly in Section 8 how one sees that $\widetilde{\Gamma}_g^{++}$ is a universal central extension.

The remainder of the paper is devoted to applications of our results to TQFT. As already said, we use Walker's [W] and Turaev's [T3] approach to TQFT, where one consider surfaces equipped with the extra structure of a lagrangian subspace of their first homology, and 3-manifolds equipped with an integer weight. These are called extended manifolds, and the resulting extended cobordism category is used to resolve the anomalies that arise in TQFT. We believe that the skein theory approach of [BHMV2] modified by substituting extended manifolds for manifolds with p_1-structures is the most concrete and computable approach to the TQFTs associated to $SU(2)$, and $SO(3)$. The reason for this precision is that a lagrangian subspace may be specified algebraically while a p_1-structure is harder to specify.

In Section 9, we explain how this works in practice for the mapping class group representations. See for instance, Theorem 9.2, where we state precisely how the action of an element (f, n) of the extended mapping class group $\widetilde{\Gamma}_g$ on the TQFT-module associated to the surface Σ depends on the chosen lagrangian λ. We then use this to do some explicit computations (see Prop. 9.7) which were used in [GM2]. The beginning of Section 9 is written so as to provide a further and more detailed introduction to the TQFT-aspects of our results.

In the last section, we briefly consider the integral TQFT that we have been studying in [G, GM1, GM2] using the precision afforded by using extended manifolds. In Corollary 10.4, we show that the representations coming from integral TQFT when restricted to $\widetilde{\Gamma}_g^{++}$ induce modular representations of the ordinary mapping class group Γ_g.

2. Maslov index, extended manifolds and extended surgery

Extended surfaces and 3-manifolds were introduced by Walker [W] and further developed by Turaev [T3]. We begin by briefly describing these notions to fix our conventions, and sketch the background.

Let V be a rational vector space with a nonsingular skew-symmetric form $\cdot$: $V \times V \to \mathbb{Q}$. A subspace $\lambda \subset V$ is called lagrangian if $\lambda = \lambda^\perp$ where $\lambda^\perp = \{x \in V \,|\, x \cdot y = 0,\ \forall y \in \lambda\}$. It is easy to see that λ is lagrangian if and only if $\lambda \subset \lambda^\perp$ and λ has dimension $(1/2)\dim(V)$. Recall the Maslov index of an ordered triple of lagrangians $\lambda_1, \lambda_2,\ \lambda_3$ in V. The Maslov index $\mu(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}$ is defined to be the signature of the bilinear symmetric form $\odot$ on $(\lambda_1 + \lambda_2) \cap \lambda_3$ defined by $(a_1 + a_2) \odot (b_1 + b_2) = a_2 \cdot b_1$. We will need the following well-known property of $\mu(\lambda_1, \lambda_2, \lambda_3)$.

Lemma 2.1. *The Maslov index changes sign under an odd permutation of the three lagrangians. In particular,* $\mu(\lambda_1, \lambda_2, \lambda_3) = 0$ *if two of the lagrangians are the same.*

Recall that the first homology of a closed oriented 2-dimensional manifold Σ has a skew-symmetric intersection form $\cdot : H_1(\Sigma; \mathbb{Q}) \times H_1(\Sigma; \mathbb{Q}) \to \mathbb{Q}$. By a lagrangian of Σ, we mean a lagrangian for $H_1(\Sigma; \mathbb{Q})$ with this pairing.

An extended surface Σ is a closed oriented 2-dimensional manifold equipped with a lagrangian subspace $\lambda(\Sigma) \subset H_1(\Sigma; \mathbb{Q})$. It is clear how to take the disjoint union of extended surfaces.

An extended 3-manifold N is a compact oriented 3-dimensional manifold equipped with a weight $w(N) \in \mathbb{Z}$, and whose oriented boundary ∂M has been given the structure of an extended surface with a lagrangian $\lambda(\partial M)$. In this case ∂M also has a lagrangian given by kernel(i_*), where $i : \partial M \to M$ is the inclusion. We denote this lagrangian by $\lambda_M(\partial M)$. We insist that $\lambda(\partial M)$ could be chosen arbitrarily and will usually be different from $\lambda_M(\partial M)$.

If M is an extended 3-manifold and Σ is a connected component of ∂M, then $\lambda(\partial M) \cap H_1(\Sigma; \mathbb{Q})$ may or may not be a lagrangian for Σ. If it is a lagrangian for Σ, we denote this lagrangian by $\lambda(\Sigma)$ and we call Σ a boundary surface of the extended 3-manifold M.

Extended 3-manifolds can be glued along boundary surfaces. To describe this 'extended' gluing, we need one more notation. First, observe that if Σ is a boundary surface of M, then $\Sigma' = \partial M \setminus \Sigma$ is also a boundary surface, and ∂M is the disjoint union of Σ and Σ' as extended surfaces. Now let i_Σ and $i_{\Sigma'}$ denote the inclusions of Σ and Σ' into M, and define $\lambda_M(\Sigma)$ to be $i_\Sigma^{-1}\left(i_{\Sigma'}(\lambda(\Sigma'))\right)$. In other words, we restrict the given lagrangian $\lambda(\partial M)$ to Σ', and then 'transport' it over to Σ, using M. Note that if Σ is the whole boundary of M, so that $\Sigma' = \emptyset$, this agrees with the earlier definition of $\lambda_M(\partial M)$. As before, we insist that $\lambda_M(\Sigma)$ will in general be different from $\lambda(\Sigma)$.

Throughout this paper, we denote orientation reversal by an overbar. If Σ is an extended surface, $\overline{\Sigma}$ denotes the same surface with opposite orientation and with the same lagrangian $\lambda(\overline{\Sigma}) = \lambda(\Sigma)$. If M is an extended 3-manifold, $\overline{M}$ denotes the same manifold with opposite orientation and weight $w(\overline{M}) = -w(M)$.

We can now spell out the gluing formula. Let M and M' be two extended 3-manifolds and assume that Σ is a boundary surface of M and $\overline{\Sigma}$ is a boundary surface of M'. Then we may glue M and M' (by the orientation reversing identity map from Σ to $\overline{\Sigma}$) together to form a new extended 3-manifold $M \cup_\Sigma M'$. The weight of $M \cup_\Sigma M'$ is defined as

$$w(M \cup_\Sigma M') = w(M) + w(M') - \mu_\Sigma\left(\lambda_M(\Sigma), \lambda(\Sigma), \lambda_{M'}(\overline{\Sigma})\right). \tag{1}$$

We write μ_Σ to indicate that this Maslov index is to be computed using the intersection form of Σ, rather than $\overline{\Sigma}$. We note that $\lambda_{M'}(\overline{\Sigma})$ is a lagrangian for both Σ and $\overline{\Sigma}$ as the notion of lagrangian does not depend on the orientation of the surface. The minus sign in the above formula is needed to make Lemma 2.2 hold.

We would get the same number computing:

$$w(M' \cup_{\overline{\Sigma}} M) = w(M') + w(M) - \mu_{\overline{\Sigma}}\left(\lambda_{M'}(\overline{\Sigma}), \lambda(\overline{\Sigma}), \lambda_M(\Sigma)\right),$$

as the intersection pairings differ by a sign but an odd permutation of the lagrangians has been introduced.

Thus gluing of extended manifolds is 'commutative'. In other words, it does not matter whether we think we are gluing M to M' or M' to M. Gluing is also 'associative', meaning that if we have a collection of extended 3-manifolds that we wish to glue together along boundary surfaces, it does not matter in what order we do the gluing. This follows from the geometric interpretation of weights in terms of signatures of associated 4-manifolds given by Walker, as well as by the more algebraic approach given in Turaev's book.

We now wish to define the notion of extended surgery to an extended manifold M along a framed knot K in M. The resulting extended manifold will be denoted by M_K. Its underlying manifold is obtained by the usual surgery procedure: we use the framing and the orientation of M to identify a closed tubular neighborhood $\nu(K)$ of K with $\overline{S^1} \times D^2$; we then cut out the tubular neighborhood, and replace it with $D^2 \times S^1$. (Note that $\partial(\overline{S^1} \times D^2) = S^1 \times S^1 = \partial(D^2 \times S^1)$.) Now, to make M_K into an extended manifold, we do the same thing but use extended gluing, where the extended structure is as follows: We give $M \setminus \mathrm{Int}(\nu(K))$ the weight of M, the weight of $D^2 \times S^1$ is zero, and we equip $S^1 \times S^1$ with the lagrangian generated by the homology class of the meridian of the knot K, *i.e.*, $\mathrm{pt} \times S^1$. We remark that this is a natural choice for the lagrangian, as with this choice the result of extended gluing of $M \setminus \mathrm{Int}(\nu(K))$ with $\nu(K)$ (equipped with zero weight) is M with its original weight. (This follows from Lemma 2.1.)

Note that since K is a knot, we have $|w(M_K) - w(M)| \leq 1$, as the contribution from the Maslov index to the weight of M_K is computed from a symmetric bilinear form on a space of dimension at most one. If we have a framed link L in M, we may do a sequence of such extended surgeries or perform the surgeries all at once, and we would get the same result (by the above-mentioned 'associativity' of gluing). The resulting extended manifold is denoted M_L and is called extended surgery along L.

If L is a framed ordered oriented link in S^3, let $\sigma(L) = b_+(L) - b_-(L)$, where $b_\pm(L)$ is the number of positive (negative) eigenvalues (counted with multiplicity) of the *linking matrix* of L, that is, the symmetric integral matrix whose off-diagonal entries are the linking numbers of the components of L, and whose diagonal entries are the framings. The number $\sigma(L)$ is the signature of the linking matrix of L and should not be confused with what is usually called the signature of the *link* L in knot theory. Changing the order or the orientation of L does not effect $\sigma(L)$, $b_+(L)$, or $b_-(L)$.

The 4-manifold interpretation of weights [W] yields the following basic fact.

Lemma 2.2. *If S^3 is equipped with weight $w(S^3) = 0$, then $w((S^3)_L) = \sigma(L)$.*

3. Central extensions of mapping class groups

We will realize our central extensions of the mapping class group as subgroups of a certain extended cobordism category $\mathcal{C}$. The objects of $\mathcal{C}$ are extended surfaces. A morphism in $\mathcal{C}$ from Σ to Σ' is given by an extended cobordism, that is, an extended 3-manifold M whose boundary has been partitioned into the disjoint union of two boundary surfaces, one of which is identified with Σ by an orientation reversing diffeomorphism, and the other is identified with Σ' by an orientation preserving diffeomorphism. We denote such a cobordism by $M : \Sigma \rightsquigarrow \Sigma'$. We refer to Σ as the source and Σ' as the target of the cobordism. If we also have another cobordism $M' : \Sigma' \rightsquigarrow \Sigma''$, we can form $M' \circ M : \Sigma \rightsquigarrow \Sigma''$ by extended gluing M to M' along Σ'. Thus, $M' \circ M$ means *first M, then M'.* This convention is needed to make formula (2) below hold.

Two extended cobordisms from Σ to Σ' are considered equivalent if they have the same weight and if there is an orientation preserving diffeomorphism between them which preserves their boundary identifications. Composition of extended cobordisms is associative (on equivalence classes). Therefore we define the morphisms of $\mathcal{C}$ from Σ to Σ' to be equivalence classes of extended cobordisms. However, from now on we will treat equivalent cobordisms as if they are identical. When it should cause no confusion, we will act as if the boundary identifications of a cobordism are identity maps.

Sometimes we will need to discuss extended manifolds whose extended structure we have forgotten, then we will denote them by $\underline{M}$, $\underline{\Sigma}$ *etc.* Thus, forgetting the extended structure will be denoted by an underbar. We have a forgetful functor $\mathcal{C} \to \underline{\mathcal{C}}$, where $\underline{\mathcal{C}}$ denotes the usual cobordism category, with composition given by the usual gluing.

We now set out to define the extended mapping class group $\widetilde{\Gamma}(\Sigma)$ of a closed oriented surface equipped with a fixed lagrangian $\lambda(\Sigma)$. *Here and whenever we discuss a mapping class group of a surface in this paper, we assume that the surface is connected.* First of all, we denote by $\Gamma(\Sigma)$ the ordinary mapping class group of the underlying surface $\underline{\Sigma}$. (The group $\Gamma(\Sigma)$ should perhaps be denoted by $\Gamma(\underline{\Sigma})$, but we find this notation too clumsy.) Thus, $\Gamma(\Sigma)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\underline{\Sigma}$. Abusing notation, we will write f for a diffeomorphism, and its isotopy class.

If $f \in \Gamma(\Sigma)$ and $n \in \mathbb{Z}$, we let $C(f, n)$ denote the extended cobordism given by the mapping cylinder of f with weight n, where both the source and target are Σ

equipped with the lagrangian $\lambda(\Sigma)$. We call $C(f,n)$ an extended mapping cylinder. It is a morphism of $\mathcal{C}$. Its underlying cobordism is the usual mapping cylinder of f, that is, the cobordism formed from $\mathrm{I} \times \underline{\Sigma}$ by identifying $\overline{\{0\} \times \underline{\Sigma}}$ with the source surface $\underline{\Sigma}$ via the identity (which is in this case is orientation reversing) and identifying $\{1\} \times \underline{\Sigma}$ with the target surface $\underline{\Sigma}$ via f.

It follows from (1) that composition of extended mapping cylinders is given by

$$\begin{aligned} C(g,n) \circ C(f,m) &= C(g \circ f, n+m - \mu\left(f_*\lambda(\Sigma), \lambda(\Sigma), g_*^{-1}\lambda(\Sigma)\right)) \\ &= C(g \circ f, n+m + \mu\left(\lambda(\Sigma), g_*\lambda(\Sigma), (g\circ f)_*\lambda(\Sigma)\right)) \end{aligned} \tag{2}$$

Definition 3.1. *(Walker) The extended mapping class group is*

$$\widetilde{\Gamma}(\Sigma) = \{C(f,n) \mid f \in \Gamma(\Sigma),\ n \in \mathbb{Z}\}$$

with multiplication given by (2).

