ON THE STIEFEL-WHITNEY CLASSES OF A MANIFOLD.* [1]

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1. Introduction. It has been well known for many years that various relations must hold between the Stiefel-Whitney classes of the tangent bundle of a manifold which do not hold for the Stiefel-Whitney classes of an arbitrary sphere bundle. For example, Whitney [6] showed that the 3-dimensional Stiefel-Whitney class of an orientable 4-manifold is always zero. The three main theorems of this paper are results of this kind. They assert that for certain integers n and k, the k-dimensional Stiefel-Whitney class (or dual Stiefel-Whitney class) of a compact n-manifold (or a compact orientable n-manifold) is always zero.

2. Statement of results. Throughout this paper we will use only the ring of integers mod 2, Z_2, for coefficients of any cohomology groups or cohomology classes considered. The notation M^n will be consistently used to denote a compact, connected, n-dimensional manifold, $w_i \in H^i(M^n, Z_2)$ will denote the i-th Stiefel-Whitney class of its tangent bundle, and $\bar{w}_i \in H^i(M^n, Z_2)$ will denote the i-th dual Stiefel-Whitney class. The Stiefel-Whitney classes and dual Stiefel-Whitney classes are related by the following formula:

$$(2.1) \qquad (\sum_i w_i)(\sum_j \bar{w}_j) = 1.$$

According to the Whitney duality theorem, the $\bar{w}_i$ are the Stiefel-Whitney classes of the normal bundle for any differentiable imbedding of M^n in a Euclidean space of any dimension.[2]

The following three elementary properties of the Stiefel-Whitney classes of an n-manifold M^n are well known (see Wu [7]):

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[2] In his thesis [5], R. Thom showed how to define the w_i and $\bar{w}_i$ when differentiability hypotheses are lacking.

(2.2) If n is odd, $w_n = 0$.

(2.3) $w_1 = 0$ if and only if M^n is orientable.

(2.4) For any n, $\bar{w}_n = 0$.

Our theorems extend these results.

THEOREM I. *Let M^n be a compact, n-manifold and let q be an integer such that $0 < q < n$. If $w_{n-q} \neq 0$, then there exist integers $h_1, \cdots, h_q$ such that $h_1 \geqq h_2 \geqq \cdots \geqq h_q \geqq 0$ and*

$$n = 2^{h_1} + 2^{h_2} + \cdots + 2^{h_q}.$$

Moreover, if M^n is orientable, the following additional restrictions must be imposed:

(a) $q \neq 1$,

(b) *If $n \equiv 2 \bmod 4$, then $h_q \neq 1$,*

(c) *An odd number of the h_i's are not equal to $h_q + 1$.*

The proof of this theorem will be given in § 4. For the present, we will list the following corollaries.[3]

COROLLARY 1. *If $\bar{w}_{n-1} \neq 0$, then n is a power of 2 and M^n is non-orientable.*

This is the case $q = 1$ of the theorem. Note that for n-dimensional real projective space one actually has $\bar{w}_{n-1} \neq 0$ if n is a power of 2. To prove this, one can use the determination of the Stiefel-Whitney classes of n-dimensional real projective space by E. Stiefel [4] and formula (1) above.

COROLLARY 2. *If $\bar{w}_{n-2} \neq 0$, then $n = 2^k(2^h + 1)$ for non-negative integers h and k. In addition, if M^n is orientable, the cases $n = 2(2^h + 1)$ for $h > 0$ and $n = 3.2^k$ are not possible.*

This is the case $q = 2$ of the theorem. $2^k(2^h + 1) = 2^{k+h} + 2^k$, so let $h_1 = k + h$ and $h_2 = k$. The two excluded cases correspond to cases (b) and (c) respectively of the main theorem.

COROLLARY 3. *If $n = 2^r - 1$, then $\bar{w}_i = 0$ for $i > n - r$.*

Proof. Since $2^r - 1 = 1 + 2 + 2^2 + \cdots + 2^{r-1}$, the minimum value of q which can occur in Theorem I is $q = r$.

