THE MASLOV INDEX AND THE WALL SIGNATURE NON-ADDITIVITY INVARIANT

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The algebraic theory of surgery is used here to identify the Maslov index with the Wall invariant for the non-additivity of the signature.

A *lagrangian* of a nonsingular symplectic form $(V,\phi : V\times V\to\mathbb{R})$ on a finite-dimensional real vector space V is a subspace $L\subset V$ of half the rank which is maximally isotropic, i.e. such that

$$L ~=~ \{x\in V\,|\,\phi(x)(y)=0\in\mathbb{R}\text{ for all }y\in L\}~.$$

The *Maslov index* is defined for any configuration of three lagrangians L_1,L_2,L_3 in a nonsingular symplectic form (V,ϕ) to be the signature

$$M(L_1,L_2,L_3) ~=~ \sigma(L_1\oplus L_2\oplus L_3,\theta)\in\mathbb{Z}$$

of the singular symmetric form defined by

$$\begin{array}{l}\theta ~:~ L_1\oplus L_2\oplus L_3\times L_1\oplus L_2\oplus L_3\to\mathbb{R}~;\\ ((x_1,x_2,x_3),(y_1,y_2,y_3))\mapsto \sum\limits_{i\neq j}(-)^{i+j}\phi(x_i)(y_j)~.\end{array}$$

The *non-additivity invariant* of Wall [5] is defined for any three lagrangians L_1,L_2,L_3 in a nonsingular symplectic form (V,ϕ) to be the signature

$$W(L_1,L_2,L_3) ~=~ \sigma(W,\psi)\in\mathbb{Z}$$

of the nonsingular symmetric form defined by

$$\begin{array}{l}W ~=~ \dfrac{\{(x_1,x_2,x_3)\in L_1\oplus L_2\oplus L_3|x_1+x_2+x_3=0\in V\}}{\mathrm{im}(L_1\cap L_2+L_2\cap L_3+L_3\cap L_1)}~,\\ \psi ~:~ W\times W\to\mathbb{R}~;~((x_1,x_2,x_3),(y_1,y_2,y_3))\mapsto\phi(x_1)(y_2)~.\end{array}$$

The signature of an oriented $4k$-dimensional manifold with boundary $(X,\partial X)$ is the signature of the symmetric intersection form on $H^{2k}(X;\mathbb{R})$

$$\sigma(X) ~=~ \sigma(H^{2k}(X;\mathbb{R}))\in\mathbb{Z}~.$$

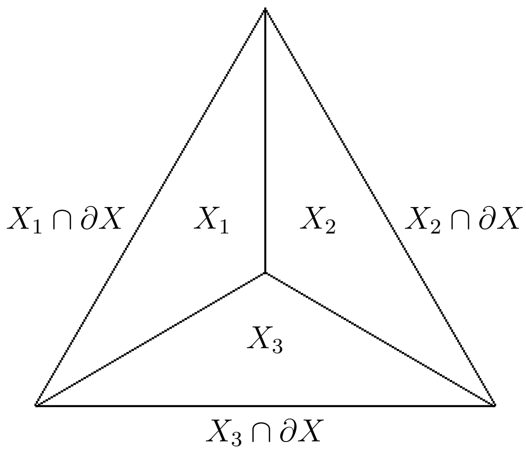
Given a decomposition as a union

$$X ~=~ X_1\cup X_2\cup X_3$$

of three codimension 0 manifolds with boundary meeting transversely with

$$X_1 \cap X_2 \cap \partial X \;=\; X_2 \cap X_3 \cap \partial X \;=\; X_3 \cap X_1 \cap \partial X \;=\; \emptyset$$

as in the diagram



the nonsingular symplectic intersection form on $H^{2k-1}(X_1 \cap X_2 \cap X_3; \mathbb{R})$ comes equipped with three lagrangians

$$\begin{aligned} L_1 &= \mathrm{im}(H^{2k-1}(X_2 \cap X_3; \mathbb{R}) \to H^{2k-1}(X_1 \cap X_2 \cap X_3; \mathbb{R})) \\ L_2 &= \mathrm{im}(H^{2k-1}(X_1 \cap X_3; \mathbb{R}) \to H^{2k-1}(X_1 \cap X_2 \cap X_3; \mathbb{R})) \\ L_3 &= \mathrm{im}(H^{2k-1}(X_1 \cap X_2; \mathbb{R}) \to H^{2k-1}(X_1 \cap X_2 \cap X_3; \mathbb{R})) \ . \end{aligned}$$

The main result of Wall [5] identifies the corresponding W-invariant with the defect of the Novikov additivity for the signature of the triple union

$$W(L_1, L_2, L_3) \;=\; \sigma(X_1) + \sigma(X_2) + \sigma(X_3) - \sigma(X) \in \mathbb{Z} \ .$$

Strictly speaking, only the case of the union of two manifolds with boundary was considered in [5], but the triple union case is a direct consequence - see the proof of Lemma 6 below.

