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On central extensions of mapping class groups

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1 Introduction

The mapping class group representations related to quantum invariants (à la Jones, Witten [35], Reshetikhin-Turaev [26], Lickorish [17, 18], Blanchet et al. [5], Wenzl [34], Turaev-Wenzl [31], and probably many more) are naturally only projective representations, as argued by Atiyah in [3]. A particularly nice way to observe this is through the approach from the skein theory of the Kauffman bracket, where a simple existence proof of projective actions was given in Roberts [27]. (Related projective actions had previously been constructed by other authors, e.g. Kohno [16], using other methods.)

The aim of this paper is to describe the central extensions of the mapping class group generated by these projective actions, and to see how the signature cocycle arises.

Fix a standard genus g Heegaard splitting of the 3-sphere. The genus g mapping class group, denoted by Γ in this paper, is generated by the associated two handlebody subgroups, which act naturally on the skein modules of the two handlebodies. This was used in [27] to construct a projective representation of the mapping class group. We call this the *geometric action* since it is obtained by moving around links in handlebodies.

In genus one the situation is particularly simple. Here the mapping class group Γ is $Sl(2,\mathbf{Z})$, and, as is well known, the projective action can be lifted to a linear action. (This is no longer true in higher genus.) Nevertheless, the *natural geometric actions* of the two Dehn twists corresponding to the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ do generate a central extension, in fact, they generate the braid group B_3. The starting point for this paper was the question of how this generalises to higher genus.

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As this example already indicates, we do not consider a projective action as a homomorphism into a projective linear group, but as an explicit lift of that homomorphism on a generating set. In our case, such an explicit lift is naturally given on the union of the two handlebody groups. There is then a natural notion of the central extension generated by such a projective action. In our case, we have a series of projective actions corresponding to different roots of unity, and the extensions are naturally quotients of a common central extension of the mapping class group Γ by $\mathbf{Z}$. This extension is denoted by $\tilde{\Gamma}_2$.

We describe this extension in two ways. The first is through a presentation. We show that the restriction of the original geometric action to Dehn twists defines an extension $\tilde{\Gamma}_1$ which is of index two in $\tilde{\Gamma}_2$. The presentation of $\tilde{\Gamma}_1$ induced by the geometric action looks just like Harer's [11] presentation of the mapping class group except that one relator is omitted. This presentation is given in Theorem 3.8. It is nice to observe that this is a straightforward generalisation of the genus one situation: the geometric action of Dehn twists defines an extension which is related to the mapping class group just as the braid group B_3 is to $Sl(2, \mathbf{Z})$.

Second, we relate the extension class of $\tilde{\Gamma}_2$ in $H^2(\Gamma; \mathbf{Z})$ to the signature extension in the following way. Using a skein-theoretical interpretation of the geometric action, the latter can be naturally extended to all Dehn twists. With hindsight from 'Topological Quantum Field Theory', we define a 'corrected' action by multiplying the action on Dehn twists by a certain constant factor. This corrected action defines an extension $\tilde{\Gamma}_4$, and $\tilde{\Gamma}_2$ is identified with a subgroup of index two in $\tilde{\Gamma}_4$. We show that $\tilde{\Gamma}_4$ is isomorphic to the signature extension, as described by Atiyah [3]. Hence twice the extension class of $\tilde{\Gamma}_2$ in $H^2(\Gamma; \mathbf{Z})$ is represented by the signature cocycle.

Note. The notations $\tilde{\Gamma}_i$ ($i = 1, 2, 4$) are motivated by the fact that the extension class group $H^2(\Gamma; \mathbf{Z})$ is cyclic in all genus [11, 12], and the class of $\tilde{\Gamma}_i$ is i times a generator. In genus at least three, the mapping class group is perfect, and the extension $\tilde{\Gamma}_1$ is its universal central extension.

1.1 Motivation: the genus one case

The starting point for this paper was the situation in genus one which is particularly simple, and which we describe first.

Consider the standard decomposition

$$S^3 = T \bigcup_{S^1 \times S^1} T'$$

of the 3-sphere into two solid tori T, T'. Let $\mathscr{S}(T)$ (resp. $\mathscr{S}(T')$) denote the (Jones-Kauffman-) *skein module* of the solid torus T (resp. T').[1] There is a bilinear form

[1] The skein module $\mathscr{S}(M)$ of a compact oriented 3-manifold M is the $\mathbf{Z}[A, A^{-1}]$-module generated by isotopy classes of banded links in M, modulo the Kauffman [14] bracket relations (see for example [18] where $\mathscr{S}(T)$ is denoted by $\mathscr{A}$.) The Kauffman relations imply that $\mathscr{S}(S^3) \approx \mathbf{Z}[A, A^{-1}]$; the isomorphism is called the Kauffman bracket and denoted by $\langle\ \rangle$. It is normalised so that the bracket of the empty link is 1.

$$\langle\ ,\ \rangle : \mathscr{S}(T) \times \mathscr{S}(T') \to \mathbf{Z}[A, A^{-1}]$$

which associates to a pair of banded links L, L' in T, T' the Kauffman bracket of the banded link $L \cup L'$ in S^3.

This form was studied for Kauffman's skein variable A a primitive $4r$-th root of unity in a series of papers by Lickorish [17, 18], and in the general case in Blanchet, Habegger, Masbaum and Vogel [5] where an orthogonal basis (over $\mathbf{Z}[A, A^{-1}]$) is described. It turns out that if one changes coefficients to the cyclotomic field $\mathbf{Q}(\zeta_{2p})$ (i.e. if one replaces A by a primitive $2p$-th root of unity ζ_{2p}) for some integer $p \geqq 3$, then the form $\langle\ ,\ \rangle$ becomes degenerate, and the quotient of $\mathscr{S}(T)$ by its left kernel is a vector space V_p of dimension $n = [(p-1)/2]$.

Let t (resp. t') denote the self-map of $\mathscr{S}(T)$ (resp $\mathscr{S}(T')$) induced by a positive Dehn twist. It turns out that t descends to V_p, and that t' has an *adjoint* $t^\star$, defined by requiring $\langle t^\star(x), y\rangle = \langle x, t'^{-1}(y)\rangle$ for all $x, y \in V_p$. With the methods of [18] or [5], it is quite easy to check that the endomorphisms t and $t^\star$ satisfy the following two relations in $End(V_p)$:

$$tt^\star t = t^\star t t^\star \tag{1}$$

$$(tt^\star t)^2 = \zeta_{2p}^{-6-p(p+1)/2} . id_{V_p} \tag{2}$$

Observe that this is a projective representation of the mapping class group of the torus $S^1 \times S^1$. Indeed, the latter is the group $Sl(2, \mathbf{Z})$, which is generated by the matrices $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, with relations

$$aba = bab \tag{3}$$

$$(aba)^4 = 1\ . \tag{4}$$

Because of formula (2), the smallest extension of $Sl(2, \boldsymbol{Z})$ which resolves the projective action on V_p to a linear action is a central extension by a cyclic group of order growing with p. To obtain an extension which acts linearly on *all* of the V_p, we must suppress relation (2) and keep only the braid relation (1). The group so presented is the braid group B_3.

Thus, we have a series of projective actions of $Sl(2, \mathbf{Z})$ which combine to 'generate' a central extension of $Sl(2, \mathbf{Z})$ by $\mathbf{Z}$, and the class of this extension (the group B_3) is well known to be a generator of $H^2(Sl(2, \mathbf{Z}); \mathbf{Z}) \approx \mathbf{Z}/12$.

We shall see that the above generalises quite nicely to higher genus.