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[3] I have been informed by A. Shapiro that the result stated in corollary 1 has been obtained independently by A. Dold.

We leave it to the reader to derive other consequences of Theorem I. In doing this it is often useful to observe that the following two conditions are equivalent: (a) $n = 2^{h_1} + 2^{h_2} + \cdots + 2^{h_q}$ for non-negative integers $h_1, \cdots, h_q$. (b) In the dyadic expansion of the integer n, the digit 1 does not occur more than q times.

THEOREM II. *If n is even and M^n is orientable, then $w_{n-1} = 0$.*

Wu indicates a proof of this result in case $n \equiv 2 \bmod 4$ (see [7]). The proof for the case $n \equiv 0 \bmod 4$ if given in § 5.

THEOREM III. *If $n \equiv 3 \bmod 4$ and M^n is orientable, then $w_n = w_{n-1} = w_{n-2} = 0$.*

This theorem is an easy consequence of Wu's formulas [7]. The proof is given in § 3.

In a certain sense, Theorems II and III together with statements (2.2) and (2.3) are the best that one can hope for in this direction. This may be seen by consideration of certain examples, as follows:[4]

In the case of non-orientable manifolds, statement (2.2) above is the best possible. For if n is even, $n = 2k$, then $M^n = (P_2)^k$ (the Cartesian product of k copies of the real projective plane, P_2) has non-vanishing Stiefel-Whitney classes in all dimensions, while if n is odd, $n = 2k + 1$, then $M^n = (P_2)^k \times S^1$ (where S^1 denotes a 1-sphere) has non-vanishing Stiefel-Whitney classes in all dimensions $< n$.

The case of orientable manifolds is more complicated. First consider the case where $n = 4k$. Let $P_2(C)$ denote the complex projective plane (a 4-dimensional manifold), and let $M^n = [P_2(C)]^k$, the Cartesian product of k copies. Then $w_i \neq 0$ for all even integers $i \leqq n$; in particular, $w_{n-2} \neq 0$, so Theorem II can not be improved if $n = 4k$. If $n = 4k + 1$, one may obtain examples of an M^n for which $w_{n-1} \neq 0$ by taking $M^n = [P_2(C)]^k \times S^1$, or $M^n = P(1, 2k)$, a manifold considered by A. Dold in [1]. For $n = 4k + 2$, one may obtain an example of an M^n for which $w_{n-2} \neq 0$ by taking M^n

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[4] For the proof of the assertions made in the following paragraphs about these example, the following result is needed. Let M and M' be compact manifolds. Identify the cohomology ring of the product space, $H^*(M \times M', Z_2)$, with the tensor product $H^*(M, Z_2) \otimes H^*(M', Z_2)$ as usual. If $w = 1 + w_1 + w_2 + \cdots$ and $w' = 1 + w'_1 + w'_2 + \cdots$ denote the total Stiefel-Whitney classes of M and M' respectively, then $w \otimes w'$ is the total Stiefel-Whitney class of $M \times M'$. For the proof, see Thom, [5], pp. 142-143. One also needs to know that for the real projective plane, P_2, $w_1 \neq 0$ and $w_2 \neq 0$ (see [4]); for the circle, S^1, $w_1 = 0$; and for the complex projective plane, $P_2(C)$, $w_2 \neq 0$ and $w_4 \neq 0$. The Stiefel-Whitney classes of Dold's manifolds $P(m, n)$ have been computed by Dold [1].

$= [P_2(C)]^k \times S^1 \times S^1$ or $M^n = P(1,2k) \times S^1$; therefore Theorem II can not be improved in this case either. Similarly, for $n = 4k + 3$ one obtains examples where $w_{n-3} \neq 0$ by taking $M^n = [P_2(C)]^k \times [S^1]^3$ or $M^n = P(1,2k) \times S^1 \times S^1$. Thus Theorem III can not be improved. Whether or not Theorem I is a best possible theorem in this sense seems like a much more difficult question.