The only object of this paper is to give a new proof of the following identification :

Theorem 1. (Cappell, Lee and Miller [1, §12]) *The Maslov index coincides with the Wall non-additivity invariant*

$$M(L_1, L_2, L_3) \;=\; W(L_1, L_2, L_3) \in \mathbb{Z} \ .$$

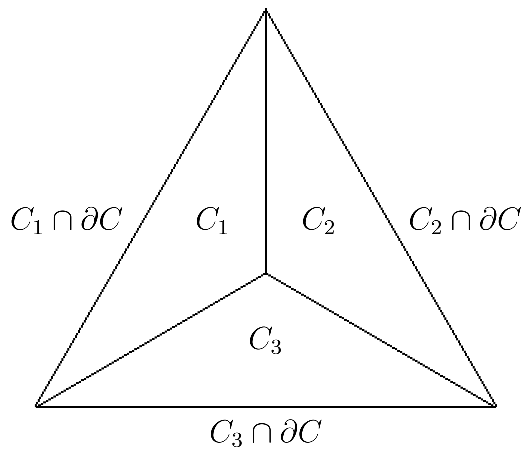
□

Theorem 1 was proved in [1] by direct manipulations of the quadratic forms, but it is more conceptual to use chain complexes with Poincaré duality in the style of Ranicki [2] and to formulate the Wall invariant for such cycles.

Definition 2. A *triple Poincaré cycle* $\Gamma = (C; C_1, C_2, C_3)$ is a chain complex C of finite-dimensional real vector spaces which is equipped with subcomplexes ∂C, C_i, ∂C_i $(i = 1, 2, 3)$ such that

$$\begin{aligned}
C &= C_1 + C_2 + C_3 \ , \\
C_1 \cap C_2 \cap \partial C &= C_2 \cap C_3 \cap \partial C = C_3 \cap C_1 \cap \partial C = 0 \ , \\
\partial C &= (C_1 \cap \partial C) \oplus (C_2 \cap \partial C) \oplus (C_3 \cap \partial C) \ , \\
\partial C_1 &= (C_1 \cap \partial C) \oplus (C_1 \cap C_2 + C_1 \cap C_3) \ , \\
\partial C_2 &= (C_2 \cap \partial C) \oplus (C_2 \cap C_3 + C_1 \cap C_1) \ , \\
\partial C_3 &= (C_3 \cap \partial C) \oplus (C_3 \cap C_1 + C_3 \cap C_2) \ ,
\end{aligned}$$

as in the diagram



It is required that $(C, \partial C)$ be a $4k$-dimensional algebraic Poincaré pair in the sense of Ranicki [3], with the chain equivalence $C^{4k-*} \simeq C/\partial C$ respecting the decomposition of C in such a way that :

(i) $(C_i, \partial C_i)$ $(1 \leqslant i \leqslant 3)$ is a $4k$-dimensional Poincaré pair,
(ii) $(C_i \cap C_j, C_1 \cap C_2 \cap C_3)$ $(1 \leqslant i < j \leqslant 3)$ is a $(4k-1)$-dimensional Poincaré pair,
(iii) $C_i \cap \partial C$ $(1 \leqslant i \leqslant 3)$ is a $(4k-1)$-dimensional Poincaré complex.

In particular, it follows that the triple intersection $C_1 \cap C_2 \cap C_3$ is a $(4k-2)$-dimensional Poincaré complex. $\square$

A triple Poincaré cycle Γ is a special case of a Poincaré cycle in the sense of Ranicki [4] over the simplicial complex defined by the above diagram.

Example 3. An oriented $4k$-dimensional manifold with boundary $(X, \partial X)$ which is expressed as a union $X = X_1 \cup X_2 \cup X_3$ as above determines a triple Poincaré cycle $\Gamma(X) = (C; C_1, C_2, C_3)$ with

$$\begin{aligned} (C, \partial C) &= (C(X;\mathbb{R}), C(\partial X;\mathbb{R})) , \\ (C_i, \partial C_i) &= (C(X_i;\mathbb{R}), C(\partial X_i;\mathbb{R}))\ (1 \leqslant i \leqslant 3) . \end{aligned}$$