Remark 1.1 As pointed out by Atiyah [3], in genus one the projective action can be renormalised to a linear action. (This does not generalise to higher genus.) In fact, one can renormalise to a linear action of $PSl(2, \mathbf{Z})$. In formulas, the renormalised action is given by

$$\tilde{t} = \kappa_p^{-1} t, \quad \tilde{t}^\star = \kappa_p^{-1} t^\star$$

where κ_p is such that

$$\kappa_p^6 = \zeta_{2p}^{-6-p(p+1)/2} \tag{5}$$

Observe that relation (1) is unaffected, but relation (2) becomes

$$(\tilde{t}\tilde{t}^{\star}\tilde{t})^2 = id_{V_p} .$$

Since $(aba)^2$ is the mapping class π represented by the central element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in Sl(2, \mathbf{Z})$, this is indeed a linear action of $PSl(2, \mathbf{Z})$. (It is nice to observe that the mapping class π extends to the solid torus T, and its geometric action on the skein module $\mathscr{S}(T)$, hence on V_p, is indeed the identity.)

Note. This renormalised action of $PSl(2, \mathbf{Z})$ was used by Freed and Gompf [8] to compute the Witten [35] invariant for certain Seifert manifolds. Here is the precise relationship: Assume $p = 2k + 4$ is even. Put $\zeta_{2p} = -e^{2\pi i/2p}$ and $\kappa_p = e^{2\pi ic/24}$ where $c = 3k/(k+2)$ is the so-called *central charge.* Then $\tilde{t}$ and $\tilde{t}\tilde{t}^{\star}\tilde{t}$ are the matrices $\tilde{T}$ and $\tilde{S}$ used in [8].

2 The projective actions ρ_p

2.1 The geometric action of the handlebody groups

The representation space $V_p(g)$ of the geometric action ρ_p is constructed as follows. Consider the standard genus g Heegaard splitting $S^3 = H \cup_{\Sigma} H'$. As in the case of genus one, the Kauffman bracket gives a bilinear form

$$\langle\ ,\ \rangle : \mathscr{S}(H) \times \mathscr{S}(H') \to \mathscr{S}(S^3) = \mathbf{Q}(\zeta_{2p})$$

(Here we work with coefficients in the cyclotomic field $\mathbf{Q}(\zeta_{2p})$, with Kauffman's skein variable A replaced by a primitive $2p$-th root of unity ζ_{2p}.)

Definition 2.1 *$V_p(g)$ is the quotient of the skein module $\mathscr{S}(H)$ by the left kernel of the form $\langle\ ,\ \rangle$.*

The vector space $V_p(g)$ is finite-dimensional. This was shown in 1991 by Blanchet et al., as part of their construction of TQFT's from the Kauffman bracket [6]. A purely skein-theoretical proof of the finite-dimensionality of $V_p(g)$ was later independently given by Lickorish [19] in the case where Kauffman's skein variable A is a $4r$-th root of unity (i.e. when p is even.)

Let K, K' be the *handlebody subgroups*, i.e. K (resp. K') is the subgroup of the *mapping class group* $\Gamma = \pi_0(\mathrm{Diff}^+(\Sigma))$ consisting of those mapping classes which extend to H (resp. H'.) Clearly K acts on $\mathscr{S}(H)$ and K' acts on $\mathscr{S}(H')$.

Theorem 2.2 [27] *The actions of the handlebody groups K on $\mathscr{S}(H)$ and K' on $\mathscr{S}(H')$ induce a linear action ρ_p of Free$(K \cup K')$ on $V_p(g)$ which descends to a projective action ρ_p of the mapping class group Γ.*

This is proved directly in [27] purely within skein theory in the case of a $4r$-th root of unity. The result can also be deduced from the general theory in [6], and is true for $2p$-th roots of unity with odd p as well.

We call ρ_p the *geometric action*, since it is obtained by moving links around in handlebodies.

Here is a description of the action. If $x \in \mathcal{S}(H)$ and $f \in K$, denote by $f_\star(x)$ the result of moving x with the unique (up to isotopy) diffeomorphism of H extending f. Put $\rho_p(f)(x) = f_\star(x)$. If $f \in K'$, define $\rho_p(f) \in End(V_p(g))$ by

$$\langle \rho_p(f)(x),\, y\rangle = \langle x,\, f_\star^{-1}(y)\rangle$$

for all $x \in \mathcal{S}(H)$, $y \in \mathcal{S}(H')$.

Note that $\rho_p(f_1 f_2) = \rho_p(f_1)\rho_p(f_2)$, where, as usual, $f_1 f_2$ means first apply f_2 then f_1, and so ρ_p is a left action. Note also that the linear actions ρ_p of K and K' on $V_p(g)$ coincide on $K \cap K'$.

Language. An element of a skein module $\mathcal{S}(M)$ of a 3-manifold M will be called a *skein element* in M. The manifold $\Sigma \times \mathbf{I}$ will be called a *shell.*

Notation. If s is a skein element in the shell $\Sigma \times \mathbf{I}$, denote by $Add(s)$ the endomorphism of $\mathcal{S}(H)$ given by adding s in the shell $\Sigma \times \mathbf{I}$ (viewed as an external collar of H) and pushing the result back into H.

Remark 2.3 Here is an outline of Roberts' proof of Theorem 2.2. One observes that

$$\rho_p(f)Add(s)\rho_p(f^{-1}) = Add(f_\star(s)) \tag{6}$$

(where $f_\star(s)$ is s transported by f extended to $\Sigma \times \mathbf{I}$.) If a word $r \in Free(K \cup K')$ represents a relator in Γ, then $r_\star(s) = s$. Hence $\rho_p(r)$ commutes with $Add(s)$ for all skein elements s in $\Sigma \times \mathbf{I}$. One now shows that $End\, V_p(g)$ is generated by endomorphisms of the form $Add(s)$. Hence, $\rho_p(r)$ is central in $End(V_p(g))$, i.e. $\rho_p(r)$ is multiplication by a scalar. This proves that ρ_p is a projective action of Γ on $V_p(g)$.

2.2 A skein-theoretical description of the action

Notation. Let $\mathscr{C}$ be the set of isotopy classes of unoriented simple closed curves on Σ. Let $\mathscr{D}$ be the corresponding set of positive Dehn twists. If $\alpha \in \mathscr{C}$, let $t_\alpha \in \mathscr{D}$ be the positive Dehn twist about α.

Convention. We shall identify $\mathscr{C} = \mathscr{D}$.

Notation. For $\alpha \in \mathscr{C}$, let $\alpha^{(+)}$ (resp. $\alpha^{(-)}$) be the banded circle (annulus) in the shell $\Sigma \times \mathbf{I}$ whose underlying circle is the curve α 'drawn' on $\Sigma \times \frac{1}{2}$, and such that the band has framing $+1$ (resp. -1) relative to $\Sigma \times \frac{1}{2}$.

From now on we work over the ring $\mathbf{Q}(\zeta_{2p})[\kappa_p]/(\kappa_p^6 - \zeta_{2p}^{-6-p(p+1)/2})$ (compare this with formula (5) in Remark 1.1).

Lemma 2.4 *There is a skein element ω_p in the solid torus such that if $t_\alpha \in \mathscr{D} \cap (K \cup K')$ then*

$$\rho_p(t_\alpha^{\pm 1}) = \kappa_p^{\pm 3} Add(\alpha^{(\mp)}[\omega_p]) \tag{7}$$

where $\alpha^{(\mp)}[\omega_p]$ is the skein element in $\Sigma \times \mathbf{I}$ obtained by cabling $\alpha^{(\mp)}$ by ω_p.

Remark 2.5 This result is well known. The skein element ω_p is up to scalar multiples the one used by many authors (e.g. [17, 18] [5] [24]) to construct 3-manifold invariants. The precise normalisation of ω_p is as in [6], Sect. 2. Formula (7) in genus one is an equation which determines ω_p completely (not as a skein element, but as an element of $V_p = V_p(1)$.)