Of course there are other directions in which one could try to extend these theorems. For example, one could try to determine more general kinds of relations between Stiefel-Whitney classes of a manifold. An example is the relation $w_1 w_2 = 0$ which holds in all manifolds of dimension $\leqq 5$ (see Wu [7]). This important problem seems very difficult, and outside of the case considered by A. Dold in [2], very little is known about it. One of the most pertinent problems in this connection is the following: Can any relation which holds between the Stiefel-Whitney classes of every n-manifold (or every orientable n-manifold) be derived from the formulas of Wu ([7] and [8])?

It should be pointed out that Theorem I may have implications for the problem of determining the lowest dimensional Euclidean space in which it is possible to imbed a given manifold. Whitney has proved that it is possible to imbed any n-dimensional smooth manifold differentiably in $2n$-dimensional Euclidean space. Moreover, if $n = 2^k$, then it is possible to give an example of an n-manifold which can not be imbedded in Euclidean $(2n-1)$-space: n-dimnsional real projective space P_n is such an example. To prove that P_n can not be imbedded in Euclidean $(2n-1)$-space if $n = 2^k$ one uses the fact that $\bar{w}_{n-1} \neq 0$. On the other hand, Corollary 1 of Theorem I shows that $\bar{w}_{n-1} = 0$ for any n-manifold if n is not a power of 2. Thus it is natural to ask the following question: If n is not a power of 2, can any n-manifold be imbedded in Euclidean $(2n-1)$-space? If the answer to this question is "no," it will require new methods to prove the existence of a counter-example.[5]

3. Notation and preliminary results. We will use the following notation and ideas in what follows. They are due to W. T. Wu [7].

(a) $U_i \in H^i(M^n, Z_2)$ is the unique cohomology class such that

$$x \cdot U_i = Sq^i(x) \tag{3.1}$$

for any $x \in H^{n-i}(M^n, Z_2)$. The existence and uniqueness of the U_i follow from the Poincaré duality theorem. Note that $U_0 = 1$, and $U_i = 0$ if $i > \frac{1}{2}n$.

[5] As a matter of fact, it was the search for examples of n-manifolds with $w_{n-1} \neq 0$ which led the author to the discovery of Theorem I.

(b) The cohomology classes $\bar{U}_i \in H^i(M^n, Z_2)$ are defined inductively by the equation

$$(3.2) \qquad (\sum_i U_i) \cdot (\sum_i \bar{U}_i) = 1.$$

Here again $\bar{U}_0 = 1$. However it is *not* true in general that $\bar{U}_i = 0$ for $i > \frac{1}{2}n$.

Wu proved the Stiefel-Whitney classes and dual Stiefel-Whitney classes may be expressed in terms of the U_i and $\bar{U}_i$ respectively as follows:

$$(3.3) \qquad w_k = \sum_i Sq^{k-i} U_i,$$

$$(3.4) \qquad \bar{w}_k = \sum_i Sq^{k-i} \bar{U}_i.$$

These formulas are basic for all later computations.

In the following lemmas we record for later use some well known facts.

LEMMA 1. *A compact n-manifold, M^n, is orientable if and only if the homomorphism $Sq^1 : H^{n-1}(M^n, Z_2) \to H^n(M^n, Z_2)$ is trivial.*

This lemma is easily proved by using the fact that the homomorphism Sq^1 is composition of the Bockstein homomorphism together with reduction mod 2, plus the known structure of the integral cohomology group in dimension n of an n-manifold.

LEMMA 2. *If M^n is orientable, then $Sq^i : H^{n-i}(M^n, Z_2) \to H^n(M^n, Z_2)$ is zero for i odd.*

This follows from the known fact that $Sq^i = Sq^1 Sq^{i-1}$ for i odd, together with Lemma 1.