□

The duality conditions in a triple Poincaré cycle $\Gamma = (C, C_1, C_2, C_3)$ ensure that $H^{2k}(C)$ is equipped with a form, and that $V = H^{2k-1}(C_1 \cap C_2 \cap C_3)$ is equipped with a nonsingular symplectic form, with three lagrangians

$$\begin{aligned} L_1 &= \mathrm{im}(H^{2k-1}(C_2 \cap C_3) \to H^{2k-1}(C_1 \cap C_2 \cap C_3)) , \\ L_2 &= \mathrm{im}(H^{2k-1}(C_1 \cap C_3) \to H^{2k-1}(C_1 \cap C_2 \cap C_3)) , \\ L_3 &= \mathrm{im}(H^{2k-1}(C_1 \cap C_2) \to H^{2k-1}(C_1 \cap C_2 \cap C_3)) . \end{aligned}$$

Definition 4. (i) The *Maslov index* of a triple Poincaré cycle Γ is defined by

$$M(\Gamma) = M(L_1, L_2, L_3) .$$

(ii) The *Wall non-additivity invariant* of a triple Poincaré cycle Γ is defined by

$$W(\Gamma) = W(L_1, L_2, L_3) .$$

□

The Theorem will be proved by identifying $M(\Gamma) = W(\Gamma)$ for every Γ.

The signature of a $4k$-dimensional Poincaré pair $(C, \partial C)$ over $\mathbb{R}$ is the signature of the singular form on $H^{2k}(C, \partial C)$

$$\sigma(C) = \sigma(H^{2k}(C, \partial C)) \in \mathbb{Z} .$$

For simplicity, the terminology makes no special mention of the actual symmetric bilinear pairing

$$H^{2k}(C, \partial C) \times H^{2k}(C, \partial C) \to \mathbb{R} ,$$

but it is understood to be the one induced by the given chain equivalence $C^{4k-*} \simeq C/\partial C$.

Lemma 5. *The Wall non-additivity invariant of a triple Poincaré cycle* Γ *is such that*

$$W(\Gamma) \;=\; \sigma(C_1)+\sigma(C_2)+\sigma(C_3)-\sigma(C)\in\mathbb{Z}\ .$$

Proof The proof of the corresponding theorem for manifolds in Wall [5] only uses the underlying Poincaré triple structure. In greater detail, consider first the union $4k$-dimensional Poincaré pair C_1+C_2 with boundary

$$\partial(C_1+C_2) \;=\; (\partial C_1\cap\partial C)\oplus(C_1+C_2)\cap C_3\oplus(\partial C_2\cap\partial C)\ .$$

A direct application of the main result of [5] gives

$$W(\Gamma) \;=\; \sigma(C_1)+\sigma(C_2)-\sigma(C_1+C_2)\in\mathbb{Z}$$

and Novikov additivity for $C=(C_1+C_2)+C_3$ gives

$$\sigma(C) \;=\; \sigma(C_1+C_2)+\sigma(C_3)\in\mathbb{Z}\ .$$

Elimination of $\sigma(C_1+C_2)$ from these two identities gives the required expression for $W(\Gamma)$. □

Lemma 6. *Given a nonsingular symplectic form* (V,ϕ) *and lagrangians* L_1,L_2,L_3 *there exists a triple Poincaré cycle* $\Gamma=(C;C_1,C_2,C_3)$ *with*

$$\begin{aligned}
V &= H^{2k-1}(C_1\cap C_2\cap C_3)\ ,\\
L_1 &= \mathrm{im}(H^{2k-1}(C_2\cap C_3)\to H^{2k-1}(C_1\cap C_2\cap C_3))\ ,\\
L_2 &= \mathrm{im}(H^{2k-1}(C_1\cap C_3)\to H^{2k-1}(C_1\cap C_2\cap C_3))\ ,\\
L_3 &= \mathrm{im}(H^{2k-1}(C_1\cap C_2)\to H^{2k-1}(C_1\cap C_2\cap C_3))\ ,
\end{aligned}$$

such that

$$M(L_1,L_2,L_3) \;=\; W(\Gamma) \;=\; W(L_1,L_2,L_3)\in\mathbb{Z}\ .$$

Proof See Ranicki [3, §1] for the glueing of algebraic Poincaré complexes in general, and of forms and formations in particular.
For any $(4k-1)$-dimensional Poincaré complex D product with the simplicial chain complex of $(I;\{0\},\{1\})$ defines a $4k$-dimensional Poincaré cobordism $D\otimes(I;\{0\},\{1\})$ between two copies of D, with signature

$$\sigma(D\otimes I) \;=\; 0\in\mathbb{Z}\ .$$

Let E be the $(4k-2)$-dimensional Poincaré complex defined by

$$E_r \;=\; \begin{cases} V^* & \text{if } r=2k-1\\ 0 & \text{otherwise}\ ,\end{cases}$$

and use the given lagrangians to define $(4k-1)$-dimensional null-cobordisms (F_i, E) $(i = 1, 2, 3)$ with

$$(F_i)_r = \begin{cases} L_i^* & \text{if } r = 2k-1 \\ 0 & \text{otherwise .} \end{cases}$$