Definition 2.6 *Define a new projective action $\hat{\rho}_p$ of Γ on $V_p(g)$ by setting*

$$\hat{\rho}_p(t_\alpha^{\pm 1}) = Add(\alpha^{(\mp)}[\omega_p]) \tag{8}$$

for all Dehn twists $t_\alpha \in \mathscr{D}$. Also, extend the original projective action ρ_p to all of $\mathscr{D}$ by setting

$$\rho_p(t_\alpha^{\pm 1}) = \kappa_p^{\pm 3} \hat{\rho}_p(t_\alpha^{\pm 1}) \tag{9}$$

2.3 The main lemma

Consider a word $w = \prod_{i=1}^n t_{\alpha_i}^{\varepsilon_i} \in Free(\mathscr{D})$, where $\alpha_i \in \mathscr{C}$ and $\varepsilon_i \in \{\pm 1\}$. Let $L(w)$ be the banded link in

$$H \bigcup_\Sigma \left(\Sigma \times \mathbf{I} \bigcup_\Sigma \dots \bigcup_\Sigma \Sigma \times \mathbf{I} \right) \bigcup_\Sigma H' = S^3$$

(n copies of $\Sigma \times \mathbf{I}$) given by inserting $\alpha_i^{(-\varepsilon_i)}$ into the $(n-i)$-th shell (counting from the left.) We define a "signature" function $\sigma_b : Free(\mathscr{D}) \to \mathbf{Z}$ as follows.

Definition 2.7 $\sigma_b(w) =$ signature $L(w)$, *where* signature $L(w)$ *is the signature of the 4-manifold obtained by attaching 2-handles to the 4-ball along $L(w)$. (The subscript b in σ_b stands for 'ball'.)*

Notation. Let $\mathscr{R} = ker(Free(\mathscr{D}) \to \Gamma)$ be the set of all relations between Dehn twists in Γ. For a mapping class $f \in K \cup K'$, let $w_f \in Free(\mathscr{D})$ be a word in Dehn twists representing f. Let $e(w)$ be the exponent sum of a word $w \in Free(\mathscr{D})$.

Lemma 2.8 (Main lemma) *Recall that $\rho_p : Free(\mathscr{D} \cup K \cup K') \to End(V_p)$ is well-defined. Then we have*
(i) $\rho_p(r) = \kappa_p^{3(e(r)+\sigma_b(r))}$ *for all $r \in \mathscr{R}$,*
(ii) $\rho_p(w_f f^{-1}) = \rho_p(f^{-1} w_f) = \kappa_p^{3(e(w_f)+\sigma_b(w_f))}$ *for all words w_f representing $f \in K \cup K'$,*
(iii) $\hat{\rho}_p(r) = \kappa_p^{3\sigma_b(r)}$ *for all $r \in \mathscr{R}$.*

Proof. Observe that (i) is a special case of (ii), and that (i) implies (iii) by the definitions of ρ_p and $\hat{\rho}_p$. Hence, it suffices to show (ii).

Let $\hat{s}(w_f)$ be the skein element in the shell $\Sigma \times \mathbf{I}$ obtained by cabling the link $L(w_f)$ (which lies in the shell) by ω_p. Let $s(w_f) = \kappa_p^{3e(w_f)}\hat{s}(w_f)$. So

$$\rho_p(w_f) = Add(s(w_f)).$$

Because the word $w_f f^{-1}$ represents the identity mapping class, and ρ_p is a projective representation of the mapping class group, $\rho_p(w_f f^{-1})$ is a scalar multiple of the identity. This scalar may be computed as follows. Let $v_\emptyset$ denote the element of $V_p(g)$ represented by the empty link in H. Using the bilinear form $\langle\ ,\ \rangle$, the empty link in H' defines a linear form on $V_p(g)$ which will be denoted by $\langle\ \ , v_\emptyset\rangle$. Let $\langle s(w_f)\rangle$ be the bracket of $s(w_f)$, considered as a skein element in $S^3 = H \cup_\Sigma (\Sigma \times I) \cup_\Sigma H'$. It is clear from the definition of $V_p(g)$ that

$$\langle \rho_p(w_f)(v_\emptyset), v_\emptyset\rangle = \langle s(w_f)\rangle\ .$$

On the other hand,

$$\langle \rho_p(f)(v_\emptyset), v_\emptyset\rangle = \langle v_\emptyset, v_\emptyset\rangle = 1$$

since any $f \in K$ fixes the empty vector $v_\emptyset$, any $f \in K'$ fixes the linear form $\langle\ , v_\emptyset\rangle$, and by convention the number $\langle v_\emptyset, v_\emptyset\rangle$, which is the bracket of the empty link in S^3, is 1. Hence

$$\rho_p(f^{-1}w_f) = \rho_p(w_f f^{-1}) = \langle s(w_f)\rangle\ .$$

Since $s(w_f) = \kappa_p^{3e(w_f)}\hat{s}(w_f)$, it remains to show that

$$\langle \hat{s}(w_f)\rangle = \kappa_p^{3\sigma_b(w_f)}\ . \tag{10}$$

The reason behind formula (10) is that the skein element ω_p is the one that gives 3-manifold invariants, and is correctly normalised. Here are the details.

Notation. For $f \in \Gamma$, let

$$C(f) = \Sigma \times \left[0, \tfrac{1}{2}\right] \bigcup_{(x,\frac{1}{2})\sim(f(x),\frac{1}{2})} \Sigma \times \left[\tfrac{1}{2}, 1\right]$$

be its *mapping cylinder.* Let M_f denote the closed 3-manifold

$$H \bigcup_{\Sigma} C(f) \bigcup_{\Sigma} H'$$

Thus, M_f has *Heegaard splitting* (Σ, f). Observe that if f is in K or K', then M_f is diffeomorphic to S^3.

The following is well known.

Lemma 2.9 *Let $\alpha \in \mathscr{C}$. The result of surgery (rel. boundary) on the banded link $\alpha^{(\mp)} \subset \Sigma \times \mathbf{I}$ is the mapping cylinder $C(t_\alpha^{\pm 1})$.*

It follows that if $f \in \Gamma$ is represented by a word $w = \prod_{i=1}^n t_{\alpha_i}^{\varepsilon_i} \in Free(\mathscr{D})$, then the result of surgery along the banded link $L(w)$ is precisely the manifold M_f.

Proof of formula (10). By Kirby's theorem about surgery presentations of 3-manifolds and the fundamental property of ω_p, the number $\kappa_p^{-3\sigma_b(w)}\langle \hat{s}(w)\rangle$ is a topological invariant $I_p(M_f)$ of the closed 3-manifold M_f. (This invariant is just a renormalised version of the invariant obtained in [18] and in [5].) In our case, $f \in K \cup K'$, hence $M_f = S^3$. But $I_p(S^3) = 1$ (since S^3 is also obtained by surgery on the empty link.) The result follows.

3 Three central extensions of the mapping class group

3.1 *Extensions generated by projective actions*

Let G be a group and V a free k-module where k is a commutative ring with unit. It is well-known that a projective representation of G on V, thought of as a homomorphism $\bar{\rho} : G \to PGL_k(V)$, may be 'resolved' to a linear action of a central extension $\tilde{G}_{\bar{\rho}}$ of G by $k^\star$.

This may be thought of in several ways. From a formal point of view, $\tilde{G}_{\bar{\rho}}$ may be constructed by pulling back the 'tautological' central extension $k^\star \to GL_k(V) \to PGL_k(V)$ by the homomorphism $\bar{\rho}$.

By the central extension *generated by a projective action*, we mean a slightly different concept. We define a *projective action* of G on V to be a pair (S, ρ) such that $S \subset G$ is a generating set, and $\rho : S \to GL_k(V)$ is a map such that the induced homomorphism $\rho : \mathit{Free}(S) \to \mathit{Aut}(V)$ satisfies $\rho(r) \in k^\star$ for all relators r in G, i.e. for all $r \in R = \mathit{ker}(\mathit{Free}(S) \to G)$. Note that (S, ρ) induces a homomorphism $\bar{\rho} : G \to PGL_k(V)$.