LEMMA 3. *If M^n is orientable, then $U_i = \bar{U}_i = 0$ for i odd.*

The fact that $U_i = 0$ for i odd follows from Lemma 2 and the definition of U_i. Then one can prove that $\bar{U}_i = 0$ for i odd by using formula (3.2).

In our proofs we need to make use of known properties of Steenrod squares and iterated Steenrod squares. For the sake of convenience, we will use the terminology and notation of Serre [3]. We assume the reader is familar with the properties of Steenrod squares as listed in § 2 of Serre's paper. Especially frequent use will be made of the properties of the homomorphism Sq^1. According to Cartan's formula,

$$(3.5) \qquad Sq^1(x \cdot y) = (Sq^1 x) \cdot y + x \cdot (Sq^1 y),$$

i. e., Sq^1 is a derivation of the algebra $H^*(X, Z_2)$. In particular,

$$(3.6) \qquad Sq^1(x^k) = k x^{k-1} \cdot Sq^1 x$$

for any positive integer k. Note also that

(3.7) $$Sq^1 Sq^1 = 0.$$

This implies that for any odd integer i,

(3.8) $$Sq^1 Sq^i = 0.$$

We conclude this section by proving Theorem II. For an orientable manifold M^n of dimension $n = 4k + 3$, $U_i = 0$ unless i is even and $0 \leqq i \leqq 2k$ (see Lemma 3). From this and (3.3) it follows that $w_i = 0$ for $i > 4k$ as desired.

4. Proof of the Theorem I. In the proof of Theorem I, frequent use will be made of the properties of iterated Steenrod squares. If $I = (i_1, i_2, \cdots, i_r)$ is any sequence of positive integers, then the notation Sq^I denotes the iterated Steenrod square $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_r}$. Such a sequence $I = (i_1, i_2, \cdots, i_r)$ is *admissible* if $i_1 \geqq 2i_2, i_2 \geqq 2i_3, \cdots, i_{r-1} \geqq 2i_r$. Every iterated Steenrod square may be expressed as a sum of admissible iterated Steenrod squares by repeated use of Adem's relations (see Serre [3], § 32).

With any admissible sequence of positive integers $I = (i_1, i_2, \cdots, i_r)$ one may associate a sequence of non-negative integers $(\alpha_1, \alpha_2, \cdots, \alpha_r)$ by the formulas

(4.1) $$\alpha_1 = i_1 - 2i_2,\ \alpha_2 = i_2 - 2i_3, \cdots, \alpha_{r-1} = i_{r-1} - 2i_r,\ \alpha_r = i_r.$$

It is clear that the sequence $(\alpha_1, \cdots, \alpha_r)$ determines without ambiguity the sequence $(i_1, \cdots, i_r)$. The integer $n(I) = i_1 + \cdots + i_r$ is called the *degree* of I, and $e(I) = \alpha_1 + \cdots + \alpha_r$ is called the *excess* of I.

LEMMA 4. *For any* mod 2 *cohomology class x, $Sq^I(x) = 0$ if the degree of x is less than the excess of I.*

The proof depends on the fact that $Sq^k(x) = 0$ if k is greater than the degree of x. The details are left to the reader.

LEMMA 5. *Let $I = (i_1, \cdots, i_r)$ be an admissible sequence of excess $e(I)$. Then there exists a unique admissible sequence $J = (j_1, \cdots, j_s)$ and a power of 2, $m = 2^k$, such that for any cohomology class x of degree $e(I)$,*

$$Sq^I(x) = (Sq^J x)^m$$

and $e(J) < e(I)$.

For the proof, see the proof of Lemma 1, p. 204, of Serre [3].