We have a short exact sequence of groups (see Remark 3.2 below):

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma}(\Sigma) \longrightarrow \Gamma(\Sigma) \longrightarrow 0.$$

The map $\widetilde{\Gamma}(\Sigma) \to \Gamma(\Sigma)$ is given by $C(f,n) \mapsto f$. This is a central extension. The kernel is generated by $C(\mathrm{Id}_\Sigma, 1) \in \widetilde{\Gamma}(\Sigma)$. We denote this central generator by W.

Remark 3.2. In Definition 3.1, we realize $\widetilde{\Gamma}(\Sigma)$ as a subset of the endomorphisms of Σ in the extended cobordism category $\mathcal{C}$. But notice that the extended mapping cylinder $C(f,n)$ (which we view as an equivalence class of morphisms in $\mathcal{C}$) determines $(f,n) \in \Gamma(\Sigma) \times \mathbb{Z}$, because of the following fact: One has that $f = g$ in $\Gamma(\Sigma)$ if and only if the (ordinary) mapping cylinders of f and g are equivalent as morphisms of $\underline{\mathcal{C}}$. (For the 'if' part, one can use a result of Baer [FM, Theorem(1.9)].) In later sections, we will therefore think of $\widetilde{\Gamma}(\Sigma)$ as the set of pairs $(f,n) \in \Gamma(\Sigma) \times \mathbb{Z}$ with multiplication given by (2). But in this and the next section, it will be convenient to think of elements of $\widetilde{\Gamma}(\Sigma)$ as extended mapping cylinders.

Remark 3.3. The multiplication in (2) depends on $\lambda(\Sigma)$. Nevertheless, if Σ and Σ' have the same underlying surface $\underline{\Sigma} = \underline{\Sigma}'$, then $\widetilde{\Gamma}(\Sigma)$ and $\widetilde{\Gamma}(\Sigma')$ are canonically isomorphic. The isomorphism is given by conjugating by $\mathrm{I} \times \underline{\Sigma}$ with identity boundary identifications, but with the source and target being respectively Σ and Σ'.

Definition 3.4. *(Gilmer* [G]*)*

$$\widetilde{\Gamma}(\Sigma)^+ = \{C(f,n) \mid f \in \Gamma(\Sigma),\ n \equiv \mathrm{genus}(\Sigma) + \dim\bigl(\lambda(\Sigma) \cap f_*(\lambda(\Sigma))\bigr) \pmod 2\}$$

Theorem 3.5. [G, §7] *The set $\widetilde{\Gamma}(\Sigma)^+$ is an index two subgroup of $\widetilde{\Gamma}(\Sigma)$.*

Proof. (Sketch) A subcategory of $\mathcal{C}$ called the even subcategory is constructed in [G, §7]. The even subcategory intersected with $\widetilde{\Gamma}(\Sigma)$ is $\widetilde{\Gamma}(\Sigma)^+$, as defined above. Thus $\widetilde{\Gamma}(\Sigma)^+$ is a subgroup of $\widetilde{\Gamma}(\Sigma)$. □

Remark 3.6. This theorem can also be obtained as a consequence of results shown in the present paper, see Remark 5.10.

The main object of the paper is the study of a further subgroup $\widetilde{\Gamma}(\Sigma)^{++}$, of which we will give several descriptions. The first description (which will be our definition) is as follows. Recall that the mapping class group $\Gamma(\Sigma)$ is generated by

Dehn twists. If α is an unoriented simple closed curve in Σ, let $D(\alpha)$ denote the Dehn twist along α. Our Dehn twists are defined as in Birman [Bi] (*i.e.,* they 'turn right'); this is the opposite convention from the one in [FM]. Let α_- denote the framed knot in $\mathrm{I} \times \Sigma$ given by $\frac{1}{2} \times \alpha$ with framing -1 with respect to the 'surface framing' that this knot has as a subset of the surface $\frac{1}{2} \times \Sigma$.

Lemma 3.7. *Let α be a simple closed curve in Σ. Let $C(\alpha) \in \widetilde{\Gamma}(\Sigma)$ be the result of extended surgery along the framed knot α_- on the identity cobordism $\mathrm{I} \times \Sigma$ (with weight $w(\mathrm{I} \times \Sigma) = 0$, and both ends equipped with $\lambda(\Sigma)$.) Then*

(i) the underlying cobordism is the mapping cylinder of the Dehn twist $D(\alpha)$.

(ii) Moreover, the weight of $C(\alpha)$ is given by

$$w(C(\alpha)) = \begin{cases} -1 & \text{if } [\alpha] \in \lambda(\Sigma) \\ 0 & \text{if } [\alpha] \notin \lambda(\Sigma) \end{cases} \tag{3}$$

Here, $[\alpha] \in H_1(\Sigma; \mathbb{Q})$ denotes the homology class of α with an arbitrary orientation. Note that in the formulae above, replacing $[\alpha]$ by $-[\alpha]$ has no effect.

Proof. Statement (i) of the lemma is well-known, see *e.g.* [MR]. Statement (ii) will be proven in Section 4. □

It will be convenient to shift the weight of $C(\alpha)$ by one. Therefore we define $W(\alpha) = W \circ C(\alpha) \in \widetilde{\Gamma}(\Sigma)$, so that

$$w(W(\alpha)) = \begin{cases} 0 & \text{if } [\alpha] \in \lambda(\Sigma) \\ 1 & \text{if } [\alpha] \notin \lambda(\Sigma) \end{cases}$$

Definition 3.8. *The group $\widetilde{\Gamma}(\Sigma)^{++}$ is defined to be the subgroup of $\widetilde{\Gamma}(\Sigma)$ generated by the $W(\alpha)$, for all (isotopy classes of) simple closed curves α on Σ.*

Note that if the curve α bounds a disk, then $C(\alpha) = W^{-1}$ but $W(\alpha) = 1$ is the identity element of $\widetilde{\Gamma}(\Sigma)^{++}$.

Theorem 3.9. *The group $\widetilde{\Gamma}(\Sigma)^{++}$ is a central extension of the mapping class group $\Gamma(\Sigma)$. It fits into the following commutative diagram with exact rows, where the vertical maps are all inclusions*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 4\mathbb{Z} & \longrightarrow & \widetilde{\Gamma}(\Sigma)^{++} & \longrightarrow & \Gamma(\Sigma) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & 2\mathbb{Z} & \longrightarrow & \widetilde{\Gamma}(\Sigma)^{+} & \longrightarrow & \Gamma(\Sigma) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\Gamma}(\Sigma) & \longrightarrow & \Gamma(\Sigma) & \longrightarrow & 0.
\end{array}$$

The proof of Theorem 3.9 will be given in Section 4. Note that since $\Gamma(\Sigma)$ is generated by Dehn twists, it is clear from Lemma 3.7 that $\widetilde{\Gamma}(\Sigma)^{++}$ maps onto $\Gamma(\Sigma)$. The non-trivial statement is that $\widetilde{\Gamma}(\Sigma)^{++}$ is an index two subgroup of $\widetilde{\Gamma}(\Sigma)^{+}$, and (hence) an index 4 subgroup of $\widetilde{\Gamma}(\Sigma)$. In other words, the kernel of $\widetilde{\Gamma}(\Sigma)^{++} \to \Gamma(\Sigma)$ is generated by W^4.

Assuming Theorem 3.9, the following definition makes sense.

Definition 3.10. *Let* λ *denote the lagrangian* $\lambda(\Sigma)$. *For* $f \in \Gamma(\Sigma)$, *we define* $n_\lambda(f) \in \mathbb{Z}/4$ *to be the unique integer* (mod 4) *so that*

$$\widetilde{\Gamma}(\Sigma)^{++} = \{C(f,n) \mid f \in \Gamma(\Sigma),\ n \equiv n_\lambda(f) \pmod 4\}$$

In Section 4, together with the proof of Theorem 3.9, we will give an explicit description of the elements of $\widetilde{\Gamma}(\Sigma)^{++}$ lying over a given mapping class $f \in \Gamma(\Sigma)$. This gives in particular a surgery formula for $n_\lambda(f)$. In Section 5, we will then give a different and completely algebraic description of $n_\lambda(f)$, from which it will follow that $n_\lambda(f)$ only depends on λ and the action of f on the homology of the surface.

Remark 3.11. In the above, we assumed that Σ is closed. But everything generalizes easily to mapping class groups of surfaces with boundary. This will be explained in Section 7.

Remark 3.12. If the genus of Σ is at least four, then $\widetilde{\Gamma}(\Sigma)^{++}$ is a universal central extension of $\Gamma(\Sigma)$. This will be explained in Section 8.

Remark 3.13. The three extensions $\widetilde{\Gamma}(\Sigma)^{++} \subset \widetilde{\Gamma}(\Sigma)^{+} \subset \widetilde{\Gamma}(\Sigma)$ of $\Gamma(\Sigma)$ are abstractly isomorphic to the extensions $\widetilde{\Gamma}_1 \subset \widetilde{\Gamma}_2 \subset \widetilde{\Gamma}_4$ constructed in [MR]. But our point of view here is different, as there were no lagrangians in [MR].

4. Surgery description of $\widetilde{\Gamma}(\Sigma)^{++}$

Let Σ be an extended surface with lagrangian $\lambda = \lambda(\Sigma)$. We can embed Σ in S^3 so that it is the boundary of a handlebody $\mathcal{H}$ in S^3 such that $\lambda(\Sigma) =$ kernel $(H_1(\Sigma;\mathbb{Q}) \to H_1(\mathcal{H};\mathbb{Q}))$ and such that the complement $S^3 \setminus \mathrm{Int}(\mathcal{H})$ is another handlebody $\mathcal{H}'$. If these conditions are satisfied, we say that Σ is *well placed* with respect to λ.

Proof of Lemma 3.7. Let α be a simple closed curve on Σ. It remains to prove formula (3) giving the weight of the extended cobordism $C(\alpha)$. To do this, assume Σ is well placed in S^3 with respect to λ. Decompose $S^3 = \mathcal{H} \cup (\mathrm{I} \times \Sigma) \cup \mathcal{H}'$ where $\mathrm{I} \times \Sigma$ is a collar on the boundary. Each piece of this decomposition should have weight zero. Then it follows from Lemma 2.1 that the weight $w(S^3) = 0$ as well.

Let g denote the genus of Σ. Choose simple closed oriented curves $m_1, \dots m_g$, $\ell_1, \dots \ell_g$ such that each $m_i \cap \ell_i$ consists of a point (and $m_i \cdot \ell_i = 1$ for the given orientation of Σ) but the m_i and ℓ_j are otherwise disjoint. Moreover the m_i should bound disjoint disks in $\mathcal{H}$, and the ℓ_j should bound disjoint disks in $\mathcal{H}' = S^3 \setminus \mathrm{Int}\,\mathcal{H}$. We refer to the m_i as the meridians of $\mathcal{H}$. See Figure 1.

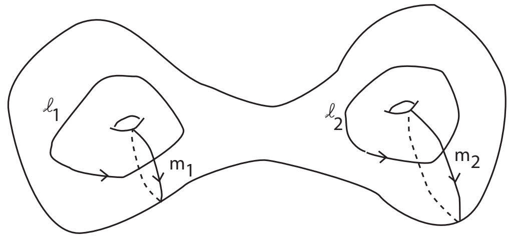

Figure 1. m_1, l_1, m_2, l_2 on Σ of genus two. Suppose in the figures below that $\lambda(\Sigma)$ is spanned by m_1 and m_2.

Consider the zero-framed unlink U with g components obtained by pushing the meridians $m_1, \dots m_g$ of $\mathcal{H}$ up into $\mathcal{H}'$ in S^3. Let $L^0(\alpha)$ be the $(g+1)$-component link in S^3 whose first component is α_- sitting in $\mathrm{I} \times \Sigma \subset S^3$, and whose later components are given by U.

Lemma 4.1. *We have* $w(C(\alpha)) = \sigma(L^0(\alpha))$.

Assuming Lemma 4.1, the proof of Lemma 3.7 proceeds as follows. Suppose the homology class $[\alpha] = \sum_{i=1}^{g}(a_i[m_i] + b_i[\ell_i])$, with integers a_i and b_i $(i = 1, \dots, g)$. (Here, we have picked an arbitrary orientation of the curve α.) Let α' be a parallel copy of α on one of the layers $\{t\} \times \Sigma$ (for $t \neq \frac{1}{2}$). Then the linking number $\mathrm{Lk}(\alpha, \alpha') = \sum_i a_i b_i$. Thus the framing of the first component of $L^0(\alpha)$ is $-1 + \sum_i a_i b_i$. The linking number of the first component of $L^0(\alpha)$ and the $(i+1)$th component of L is b_i. The lower right $g \times g$ block of the linking matrix of $L^0(\alpha)$ consists of zeros. Note that by construction, the lagrangian λ is the span of the meridians m_i. If $[\alpha] \in \lambda$, then all the b_i's are zero, and $\sigma(L^0(\alpha)) = -1$. If $[\alpha] \notin \lambda$, then some $b_i \neq 0$, and $\sigma(L^0(\alpha)) = 0$. This proves Lemma 3.7. □

See Figures 2 and 3 for an example.

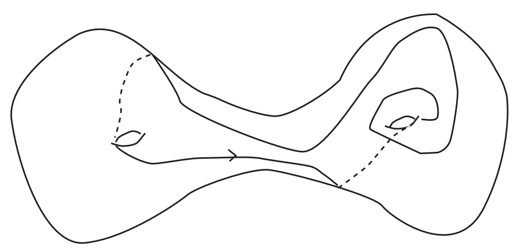

FIGURE 2. A curve α with $[\alpha] = m_1 + \ell_1 + m_2 + 2\ell_2$. One has $\mathrm{Lk}(\alpha, \alpha') = 3$, so that the framing specified by the surface is the '3-framing' in this case.