We define the *central extension of G generated by the projective action* (S, ρ), denoted by $\tilde{G}(S, \rho)$, to be the smallest quotient of $\mathit{Free}(S)$ that (i) maps homomorphically to Γ and (ii) resolves the projective action to a linear action. The group $\tilde{G}(S, \rho)$ is the quotient of $\mathit{Free}(S)$ by R_ρ, where $R_\rho = \{r \in R | \rho(r) = 1\}$. (Note that R_ρ is a normal subgroup of $\mathit{Free}(S)$.) The map ρ induces a homomorphism $\tilde{G}(S, \rho) \to Gl_k(V)$ which is again denoted by ρ.

Note. The extension $\tilde{G}(S, \rho)$ is a central extension of G by the subgroup $\rho(R) \subset k^\star$, i.e. it is naturally identified with a *subgroup* of $\tilde{G}_{\bar{\rho}}$. (In fact the kernel is generated by the set of values $\rho(r)$, for a set of relators in a presentation of G by the set S.)

3.2 *Definition of the extensions $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, and $\tilde{\Gamma}_4$*

Recall that we have defined three projective actions of the mapping class group on $V_p(g)$: the original geometric action $(K \cup K', \rho_p)$, its skein-theoretical version $(\mathscr{D}, \rho_p)$ on Dehn twists, and the 'corrected' action $(\mathscr{D}, \hat{\rho}_p)$. They all induce the same homomorphism $\Gamma \to PGl(V_p(g))$, but the extensions generated by them will turn out to differ slightly.

By the main lemma, for all three of these projective actions, any relator acts by a power of κ_p^3 which is independent of p. Hence in all three cases the projective actions for different p fit together to generate a common central extension. The latter will be an extension by $\mathbf{Z}$ since the order of the root of unity κ_p^3 is not bounded as $p \to \infty$.

Definition 3.1 *We denote by $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, and $\tilde{\Gamma}_4$ the central extensions of the mapping class group Γ generated by the three projective actions referred to above, i.e. by a slight abuse of notation*

$$\begin{aligned}\tilde{\Gamma}_1 &= \tilde{\Gamma}(\mathscr{D}, \rho)\\ \tilde{\Gamma}_2 &= \tilde{\Gamma}(K \cup K', \rho)\\ \tilde{\Gamma}_4 &= \tilde{\Gamma}(\mathscr{D}, \hat{\rho})\end{aligned}$$

Let us denote the isotopy class of a small (null-homotopic) unknot in Σ by $T \in \mathscr{C}$. Continuing our convention $\mathscr{C} = \mathscr{D}$, let T denote also the corresponding Dehn twist in $\mathscr{D}$. Observe that $\hat{\rho}_p(T) = \kappa_p^{-3}$ and $\sigma_b(T) = -1$. (Note that T is the identity as a mapping class, but does not map to 1 under $\hat{\rho}_p$.) The main Lemma 2.8 immediately gives the following:

Theorem 3.2 (i) *The group $\tilde{\Gamma}_4 = \tilde{\Gamma}(\mathscr{D}, \hat{\rho})$ is a central extension of Γ by $\mathbf{Z}$. The kernel of $\tilde{\Gamma}_4 \to \Gamma$ is generated by T. Under $\hat{\rho}_p$, it acts as multiplication by κ_p^{-3} on $V_p(g)$.*
(ii) *The group $\tilde{\Gamma}_4$ has a presentation*

$$\tilde{\Gamma}_4 = \langle \mathscr{D} | \{ r T^{\sigma_b(r)} : r \in \mathscr{R} \} \rangle = \langle \mathscr{D} | \{ r : r \in \mathscr{R}, \sigma_b(r) = 0 \} \rangle\, .$$

Note. The group $\tilde{\Gamma}(\mathscr{D}, \hat{\rho}_p)$ is the quotient of $\tilde{\Gamma}_4$ by the relation $T^{\nu_p} = 1$ where ν_p is the order of the root of unity κ_p^3.

3.3 Comparing the extensions

The three central extensions are related in the following way. Let $\mathscr{D}_0 = \mathscr{D} \cap (K \cup K')$ be the set of Dehn twists about curves that bound a disc in H or H' (this set still generates Γ). One has the following commutative diagram of inclusions:

$$\begin{array}{ccc} \tilde{\Gamma}(\mathscr{D}_0, \rho) & \hookrightarrow & \tilde{\Gamma}_2 \\ \downarrow\approx & & \downarrow\varphi \\ \tilde{\Gamma}_1 & \xrightarrow{\varphi} & \tilde{\Gamma}_4\, . \end{array}$$

Here, the maps $\tilde{\Gamma}(\mathscr{D}_0, \rho) \hookrightarrow \tilde{\Gamma}_2$ and $\tilde{\Gamma}(\mathscr{D}_0, \rho) \hookrightarrow \tilde{\Gamma}_1$ are induced by the inclusions $\mathscr{D}_0 \subset K \cup K'$ and $\mathscr{D}_0 \subset \mathscr{D}$. Observe that every Dehn twist is conjugate to one in K, and ρ_p is the identity on conjugation relators by Roberts' argument (see Remark 2.3, putting $s = \beta^{(-)}[\omega_p]$.) Thus extending the geometric actions ρ_p from $\mathscr{D}_0$ to $\mathscr{D}$ does not produce any new projective factors. Hence the map $\tilde{\Gamma}(\mathscr{D}_0, \rho) \hookrightarrow \tilde{\Gamma}_1$ is an isomorphism, as indicated in the diagram.

Thus, $\tilde{\Gamma}_1$ is naturally a subgroup of $\tilde{\Gamma}_2$, which justifies the notation φ for both the maps $\tilde{\Gamma}_2 \hookrightarrow \tilde{\Gamma}_4$ and $\tilde{\Gamma}_1 \hookrightarrow \tilde{\Gamma}_4$. They are defined as follows:

$$\begin{aligned}\varphi(f) &= T^{\sigma_b(w_f)} w_f\\ \varphi(t_\alpha) &= T^{-1} t_\alpha\end{aligned}$$

where $f \in K \cup K'$ is represented by $w_f \in Free(\mathcal{D})$, and $t_\alpha \in \mathcal{D}$. (To see that this gives well-defined inclusions maps it suffices to check e.g. $\hat{\rho}_p(\varphi(f)) = \rho_p(f)$ which is trivial using the main lemma 2.8.)

Theorem 3.3 (i) *The injection* $\varphi : \tilde{\Gamma}_2 \hookrightarrow \tilde{\Gamma}_4$ *has index two.*
(ii) *The injection* $\varphi : \tilde{\Gamma}_1 \hookrightarrow \tilde{\Gamma}_4$ *has index four.*

Proof. The higher genus generalisation of the element $\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in Sl(2, \mathbf{Z})$ is a 180 degree twist along the (separating) waist curve of a 1-handle of H. Denote this mapping class again by π. It lies in $K \cap K'$. In Γ, π may be written as $w_\pi = (ML)^3$ where M and L are Dehn twists along the meridian and the longitude of the handle. We now use the relator $\pi^{-1}(ML)^3$ to show that the index of the inclusions $\tilde{\Gamma}_2 \hookrightarrow \tilde{\Gamma}_4$ (resp. $\tilde{\Gamma}_1 \hookrightarrow \tilde{\Gamma}_4$) is at most two (resp. four.)