Lemmas 4 and 5 together show that when considering iterated Steenrod squares operating on cohomology classes x of a fixed degree q, we can restrict our attention to those iterated squares Sq^I such that $e(I) \leqq q-1$. In this case it is convenient (following Serre [3], p. 212) to let $\alpha_0 = q-1-e(I)$. Then one can derive the following formulae in case x is any mod 2 cohomology class of degree q:

$$\begin{aligned} \text{degree } (Sq^I x) &= n(I) + q \\ &= \sum_{i=1}^{r} (2^i - 1)\alpha_i + q = \sum_{i=1}^{r} 2^i \alpha_i - e(I) + q \\ &= \sum_{i=1}^{r} 2^i \alpha_i + \alpha_0 + 1 = 1 + \sum_{i=0}^{r} 2^i \alpha_i. \end{aligned} \tag{4.2}$$

Since $\sum_{i=0}^{r} \alpha_i = \alpha_0 + e(I) = q-1$, there are in all $(q-1)$ powers of 2 in formula (4.2). Therefore we can rewrite this formula as follows,

$$\text{degree}(Sq^I x) = 1 + 2^{h_1} + 2^{h_2} + \cdots + 2^{h_{q-1}}, \tag{4.3}$$

where $h_1 \geqq h_2 \geqq \cdots \geqq h_{q-1} \geqq 0$, and 2^i occurs α_i times in this sum (this is formula (17.5) of Serre [3], p. 212).

Next we will prove a couple of lemmas which are needed in the proof of Theorem I.

Lemma 6. *For any* $x \in H^k(M^n, Z_2)$, $0 < k < n$, $x \cdot \bar{w}_{n-k} = \sum_{i>0} (Sq^i x)\bar{w}_{n-k-i}$.

Proof. By equation (3.4),

$$\begin{aligned} \bar{w}_{n-k} &= \sum_{i \geqq 0} Sq^i \bar{U}_{n-k-i} \\ &= \bar{U}_{n-k} + \sum_{i>0} Sq^i \bar{U}_{n-k-i}. \end{aligned}$$

By equation (3.2),

$$\bar{U}_{n-k} = \sum_{i>0} U_i \bar{U}_{n-k-i},$$

hence

$$\bar{w}_{n-k} = \sum_{i>0} (U_i \bar{U}_{n-k-i} + Sq^i \bar{U}_{n-k-i}).$$

Therefore if $x \in H^k(M^n, Z_2)$,

$$x \cdot \bar{w}_{n-k} = \sum_{i>0} (x \cdot U_i \bar{U}_{n-k-i} + x \cdot Sq^i \bar{U}_{n-k-i}).$$

But

$$\begin{aligned} xU_i \bar{U}_{n-k-i} = U_i(x\bar{U}_{n-k-i}) &= Sq^i{}_{n-k-i})\,(x\bar{U}_{n-k-i}) \\ &= \sum_{r=0}^{i} (Sq^r x)\,(Sq^{i-r} \bar{U}_{n-k-i}), \end{aligned}$$

from which it follows that

$$\begin{aligned} x \cdot \bar{w}_{n-k} &= \sum_{i>0} \sum_{r=1}^{i} (Sq^r x)(Sq^{i-r} \bar{U}_{n-k-i}) \\ &= \sum_{0<r\leqq i} (Sq^r x)(Sq^{i-r} \bar{U}_{n-k-i}) \\ &= \sum_{r>0} [(Sq^r x) \sum_{j\geqq 0} Sq^j \bar{U}_{n-k-r-j}] \\ &= \sum_{r>0} (Sq^r x) \bar{w}_{n-k-r}. \end{aligned}$$

as was to be proved.

LEMMA 7. *The homomorphism* $H^k(M^n, Z_2) \to H^n(M^n, Z_2)$ *defined by* $x \to x \cdot \bar{w}_{n-k}$ *is a sum of iterated Steenrod squares.*

In view of Lemma 6, this lemma is obvious: one applies Lemma 6 repeatedly until the desired reduction to a sum of iterated Steenrod squares is obtained.