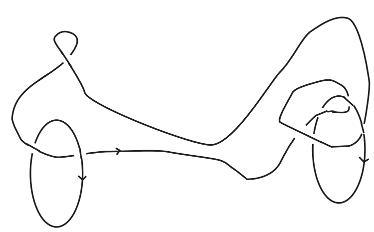

FIGURE 3. The framed link $L^0(\alpha)$ indicated with the 'blackboard framing' convention. The framing of α_- is 2. One has $\sigma(L^0(\alpha)) = 0$.

It remains to give the

Proof of Lemma 4.1. Let Y be the result of extended gluing $\mathcal{H} \cup C(\alpha) \cup \overline{\mathcal{H}}$, where the source surface of the extended mapping cylinder $C(\alpha)$ is glued to the boundary of $\mathcal{H}$, and the target surface of $C(\alpha)$ is glued to the boundary of $\overline{\mathcal{H}}$. Since $w(\mathcal{H}) = 0$, we also have $w(\overline{\mathcal{H}}) = 0$, and hence

$$\begin{aligned} w(Y) &= w(\mathcal{H}) + w(C(\alpha)) + w(\overline{\mathcal{H}}) + \mu(\lambda, \lambda, D(\alpha)_\star^{-1}(\lambda)) + \mu(D(\alpha)_\star(\lambda), \lambda, \lambda) \\ &= w(C(\alpha)) \ . \end{aligned}$$

Here, the two Maslov index terms are zero, because in both cases two of the three lagrangians coincide (see Lemma 2.1).

Let $(\mathcal{H}')_U$ denote the result of extended surgery on $\mathcal{H}'$ along the zero-framed unlink U. Then extended gluing $\mathcal{H} \cup (\mathcal{H}')_U$ gives $(S^3)_U$, which is $\#^g S^1 \times S^2$ (the connected sum of g copies of $S^1 \times S^2$) with weight zero (use Lemma 2.2 for the weight computation). On the other hand, extended gluing $\mathcal{H} \cup \overline{\mathcal{H}}$ is also $\#^g S^1 \times S^2$ with weight zero, as follows from a Maslov index computation as the one for $w(Y)$ given above. This shows that the standard identification of $(\mathcal{H}')_U$ with $\overline{\mathcal{H}}$ holds true as extended manifolds. Thus

$$\begin{aligned} w(C(\alpha)) &= w(\mathcal{H} \cup C(\alpha) \cup \overline{\mathcal{H}}) = w(\mathcal{H} \cup C(\alpha) \cup (\mathcal{H}')_U) = w((S^3)_{L^0(\alpha)}) \\ &= \sigma(L^0(\alpha)) \end{aligned}$$

where we have again used Lemma 2.2 in the last equality. This completes the proof of Lemma 4.1. □

We now generalize the above construction of the framed link $L^0(\alpha)$ to give an explicit description of the elements of $\widetilde{\Gamma}(\Sigma)^{++}$ mapping to a given $f \in \Gamma(\Sigma)$. Consider a word $\mathfrak{w} = \prod_{i=1}^n {\alpha_i}^{\varepsilon_i}$, where $\varepsilon_i = \pm 1$, and the α_i are unoriented simple closed curves in Σ. Let $D(\mathfrak{w}) = \prod_{i=1}^n D(\alpha_i)^{\varepsilon_i} \in \Gamma(\Sigma)$. Since Dehn twists generate $\Gamma(\Sigma)$, every mapping class f is of the form $D(\mathfrak{w})$ for some word $\mathfrak{w}$.

We construct an n-component framed link $L(\mathfrak{w})$ and an $(n+g)$-component framed link $L^0(\mathfrak{w})$ in S^3. Both links depend on a choice of a decomposition $S^3 = \mathcal{H} \cup (\mathrm{I} \times \Sigma) \cup \mathcal{H}'$, as in the proof of Lemma 3.7. We choose this decomposition so that Σ is well placed with respect to the lagrangian $\lambda(\Sigma)$. Recall that this simply means that $\lambda(\Sigma)$ is the kernel of $H_1(\Sigma; \mathbb{Q}) \to H_1(\mathcal{H}; \mathbb{Q})$.

The framed link $L(\mathfrak{w})$ lies in $\mathrm{I} \times \Sigma \subset S^3$. It is constructed, as in [MR, 2.7], by layering $-\varepsilon_i$-framed (with respect to the surface framing) copies of α_i, starting with α_n near $\{0\} \times \Sigma$, then α_{n-1} and so on, moving outward until α_1 is inserted near $\{1\} \times \Sigma$. [1] (Here, the orientation of the individual link components is chosen arbitrarily. It will not play a role in what follows.) The framed link $L^0(\mathfrak{w})$ is $L(\mathfrak{w}) \cup U \subset S^3$, where U is the unlink consisting of the meridians of $\mathcal{H}$ pushed into $\mathcal{H}'$, as in the proof of Lemma 3.7. For example, if $\mathfrak{w} = \alpha$, then $L(\mathfrak{w})$ is simply α_- sitting in $\mathrm{I} \times \Sigma \subset S^3$, and $L^0(\mathfrak{w})$ is $L^0(\alpha)$.

Recall that $\widetilde{\Gamma}(\Sigma)^{++}$ is generated, as a subgroup of $\widetilde{\Gamma}(\Sigma)$, by the extended mapping cylinders $W(\alpha) = W \circ C(\alpha)$, where $W = C(\mathrm{Id}_\Sigma, 1)$ is the central generator. Given a word $\mathfrak{w} = \prod_{i=1}^n {\alpha_i}^{\varepsilon_i}$, we denote its exponent sum by $e(\mathfrak{w}) = \sum_{i=1}^n \varepsilon_i$, and we let $W(\mathfrak{w}) \in \widetilde{\Gamma}(\Sigma)^{++}$ be the product of the $W(\alpha_i)^{\varepsilon_i}$:

$$W(\mathfrak{w}) = \prod_{i=1}^n W(\alpha_i)^{\varepsilon_i} = W^{e(\mathfrak{w})} \prod_{i=1}^n C(\alpha_i)^{\varepsilon_i} \ .$$

Here, the product structure is composition of mapping cylinders as defined in (2).

[1] The reason for inserting the α_i in this order is that the composition of mapping cylinders *first* $\underline{C}(f)$*, then* $\underline{C}(g)$ is $\underline{C}(g \circ f)$.

Theorem 4.2. *The number $n_\lambda(\mathfrak{w}) = e(\mathfrak{w}) + \sigma(L^0(\mathfrak{w}))$ only depends on the word $\mathfrak{w}$ and the lagrangian $\lambda = \lambda(\Sigma)$. Moreover, in $\widetilde{\Gamma}(\Sigma)^{++} \subset \widetilde{\Gamma}(\Sigma)$, one has*

$$W(\mathfrak{w}) = C(D(\mathfrak{w}), n_\lambda(\mathfrak{w})) \ . \tag{4}$$

Proof. Let $C(\mathfrak{w}) = \prod C(\alpha_i)^{\varepsilon_i}$. Then $C(\mathfrak{w})$ is the result of extended surgery on $\mathrm{I}\times\Sigma$ along $L(\mathfrak{w})$ (where $L(\mathfrak{w})$ is now viewed as a framed link in $\mathrm{I} \times \Sigma$). This follows from the associativity of extended gluing. The same argument as in the proof of Lemma 3.7 now shows that the weight

$$w(C(\mathfrak{w})) = \sigma(L^0(\mathfrak{w})) \ . \tag{5}$$

This proves (4). The fact that $n_\lambda(\mathfrak{w})$ only depends on $\mathfrak{w}$ and the lagrangian $\lambda = \lambda(\Sigma)$ (and not on the handlebodies $\mathcal{H}$ and $\mathcal{H}'$, as long as they are chosen such that Σ is well placed with respect to λ) follows from the interpretation of $\sigma(L^0(\mathfrak{w}))$ as the weight of the product of the $C(\alpha_i)^{\varepsilon_i}$ in the group $\widetilde{\Gamma}(\Sigma)$ (since multiplication in this group only depends on λ). □

In order to prove Theorem 3.9, we will need the following observation.

Lemma 4.3. *If $\mathfrak{w}$ is a relator in the mapping class group (i.e., if $D(\mathfrak{w}) = \mathrm{Id}$), then $\sigma(L^0(\mathfrak{w})) = \sigma(L(\mathfrak{w}))$.*

Proof. Recall that $\mathcal{H} \cup (\mathrm{I} \times \Sigma) \cup \mathcal{H}'$ is S^3 with weight zero. Now consider the extended gluing $X = \mathcal{H} \cup C(\mathfrak{w}) \cup \mathcal{H}'$. Since $D(\mathfrak{w}) = \mathrm{Id}$, we have that X is S^3 with some weight. We compute this weight in two ways. On the one hand, X is extended surgery on S^3 along the link $L(\mathfrak{w})$, hence

$$w(X) = \sigma(L(\mathfrak{w})) \tag{6}$$

by Lemma 2.2. On the other hand, the fact that $D(\mathfrak{w}) = \mathrm{Id}$ implies that we have strict additivity when computing the weight of the gluing in X (the two Maslov index terms are zero, because for both of them two of the three lagrangians coincide). Thus

$$w(X) = w(\mathcal{H}) + w(C(\mathfrak{w})) + w(\mathcal{H}') = w(C(\mathfrak{w})) = \sigma(L^0(\mathfrak{w})) \ , \tag{7}$$

where we used (5) in the last equality. Comparing (6) and (7), the lemma follows. □

We can now give the

Proof of Theorem 3.9. Let us first show that $\widetilde{\Gamma}(\Sigma)^{++} \subset \widetilde{\Gamma}(\Sigma)^{+}$. Recall that

$$w(W(\alpha)) = \begin{cases} 0 & \text{if } [\alpha] \in \lambda(\Sigma) \\ 1 & \text{if } [\alpha] \notin \lambda(\Sigma) \end{cases}$$

In the first case, $D(\alpha)_\star(\lambda(\Sigma)) = \lambda(\Sigma)$. In the second case,

$$\dim\left(\lambda(\Sigma) \cap D(\alpha)_\star(\lambda(\Sigma))\right) = \dim\left(\lambda(\Sigma)\right) - 1 \ .$$

In both cases, the congruence which defines $\widetilde{\Gamma}(\Sigma)^{+}$ is satisfied, hence $W(\alpha) \in \widetilde{\Gamma}(\Sigma)^{+}$. Thus $\widetilde{\Gamma}(\Sigma)^{++} \subset \widetilde{\Gamma}(\Sigma)^{+}$.

It remains to see that the kernel of the forgetful map $\widetilde{\Gamma}(\Sigma)^{++} \to \Gamma(\Sigma)$ is the subgroup generated by W^4. By Theorem 4.2, this kernel is

$$\{W^{n_\lambda(\mathfrak{w})} | D(\mathfrak{w}) = \mathrm{Id}_\Sigma\} \ .$$

But we know from Lemma 4.3 that if $D(\mathfrak{w}) = \mathrm{Id}_\Sigma$, then

$$n_\lambda(\mathfrak{w}) = e(\mathfrak{w}) + \sigma(L^0(\mathfrak{w})) = e(\mathfrak{w}) + \sigma(L(\mathfrak{w})) \ .$$

By [MR, Prop 3.4 (ii)], if $D(\mathfrak{w}) = \mathrm{Id}_\Sigma$ (in other words, if $\mathfrak{w}$ is a relator for the presentation of $\Gamma(\Sigma)$ with generators all Dehn twists), then $e(\mathfrak{w}) + \sigma(L(\mathfrak{w})) \equiv 0 \pmod 4$. Thus the kernel is included in the subgroup generated by W^4. Moreover, it is shown in the proof of [MR, Prop 3.4 (ii)] that for a certain relator c, one has $e(c) + \sigma(L(c)) = 4$. Thus the kernel is equal to the subgroup generated by W^4. □

Remark 4.4. The number $\sigma(L(\mathfrak{w}))$ was denoted by $\sigma_b(\mathfrak{w})$ in [MR, Def. 2.7]. The proof of [MR, Prop 3.4 (ii)] depends on a presentation of the mapping class group, as it consists in computing $e(\mathfrak{w}) + \sigma(L(\mathfrak{w}))$ for every relator $\mathfrak{w}$. It is no accident that we are led to the same computation as the one in [MR], although there were no lagrangians in [MR]. The reason is that if $\mathfrak{w}$ is a relator, then $\sigma(L(\mathfrak{w}))$ does not depend on the lagrangian λ. This can be seen as follows. If $\mathfrak{w}$ is a relator, then $\sigma(L(\mathfrak{w}))$ is the weight of $C(\mathfrak{w})$. Changing the lagrangian amounts to conjugating in the way explained in Remark 3.3. Since $C(\mathfrak{w})$ has underlying manifold $\mathrm{I} \times \Sigma$, its weight is not changed by conjugating.

Recall that we defined $n_\lambda(f) \in \mathbb{Z}/4$ so that

$$\widetilde{\Gamma}(\Sigma)^{++} = \{C(f,n) \mid f \in \Gamma(\Sigma),\ n \equiv n_\lambda(f) \pmod 4\} \ .$$

The following corollary of Theorem 4.2 gives a surgery description of $n_\lambda(f)$.

Corollary 4.5. *Let $f = D(\mathfrak{w}) \in \Gamma(\Sigma)$, and let λ be a lagrangian. Then*

$$n_\lambda(f) \equiv e(\mathfrak{w}) + \sigma(L^0(\mathfrak{w})) \pmod 4 \ .$$

In particular, $e(\mathfrak{w}) + \sigma(L^0(\mathfrak{w})) \pmod 4$ only depends on f and λ, but not on the choice of word $\mathfrak{w}$ representing f.

Proof. This follows from Theorem 4.2 and the formula $n_\lambda(\mathfrak{w}) = e(\mathfrak{w}) + \sigma(L^0(\mathfrak{w}))$. □

5. Algebraic description of $\widetilde{\Gamma}(\Sigma)^{++}$

In this section, we give a purely homological description of $n_\lambda(f)$. When it should cause no confusion, we use the same letter to denote a mapping class group element and its induced map on the rational first homology of the surface. Unless otherwise stated, all homology groups are with rational coefficients. As before, we write $\cdot$ for the intersection form on $H_1(\Sigma)$. We denote the lagrangian $\lambda(\Sigma)$ simply by λ.