The number $\sigma_b((ML)^3)$ is the signature of the linking matrix of the link of six -1-framed unknots in S^3 corresponding to the word $w_\pi = (ML)^3$. This is easily computed to be -4, either directly from the matrix or by doing a few handleslides on the link first. Since $e((ML)^3) + \sigma_b((ML)^3) = 6 - 4 = 2$, the main lemma 2.8 implies

$$\varphi(\pi^{-1}(ML)^3) = T^{-2} . \tag{11}$$

Thus, the index of $\varphi : \tilde{\Gamma}_2 \hookrightarrow \tilde{\Gamma}_4$ is at most two. Since π^2 is a Dehn twist in $\mathcal{D}$, and $\varphi(\pi^{-2}(ML)^6) = T^{-4}$, it also follows that the index of $\varphi : \tilde{\Gamma}_1 \hookrightarrow \tilde{\Gamma}_4$ is at most four.

Theorem 3.3 now follows from the main lemma and the following result which is worth stating independently.

Proposition 3.4 (i) *If* $f \in K \cup K'$ *is represented by* $w_f \in Free(\mathcal{D})$, *then* $e(w_f) + \sigma_b(w_f)$ *is even.*
(ii) *If* $r \in \mathcal{R}$, *then* $e(r) + \sigma_b(r) \equiv 0$ *mod* 4.

Proof. For (i), observe that taken modulo 2, the exponent $e(w_f)$ is the size of the linking matrix of $L(w_f)$, and the signature is its number of non-zero diagonal entries once diagonalised. Thus $\sigma(w_f) + e(w_f)$ equals (mod 2) the number of zero diagonal entries, that is the first Betti number of the 3-manifold M_{w_f}. But $M_{w_f} = S^3$, since $w_f \in K$, hence this is zero.

We defer the proof of (ii) until subsection 3.5, where we will make use of the presentation of the mapping class group which we have so far avoided.

Note. It is possible to prove (ii) without using a presentation (in fact using Meyer's [23] result that the signature of a surface bundle over a closed surface is divisible by 4).

3.4 A presentation of the mapping class group

The first explicit presentation of Γ for all genus was written down by Wajnryb [32], following the work of Hatcher and Thurston [13] and Harer [11]. For our purposes, the following (infinite) presentation, implicit in Harer's work and easily derived from Wajnryb's (finite) presentation, is sufficient.

Let $\mathscr{C}_{ns} \subset \mathscr{C}$ be the set of isotopy classes of unoriented simple closed curves on Σ which are *non-separating* (i.e. which are essential in homology.) Let $\mathscr{D}_{ns} \subset \mathscr{D}$ be the corresponding set of Dehn twists. The positive Dehn twist about a curve δ will be denoted by t_δ. Observe that twists about non-separating curves α, β satisfy the relation

$$t_\alpha t_\beta (t_\alpha)^{-1} = t_{\beta^\alpha} \tag{12}$$

in Γ, where β^α denotes the curve β translated by t_α. The corresponding relator of the mapping class group is called a *conjugation relator*, and denoted by $r_{\alpha,\beta} \in Free(\mathscr{D}_{ns})$.

Note. It is nice to think of the set of non-separating curves in the surface as acting on itself in this way, giving it the structure of a *wrack* (or *rack*) (see e.g. [7].)

Theorem 3.5 [11, 32] *The mapping class group Γ is presented as*

$$\langle \mathscr{D}_{ns} | R_{conj} \cup \{c, d\} \rangle$$

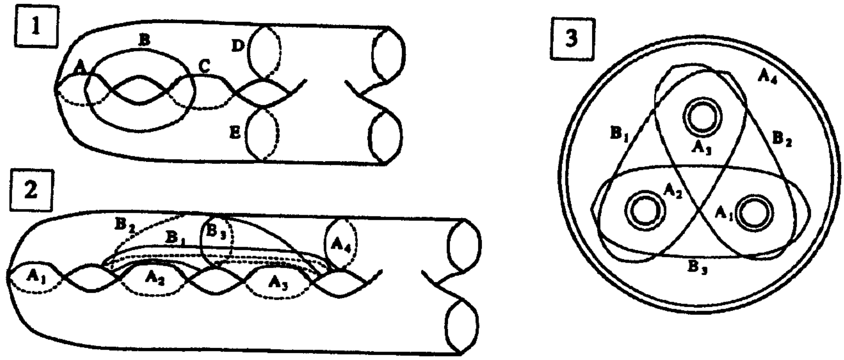
where $R_{conj} = \{r_{\alpha,\beta} | \alpha, \beta \in \mathscr{C}_{ns}\}$ are the conjugation relations between elements of $\mathscr{D}_{ns}$, and c, d are two exceptional words described below. (In genus one and two, relator d is omitted.)

The exceptional words are $c = (ABA)^4$ in genus one, $c = (ABC)^4 D^{-1} E^{-1}$ in genus $\geqq 2$, and $d = A_1 A_2 A_3 A_4 B_3^{-1} B_2^{-1} B_1^{-1}$ in genus $\geqq 3$. Here the $A, B, \ldots$ are twists around the curves shown in Figs. 1 and 2. More precisely, the word c is obtained from an embedding of the 2-holed torus into Σ such that the five curves A, B, C, D, E are non-separating in Σ; Fig. 1 shows a standard way of doing this, from which it is clear that in genus two, one has $D = E$, and in genus one, one has $A = C$, and D and E are suppressed. The word d is obtained from an embedding of the 3-holed disk shown in Fig. 3 such that the seven curves $A_1, A_2, A_3, A_4, B_3, B_2, B_1$ are non-separating in Σ; again, Fig. 2 shows a standard way of doing this and shows that d exists only in genus $\geqq 3$. (Observe that the choice of the embeddings is actually irrelevant, because any two embeddings of the 2-holed torus or the 3-holed disk with the above properties are related by a diffeomorphism of Σ.)

The following well known result will be needed later.

Lemma 3.6 *The group presented as $\langle \mathscr{D}_{ns} | R_{conj} \rangle$ is a central extension of Γ.*

Proof. Let the word $r \in Free(\mathscr{D}_{ns})$ be a relator in Γ. Let t_δ be any twist in $\mathscr{D}$, then by applying a series of conjugation relations, $r t_\delta r^{-1} = t_{\delta^r}$, where δ^r



Figs. 1–3

denotes δ translated by the mapping class represented by r. Since r is a relator, $\delta^r = \delta$. Thus r commutes with all Dehn twists, hence is central in $\langle \mathscr{D}_{ns} | R_{conj} \rangle$. The result follows.

Remark 3.7 Relator d is the so-called *lantern relator* (usually pictured as in Fig. 3.) Its existence shows that Γ is perfect in genus $\geqq 3$, and (hence) has a *universal central extension.* According to Harer [11], p. 238, the latter is presented as $\langle \mathscr{D}_{ns} | R_{conj} \cup \{d\} \rangle$.

Note. Gervais [9, 10] has shown that one may replace R_{conj} by the subset of those conjugation relators $r_{\alpha,\beta}$ where the curves α and β intersect in at most one point.

3.5 Proof of Proposition 3.4(ii)

Here is the proof of part (ii) of Proposition 3.4. We must show that $e(r) + \sigma_b(r) \equiv 0 \bmod 4$ for all $r \in \mathscr{R} = ker(Free(\mathscr{D}) \to \Gamma)$: equivalently, that $\rho_p(r)$ is always a power of κ_p^{12}. It is sufficient to show this

(i) for relators c, d and the conjugation relators $r_{\alpha,\beta}$ in Harer's presentation of Γ as quotient of $Free(\mathscr{D}_{ns})$, and

(ii) for relators of the form $(t_\delta)^{-1} w_\delta$ where w_δ is a word in $Free(\mathscr{D}_{ns})$ representing a Dehn twist t_δ along a *separating* curve δ.