We are now in a position to prove Theorem I. Assume that $\bar{w}_{n-q} \in H^{n-q}(M^n, Z_2)$ is non-zero. By the Poincaré duality theorem, the homomorphism $H^q(M^n, Z_2) \to H^n(M^n, Z_2)$ defined by $x \to x \cdot \bar{w}_{n-q}$ is also non-zero. By Lemma 7, this homomorphism is a sum of iterated Steenrod squares, which we may assume to be admissible on account of Adem's relations. Hence the hypothesis of the theorem implies the following statement: There exists a non-zero admissible iterated Steenrod square

$$Sq^I \colon H^q(M^n, Z_2) \to H^n(M^n, Z_2),$$

where $I = (i_1, \cdots, i_r)$. By Lemma 4, $e(I) \leqq q$. Moreover, if $e(I) = q$, it follows from Lemma 5 that there exists an admissible sequence $J = (j_1, \cdots, j_s)$ and a power of 2, $m = 2^k$, such that

$$Sq^I(x) = [Sq^J(x)]^m$$

and $e(J) < q$. Therefore

$$\begin{aligned} n &= \text{degree}(Sq^I x) = 2^k \cdot \text{degree}(Sq^J x) \\ &= 2^k(2^{k_1} + 2^{k_2} + \cdots + 2^{k_{q-1}} + 1) \end{aligned} \tag{4.4}$$

by equation (4.3). Here $k_1, k_2, \cdots$ are integers such that $k_1 \geqq k_2 \geqq \cdots \geqq k_{q-1} \geqq 0$. If now we let

$$h_1 = k_1 + k, h_2 = k_2 + k, \cdots, h_{q-1} = k_{q-1} + k, h_q = k, \tag{4.5}$$

then (4.4) takes the form

(4.6) $$n = 2^{h_1} + 2^{h_2} + \cdots + 2^{h_q}$$

with $h_1 \geqq h_2 \geqq \cdots \geqq h_q \geqq 0$, and the first part of the theorem is proved.

Next, we will assume that M^n is orientable and prove the remaining parts of the theorem. In this case we can apply the results of Lemmas 1 and 2.

First assume that $q = 1$. Then $n = 2^{h_1}$ from what we have just proved, and $h_1 = k$ according to equation (4.5). Therefore the only non-zero iterated Steenrod square

$$Sq^I\colon H^1(M^n, Z_2) \to H^n(M^n, Z_2)$$

would be of the form $Sq^I(x) = x^n$ with $n = 2^k$. Since n is even, $x^n = Sq^1(x^{n-1})$ by equation (3.6). By use of Lemma 1, we see that if $x^n \neq 0$, then M^n is non-orientable, as was to be proved.

Next we will consider the case were $n \equiv 2 \bmod 4$, i.e., $n = 4l + 2$, and $h_q = 1$. Then it follows from (4.5) that $k = 1$. Therefore $Sq^I(x) = [Sq^J(x)]^2$; and by equation (4.4),

$$n = \text{degree}(Sq^I x) = 2 \cdot \text{degree}(Sq^J x).$$

Hence $\text{degree}(Sq^J x) = n/2 = 2l + 1$. Thus

$$Sq^I(x) = Sq^{2l+1}[Sq^J(x)]$$

which is zero by Lemma 2. But this is a contradiction. Thus part (b) of Theorem I is proved.