Our formula for $n_\lambda(f)$ uses the bilinear form $\star_f$ described in the following lemma. This form was introduced by Turaev [T1, T2].

Lemma 5.1. *(Turaev [T2, 2.1,2.2]) If $f \in \Gamma(\Sigma)$, then*

$$a \star_f b = (f-1)^{-1}(a) \cdot b$$

is a well-defined non-singular bilinear form on $(f-1)H_1(\Sigma)$.

Here, $(f-1)^{-1}(a) \cdot b$ means $x \cdot b$ where x is any element of $(f-1)^{-1}(a)$.

Proof. Suppose that $x_1, x_2 \in (f-1)^{-1}(a)$. We let $x = x_1 - x_2$. To see that $\star_f$ is well-defined on $(f-1)H_1(\Sigma)$, we need to see that

$$x \cdot b = 0 \tag{8}$$

provided $b = (f-1)(y)$ for some $y \in H_1(\Sigma)$. This is shown as follows. Since $(f-1)(x) = 0$, we have $f(x) = x$, and hence also $f^{-1}(x) = x$. Using that the intersection form $\cdot$ is preserved by f^{-1}, the following computation proves (8):

$$x \cdot b = x \cdot f(y) - x \cdot y = f^{-1}(x) \cdot y - x \cdot y = x \cdot y - x \cdot y = 0 \ .$$

To show non-singularity of the form $\star_f$, observe that

$$(f-1)(a) \cdot b = f(a) \cdot b - a \cdot b = f(a) \cdot b - f(a) \cdot f(b) = -f(a) \cdot (f-1)(b)$$

for all $a, b \in H_1(\Sigma)$. Hence the kernel of $f-1$ is contained in the annihilator (with respect to $\cdot$) of $(f-1)H_1(\Sigma)$. Counting dimensions, it follows that the kernel of $f-1$ is equal to this annihilator. This proves that $\star_f$ is non-singular on $(f-1)H_1(\Sigma)$. □

Definition 5.2. *(Turaev) We define* Sign[det($\star_f$)] *to be the sign of the determinant of a matrix for $\star_f$ with respect to a basis of $(f-1)H_1(\Sigma)$.*

Note that Sign[det($\star_f$)] does not depend on the choice of the basis. The form $\star_f$ is neither symmetric or skew-symmetric in general, but the definition of Sign[det($\star_f$)] makes sense. Since $\star_f$ is non-singular, we have

$$\mathrm{Sign}[\det(\star_f)] = \pm 1 \ .$$

Here, let us agree that $\mathrm{Sign}[\det(\star_{\mathrm{Id}})] = 1$ (*i.e.,* the determinant of a 0×0 matrix should be taken to be one.)

Remark 5.3. Turaev [T1, T2] denotes Sign[det($\star_f$)] by $\varepsilon(f)$.

To state our formula for $n_\lambda(f)$, we need the following simple observation.

Lemma 5.4. *For every lagrangian $\lambda \subset H_1(\Sigma)$, the restriction of the form $\star_f$ to $\lambda \cap (f-1)H_1(\Sigma)$ is symmetric.*

Proof. Suppose that $(f-1)x = a \in \lambda$, and $(f-1)y = b \in \lambda$, then

$$a \star_f b - b \star_f a = x \cdot (f(y) - y) - y \cdot (f(x) - x) = x \cdot f(y) + f(x) \cdot y - 2x \cdot y$$

On the other hand,

$$0 = b \cdot a = (f(y) - y) \cdot (f(x) - x) = x \cdot f(y) + f(x) \cdot y - 2x \cdot y$$

Thus $a \star_f b = b \star_f a$, as asserted. □

Definition 5.5. *Let $\star_{f,\lambda}$ denote the restriction of the form $\star_f$ to $\lambda \cap (f-1)H_1(\Sigma)$. We denote the signature of this form by* Sign($\star_{f,\lambda}$)*.*

We can now state the main result of this section. Recall that an extended mapping cylinder $C(f,n)$ lies in $\widetilde{\Gamma}(\Sigma)^{++}$ if and only if $n \equiv n_\lambda(f) \pmod 4$. The following gives an algebraic characterization of $n_\lambda(f) \in \mathbb{Z}/4$.

Theorem 5.6. *We have*

$$n_\lambda(f) \equiv \mathrm{Sign}(\star_{f,\lambda}) - \dim((f-1)H_1(\Sigma)) - \mathrm{Sign}[\det(\star_f)] + 1 \pmod 4 \tag{9}$$

The proof of Theorem 5.6 will be given in the next section.

If we reduce formula (9) for $n_\lambda(f)$ modulo 2, we get the following alternative definition of the group $\widetilde{\Gamma}(\Sigma)^+$ (see Theorem 3.9).

Corollary 5.7. *We have*

$$\widetilde{\Gamma}(\Sigma)^+ = \{C(f,n) \mid f \in \Gamma(\Sigma),\ n \equiv \mathrm{Sign}(\star_{f,\lambda}) + \dim((f-1)H_1(\Sigma)) \pmod 2\ \}$$

Proof. By Theorem 3.9, we have that $C(f,n)$ lies in $\widetilde{\Gamma}(\Sigma)^+$ if and only if $n \equiv n_\lambda(f)$ (mod 2). ☐

The following proposition states that the characterization of $\widetilde{\Gamma}(\Sigma)^+$ in Corollary 5.7 defines the same subset of $\widetilde{\Gamma}(\Sigma)$ as Definition 3.4. Thus this proposition follows from Corollary 5.7. However we prefer to give a different proof for the reasons given in Remarks 5.9 and 5.10 below. Note that this proposition is really a statement about automorphisms of a non-degenerate symplectic form, and the proof that we give takes place completely within this context.

Proposition 5.8. *For every f and λ, we have*

$$\mathrm{Sign}(\star_{f,\lambda}) + \dim((f-1)H_1(\Sigma)) \equiv \dim(\lambda) + \dim(\lambda \cap f(\lambda)) \pmod 2\ . \tag{10}$$

Proof. Let us write $V = (f-1)H_1(\Sigma)$ and $E = \ker(f-1)$. As shown in the proof of Lemma 5.1, we have $V = E^\perp$ with respect to the form $\cdot$ on $H_1(\Sigma)$. The domain of definition of the form $\star_{f,\lambda}$ is $\lambda \cap V$. The radical of the form $\star_{f,\lambda}$ is given by

$$\mathrm{rad}(\star_{f,\lambda}) = \lambda \cap (f-1)(\lambda)\ . \tag{11}$$

To see this, notice that an element $a = (f-1)(x) \in \lambda \cap V$ is in the radical of $\star_{f,\lambda}$ iff $x \cdot b = 0$ for all $b \in \lambda \cap V$. Thus

$$\mathrm{rad}(\star_{f,\lambda}) = \lambda \cap (f-1)\big((\lambda \cap V)^\perp\big)\ .$$

But $(\lambda \cap V)^\perp = \lambda^\perp + V^\perp = \lambda + E$ and $(f-1)(E) = 0$. This proves (11). Now observe that

$$\mathrm{Sign}(\star_{f,\lambda}) \equiv \mathrm{rank}(\star_{f,\lambda}) = \dim(\lambda \cap V) - \dim \mathrm{rad}(\star_{f,\lambda}) \pmod 2\ . \tag{12}$$

Using (11), the exact sequence

$$0 \longrightarrow \lambda \cap E \longrightarrow \lambda \cap f(\lambda) \xrightarrow{1-f^{-1}} \lambda \cap (f-1)(\lambda) \longrightarrow 0$$

shows that

$$\dim \mathrm{rad}(\star_{f,\lambda}) = \dim(\lambda \cap f(\lambda)) - \dim(\lambda \cap E)\ . \tag{13}$$

But $\lambda \cap E = \lambda^\perp \cap V^\perp = (\lambda + V)^\perp$, hence

$$\dim(\lambda \cap E) \equiv \dim(\lambda + V) \pmod 2\ . \tag{14}$$

Putting together (12), (13), and (14), we have

$$\mathrm{Sign}(\star_{f,\lambda}) \equiv \dim(\lambda \cap V) + \dim(\lambda \cap f(\lambda)) + \dim(\lambda + V) \pmod 2\ .$$

This implies Proposition 5.8 because of the equality

$$\dim(\lambda \cap V) + \dim(\lambda + V) = \dim V + \dim \lambda\ .$$ ☐

Remark 5.9. The group $\widetilde{\Gamma}(\Sigma)^+$ is not used in the proof of Theorem 5.6. One can use the characterization of $\widetilde{\Gamma}(\Sigma)^+$ in Corollary 5.7 as the definition of $\widetilde{\Gamma}(\Sigma)^+$, and then use Proposition 5.8 to deduce the original definition of $\widetilde{\Gamma}(\Sigma)^+$ (see Definition 3.4) as a corollary. If we define $\widetilde{\Gamma}(\Sigma)^+$ in this way, the fact that $\widetilde{\Gamma}(\Sigma)^+$ is an index two subgroup of $\widetilde{\Gamma}(\Sigma)$ is easy to see. Indeed, our new definition of $\widetilde{\Gamma}(\Sigma)^+$ is as the set $\{C(f,n)\,|\,n \equiv n_\lambda(f) \pmod 2\}$. We only need to see that this set is closed under multiplication. But this follows from the fact that $\{C(f,n)\,|\,n \equiv n_\lambda(f) \pmod 4\}$ (which is equal to $\widetilde{\Gamma}(\Sigma)^{++}$) is closed under multiplication.

Remark 5.10. The fact that $\{C(f,n)\,|\,n \equiv n_\lambda(f) \pmod 4\}$ is closed under multiplication is equivalent to the fact that $\widetilde{\Gamma}(\Sigma)^{++}$ is an index four subgroup of $\widetilde{\Gamma}(\Sigma)$. The proof of this fact in Section 4 does not use anything about $\widetilde{\Gamma}(\Sigma)^+$. Therefore the preceding remark together with Proposition 5.8 gives a new proof that Definition 3.4 defines an index two subgroup of $\widetilde{\Gamma}(\Sigma)$.

6. Proof of Theorem 5.6.

We begin the proof with some preliminary material. The set-theoretical section $f \mapsto C(f,0)$ of $\widetilde{\Gamma}(\Sigma) \to \Gamma(\Sigma)$ can be used to define a 2-cocycle on $\Gamma(\Sigma)$ whose cohomology class in $H^2(\Gamma(\Sigma);\mathbb{Z})$ classifies the extension. The extension and thus the cocycle depend on $\lambda(\Sigma)$, which we are denoting by λ. We will denote the cocycle by m_λ and call it the *Maslov cocycle*. In general, the 2-cocycle defined by a section s is given by $(g,f) \mapsto s(g)s(f)s(g\circ f)^{-1}$. By (2), we have that

$$m_\lambda(g,f) = \mu_\Sigma\left(\lambda, g(\lambda), (g\circ f)(\lambda)\right) = -\mu_\Sigma\left((g\circ f)(\lambda), g(\lambda), \lambda\right) \ . \tag{15}$$

Next, we recall the well-known 4-manifold interpretation of the Maslov index $\mu_\Sigma(\lambda_1,\lambda_2,\lambda_3)$ of three lagrangians. See for example [CLM, Section 12]. Let $\mathcal{H}_i$ denote a handlebody with boundary Σ such that λ_i is the kernel of the map $H_1(\Sigma) \to H_1(\mathcal{H}_i)$ induced by inclusion. Consider the 4-manifold $U(\lambda_1,\lambda_2,\lambda_3)$ obtained by gluing to $D^2\times\Sigma$ (with the product orientation using the standard orientation on D^2) three thickened handlebodies $\mathrm{I}\times\mathcal{H}_i$, in the cyclic order indicated in Figure 4.

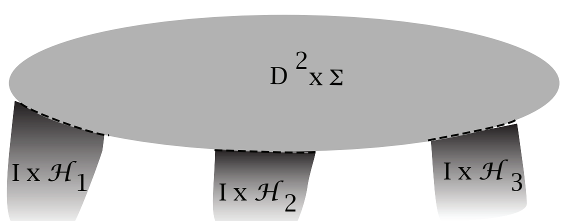

Figure 4. A picture of $U(\lambda_1,\lambda_2,\lambda_3)$. The large oval disk represents $D^2\times\Sigma$. The three dotted lines represent copies of $\mathrm{I}\times\Sigma$ along which are glued the thickened handlebodies $\mathrm{I}\times\mathcal{H}_i$. The boundary of $U(\lambda_1,\lambda_2,\lambda_3)$ has three connected components. To indicate this, the thickened handlebodies are drawn fading away.

Careful consideration of the Wall non-additivity formula shows that

$$\operatorname{Sign} U(\lambda_1,\lambda_2,\lambda_3) = \mu_\Sigma(\lambda_1,\lambda_2,\lambda_3). \tag{16}$$

Note that in contrast with (1), there is no minus sign in (16).

Proposition 6.1. $m_\lambda(g, f) = -\operatorname{Sign} U((g \circ f)(\lambda), g(\lambda), \lambda)$.

Proof. This follows from (15) and (16). □

For the rest of this section, $\mathcal{H}$ will be a handlebody with $\partial\mathcal{H} = \Sigma$ such that the kernel of $H_1(\Sigma) \to H_1(\mathcal{H})$ is the given lagrangian $\lambda = \lambda(\Sigma)$.

One can obtain $U((g \circ f)(\lambda), g(\lambda), \lambda)$ from $U(\lambda, \lambda, \lambda)$ by cutting along two arcs running across the disk and regluing using f and g as in Figure 5. In this figure (and in similar figures below) the arrow labelled f indicates the direction of gluing: a point x on the boundary component corresponding to the tail side of the arrow is glued to the point $f(x)$ on the boundary component corresponding to the head of the arrow.