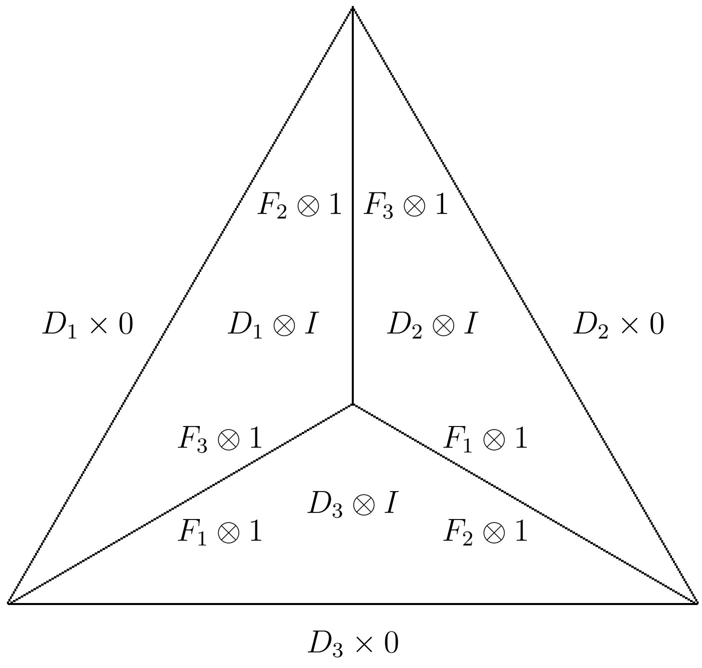
For each cyclic permutation (hij) of (123) define the union $4k$-dimensional Poincaré complex

$$D_h = F_i \cup_E F_j ,$$

the complex associated to the symplectic formation $(V; L_i, L_j)$ in Ranicki [10] with

$$(D_h)_r = \begin{cases} V^* & \text{if } r = 2k \\ L_i^* \oplus L_j^* & \text{if } r = 2k-1 \\ 0 & \text{otherwise .} \end{cases}$$

The required triple Poincaré cycle Γ is obtained by constructing the union of the $4k$-dimensional Poincaré cobordisms $D_i \otimes (I; \{0\}, \{1\})$ $(i = 1, 2, 3)$ with the identifications as in the diagram



The chain complex C is given up to chain equivalence by

$$d_C = \begin{pmatrix} j_1^* & j_1^* \\ j_2^* & 0 \\ 0 & j_3^* \end{pmatrix} : C_{2k+1} = V^* \oplus V^* \to C_{2k} = L_1^* \oplus L_2^* \oplus L_3^* ,$$

$$C_r = 0 \text{ for } r \neq 2k, 2k+1$$

with $j_i : L_i \to V$ $(i = 1, 2, 3)$ the inclusions. The signature of C is the Maslov index of Γ

$$\sigma(C) \;=\; \sigma(H^{2k}(C)) \;=\; \sigma(C^{2k}) \;=\; M(\Gamma) \in \mathbb{Z} \;,$$

since the composite $C^{4k-*} \to (C/\partial C)^{4k-*} \simeq C$ is given by the adjoint of the symmetric form

$$\theta \;=\; \begin{pmatrix} 0 & j_1^*\phi j_2 & j_1^*\phi j_3 \\ -j_2^*\phi j_1 & 0 & j_2^*\phi j_3 \\ -j_3^*\phi j_1 & -j_3^*\phi j_2 & 0 \end{pmatrix} \;:$$

$$C^{2k} \;=\; L_1 \oplus L_2 \oplus L_3 \to C_{2k} \;=\; L_1^* \oplus L_2^* \oplus L_3^*$$

used to define the Maslov index

$$M(\Gamma) \;=\; M(L_1, L_2, L_3) \;=\; \sigma(L_1 \oplus L_2 \oplus L_3, \theta) \in \mathbb{Z} \;.$$

The Wall invariant is given by Lemma 5 to be

$$\begin{aligned} W(L_1, L_2, L_3) \;&=\; W(\Gamma) \\ &=\; \sigma(D_1 \otimes I) + \sigma(D_2 \otimes I) + \sigma(D_3 \otimes I) - \sigma(C) \\ &=\; -\sigma(C) \;=\; -M(L_1, L_2, L_3) \in \mathbb{Z} \;. \end{aligned}$$

□

Remark 7. See Wall [6, pp.72-73] for the connection between the non-additivity invariant of Wall [5] and the proof in [6, §6] that the odd-dimensional surgery obstruction groups are abelian. □

References

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