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COUNTING HOMOTOPY TYPES OF MANIFOLDS

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THEOREM (1). *Let C be a topological space dominated by a finite CW-complex K. Then $C \times S^1$ has the homotopy type of a finite CW-complex.*

Proof. Replace the given space C by the mapping cylinder of the given map $K \to C$, which has the same homotopy type as C. Then the map $C \to K$ becomes a map $f: C \to C$ whose image lies in K embedded in C, and which is homotopic to the identity. We may suppose that $f|K$ is cellular.

Define the *mapping torus* $T(f)$ of f by taking $C \times I$ and identifying $c \times 1$ with $f(c) \times 0$ for each $c \in C$. As with a mapping cone, $1 \simeq f$ implies $T(1) \simeq T(f)$. $T(1)$ is $C \times S^1$, so $C \times S^1 \simeq T(f)$. Define a homotopy $h_t: T(f) \to T(f)$ by:

$$\begin{aligned} h_t(c \times s) &= c \times (s + t) \quad \text{for} \quad s + t \leqslant 1 \\ &= f(c) \times (s + t - 1) \quad \text{for} \quad s + t \geqslant 1. \end{aligned}$$

This can be visualised as pushing the mapping torus through an angle $2\pi t$. This homotopy is a weak retraction of $T(f)$ to $T(f|K)$, naturally embedded in $T(f)$. Hence $C \times S^1 \simeq T(f) \simeq T(f|K)$.

But $T(f|K)$ is a finite CW-complex, so the theorem is proved.

THEOREM (2). *The set of homotopy types of spaces dominated by finite CW-complexes is countable.*

Proof. Let C be any such space. Then, by Theorem (1), $C \times S^1$ is homotopy equivalent to a finite CW-complex K. But the set of homotopy types of finite CW-complexes is countable. Hence we need only prove the theorem for spaces C such that $C \times S^1 \simeq K$.

Choose a particular homotopy equivalence $h: C \times S^1 \to K$ for each such space C. (We suppose that all spaces have base points, which are preserved by maps but not by homotopies.) Now C is homotopy equivalent to $C \times R$, which is the covering space of $C \times S^1$ determined by the subgroup $\pi_1(C)$ of $\pi_1(C \times S^1)$. It follows that C is also homotopy equivalent to the covering space of K determined by the subgroup $h_*\pi_1(C)$ of $\pi_1(K)$. But $\pi_1(C)$ is finitely generated and $\pi_1(K)$ countable, so that there are only a countable number of such subgroups. This proves the theorem.

COROLLARY. *The set of homotopy types of compact topological manifolds is countable.*

Proof. Any such manifold is a compact *ANR* [1, Theorem (3.3)], and so is dominated by a finite *CW*-complex [1, Theorem (6.3)]. Hence we may apply Theorem (2).

I am grateful to Dr. D. B. A. Epstein for suggesting that this corollary follows from Theorem (1).

REFERENCE

1. O. Hanner: Some theorems on absolute neighbourhood retracts, *Ark. Math.* **1** (1950), 389–408.

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