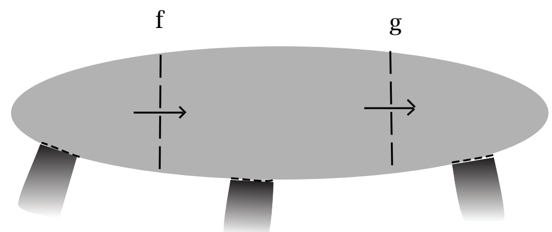

FIGURE 5. The three thickened handlebodies $\mathrm{I} \times \mathcal{H}$ are attached first to form $U(\lambda, \lambda, \lambda)$. Then the manifold $U(\lambda, \lambda, \lambda)$ is cut along the dotted seams and reglued using f and g. The result is oriented diffeomorphic to $U((g \circ f)(\lambda), g(\lambda), \lambda)$.

The boundary of $U((g \circ f)(\lambda), g(\lambda), \lambda)$ consists of three *Heegaard manifolds* defined as follows. The Heegaard manifold of $f \in \Gamma(\Sigma)$, denoted $\mathcal{H}(f)$, is the quotient space $(\mathcal{H} \sqcup \overline{\mathcal{H}})/\sim$ where $\sim$ is the equivalence relation given by identifying $x \in \partial\mathcal{H}$ with $f(x) \in \partial\overline{\mathcal{H}}$. Here, $\mathcal{H}$ is oriented so that it induces the given orientation on Σ. Using that $\mathcal{H}(f^{-1}) = \overline{\mathcal{H}(f)}$, one easily checks the following

Proposition 6.2. *The oriented boundary of* $U((g \circ f)(\lambda), g(\lambda), \lambda)$ *is*

$$\overline{\mathcal{H}(g \circ f)} \sqcup \mathcal{H}(f) \sqcup \mathcal{H}(g)\ .$$

As we construct further oriented 4-manifolds that we will use as building blocks, we will continue to pay close attention to which 3-manifolds form their oriented boundaries. This will make it easier in the proof of Theorem 6.10 to see that a certain collection of 4-manifolds fit together to form the boundary of a 5-manifold.

Meyer [M] defined a 2-cocycle for $\Gamma(\Sigma)$, now called the *signature cocycle.* We denote this cocycle by τ. This cocycle does not require a choice of a lagrangian. In fact,

$$\tau(f, g) = \operatorname{Sign} W(f, g) \tag{17}$$

where $W(f, g)$ is the 4-manifold which fibers over a two-holed disk, with fiber Σ, and whose monodromy around the two holes is given by f and g. See Figure 6 and [A, 4.1]. Note that $W(f, g) \approx W(g, f)$, and so $\tau(f, g) = \tau(g, f)$.

The boundary of Meyer's manifold $W(f, g)$ consists of three *mapping tori.* Here, the mapping torus of $f \in \Gamma(\Sigma)$, denoted $T(f)$, is the quotient space $(\mathrm{I} \times \Sigma)/\sim$

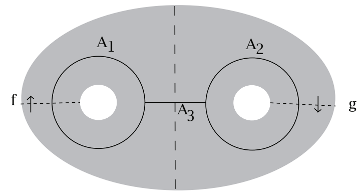

Figure 6. A picture of $W(f,g)$. It is the result of cutting a two-holed disk times Σ along the two seams $\mathrm{I}\times\Sigma$ given by the horizontal dotted lines and regluing by f or g, as indicated. The solid lines labelled A_1, A_2 and A_3 indicate 2-chains that will be used in the proof of Proposition 6.5. The vertical dotted line will also be used in the proof of this proposition.

where $\sim$ is the equivalence relation generated by $(1,x) \sim (0, f(x))$. One easily checks the following

Proposition 6.3. *As an oriented manifold, the boundary of* $W(f,g)$ *is*

$$\overline{T(g\circ f)} \sqcup T(f) \sqcup T(g) \ .$$

Turaev [T1, T2] independently defined and studied a cocycle φ which turns out to be equal to $-\tau$ (see Prop. 6.5 below). Meyer defined τ as a cocycle for $\mathrm{Sp}(g(\Sigma),\mathbb{Z})$, and Turaev considered his cocycle as a cocycle for the symplectic group $\mathrm{Sp}(g(\Sigma),\mathbb{R})$, but for our purposes, we just consider it as a 2-cocycle for $\Gamma(\Sigma)$. Turaev modeled the construction of $W(f,g)$ algebraically and defined $\varphi(f,g)$ as the signature of the bilinear form $\star_{f,g}$ given as follows.

Proposition 6.4 (Turaev)**.** *If* V *is a rational vector space with a nonsingular skew symmetric inner product* $\cdot$ *and* f *and* g *are automorphisms which preserve* $\cdot$*, then*

$$a \star_{f,g} b = \big((f-1)^{-1}a + (g-1)^{-1}a + a\big)\cdot b \tag{18}$$

defines a symmetric bilinear form on $(f-1)V \cap (g-1)V$.

(Turaev's result actually dealt with real vector spaces.) Turaev then proceeded to study φ algebraically. As Turaev used $W(f,g)$ only for motivation or inspiration, he did not need to include a proof of the following proposition, which he must have known.

Proposition 6.5. *If* $f,g\in\Gamma(\Sigma)$*, the intersection form on* $H_2(W(f,g))$ *divided by part of its radical is isomorphic to* minus *the form* $\star_{f,g}$ *on* $(f-1)H_1(\Sigma)\cap(g-1)H_1(\Sigma)$ *defined in (18). In particular*

$$\tau(f,g) = \mathrm{Sign}(W(f,g)) = -\,\mathrm{Sign}(\star_{f,g}) = -\varphi(f,g) \ .$$

Remark 6.6. This gives a topological proof that the form $\star_{f,g}$ is symmetric.

Proof of Proposition 6.5. If we cut $W(f,g)$ along the $\mathrm{I}\times\Sigma$ indicated by the vertical dotted line in Figure 6, we obtain the disjoint union of $\mathrm{I}\times T(f)$ and $\mathrm{I}\times T(g)$. This gives the long exact Mayer-Vietoris sequence:

$$H_2(T(f))\oplus H_2(T(g)) \to H_2(W(f,g)) \to H_1(\Sigma) \to H_1(T(f))\oplus H_1(T(g))$$

The image of the first arrow is contained in the radical of the intersection form. Thus, on the cokernel of this map, there is an induced bilinear symmetric form whose signature is $\mathrm{Sign}(W(f,g))$. On the other hand, this cokernel is isomorphic to the kernel of the last arrow which can be identified with $(f-1)H_1(\Sigma)\cap(g-1)H_1(\Sigma)$. We only need to see that the middle arrow (which is the Mayer-Vietoris boundary map) sends the intersection form on $W(f,g)$ to minus the form $\star_{f,g}$.

We may describe a homology class in $H_2(W(f,g))$ which maps to an element $a \in (f-1)H_1(\Sigma)\cap(g-1)H_1(\Sigma) \subset H_1(\Sigma)$ as follows. Suppose α_1 and α_2 are closed oriented curves in Σ such that $(f-1)[\alpha_1] = a$, and $(g-1)[\alpha_2] = a$. Then α_1 sweeps out in $T(f)$ a cylinder A_1 which projects onto the solid circle labeled A_1 in Figure 6. We think of this cylinder as a 2-chain with boundary lying in the copy of Σ lying over the point where the lines labelled A_1 and A_3 meet. We will denote this copy of Σ by Σ_1. By construction, we have

$$[\partial A_1] = [f(\alpha)] - [\alpha] = a \in H_1(\Sigma_1)\ .$$

There is also a similar 2-chain A_2 in $T(g)$ with boundary ∂A_2 representing a in $H_1(\Sigma_2)$, where Σ_2 is another copy of Σ lying over the point where the lines labelled A_2 and A_3 meet. We connect the boundaries of the 2-chains A_1 and $-A_2$ by a 2-chain A_3 in the copy of $\mathrm{I}\times\Sigma$ joining Σ_1 and Σ_2 (lying over the arc labelled A_3 in the figure) so that

$$\partial A_3 = \partial A_2 - \partial A_1\ .$$

Then $A_1 + A_3 - A_2$ gives a 2-cycle in $W(f,g)$ representing a homology class which maps to a under the Mayer-Vietoris boundary map. (The minus sign in front of A_2 is necessary from the definition of the Mayer-Vietoris boundary map.)

If we have another such 2-chain $B_1 + B_3 - B_2$ mapping to $[b] \in (f-1)H_1(\Sigma)\cap(g-1)H_1(\Sigma)$, but placed further inside and rotated slightly, Figure 7 indicates why

$$\begin{aligned}[A_1 + A_3 - A_2] \cap [B_1 + B_3 - B_2] &= -\ ((f-1)^{-1}a + a + (g-1)^{-1}a)\cdot b \\ &= -\ a \star_{f,g} b\ .\end{aligned}$$

The reason for the minus sign in this equation is that a point of intersection x of the two 2-chains corresponding to a positive intersection point p in the base (with frame (e_1,e_2), say) and a positive intersection point q in the fiber over p (with frame (e_3,e_4), say) should be counted negatively, since the frame (e_1,e_3,e_2,e_4) at x differs from the standard frame (e_1,e_2,e_3,e_4) by a transposition. ☐

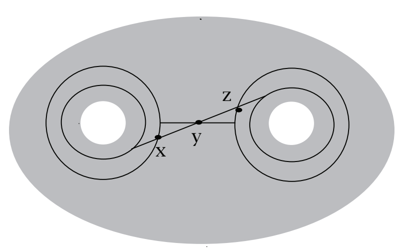

FIGURE 7. The point x indicates $A_1 \cap B_3$ which contributes $-(f-1)^{-1}(a)\cdot b$. The point y indicates $A_3 \cap B_3$ which contributes $-a\cdot b$. The point z indicates $-A_2 \cap B_3$ which contributes $-(g-1)^{-1}(a)\cdot b$.

We will need [T1, T2, Theorem 2] where Turaev showed that the cocycle φ is a coboundary (mod 4). Before stating this result, we recall that, with $\Gamma(\Sigma)$ acting trivially on $\mathbb{Z}$, the coboundary δc of a 1-cochain $c : \Gamma(\Sigma) \to \mathbb{Z}$ is given by $(\delta c)(g, h) = c(g) + c(h) - c(gh)$.

Theorem 6.7 (Turaev [T1, T2])**.** *The* 1*-cochain k on $\Gamma(\Sigma)$ which assigns to f*

$$k(f) = \dim\left((f-1)H_1(\Sigma)\right) + \mathrm{Sign}[\det(\star_f)] - 1 \tag{19}$$

has coboundary δk satisfying

$$\delta k \equiv \varphi \pmod 4.$$

Remark 6.8. The reason that we gave a proof of Proposition 6.5 is that Turaev proves Theorem 6.7 for the cocycle φ given by $\mathrm{Sign}(\star_{f,g})$, and we will use the cocycle τ described by $\mathrm{Sign}(W(f, g))$. So we need to know how exactly they are related.

Walker [W, p. 124] defines a 1-cochain j_λ on $\Gamma(\Sigma)$ which assigns to f the signature of the 4-manifold $J_\lambda(f)$ obtained by gluing $\mathrm{I} \times \mathcal{H}$ along $\mathrm{I} \times \Sigma$ in the boundary of $\mathrm{I} \times T(f)$ as indicated in Figure 8.[2] Here, as above, λ is the kernel of $H_1(\Sigma) \to H_1(\mathcal{H})$. Note that the boundary of $J_\lambda(f)$ is $\overline{\mathcal{H}(f)} \sqcup T(f)$.

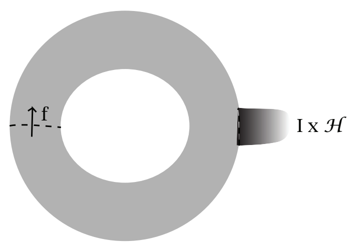

FIGURE 8. A thickened handlebody attached to a thickened mapping torus of f. Its signature is $j_\lambda(f)$.

We have a formula for $j_\lambda(f)$ which is similar to Turaev's formula for $\varphi(f, g)$. Recall that $\star_{f,\lambda}$ is the symmetric bilinear form on $\lambda \cap (f-1)H_1(\Sigma)$ obtained as the restriction of the non-symmetric form $\star_f$.

Proposition 6.9. *One has*

$$\mathrm{Sign}(J_\lambda(f)) = j_\lambda(f) = -\,\mathrm{Sign}(\star_{f,\lambda}) \tag{20}$$

Proof. We have a Mayer-Vietoris sequence:

$$H_2(T(f)) \oplus H_2(\mathcal{H}) \to H_2(J_\lambda(f)) \to H_1(\Sigma) \to H_1(T(f)) \oplus H_1(\mathcal{H})$$

The image of the first arrow is contained in the radical of the intersection form. Thus, on the cokernel of this map, there is an induced bilinear symmetric form whose signature is $\mathrm{Sign}(J_\lambda(f))$. On the other hand, this cokernel is isomorphic to the kernel of the last arrow which can be identified with $\lambda \cap (f-1)H_1(\Sigma)$. We need to show that the middle arrow sends the intersection form on $J_\lambda(f)$ to minus the form $\star_{f,\lambda}$.

[2] Walker actually draws the arrow for f in the other direction, and this has the effect that his j is minus our j. Similarly Walker's $d(f, g)$ is minus our $\tau(f, g)$.

The proof is very much like the proof of Proposition 6.5. We describe a homology class in $H_2(J_\lambda(f))$ which maps to an element $a \in \lambda \cap (f-1)H_1(\Sigma)$. Suppose α is a curve in Σ such that $(f-1)[\alpha] = a$. Then α sweeps out in $T(f)$ a 2-chain A_1 in $T(f)$ with boundary representing a on Σ_1, a copy of Σ. Moreover there is a 2-chain A_2 in $\mathcal{H}$ with boundary representing a. Then $A_1 - A_2$ gives a 2-cycle in $J_\lambda(f)$ representing a class which maps to a under the Mayer-Vietoris boundary map.