We first show (ii). Since any Dehn twist is conjugate to one in K, we may assume $t_\delta \in K$ is a twist around a waist curve δ. Since δ is separating, we may write $t_\delta = f^2$ for an $f \in K$ (corresponding to a 180 degree flip of one half of the surface). Thus, we may take $w_\delta = w_f^2$ where $w_f \in Free(\mathscr{D}_{ns})$ represents f. Observe that $\varphi((t_\delta)^{-1} w_f^2) = \varphi(f^{-1} w_f)^2$. Now we know already that $e(w_f) + \sigma_b(w_f)$ is even, hence $\varphi(f^{-1} w_f)$ is a power of T^2. It follows that $\varphi(f^{-1} w_f)^2$ is a power of T^4, whence the result for t_δ.

Next, we show (i). For a conjugation relator, the result is clear since we know that $\rho_p(r_{\alpha,\beta}) = 1$ from Roberts' argument (see Remark 2.3.) For d, observe that $e(d) = 1$ and $\sigma_b(d) = -1$ (for the last statement, observe that the seven curves in the word d are mutually unlinked unknots.) For relator c, the result is $e(c)+\sigma_b(c)=4$, which is equivalent to $\rho_p(c)=\langle s(c)\rangle=\kappa_p^{12}$. To see this, observe that $L(c)$ is a 14-component link lying in a neighborhood of a torus in S^3. It follows that it is sufficient to do the computation in the case of genus one, when $A=C$ and $\langle s(c)\rangle=\langle s((AB)^6)\rangle$. But the word $(AB)^3$ represents the mapping class $\pi\in K$, and by formula (11) we know that $\rho_p(\pi^{-1}(AB)^3)=\langle s((AB)^3)\rangle=\kappa_p^6$. Hence $\langle s(c)\rangle=\kappa_p^{12}$, as asserted. This completes the proof.

3.6 Presentations of the extended groups

We can now spell out presentations of the extended groups. Recall that the mapping class group is presented as $\Gamma=\langle\mathscr{D}_{ns}|R_{conj}\cup\{c,d\}\rangle$.

Theorem 3.8 *For genus $g\geqq 3$, the extended groups have presentations*
(i) $\tilde{\Gamma}_1=\langle\mathscr{D}_{ns}|R_{conj}\cup\{d\}\rangle$
(ii) $\tilde{\Gamma}_2=\langle\mathscr{D}_{ns}\cup\{U\}|R_{conj}\cup\{d,cU^{-2}\}\cup\{U\ central\}\rangle$
(iii) $\tilde{\Gamma}_4=\langle\mathscr{D}_{ns}\cup\{T\}|R_{conj}\cup\{dT^{-1},cT^{-6}\}\cup\{T\ central\}\rangle$

Proof. In the preceding subsection, we have seen that $\rho_p(d)=1$, $\rho_p(c)=\kappa_p^{12}$, and $\rho_p(r_{\alpha,\beta})=1$ for all $\alpha,\beta\in\mathscr{C}_{ns}$. It follows that $\tilde{\Gamma}(\mathscr{D}_{ns},\rho)$ is presented as $\langle\mathscr{D}_{ns}|R_{conj}\cup\{d\}\rangle$. Observe that we do not need relators expressing that c is central because the group presented as $\langle\mathscr{D}_{ns}|R_{conj}\rangle$ is already a central extension of Γ (see Lemma 3.6.) Also, part (ii) of the proof of Proposition 3.4(ii) shows that the inclusion $\mathscr{D}_{ns}\subset\mathscr{D}$ induces an isomorphism

$$\tilde{\Gamma}(\mathscr{D}_{ns},\rho)\xrightarrow{\approx}\tilde{\Gamma}(\mathscr{D},\rho)=\tilde{\Gamma}_1$$

This implies the result for $\tilde{\Gamma}_1$. For $\tilde{\Gamma}_2$, the result follows from Theorem 3.3(i). (Geometrically, U is the relator $\pi^{-1}(ML)^3$ used in formula (11).) For $\tilde{\Gamma}_4$, one may use Theorem 3.3(ii) (or compute $\hat{\rho}_p(d)=\kappa_p^{-3}$, $\hat{\rho}_p(c)=\kappa_p^{-18}$.)

Remark 3.9 (i) The kernels of the extensions are generated by relator c, U and T respectively. The groups act linearly on $V_p(g)$ by the representations $\rho_p,\rho_p,\hat{\rho}_p$ respectively, under which c acts as κ_p^{12}, U as κ_p^6, and T as κ_p^{-3}.

(ii) In the case of genus one or two, the relators corresponding to d are omitted. In the case of genus one, the relator cT^{-6} in $\tilde{\Gamma}_4$ must be replaced by cT^{-4} because c is defined differently.

(iii) In genus one, $\mathscr{D}_0=\mathscr{D}\cap(K\cup K')\subset\Gamma$ is identified with $\{a,b\}\subset Sl(2,\mathbf{Z})$, where $a=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $b=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$. The extension $\tilde{\Gamma}_1$ is the braid group $B_3=\langle a,b\mid aba=bab\rangle$. It is obtained from the presentation of $Sl(2,\mathbf{Z})$ by omitting the relation $(aba)^4=1$ (i.e. relator c.) It is nice to observe that in higher genus, the situation is exactly the same: the extension

$\tilde{\Gamma}_1$ is presented just as the mapping class group Γ, except that relator c is omitted.[2]

Theorem 3.10 *In $H^2(\Gamma;\mathbf{Z})$, $[\tilde{\Gamma}_4] = 2[\tilde{\Gamma}_2] = 4[\tilde{\Gamma}_1]$. The extension class $[\tilde{\Gamma}_1]$ generates $H^2(\Gamma;\mathbf{Z})$, and $[\tilde{\Gamma}_4]$ is represented by the signature cocycle.*

Remark 3.11 It is known that the group $H^2(\Gamma;\mathbf{Z})$ is cyclic in all genus: in fact $H^2(\Gamma;\mathbf{Z})$ is $\mathbf{Z}$ in genus at least three, $\mathbf{Z}/12$ in genus one, and $\mathbf{Z}/10$ in genus two, see [11, 12]. The fact that the signature cocycle represents four times a generator (in all genus) is well known (Meyer [23], p. 240.)

Proof of Theorem 3.10. The first statement follows from Theorem 3.3 by standard group cohomology. The second statement follows from the presentation of $\tilde{\Gamma}_1$ obtained in Theorem 3.8. Indeed, in genus at least three, the presentation $\tilde{\Gamma}_1 = \langle \mathcal{D}_{ns} | R_{conj} \cup \{d\}\rangle$ shows that $H_1(\tilde{\Gamma}_1;\mathbf{Z}) = 0$, and since $H_1(\Gamma;\mathbf{Z}) = 0$, it follows by elementary group cohomology that $[\tilde{\Gamma}_1]$ is a generator of $H^2(\Gamma;\mathbf{Z}) = \mathbf{Z}$. In genus one and two, the presentation $\tilde{\Gamma}_1 = \langle \mathcal{D}_{ns} | R_{conj}\rangle$ shows that $H_1(\tilde{\Gamma}_1;\mathbf{Z}) = \mathbf{Z}$. But now $H^2(\Gamma;\mathbf{Z}) = Ext(H_1(\Gamma;\mathbf{Z}),\mathbf{Z})$, and the class $[\tilde{\Gamma}_1]$ is represented by the short exact sequence $\mathbf{Z} \to H_1(\tilde{\Gamma}_1;\mathbf{Z}) \to H_1(\Gamma;\mathbf{Z})$. Since $H_1(\tilde{\Gamma}_1;\mathbf{Z}) = \mathbf{Z}$, this proves $[\tilde{\Gamma}_1]$ is a generator. As for the third statement, we shall give in subsection 4.3 a direct proof that $\tilde{\Gamma}_4$ is isomorphic to the signature-cocycle extension by using p_1-structures.

