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COMPLETION OF A SYMMETRIC UNITARY MATRIX*

ROBERT F. MATHIS†

The symmetric unitary matrix has received special attention in physics and electrical engineering because, "The scattering matrix is symmetrical and unitary for a lossless junction" [1]. The present note resulted from an attempt to extend the work of Harold F. Mathis [2] to find a simple procedure for designing a lossless junction, given any number of rows in its scattering matrix.

A given set of orthonormal vectors can be extended to a complete orthonormal set by adjoining arbitrary linearly independent vectors, using the Gram–Schmidt orthogonalization process, and normalizing. If the given set of orthonormal vectors are considered the first rows of a matrix, this method can be used to complete a unitary matrix. If the given rows satisfy a simple minimum condition, the following method yields a symmetric unitary matrix. This method proceeds, as does the Gram-Schmidt process, by constructing one row at a time.

THEOREM. *Given a square matrix A and a rectangular matrix B with the same number of rows as A, the rectangular matrix $(A:B)$ can be extended to a symmetric unitary matrix* $\begin{bmatrix} A & B \\ B^T & X \end{bmatrix}$ *if and only if* $A = A^T$ *and* $AA^* + BB^* = I$.

Proof of necessity. From the definition of a unitary matrix,

$$\begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \begin{bmatrix} A^* & \bar{B} \\ B^* & X^* \end{bmatrix} = I. \tag{1}$$

This implies that $AA^* + BB^* = I$. The symmetry of $\begin{bmatrix} A & B \\ B^T & X \end{bmatrix}$ implies $A = A^T$.

Proof of sufficiency. The matrix X to be found must be symmetric. The proof will be by induction on the number of rows of X. In particular, the first column of X will be constructed, and it will be shown that when A and B are enlarged using the elements of this column, the resulting incomplete matrix again satisfies the conditions of the theorem.

Equation (1) is expanded to give

$$AA^* + BB^* = I, \tag{2}$$

$$A\bar{B} + BX^* = 0, \tag{3}$$

$$B^TA^* + XB^* = 0, \tag{4}$$

$$B^T\bar{B} + XX^* = I. \tag{5}$$

Equation (4) is the conjugate transpose of (3), so it is only necessary to consider (3) and (5). Let X_1, B_1 and 0_1 denote the first columns of X, B and 0, respectively.

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† Department of Mathematics, The Ohio State University, Columbus, Ohio 43210.

After taking the conjugate of (3), the corresponding equations for X_1 are

$$\bar{A}B_1 + \bar{B}X_1 = 0_1, \tag{6}$$

$$B_1^T\bar{B}_1 + X_1^T\bar{X}_1 = I. \tag{7}$$

First it will be shown that it is always possible to find a column matrix a such that $BB^*a = B_1$. If BB^* is nonsingular, it is obvious that a exists. If BB^* is singular, there exist nonzero column matrices λ such that $\lambda^T BB^* = 0_1^T$. For any such λ, $\lambda^T BB^*\bar{\lambda} = 0 = (B^T\lambda)^T(\overline{B^T\lambda})$, which requires $B^T\lambda = 0_1$ and $B_1^T\lambda = \lambda^T B_1 = 0$. Consequently, BB^* and the augmented matrix formed by adding B_1 as an extra column to BB^* have the same rank. This is a well-known criterion for the existence of a.

Finally, let $X_1 = B^T(\bar{a} - A^*a) - I_1$, where I_1 denotes the first column of I. Next, using (2), it is verified that X_1 satisfies equations (6) and (7):

$$\begin{aligned}
\bar{A}B_1 + \bar{B}X_1 &= \bar{A}B_1 + \bar{B}B^T(\bar{a} - \bar{A}a) - \bar{B}I_1 \\
&= \bar{A}B_1 + \bar{B}_1 - (I - \bar{A}A)\bar{A}a - \bar{B}_1 \\
&= \bar{A}B_1 - \bar{A}(I - A\bar{A})a \\
&= \bar{A}B_1 - \bar{A}BB^*a = 0_1,
\end{aligned}$$

$$\begin{aligned}
B_1^T\bar{B}_1 + X_1^T\bar{X}_1 &= B_1^T\bar{B}_1 + (\bar{a}^T B - a^T\bar{A}B - I_1^T)(B^*a - B^*A\bar{a} - I_1) \\
&= B_1^T\bar{B}_1 + \bar{a}^T B_1 - \bar{a}^T(I - A\bar{A})A\bar{a} - \bar{a}^T B_1 \\
&\quad - a^T\bar{A}B_1 + a^T\bar{A}(I - A\bar{A})A\bar{a} + a^T\bar{A}B_1 \\
&\quad - \bar{B}_1^T a + B_1^T A\bar{a} + 1 \\
&= B_1^T\bar{B}_1 - \bar{a}^T A\bar{B}B^T\bar{a} + a^T(I - \bar{B}B^T)\bar{B}B^T\bar{a} \\
&\quad - \bar{B}_1^T a + \bar{B}_1^T A\bar{a} + 1 \\
&= B_1^T\bar{B}_1 - (BB^*a)^T\bar{B}_1 + 1 = 1.
\end{aligned}$$

Let $B = (B_1 : B_2)$, let x_{11} be the first element of X_1 and let $X_1^T = (x_{11} : X_2^T)$. Now A is changed to $\begin{bmatrix} A & B_1 \\ B_1^T & x_{11} \end{bmatrix}$ and B is changed to $\begin{bmatrix} B_2 \\ X_2^T \end{bmatrix}$. So (2) becomes

$$\begin{bmatrix} A & B_1 \\ B_1^T & x_{11} \end{bmatrix}\begin{bmatrix} A & B_1 \\ B_1^T & x_{11} \end{bmatrix}^* + \begin{bmatrix} B_2 \\ X_2^T \end{bmatrix}\begin{bmatrix} B_2 \\ X_2^T \end{bmatrix}^* = I. \tag{8}$$

Equation (8) follows directly from (2), (6) and (7).

Thus the desired column matrix X_1 can be constructed. The first row of X is X_1^T, and the first column of X is X_1. This process can be repeated until the complete matrix X is found. If the first column of B is interchanged with some other column, it is possible to begin constructing X with this column. It is obvious from the nature of the problem and the proof that X need not be unique.

Remark. In some cases the computation of X_1 is easier and more direct than in the proof of the theorem. If $B_1 = 0_1$, then $X_1 = I_1$ is a solution. If there exists a column matrix b such that $B^*b = I_1$, let $X_1 = -B^T\bar{A}b$. It can be verified that this X_1 satisfies (6) and (7).

If there exists a column matrix c such that $B^*Bc = I_1$, let $b = Bc$. If B^*B is nonsingular, it is always possible to solve the equation $B^*Bc = I_1$. Consequently, if B^*B is nonsingular, $X_1 = -B^T\bar{A}B(B^*B)^{-1}I_1$ satisfies (6) and (7).

REFERENCES

[1] R. H. Dicke, *General Microwave Circuit Theorems*, Principles of Microwave Circuits, C. G. Montgomery, R. H. Dicke and E. M. Purcell, eds., McGraw-Hill, New York, 1948, Chap. 5.

[2] H. F. Mathis, *Design of lossless junction, given one row in its scattering matrix*, IEEE Trans. Microwave Theory and Techniques, MTT-14 (1966), p. 159.