If we have another 2-chain $B_1 - B_2$ mapping to a class $[b] \in \lambda \cap (f-1)H_1(\Sigma)$ but placed further inside and rotated slightly, Figure 9 indicates why

$$[A_1 - A_2] \cap [B_1 - B_2] = -a \star_{f,\lambda} b \ .$$

□

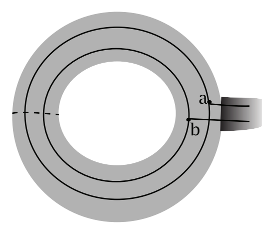

FIGURE 9. The intersection of 2-cycles is given by $-(f-1)^{-1}(a) \cdot b$.

In the following Theorem and proof, we adapt an argument of Walker [W, pp 123-125] to our definitions. Let $[m_\lambda]$ and $[\tau]$ represent the cohomology classes in $H^2(\Gamma(\Sigma);\mathbb{Z})$ represented by the cocycles m_λ and τ.

Theorem 6.10 (Walker)**.** *We have that $\delta(j_\lambda) = \tau + m_\lambda$. Thus $[m_\lambda] = -[\tau]$.*

Proof. Form a 5-manifold X by attaching $D^2 \times \mathcal{H}$ to $\mathrm{I} \times W(f,g)$ along $R \subset \{1\} \times W(f,g)$, where $R \approx D^2 \times \Sigma$ is represented by the darkest shaded region in Figure 10. The boundary ∂X is the union of copies of the oriented manifolds $\overline{W(f,g)}$, $U((g \circ f)(\lambda), g(\lambda), \lambda)$, $\overline{J_\lambda(g\circ f)}$, $J_\lambda(g)$, $J_\lambda(f)$. To see that the orientations are as stated, note that:

$$\begin{aligned}
\partial(\overline{W(f,g)}) &= T(g\circ f) \sqcup \overline{T(g)} \sqcup \overline{Tf)} \\
\partial(U((g\circ f)(\lambda), g(\lambda), \lambda)) &= \overline{\mathcal{H}(g\circ f)} \sqcup \mathcal{H}(g) \sqcup \mathcal{H}(f) \\
\partial(\overline{J_\lambda(g\circ f)}) &= \overline{T(g\circ f)} \sqcup \mathcal{H}(g\circ f) \\
\partial(J_\lambda(g)) &= T(g) \sqcup \overline{\mathcal{H}(g)} \\
\partial(J_\lambda(f)) &= T(f) \sqcup \overline{\mathcal{H}(f)}.
\end{aligned}$$

These 4-manifolds are glued along closed 3-manifolds. By Novikov additivity, the signature of ∂X is the sum of the signatures of the pieces. As the signature of a 4-manifold which is the boundary of a 5-manifold is zero, we have that:

$$-\tau(g,f) - m_\lambda(g,f) - j_\lambda(g\circ f) + j_\lambda(g) + j_\lambda(f) = 0.$$

□

We are now ready to give the

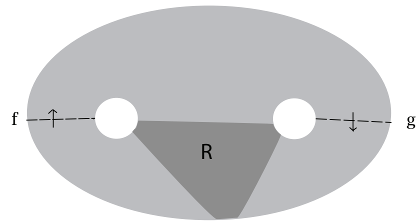

FIGURE 10. $D^2 \times \mathcal{H}$ is attached along a copy of $D^2 \times \Sigma$ indicated by the dark shaded region R.

Proof of Theorem 5.6. Using (19) and Prop. 6.9, the statement to be proved can be reformulated as follows: for every $f \in \Gamma(\Sigma)$, one has

$$n_\lambda(f) + j_\lambda(f) + k(f) \equiv 0 \pmod 4 . \tag{21}$$

To prove it, choose for every f an arbitrary lift $\tilde{n}_\lambda(f) \in \mathbb{Z}$ of $n_\lambda(f) \in \mathbb{Z}/4$. The set-theoretical section $f \mapsto C(f, \tilde{n}_\lambda(f))$ of $\widetilde{\Gamma}(\Sigma) \to \Gamma(\Sigma)$ defines a cocycle given by $m_\lambda + \delta(\tilde{n}_\lambda)$. By Theorem 3.9, this cocycle is zero (mod 4), since the section takes values in the index four subgroup $\widetilde{\Gamma}(\Sigma)^{++}$. Thus

$$m_\lambda + \delta(n_\lambda) \equiv m_\lambda + \delta(\tilde{n}_\lambda) \equiv 0 \pmod 4 .$$

By Walker's theorem 6.10, Prop. 6.5, and Turaev's theorem 6.7, we have that

$$m_\lambda = \delta(j_\lambda) - \tau = \delta(j_\lambda) + \varphi \equiv \delta(j_\lambda + k) \pmod 4 .$$

Thus

$$\delta(n_\lambda + j_\lambda + k) \equiv 0 \pmod 4 . \tag{22}$$

If $g(\Sigma) \geq 3$, it is well-known that $\Gamma(\Sigma)$ is perfect, hence (22) implies that

$$n_\lambda + j_\lambda + k \equiv 0 \pmod 4 .$$

This proves (21) if $g(\Sigma) \geq 3$.

If $g(\Sigma) < 3$, we reduce to the case $g(\Sigma) \geq 3$ by considering the effect of the stabilization which replaces the surface Σ with $\Sigma\#(S^1 \times S^1)$, the mapping class f with $f\#\mathrm{Id}_{S^1 \times S^1}$, and the lagrangian λ with $\lambda \oplus \lambda_1$ for some lagrangian λ_1 in $H_1(S^1 \times S^1)$. To complete the proof, it suffices to show that $n_\lambda(f)$, $k(f)$, and $j_\lambda(f)$ are unchanged under the stabilization. This is clear for $k(f)$ and $j_\lambda(f)$ (using the algebraic definitions in (19) and (20)). To see that $n_\lambda(f)$ is also unchanged, one may use the formula

$$n_\lambda(f) \equiv e(\mathfrak{w}) + \sigma(L^0(\mathfrak{w})) \pmod 4 \tag{23}$$

given in Corollary 4.5. If we perform the connected sum away from the curves in $\mathfrak{w}$, it is clear that the right hand side of (23) does not change. Hence $n_\lambda(f)$ is also unchanged. This completes the proof. □

7. SURFACES WITH BOUNDARY

To simplify the exposition, we delayed the discussion of surfaces with boundary. However, all the preceding results hold for the mapping class group of a surface with boundary, modulo the following modifications.

An extended surface with boundary is a compact oriented surface Σ together with a choice of lagrangian λ in $H_1(\widehat{\Sigma};\mathbb{Q})$, where $\widehat{\Sigma}$ is the closed surface obtained from Σ by attaching a disk to each boundary component. As in the case without boundary, we denote the ordinary mapping class group of Σ by $\Gamma(\Sigma)$. It is the group of orientation preserving diffeomorphisms of $\widehat{\Sigma}$ which are the identity on the attached disks, modulo isotopies which are again the identity on the attached disks.

We define the extended mapping class group $\widetilde{\Gamma}(\Sigma)$ to be the group of pairs $(f,n) \in \Gamma(\Sigma) \times \mathbb{Z}$ with multiplication

$$(g,m)\circ(f,n) = (g\circ f, m+n+\mu_{\widehat{\Sigma}}(\lambda, g_*\lambda, (g\circ f)_*\lambda)) \tag{24}$$

If Σ has no boundary, this is equivalent to the definition of $\widetilde{\Gamma}(\Sigma)$ in terms of mapping cylinders (see Remark 3.2). If Σ has boundary, we can again think of (f,n) as represented by the mapping cylinder $C(f,n)$ viewed as a cobordism from $\widehat{\Sigma}$ to itself. But the notion of equivalence of cobordisms has to be modified appropriately so that $C(f,n)$ determines (f,n).

The groups $\widetilde{\Gamma}(\Sigma)^+$ and $\widetilde{\Gamma}(\Sigma)^{++}$ are now defined exactly as in the closed case, and Theorem 3.9 continues to hold as stated. The surgery description of $\widetilde{\Gamma}(\Sigma)^{++}$ in Theorem 4.2 also holds as stated. But notice that although the curves representing Dehn twists will avoid the attached disks, we must, of course, use the closed surface $\widehat{\Sigma}$ (well-placed in S^3 with respect to the lagrangian λ) to construct the framed link $L^0(\mathfrak{w})$ associated to a word $\mathfrak{w}$ in Dehn twists. As for the algebraic description of $\widetilde{\Gamma}(\Sigma)^{++}$, Theorem 5.6 continues to hold except that in its statement, we must replace the homology group $H_1(\Sigma)$ by $H_1(\widehat{\Sigma})$.

The reason why all results in Sections 4 and 5 go through for surfaces with boundary is that all the extensions and cochains of the mapping class group $\Gamma(\Sigma)$ we consider are pull-backs from the corresponding extensions and cochains of $\Gamma(\widehat{\Sigma})$.

8. Universal central extension

In this section, let $\Sigma = \Sigma_{g,r}$ denote a connected compact oriented surface of genus g with r boundary components. We denote the ordinary mapping class group of $\Sigma_{g,r}$ by $\Gamma_{g,r}$. We also write $\widetilde{\Gamma}_{g,r}$ and $\widetilde{\Gamma}^{++}_{g,r}$ for the extended mapping class group $\widetilde{\Gamma}(\Sigma)$ and its index four subgroup $\widetilde{\Gamma}(\Sigma)^{++}$. As remarked in 3.3, although our description of these groups requires the choice of a lagrangian, they are independent of this choice up to isomorphism.

Proposition 8.1. *If $g \geq 4$, then $\widetilde{\Gamma}^{++}_{g,r}$ is a universal central extension of $\Gamma_{g,r}$.*

Proof. For $g \geq 3$, $\Gamma_{g,r}$ is perfect. Hence it has a universal central extension [B, p.96]. This is an extension by $H_2(\Gamma_{g,r};\mathbb{Z})$ satisfying a certain universal property. If $g \geq 4$, it is known that $H_2(\Gamma_{g,r};\mathbb{Z}) = \mathbb{Z}$ and $H^2(\Gamma_{g,r};\mathbb{Z}) = \mathbb{Z}$. See [KS], and the references therein. Hence, if $g \geq 4$, a universal central extension of $\Gamma_{g,r}$ is an extension by $\mathbb{Z}$, and an extension of $\Gamma_{g,r}$ by $\mathbb{Z}$ is a universal central extension if and only if its cohomology class is a generator of $H^2(\Gamma_{g,r};\mathbb{Z})$.

Meyer [M] showed that if $g \geq 3$, the cohomology class $[\tau] \in H^2(\Gamma_{g,0};\mathbb{Z})$ defines a map $H_2(\Gamma_{g,0};\mathbb{Z}) \to \mathbb{Z}$ whose image is $4\mathbb{Z}$. This implies that $[\tau]/4$ is a generator of $H^2(\Gamma_{g,0};\mathbb{Z})$ if $g \geq 4$. Since $[\tau] = -[m_\lambda]$ and the extension $\widetilde{\Gamma}^{++}_{g,0}$ is classified by $[m_\lambda]/4$, this shows that $\widetilde{\Gamma}^{++}_{g,0}$ is a universal central extension if $g \geq 4$. Finally, the same is true for $\widetilde{\Gamma}^{++}_{g,r}$ if $r > 0$, since the extension $\widetilde{\Gamma}^{++}_{g,r} \to \Gamma_{g,r}$ is a pullback of the extension $\widetilde{\Gamma}^{++}_{g,0} \to \Gamma_{g,0}$ and the natural map $H_2(\Gamma_{g,r};\mathbb{Z}) \to H_2(\Gamma_{g,0};\mathbb{Z})$ is an isomorphism in our situation. □

9. Applications to TQFT

A Topological Quantum Field Theory (TQFT) in the sense of Atiyah and Segal includes in particular representations of centrally extended mapping class groups of surfaces. The fact that one needs to consider central extensions is sometimes called the 'framing anomaly' of the TQFTs we are interested in. There are essentially four ways to describe the central extension in the literature: Atiyah's description [A] using 2-framings and the signature cocycle, Walker's description [W] using integral weights, lagrangians, and Maslov indices, as in Definition 3.1, the description using p_1-structures given in [BHMV2] (see also Gervais [Ge]), and the description in [MR] *via* an explicit computation of the projective factors arising in the TQFT-representations of the mapping class group.

In [BHMV2], a version of the Reshetikhin-Turaev $SU(2)$- and $SO(3)$-TQFT was constructed using the skein theory of the Kauffman bracket. In this section, we consider the TQFT constructed in the same way as in [BHMV2], but with integral weights and lagrangians in place of the p_1-structures. This variant of the TQFT constructed in [BHMV2] has been described and used in [G, GMW, GM1, GM2]. Our aim here is to show how to use the techniques of Section 4 to describe the representations of the extended mapping class group $\widetilde{\Gamma}(\Sigma)$ arising in this TQFT explicitly, and to do some computations with these representations which are used in [GM2] and in work in progress.

A TQFT is a functor on a certain cobordism category with values in the category of vector spaces, or, more generally, modules over a commutative ring. The cobordism category we use is an enhancement of the extended cobordism category $\mathcal{C}$ described in Section 3. The enhancement consists in allowing surfaces to contain (possibly empty) collections of colored banded points and 3-manifolds to contain a (possibly empty) colored banded trivalent graph which meets the boundary in the banded points of the boundary surfaces. As in [BHMV2], a banded point is an oriented arc through the point. A banded trivalent graph is a trivalent graph together with an oriented surface which deformation retracts to the graph. The colors are from a certain finite palette which depend on the specific TQFT under consideration. We refer to extended surfaces and 3-manifolds which are enhanced in this way simply as extended surfaces and extended 3-manifolds.