4 The relationship with Atiyah's extension and p_1-structures

4.1 2-framings and p_1-structures

Atiyah [3] has described extensions of mapping class groups in terms of 2-*framings*, and explained their relationship with the signature cocycle. He defined a 2-framing on a closed 3-manifold to be a framing of twice the tangent bundle.

As in [6], it will be convenient to replace the notion of 2-framings by the more homotopy-theoretical notion of *p_1-structures* which makes sense in all dimensions. (This is analogous to the case of spin structures, where it is sometimes convenient to think of a spin structure on an oriented manifold M as a fibre homotopy class of lifts of the stable tangent bundle to BSpin.) The formal definition is as follows.

Definition 4.1 [6] *Let X be the homotopy fibre of the map $p_1 : BO \to K(\mathbf{Z},4)$ corresponding to the first Pontrjagin class of the universal stable bundle γ over BO. Let γ_X be the pull-back of γ over X. A p_1-structure on a manifold M is a fibre map from the stable tangent bundle of M, τ_M, to γ_X.*

Thus, a fibre homotopy class of p_1-structures is the analogue of a spin structure, where the second Stiefel-Whitney class w_2 is replaced by the first

[2] It is tempting to say that this relator is 'quantized'.

Pontrjagin class p_1. (For the relevant classical algebraic topology, see for instance [29].)

Note. A *p_1-manifold* is a manifold together with an actual p_1-structure, rather than a homotopy class of such structures. This is important when it comes to gluing them together.

It is easy to see that every manifold of dimension $\leqq 3$ admits a p_1-structure. In dimension $\leqq 2$, p_1-structures are unique up to homotopy. In dimension 3, homotopy classes of p_1-structures ξ on a closed connected oriented 3-manifold M are *classified* by the $\mathbf{Z}$-valued invariant $\sigma(M,\xi)$ defined as follows. There exist compact oriented 4-manifolds W with $\partial W = M$. Given such a W, let $p_1(W,\xi) \in \mathbf{Z}$ denote the obstruction to extending ξ to W, evaluated on the fundamental class.

Definition 4.2 *If ξ is a p_1-structure in M, let*

$$\sigma(M,\xi) = 3\text{signature}(W) - p_1(W,\xi)\,,$$

where W is any compact oriented 4-manifold with boundary M. (Compare [3]: our definition corresponds to 3 times Atiyah's σ.)

The fact that this number is independent of the choice of W follows from Hirzebruch's signature theorem.

Remark 4.3 There is an obvious notion of *p_1-surgery*, that is, we demand that the trace of the surgery has a p_1-structure. If M_2^3 is obtained from M_1^3 by surgery along a framed knot or link, then every p_1-structure on M_1^3 extends over the trace of the surgery (uniquely up to homotopy), and hence determines a p_1-structure on M_2^3 (uniquely defined up to homotopy).

4.2 The extended mapping class groups $\tilde{\Gamma}_s$ and $\tilde{\Gamma}_{p_1}$

Let Σ be a connected oriented closed surface, and let Γ be the mapping class group of Σ. For $f \in \Gamma$, let $C(f)$ be its mapping cylinder. Fix a p_1-structure ξ_0 on Σ.

Definition 4.4 *The extended mapping class group $\tilde{\Gamma}_{p_1}$ is the set of pairs $(f,[\xi])$ where $f \in \Gamma$, and $[\xi]$ is a homotopy class (rel boundary) of p_1-structures ξ on the mapping cylinder $C(f)$ extending ξ_0 on $\Sigma \times 0$ and $\Sigma \times 1$, together with the obvious composition.*

If ξ is a p_1-structure on $C(f)$ extending ξ_0, let $\hat{\xi}$ be the p_1-structure on the mapping torus $T(f)$ obtained by gluing. This induces a bijection between homotopy classes of p_1-structures on the mapping cylinder (rel boundary) and the mapping torus. Thus for $(f,[\xi]) \in \tilde{\Gamma}_{p_1}$, the number $\sigma(T(f),\hat{\xi})$ is well defined. The resulting function $\tilde{\Gamma}_{p_1} \to \mathbf{Z}$ will again be denoted by σ.

The forgetful map $\tilde{\Gamma}_{p_1} \to \Gamma$ is a central extension by $\mathbf{Z}$.

Remark. Atiyah's extended group [3] is defined in terms of 2-framings on mapping tori. The assignment $\xi \mapsto \hat{\xi}$ together with the identification of homotopy classes of 2-framings with homotopy classes of p_1-structures provides an identification of the extension $\tilde{\Gamma}_{p_1}$ with Atiyah's group.

Definition 4.5 *Denote by $\tilde{\Gamma}_s$ the subset of those $x \in \tilde{\Gamma}_{p_1}$ such that $\sigma(x) \equiv 0$ mod 3. (The subscript 's' stands for signature.)*

Proposition 4.6 *$\tilde{\Gamma}_s$ is a subgroup of index 3 in $\tilde{\Gamma}_{p_1}$.*

Proof. This follows easily (by Hirzebruch's formula and additivity of the signature) from the fact that a p_1-structure ξ on a 3-manifold M can be extended to *some* compact 4-manifold if and only if its σ-invariant is divisible by 3. In other words, the cobordism group $\Omega_3^{p_1}$, of oriented 3-manifolds with p_1-structure, is isomorphic to $\mathbf{Z}/3\mathbf{Z}$.

It is clear that the kernel of the extension $\tilde{\Gamma}_{p_1} \to \Gamma$ is generated by the element $T_1 = (id, [\xi])$ where ξ satisfies $\sigma(\Sigma \times S^1, \hat{\xi}) = 1$. Thus

$$\sigma(T_1) = 1 . \tag{13}$$

Moreover, it is clear that for all $x \in \tilde{\Gamma}_{p_1}$, one has

$$\sigma(x \,.\, T_1) = \sigma(T_1 \,.\, x) = \sigma(x) + 1 . \tag{14}$$

Because of properties (13) and (14), the function $\sigma : \tilde{\Gamma}_{p_1} \to \mathbf{Z}$ corresponds to a set-theoretical section of the extension, and determines a 2-cocycle c_σ on the mapping class group Γ. (The cocycle is given by $c_\sigma(f_1, f_2) = \sigma(\tilde{f}_1 \,.\, \tilde{f}_2) - \sigma(\tilde{f}_1) - \sigma(\tilde{f}_2)$, where $\tilde{f}_1, \tilde{f}_2$ are any lifts of f_1, f_2.) Similarly, the function $\sigma/3 : \tilde{\Gamma}_s \to \mathbf{Z}$ determines a cocycle c_s on Γ. Clearly $c_\sigma = 3c_s$.

Proposition 4.7 [3] *The 2-cocycle c_s is the signature cocycle.*

This result is a direct consequence of the definition of the signature cocycle and Hirzebruch's signature theorem. Thus the cocycle c_σ is three times the signature cocycle. Hence, as observed by Atiyah [3], the extension $\tilde{\Gamma}_{p_1}$ represents twelve times a generator in $H^2(\Gamma; \mathbf{Z})$ (since the signature extension is four times a generator.)

4.3 Identifying $\tilde{\Gamma}_4$ and $\tilde{\Gamma}_s$

We now complete the proof of theorem 3.10. For a Dehn twist $t_\alpha \in \mathcal{D}$, p_1-surgery on the curve $\alpha^{(-)}$ in $\Sigma \times \mathbf{I}$ equipped with the product structure $\xi_0 \times 1$ defines a lift $\tilde{t}_\alpha \in \tilde{\Gamma}_s$. Equivalently (a simple check), define $\tilde{t}_\alpha$ by requiring $\sigma(\tilde{t}_\alpha) = -3$. Extend this definition to a homomorphism $\psi : Free(\mathcal{D}) \to \tilde{\Gamma}_s$ by $\psi(t_\alpha) = \tilde{t}_\alpha$.