Finally, we consider the case where an odd number of the h_i's are equal to $h_q + 1$. Then it follows from equation (4.5) that an odd number of the k_i's are equal to 1. Thus in (4.3), the summand 2^1 occurs an odd number of times,[6] i.e., α_1 is odd, it follows from equation (4.1) that j_1 is odd in the expression

$$Sq^I(x) = [Sq^J(x)]^m = [Sq^{j_1} \cdots Sq^{j_s}(x)]^m,$$

where $m = 2^k$. Since j_1 is odd,

$$Sq^{j_1} = Sq^1 Sq^{j_1 - 1},$$

and $Sq^1 Sq^J(x) = 0$ by equation (3.8). Therefore

$$[Sq^J(x)]^m = Sq^1\{[Sq^{j_1-1} Sq^{j_2} \cdots Sq^{j_s}(x)] \cdot [Sq^J(x)]^{m-1}\}$$

which is again zero by Lemma 1. Thus we have again reached a contradiction, and part (c) is proved.

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[6] Actually, we are here concerned with the analog of equation (4.3) which is obtained by replacing I by J and h_i by k_i for $i = 1, 2, \cdots, q - 1$.

5. Proof of Theorem II for the case $n \equiv 0 \bmod 4$. The following well-known lemma will be used in the course of the proof:

LEMMA 8. *If x is a* mod 2 *cohomology class of degree* 1, *then*

$$Sq^j(x^k) = C_j^k x^{k+j},$$

where C_j^k is the binomial coefficient reduced mod 2. *In particular, if k is a power of 2, then $Sq^j x^k = 0$ unless $j = 0$ or $j = k$.*

The proof is left to the reader.

Now assume that M^n is a compact orientable manifold of dimension $n = 4k$. Then

$$w_{n-1} = w_{4k-1} = Sq^{2k-1} U_{2k}$$

by (3.3). To prove that $w_{n-1} = 0$, it suffices to prove that $x \cdot w_{n-1} = 0$ for any $x \in H^1(M^n, Z)$. Now

$$\begin{aligned} x \cdot w_{n-1} &= x \cdot Sq^{2k-1} U_{2k} \\ &= Sq^{2k-1}(x \cdot U_{2k}) + (Sq^1 x)(Sq^{2k-2} U_{2k}). \end{aligned}$$

However the first term on the right is zero by Lemma 2, and in the second term, $Sq^1 x = x^2$. Therefore

$$x \cdot w_{n-1} = x^2 \cdot Sq^{2k-2} U_{2k} \tag{5.1}$$

We will now show that if $p = 2^q$ is a power of 2 and $2 \leqq p < 2k$, then

$$x^p \cdot Sq^{2k-p} U_{2k} = x^{2p} \cdot Sq^{2k-2p} U_{2k}. \tag{5.2}$$

To prove this, one computes as follows:

$$x^p Sq^{2k-p}\, U_{2k} = Sq^{2k-p}(x^p \cdot U_{2k}) + x^{2p} Sq^{2k-2p}\, U_{2k}. \tag{5.3}$$

Here we have used the formula for the Steenrod square of a cup product together with Lemma 8. Next, note that

$$\begin{aligned} Sq^{2k-p}(x^p\, U_{2k}) &= U_{2k-p}(x^p\, U^{2k}) \\ &= U_{2k}(x^p\, U_{2k-p}) = Sq^{2k}(x^p\, U_{2k-p}) \\ &= (x^p\, U_{2k-p})^2 = x^{2p} \cdot U^2{}_{2k-p} \\ &= Sq^1(x^{2p-1} \cdot U^2{}_{2k-p}) = 0 \end{aligned} \tag{5.4}$$

by (3.5), (3.6), and Lemma 1. Substitution of (5.4) in (5.3) gives (5.2), as desired.

One can now apply (5.2) to (5.1) repeatedly with $p = 2, 4, 8, \cdots$, in succession. If n is not a power of 2, this procedure leads to the result that $x \cdot w_{n-1} = 0$, as desired. If n is a power of 2, this same procedure shows that $x \cdot w_{n-1} = x^n$ for any $x \in H^1(M^n, Z)$. However in this case, since n is even,

$$x^n = Sq^1(x^{n-1})$$

by (3.6). But $Sq^1(x^{n-1}) = 0$ by Lemma 1, as was to be proved. The proof of Theorem II is complete.

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