The TQFT's we consider are indexed by an integer $p \geq 3$ and denoted (Z_p, V_p). The notation is such that to an extended surface Σ, there is associated a k_p-module $V_p(\Sigma)$, and to an extended cobordism $M : \Sigma \rightsquigarrow \Sigma'$, there is associated a k_p-linear map

$$Z_p(M) : V_p(\Sigma) \to V_p(\Sigma') ,$$

where k_p denotes the ring of coefficients. The module $V_p(\emptyset)$ is canonically identified with the ground ring k_p. Although our modules are not vector spaces, it is customary in TQFT to call their elements vectors. If $M : \emptyset \rightsquigarrow \Sigma$, we simply write $Z_p(M)$ for the vector $Z_p(M)(1) \in V_p(\Sigma)$. In [G, GM1], this vector is denoted by $[M]_p$.

We take the ring of coefficients to be $k_p = \mathbb{Z}[\frac{1}{p}, A, \kappa]$, where A is a primitive $2p$-th root of unity, and κ is a square root of $A^{-6-p(p+1)/2}$. Increasing the weight of an extended 3-manifold M by one multiplies the vector $Z_p(M)$ by κ. (Here we depart from the notation of [BHMV2] whose κ is a further third root of our κ.) The palette of allowed colors is $\{0,\ldots,k\}$, if $p=2k+4$ is even, and $\{0,\ldots,p-2\}$, if $p\geq 3$ is odd. Moreover, the colorings of the trivalent graphs must be p-admissible [BHMV2, p. 905]. If $p=2k+4$ then (Z_p,V_p) is a variant of the $SU(2)$-theory at level k, while for odd p it is called an $SO(3)$-theory.

A fundamental ingredient in the construction of [BHMV2] is the surgery axiom which allows one to replace surgery along a banded knot with cabling that knot with a certain skein element ω in the solid torus. Here, a skein element in a 3-manifold is a linear combination of banded links (or, more generally, colored banded graphs). In [BHMV2] the relevant notion of surgery was p_1-surgery. Here is a formulation of the surgery axiom in our present context.

Let ω denote the skein element in solid torus $\overline{S^1}\times D^2$ described in [BHMV2]. The boundary of $\overline{S^1}\times D^2$ is the torus $S^1\times S^1$, which we denote by $\mathcal{T}$. It is also the boundary of $D^2\times S^1$. As in Section 2, we make $\overline{S^1}\times D^2$ and $D^2\times S^1$ into extended manifolds by giving both of them weight zero. If $\mathcal{T}$ is made into an extended surface by equipping it with some lagrangian $\lambda(\mathcal{T})$, then the pair $(\overline{S^1}\times D^2,\omega)$ defines a vector $Z_p(\overline{S^1}\times D^2,\omega)$ in $V_p(\mathcal{T})$.

Lemma 9.1 (Surgery Axiom)**.** *Assume $\lambda(\mathcal{T})$ is the lagrangian generated by the homology class of the meridian $pt\times S^1$ of $\overline{S^1}\times D^2$. Then in $V_p(\mathcal{T})$ one has*

$$Z_p(\overline{S^1}\times D^2,\omega)=Z_p(D^2\times S^1)\ .$$

The proof of the Surgery Axiom in our current context of extended manifolds is completely analogous to the proof of this axiom in the original context of [BHMV2]. We omit the details.

Now let Σ be a connected extended surface, with lagrangian $\lambda(\Sigma)$. Consider the extended mapping cylinder $C(f,n)\in\widetilde{\Gamma}(\Sigma)$ where $(f,n)\in\Gamma(\Sigma)\times\mathbb{Z}$. Let

$$\rho_p(f,n)=Z_p(C(f,n))\ . \tag{25}$$

This defines a representation ρ_p of $\widetilde{\Gamma}(\Sigma)$ on $V_p(\Sigma)$. This representation can be described in very concrete terms, as follows. Let $\mathfrak{w}=\prod_{i=1}^n {\alpha_i}^{\varepsilon_i}$ be a word so that $D(\mathfrak{w})=f$. Let $L(\mathfrak{w})\subset \mathrm{I}\times\Sigma$ be the framed link considered in Section 4. Let $s(\mathfrak{w})$ be the skein element in $\mathrm{I}\times\Sigma$ obtained by cabling every component of this framed link with ω. We consider $\mathrm{I}\times\Sigma$ as an extended manifold by giving it weight zero.

Theorem 9.2. *One has that*

$$\rho_p(f,n)=\kappa^{n+e(\mathfrak{w})-n_\lambda(\mathfrak{w})}Z_p(\mathrm{I}\times\Sigma,s(\mathfrak{w}))\ . \tag{26}$$

Here, $e(\mathfrak{w})$ is exponent sum, λ is the given lagrangian $\lambda(\Sigma)$, and $n_\lambda(\mathfrak{w})=e(\mathfrak{w})+\sigma(L^0(\mathfrak{w}))$ is the integer defined in Theorem 4.2. The exponent in (26) can be written $n-\sigma(L^0(\mathfrak{w}))$ but we prefer the expression above as it stresses the dependence on the lagrangian λ.

Proof. As explained in the proof of Theorem 4.2, extended surgery along $L(\mathfrak{w})$ on the identity mapping cylinder $C(\mathrm{Id}_\Sigma,0)$ gives $C(f,n_0)$ where $n_0=\sigma(L^0(\mathfrak{w}))$. Therefore the surgery axiom implies that

$$Z_p(\mathrm{I}\times\Sigma,s(\mathfrak{w}))=Z_p(C(f,n_0))=\rho_p(f,n_0)\ .$$

This differs from $\rho_p(f,n)$ by the factor $\rho_p(W)^{n-n_0}$ where $W=C(\mathrm{Id}_\Sigma,1)$ is the generator of the center of $\widetilde{\Gamma}(\Sigma)$. But W acts as multiplication by κ on $V_p(\Sigma)$. This proves the result. ∎

Remark 9.3. The module $V_p(\Sigma)$ can be presented as a quotient of the skein module of a handlebody $\mathcal{H}$ with boundary Σ. In other words, the natural map which sends a skein element x in $\mathcal{H}$ to the vector $Z_p(\mathcal{H}, x)$ in $V_p(\Sigma)$, is onto. This follows from the surgery axiom as in [BHMV2, Prop. 1.9]. We remark that we can choose $\mathcal{H}$ arbitrarily here; in particular, we do *not* need to require that the given lagrangian $\lambda(\Sigma)$ be the kernel of $H_1(\Sigma) \to H_1(\mathcal{H})$. The endomorphism $Z_p(\mathrm{I} \times \Sigma, s(\mathfrak{w}))$ lifts to an endomorphism of the skein module of the handlebody $\mathcal{H}$. This endomorphism can then be computed skein-theoretically using recoupling theory [KL, MV]. Note that no further powers of κ are introduced when gluing $\mathcal{H}$ to $(\mathrm{I}\times\Sigma, s(\mathfrak{w}))$, because in the Maslov index computation, two of the three lagrangians are the same.[3] Thus the expression (26) in Theorem 9.2 gives a completely explicit description of $\rho_p(f, n)$. In particular, it explains how $\rho_p(f, n)$ depends on the lagrangian λ.

Here is an example showing how to use Theorem 9.2 to identify specific lifts of mapping classes to the extended mapping class group. Let $\mathcal{T}_c$ denote a torus equipped with one banded point colored $2c$. Assume $\mathcal{T}_c$ is presented as the boundary of a solid torus which we will denote by $\mathcal{H}$. Let m and ℓ be simple closed curves on $\mathcal{T}_c$ which avoid the banded point, and such that m is a meridian of $\mathcal{H}$ and ℓ is a longitude. Mapping classes of $\mathcal{T}_c$ must preserve the banded point, so that the ordinary mapping class group $\Gamma(\mathcal{T}_c)$ of $\mathcal{T}_c$ is the mapping class group of the one-holed torus obtained from $\mathcal{T}_c$ by removing an open disk neighborhood of the banded point. This group is generated by the Dehn twists $D(m)$ and $D(\ell)$. They satisfy

$$D(m)D(\ell)D(m) = D(\ell)D(m)D(\ell)\ , \tag{27}$$

and this is the only relation in a presentation of $\Gamma(\mathcal{T}_c)$ in terms of these generators.

In [GM2], we represented certain lifts of $D(m)$ and $D(\ell)$ to the extended mapping class group $\widetilde{\Gamma}(\mathcal{T}_c)$ by certain automorphisms t and $t^\star$ of $V_p(\mathcal{T}_c)$. Using Theorem 9.2, we can identify exactly which lifts these are by computing their weights as extended cobordisms. Of course, for this to make sense we need to choose a lagrangian λ for $\mathcal{T}_c$. We choose λ to be the lagrangian given by $\mathcal{H}$. Thus $[m] \in \lambda$ but $[\ell] \notin \lambda$.

Proposition 9.4. *As automorphisms of $V_p(\mathcal{T}_c)$, one has that*

$$t\ = \rho_p(D(m), 0) \tag{28}$$

$$t^* = \rho_p(D(\ell), 1) \tag{29}$$

Proof. The automorphisms t and $t^\star$ were defined skein-theoretically in [GM2]. We briefly review the definition. The module $V_p(\mathcal{T}_c)$ is a quotient of the 'relative' skein module of $\mathcal{H}$, where the word 'relative' indicates that the skein elements are linear combinations of banded trivalent graphs in $\mathcal{H}$ which nicely meet the banded point colored $2c$ on the boundary of $\mathcal{H}$. Let ω_+ denote κ times the skein element in the solid torus obtained by giving ω a full negative twist. (An explicit formula for ω_+, derived from [BHMV1], is given in [GM2].) Then t is the self-map of $V_p(\mathcal{T}_c)$ which sends a skein element x to x union ω_+ placed on the zero-framed meridian pushed slightly into the interior. Another definition of t is as the self-map of $V_p(\mathcal{T}_c)$ induced by a full positive twist of the solid torus $\mathcal{H}$. See Figure 11. The map t^* is defined similarly (to the first description of t) except that we use the zero-framed longitude in place of the zero-framed meridian. In order to make contact with Theorem 9.2,

[3]This is true even though there may well be a Maslov index contribution when gluing $C(f, n)$ to $\mathcal{H}$. Here one sees the strength of the surgery axiom.

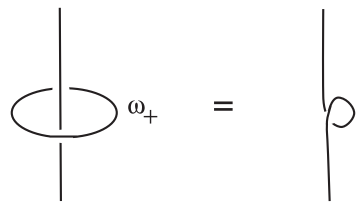

FIGURE 11. A picture for t. Encircling a strand with ω_+ has the same effect in TQFT as giving that strand a positive twist.

we denote by m_0 and ℓ_0 the meridian and longitude sitting on $\frac{1}{2} \times \mathcal{T}_c \subset \mathrm{I} \times \mathcal{T}_c$, with zero framing relative to the surface. Then the definitions of t and $t^\star$ can be reformulated as follows:

$$t = Z_p(\mathrm{I} \times \mathcal{T}_c,\ m_0 \text{ cabled by } \omega_+)$$
$$t^\star = Z_p(\mathrm{I} \times \mathcal{T}_c,\ \ell_0 \text{ cabled by } \omega_+)$$

Now [m_0 cabled by ω_+] is the same as κ times [m_- cabled by ω], where m_- is like m_0 but with -1 framing relative to the surface, as in Lemma 3.7. Thus we have a situation like on the R.H.S. of (26) in Theorem 9.2, and formula (28) for t follows from this by a signature computation. Formula (29) follows similarly. □

Remark 9.5. The following proof of (28) and (29) directly from the surgery axiom may be instructive. Since [m_0 cabled by ω_+] is the same as κ times [m_- cabled by ω], the surgery axiom gives

$$t = \kappa Z_p(C(m))$$

where $C(m)$ is extended surgery along m_- on $\mathrm{I}\times\mathcal{T}_c$ (see Lemma 3.7). By Lemma 3.7, since $[m] \in \lambda$, we have $C(m) = C(D(m), -1)$. Hence

$$t = \kappa Z_p(C(D(m), -1)) = Z_p(C(D(m), 0)) = \rho_p(D(m), 0)\ .$$

We similarly have

$$t^* = \kappa Z_p(C(\ell))$$

but this time $[\ell] \notin \lambda$, so Lemma 3.7 gives $C(\ell) = C(D(\ell), 0)$ and hence

$$t^\star = \kappa Z_p(C(D(\ell), 0)) = Z_p(C(D(\ell), 1)) = \rho_p(D(\ell), 1)\ .$$

Thus, the reason that the weights come out differently for t than for t^* is that $[m] \in \lambda$ but $[\ell] \notin \lambda$.

Remark 9.6. More generally, let α be a simple closed curve on Σ and define α_0 and α_- as above. Consider

$$W(\alpha) = W \circ C(\alpha) \in \widetilde{\Gamma}(\Sigma)^{++}$$

as defined in Section 3. We have

$$\begin{aligned} \rho_p(W(\alpha)) &= \kappa\rho_p(C(\alpha)) = \kappa Z_p(\mathrm{I} \times \Sigma,\ \alpha_- \text{ cabled by } \omega) \\ &= Z_p(\mathrm{I} \times \Sigma,\ \alpha_0 \text{ cabled by } \omega_+) \end{aligned} \tag{30}$$

This defines a representation of $\widetilde{\Gamma}(\Sigma)^{++}$ on $V_p(\Sigma)$. It is a fact that the skein element ω_+ has coefficients in the subring of k_p spanned by A and $\frac{1}{p}$; in other words, κ is not needed to define this representation. In the case $\Sigma = \mathcal{T}_c$, we have $t = \rho_p(W(m))$ and $t^* = \rho_p(W(\ell))$. In general, we have the following skein-theoretical interpretation of

$\rho_p(W(\alpha))$ for a simple closed curve α. Think of $V_p(\Sigma)$ as a quotient of the skein module of a handlebody $\mathcal{H}$ with boundary Σ. Then $\rho_p(W(\alpha))$ is the self-map of $V_p(\Sigma)$ which sends a skein element x to x union ω_+ placed on the zero-framed curve α pushed slightly into the interior. We emphasize that this is true even if $\lambda(\Sigma)$ is not equal to the kernel of $H_1(\Sigma) \to H_1(\mathcal{H})$. In analogy with [MR], we call $W(\alpha)$ the *geometric lift* of the Dehn twist $D(\alpha)$ to the extended mapping class group.