Proposition 4.8 *ψ induces an isomorphism of extensions $\psi : \tilde{\Gamma}_4 \xrightarrow{\approx} \tilde{\Gamma}_s$*

Proof. First, ψ is surjective, because $\psi(T) = T_1^{-3}$ is the central generator of $\tilde{\Gamma}_s$. If we show that $\sigma_b(w) = 0$ for any word $w \in Free(\mathcal{D})$ satisfying $\psi(w) = 1$, then the kernel of ψ is $\{r \in \mathscr{R} : \sigma_b(r) = 0\}$, and the result follows from the presentation of $\tilde{\Gamma}_4$ given in theorem 3.2.

If $\psi(w) = 1$, then by definition of ψ, p_1-surgery on the link $L(w) \subseteq (\Sigma \times \mathbf{I}, \xi_0 \times 1)$ produces $(\Sigma \times \mathbf{I}, \xi)$, where ξ is homotopic to $\xi_0 \times 1$.

Let $S^3 = H \cup (\Sigma \times \mathbf{I}) \cup H'$ with p_1-structure $\xi_0 \times 1$ on $\Sigma \times \mathbf{I}$, extended arbitrarily outside. Regard this as the boundary of the 4-ball, and perform the p_1-surgery above corresponding to w. Because ξ is homotopic to $\xi_0 \times 1$, the σ-invariant (computed in S^3) does not change under this surgery. Also the relative Pontrjagin classes involved in computing these σ-invariants are equal, by definition of a p_1-surgery. Therefore the signature of B^4 union 2-handles along $L(w)$ equals that of B^4, hence the result $\sigma_b(w) = 0$.

5 Further remarks

5.1 Cocycles cohomologous to the signature cocycle

Let X be any p_1-cobordism from (Σ, ξ_0) to itself. Let $\sigma_X : \tilde{\Gamma}_{p_1} \to \mathbf{Z}$ be defined as follows. For $(f, [\xi]) \in \tilde{\Gamma}_{p_1}$, take the corresponding structure on $C(f)$ and define the number $\sigma_X(f, [\xi]))$ to be the σ-invariant of the closed p_1-manifold obtained by gluing the mapping cylinder $C(f)$ and X together along $\Sigma \amalg -\Sigma$. The function σ_X satisfies (13) and (14), hence defines a cocycle c_X on Γ. All such cocycles (for different X) are cohomologous, since they classify the same extension.

The two isomorphic extensions $\tilde{\Gamma}_4$ and $\tilde{\Gamma}_s$ fit into this picture, as follows. First, take $X = H' \amalg H$, where $S^3 = H \cup_\Sigma H'$ is the standard Heegaard splitting, and H, H' are equipped with the restrictions of a p_1-structure ξ on S^3 such that $\sigma(S^3, \xi) = 0$. The associated function σ_X satisfies the following: If $\tilde{f} \in \tilde{\Gamma}_4$ is represented by a word $w \in Free(\mathscr{D})$, then

$$\tfrac{1}{3}\sigma_X(\psi(\tilde{f})) = \sigma_b(w)\,.$$

On the other hand, if X_0 is the identity cobordism (with product p_1-structure), the function $\frac{1}{3}\sigma_{X_0}$ is simply the function $\frac{1}{3}\sigma$. Thus, the cocycles defined from the functions σ_b and $\frac{1}{3}\sigma$ are cohomologous.

Remark 5.1 If X_1, X_2 are as above, the difference $\sigma_{X_1} - \sigma_{X_2}$ is a well-defined function on the original mapping class group Γ, and the difference $c_{X_1} - c_{X_2}$ is the coboundary of that function, considered as a 1-cochain. In fact, since Γ is perfect (in genus 3 or more), this is the *unique* such 1-cochain. One may use Wall's non-additivity of signature formula to describe this function in terms of homological data. It would be nice to know whether there is a nice formula for this function in the case of interest here. (Compare this with the case of genus one, where the signature cocycle is the coboundary of a unique (rational) cochain which is almost the Rademacher ϕ-function (see e.g. [23] [1] [4] [15]).)

5.2 Three-manifold invariants without Kirby's theorem

Let $f \in \Gamma$ be represented by a word $w_f \in Free(\mathscr{D})$, and let $M = M_f$ have Heegaard splitting (Σ, f) as in subsection 2.3. In [27], the *projective* action of Γ on $V_p(g)$ was used to give a very simple proof that the absolute value of $\langle \rho_p(w_f)(v_\emptyset), v_\emptyset \rangle$ (considered as a complex number) is a topological invariant of M. (This invariant is the absolute value of the invariant $I_p(M)$ that was used in the proof of the main lemma.) The proof in [27] used only the Reidemeister-Singer theorem about Heegaard splittings, but not Kirby's theorem on surgery presentations of 3-manifolds (which is generally considered more difficult).

In fact, one can obtain the invariant $I_p(M)$ itself in this way, using the corrected action $\hat{\rho}_p$ of the extension $\tilde{\Gamma}_4$ on $V_p(g)$. Define a section $s : \Gamma \to \tilde{\Gamma}_4$ by $s(f) = T^{\sigma_b(w_f)} w_f$. Then

$$I_p(M) = \langle \hat{\rho}_p(s(f))(v_\emptyset), v_\emptyset \rangle \, .$$

One can take this as the definition of $I_p(M)$, and apply the methods of [27] to prove its topological invariance, without using Kirby's theorem. (One needs to use a presentation of Γ, and the Suzuki generators [30] (see also Lu [20, 21]) of the handlebody groups).

Note. While ρ_p is defined over $\mathbf{Q}(\zeta_{2p})$, the corrected action $\hat{\rho}_p : \tilde{\Gamma}_4 \to Gl(V_p(g))$ depends on a choice of sign for κ_p^3. (Only κ_p^6, which is a power of the skein variable $A = \zeta_{2p}$, is determined by skein theory.) One can check that $s(f)$ lies in the subgroup of index two in $\tilde{\Gamma}_4$ if and only if $b_1(M_f)$ is even. This corresponds to the fact that making a different choice of sign for κ_p^3 multiplies the invariant $I_p(M)$ by $(-1)^{b_1(M)}$.

5.3 The relationship with the TQFT-functors of [6]

While the geometric action ρ_p has an obvious skein-theoretical meaning, the corrected action $\hat{\rho}_p$ is quite natural from the point of view of the TQFT-functors V_p of [6]. (An introduction to the concept of Topological Quantum Field Theories (TQFT) can be found in Atiyah [2].)

Here is the precise relationship. Consider a closed surface Σ equipped with the fixed p_1-structure ξ_0. The TQFT-functor V_p associates to Σ a module $V_p(\Sigma)$ (which is free of finite rank), and any cobordism M with 'structure' from Σ to itself (i.e. with a p_1-structure extending the given ξ_0, and possibly containing banded links) induces a well defined endomorphism $Z_p(M)$ of $V_p(\Sigma)$. In particular, the assignment $(f, [\xi]) \mapsto Z_p(C(f), \xi)$ is a left action of the group $\tilde{\Gamma}_{p_1}$ on $V_p(\Sigma)$.

As in the proof of proposition 4.8, fix p_1-structures $\xi_H, \xi_{H'}$ on H, H' extending ξ_0 on Σ. By the general theory in [6], the choice of (H, ξ_H) and $(H', \xi_{H'})$ determines an identification $V_p(\Sigma) = V_p(g)$, and one can show the following

Proposition 5.2 *Under this identification, the action $\hat{\rho}_p(t_\alpha)$ on $V_p(g)$ is identified with the natural action of the canonical lift $\tilde{t}_\alpha \in \tilde{\Gamma}_{p_1}$ (defined in subsection 4.3) on $V_p(\Sigma)$.*

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