The following result was stated in [GM2, Remark 4.5]. Let $q = A^2$. This is a primitive p-th root of unity.

Proposition 9.7. *As automorphisms of $V_p(\mathcal{T}_c)$, one has that $tt^*t = t^*tt^*$ and*

$$(tt^*)^6 = q^{-6+2c(c+1)-p(p+1)/2}\,\mathrm{Id}_{V_p(\mathcal{T}_c)} \ . \tag{31}$$

Proof. The first relation, in a somewhat different context, is well-known [R, MR]. Here is a proof in our context. Since $tt^*t = \rho_p(W(m\ell m))$ and $t^*tt^* = \rho_p(W(\ell m\ell))$, it is enough to show that $W(m\ell m) = W(\ell m\ell)$. This is proved as follows. We have $D(m\ell m) = D(\ell m\ell)$, $e(m\ell m) = e(\ell m\ell) = 3$, and $\sigma\left(L^0\left(m\ell m\right)\right) = \sigma\left(L^0\left(\ell m\ell\right)\right) = -2$. By Theorem 4.2, it follows that both $W(m\ell m)$ and $W(\ell m\ell)$ are equal to $C(D(m\ell m), 1)$.

For the second relation, let δ be a simple closed curve in $\mathcal{T}_c$ around the banded point colored $2c$. In the mapping class group $\Gamma(\mathcal{T}_c)$, the Dehn twist $D(\delta)$ is equal to $\big(D(m)D(\ell)\big)^6$. Let $\mathfrak{u}$ denote the word $(m\ell)^6\delta^{-1}$ which is a relator. We have $e(\mathfrak{u}) = 11$ and $\sigma\left(L(\mathfrak{u})\right) = -7$. (N.b., it is frequently efficient to begin such signature calculations with a simplification of the framed link using Kirby calculus while keeping track of signature changes.) Thus

$$W(\mathfrak{u}) = C(\mathrm{Id}_{\mathcal{T}_c}, 11-7) = C(\mathrm{Id}_{\mathcal{T}_c}, 4) \ ,$$

hence $\rho_p(W(\mathfrak{u}))$ is multiplication by κ^4. It follows that

$$(tt^*)^6 = \Big(\rho_p(W(m))\rho_p(W(\ell))\Big)^6 = \kappa^4\rho_p(W(\delta)) = q^{-6-p(p+1)/2}\rho_p(W(\delta)) \ .$$

It remains to see that

$$\rho_p(W(\delta)) = q^{2c(c+1)}\,\mathrm{Id}_{V_p(\mathcal{T}_c)} \ .$$

In view of Remark 9.6, this can be done by a skein-theoretical computation. We have to compute the effect of encircling a $2c$-colored strand by ω_+. As shown in Figure 11, this is the same as giving that strand a full positive twist. Recall from [BHMV1] that the twist eigenvalue is $\mu_c = (-A)^{c(c+2)}$. Thus $\mu_{2c} = q^{2c(c+1)}$. This completes the proof. □

We end this section with one further technique allowing one to identify specific lifts of mapping classes to the extended mapping class group. This will allow us to compute how $(tt^*)^3$ acts on $V_p(\mathcal{T}_c)$, thereby giving another proof of (31) in which the signature computation is easier. Let Σ be the boundary of a handlebody $\mathcal{H}$, which we give weight zero. Recall that every element of $V_p(\Sigma)$ can be written $Z_p(\mathcal{H}, x)$ for some skein element x in $\mathcal{H}$.

Proposition 9.8. *Assume $f \in \Gamma(\Sigma)$ is the restriction of a diffeomorphism F of $\mathcal{H}$. Assume further that the lagrangian $\lambda(\Sigma)$ is the kernel of $H_1(\Sigma) \to H_1(\mathcal{H})$. Then $\rho_p(f, 0)$ sends $Z_p(\mathcal{H}, x)$ to $Z_p(\mathcal{H}, F(x))$.*

Proof. If we glue the pair $(\mathcal{H}, x)$ to the mapping cylinder of f by identifying the boundary of $\mathcal{H}$ with the source of the mapping cylinder by the identity map, and if we forget the weights for a moment, the result is diffeomorphic, rel. boundary, to the pair $(\mathcal{H}, F(x))$. Thus the proposition holds up to a power of κ which might come from a Maslov index contribution. But in our situation f preserves $\lambda(\Sigma)$, so the Maslov index contribution is zero. This completes the proof. □

Example 9.9. Consider again the torus $\mathcal{T}_c$ equipped with one banded point colored $2c$. The Dehn twist $D(\delta)$ has a square root θ called the *half-twist*. This can be roughly described as the result of giving most of the torus (sitting in 3-space as the boundary of an unknotted solid torus) a right handed twist through an angle π around an axis passing through the banded point (and three other points on the torus) while holding a neighborhood of the banded point fixed. In the mapping class group $\Gamma(\mathcal{T}_c)$, one has $\theta = (D(m)D(\ell))^3$. Now θ extends to the solid torus, so the proposition tells us that $\rho_p(\theta, 0)$ can be computed skein-theoretically. See Remark 9.10 below for a sketch of the calculation. The result is that

$$\rho_p(\theta, 0) = (-1)^c q^{c(c+1)} \operatorname{Id}_{V_p(\mathcal{T}_c)} \ . \tag{32}$$

Remark 9.10. The calculation in (32) can be done by considering the basis $L_{c,0}z^n$ of $V_p(\mathcal{T}_c)$ in the notation of [GM2]. This basis consists of eigenvectors, all with the same eigenvalue. Moreover, since θ is a square root of the Dehn twist $D(\delta)$, the eigenvalue must be a square root of $\mu_{2c} = q^{2c(c+1)}$. Determining the sign of the square root is, however, a little subtle. One way to see the factor $(-1)^c$ in (32) is to convince onself by drawing some pictures that the eigenvalue is $\delta(2c; c, c)\mu_c$, where $\delta(2c; c, c) = A^{c^2}$ is the half-twist coefficient of [MV, Theorem 3].

Corollary 9.11. *As automorphisms of $V_p(\mathcal{T}_c)$, one has*

$$(tt^*)^3 = A^{-6-p(p+1)/2}(-1)^c q^{c(c+1)} \operatorname{Id}_{V_p(\mathcal{T}_c)} \ .$$

Proof. We have that $e((m\ell)^3) = 6$, and $\sigma(L^0((m\ell)^3)) = -4$. Thus

$$W((m\ell)^3) = C(\theta, 2)$$

by Theorem 4.2. Hence

$$(tt^\star)^3 = \rho_p(W((m\ell)^3)) = \rho_p(\theta, 2) = \kappa^2 \rho_p(\theta, 0) \ ,$$

which implies the result in view of (32). □

10. Integral TQFT and representations in characteristic p

In this section, we consider the $SO(3)$-TQFT (Z_p, V_p) where $p \geq 5$ is a prime. An integral refinement of this TQFT was defined and studied in [G, GM1]. This gives in particular rise to finite-dimensional representations of the ordinary mapping class group in characteristic p. Our aim in this section is to explain the role played by the extensions $\widetilde{\Gamma}(\Sigma)^+$ and $\widetilde{\Gamma}(\Sigma)^{++}$ in this construction.

Recall $q = A^2$ is a primitive p-th root of unity. We denote the cyclotomic ring $\mathbb{Z}[q]$ by $\mathcal{O}_p^+$. We refer the reader to section 13 of [GM1] for the definition of the (refined) integral TQFT-module $\mathcal{S}_p^+(\Sigma)$. It is a free $\mathcal{O}_p^+$-module of finite rank. The ring $\mathcal{O}_p^+$ is a Dedekind domain, and we sometimes refer to $\mathcal{S}_p^+(\Sigma)$ as a lattice. There is a canonical inclusion

$$\mathcal{S}_p^+(\Sigma) \hookrightarrow V_p(\Sigma) \ .$$

We can think of this inclusion as tensoring with k_p, the coefficient ring of $V_p(\Sigma)$. Note that k_p is obtained from $\mathcal{O}_p^+$ by adjoining p^{-1} and κ to it.

Consider the action ρ_p of the extended mapping class group $\widetilde{\Gamma}(\Sigma)$ on $V_p(\Sigma)$ defined as in (25) by $\rho_p(f,n) = Z_p(C(f,n))$. Here is one of the main results of integral TQFT.

Theorem 10.1 ([GM1]). *If $p \equiv 3 \pmod 4$, then the lattice $\mathcal{S}_p^+(\Sigma)$ is preserved by $\widetilde{\Gamma}(\Sigma)$. If $p \equiv 1 \pmod 4$, then $\mathcal{S}_p^+(\Sigma)$ is preserved by the index two subgroup $\widetilde{\Gamma}(\Sigma)^+$ of $\widetilde{\Gamma}(\Sigma)$.*

Remark 10.2. This result is stated in [GM1, Section 13]. The reason that we need to restrict to $\widetilde{\Gamma}(\Sigma)^+$ if $p \equiv 1 \pmod 4$ is that in this case $\kappa = \rho_p(\mathrm{Id}_\Sigma, 1)$ does not lie in $\mathcal{O}_p^+$. (But for $p \equiv 3 \pmod 4$, one has $\kappa \in \mathcal{O}_p^+$.) In [GM1], we therefore mainly considered the slightly bigger coefficient ring $\mathcal{O}_p = \mathcal{O}_p^+[\kappa]$ and the lattice $\mathcal{S}_p(\Sigma) = \mathcal{S}_p^+(\Sigma) \otimes \mathcal{O}_p$. (If $p \equiv 3 \pmod 4$, one has $\mathcal{O}_p = \mathcal{O}_p^+$ and $\mathcal{S}_p(\Sigma) = \mathcal{S}_p^+(\Sigma)$.) The lattice $\mathcal{S}_p(\Sigma)$ is always preserved by the extended mapping class group $\widetilde{\Gamma}(\Sigma)$.

Let h denote $1-\zeta_p$; this is a prime in $\mathcal{O}_p^+$. For every $N \geq 0$, we may consider

$$\mathcal{S}_{p,N}^+(\Sigma) = \mathcal{S}_p^+(\Sigma)/h^{N+1}\mathcal{S}_p^+(\Sigma) \ ,$$

which is a free module over the quotient ring $\mathcal{O}_p^+/h^{N+1}\mathcal{O}_p^+$. Note that for $N=0$ this ring is the finite field $\mathbb{F}_p$, so that $\mathcal{S}_{p,0}^+(\Sigma)$ is a finite-dimensional $\mathbb{F}_p$-vector space.

Definition 10.3. *Let $\rho_{p,N}$ be the representation on $\mathcal{S}_{p,N}^+(\Sigma)$ induced from ρ_p, where we restrict ρ_p to $\widetilde{\Gamma}(\Sigma)^+$ if $p \equiv 3 \pmod 4$, and to $\widetilde{\Gamma}(\Sigma)^{++}$ if $p \equiv 1 \pmod 4$.*

Note that in this definition, we have restricted to a further index two subgroup with respect to the statement in Theorem 10.1. This is needed for the following corollary to hold.

Corollary 10.4. *The representation $\rho_{p,0}$ on the $\mathbb{F}_p$-vector space $\mathcal{S}_{p,0}^+(\Sigma)$ factors through a representation of the ordinary mapping class group $\Gamma(\Sigma)$.*

Proof. The generator of the kernel of $\widetilde{\Gamma}(\Sigma)^+ \to \Gamma(\Sigma)$ acts by $\kappa^2 = A^{-6-p(p+1)/2}$. Since p is odd and A is a primitive $2p$-th root of unity, we have $A = -q^{(p+1)/2}$. It follows that κ^2 is $(-1)^{p(p+1)/2}$ times a power of q. Since $q \equiv 1 \pmod h$, it follows that $\kappa^2 \equiv (-1)^{p(p+1)/2} \pmod h$. Thus κ^2 acts trivially on $\mathcal{S}_{p,0}^+(\Sigma)$ if $p \equiv 3 \pmod 4$. But if $p \equiv 1 \pmod 4$, then κ^2 acts by -1 and only κ^4 acts trivially. $\square$

Remark 10.5. In practice, in order to compute $\rho_{p,0}(f)$ for a mapping class f, one should fix a lagrangian λ, compute $\rho_p(f,n)$ for some $n \equiv n_\lambda(f) \pmod 4$, write $\rho_p(f,n)$ as a matrix in a basis of the lattice $\mathcal{S}_p^+(\Sigma)$ (see [GM1]), and reduce coefficients modulo h. Of course, if $p \equiv 3 \pmod 4$, it suffices to take $n \equiv n_\lambda(f)$ $\pmod 2$. Another way to make sure that one uses a lift of f to the correct subgroup of the extended mapping class group is to write f as a word in Dehn twists and to use the 'geometric' lifts, as explained in Remark 9.6.

Remark 10.6. In the case $p \equiv 1 \pmod 4$, the proof of [GM1, 14.2] should be amended to read $\widetilde{\Gamma}(\Sigma)^{++}$ instead of 'the (even) extended mapping class group'. In the last sentence of [GM1, p.837], $\widetilde{\Gamma}(\Sigma)^+$ should be replaced with $\widetilde{\Gamma}(\Sigma)^{++}$.

Remark 10.7. One may think of the sequence of representations $\rho_{p,N}$ as the h-adic expansion of the representation ρ_p. Explicit matrices for this expansion in the case of a one-holed torus were given in [GM2]. Note that each $\rho_{p,N}$ factors through a finite group, since $\mathcal{S}^+_{p,N}(\Sigma)$ is of finite rank over $\mathcal{O}^+_p/h^{N+1}\mathcal{O}^+_p$, which itself is finite. Thus the h-adic expansion approximates the TQFT-representation ρ_p by representations into bigger and bigger finite groups. We believe this h-adic expansion deserves further study.

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