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## COMPLETION OF A SYMMETRIC UNITARY MATRIX\*

ROBERT F. MATHIS†

The symmetric unitary matrix has received special attention in physics and electrical engineering because, "The scattering matrix is symmetrical and unitary for a lossless junction" [1]. The present note resulted from an attempt to extend the work of Harold F. Mathis [2] to find a simple procedure for designing a lossless junction, given any number of rows in its scattering matrix.

A given set of orthonormal vectors can be extended to a complete orthonormal set by adjoining arbitrary linearly independent vectors, using the Gram-Schmidt orthogonalization process, and normalizing. If the given set of orthonormal vectors are considered the first rows of a matrix, this method can be used to complete a unitary matrix. If the given rows satisfy a simple minimum condition, the following method yields a symmetric unitary matrix. This method proceeds, as does the Gram-Schmidt process, by constructing one row at a time.

**THEOREM.** *Given a square matrix  $A$  and a rectangular matrix  $B$  with the same number of rows as  $A$ , the rectangular matrix  $(A:B)$  can be extended to a symmetric unitary matrix  $\begin{bmatrix} A & B \\ B^T & X \end{bmatrix}$  if and only if  $A = A^T$  and  $AA^* + BB^* = I$ .*

*Proof of necessity.* From the definition of a unitary matrix,

$$(1) \quad \begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \begin{bmatrix} A^* & \bar{B} \\ B^* & X^* \end{bmatrix} = I.$$

This implies that  $AA^* + BB^* = I$ . The symmetry of  $\begin{bmatrix} A & B \\ B^T & X \end{bmatrix}$  implies  $A = A^T$ .

*Proof of sufficiency.* The matrix  $X$  to be found must be symmetric. The proof will be by induction on the number of rows of  $X$ . In particular, the first column of  $X$  will be constructed, and it will be shown that when  $A$  and  $B$  are enlarged using the elements of this column, the resulting incomplete matrix again satisfies the conditions of the theorem.

Equation (1) is expanded to give

$$(2) \quad AA^* + BB^* = I,$$

$$(3) \quad A\bar{B} + BX^* = 0,$$

$$(4) \quad B^T A^* + XB^* = 0,$$

$$(5) \quad B^T \bar{B} + XX^* = I.$$

Equation (4) is the conjugate transpose of (3), so it is only necessary to consider (3) and (5). Let  $X_1$ ,  $B_1$  and  $0_1$  denote the first columns of  $X$ ,  $B$  and  $0$ , respectively.

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After taking the conjugate of (3), the corresponding equations for  $X_1$  are

$$(6) \quad \bar{A}B_1 + \bar{B}X_1 = 0_1,$$

$$(7) \quad B_1^T \bar{B}_1 + X_1^T \bar{X}_1 = I.$$

First it will be shown that it is always possible to find a column matrix  $a$  such that  $BB^*a = B_1$ . If  $BB^*$  is nonsingular, it is obvious that  $a$  exists. If  $BB^*$  is singular, there exist nonzero column matrices  $\lambda$  such that  $\lambda^T BB^* = 0_1^T$ . For any such  $\lambda$ ,  $\lambda^T BB^* \bar{\lambda} = 0 = (B^T \lambda)^T (\bar{B}^T \bar{\lambda})$ , which requires  $B^T \lambda = 0_1$  and  $B_1^T \lambda = \lambda^T B_1 = 0$ . Consequently,  $BB^*$  and the augmented matrix formed by adding  $B_1$  as an extra column to  $BB^*$  have the same rank. This is a well-known criterion for the existence of  $a$ .

Finally, let  $X_1 = B^T(\bar{a} - A^*a) - I_1$ , where  $I_1$  denotes the first column of  $I$ . Next, using (2), it is verified that  $X_1$  satisfies equations (6) and (7):

$$\begin{aligned} \bar{A}B_1 + \bar{B}X_1 &= \bar{A}B_1 + \bar{B}B^T(\bar{a} - A^*a) - \bar{B}I_1 \\ &= \bar{A}B_1 + \bar{B}_1 - (I - \bar{A}A)\bar{A}a - \bar{B}_1 \\ &= \bar{A}B_1 - \bar{A}(I - A\bar{A})a \\ &= \bar{A}B_1 - \bar{A}BB^*a = 0_1, \end{aligned}$$

$$\begin{aligned} B_1^T \bar{B}_1 + X_1^T \bar{X}_1 &= B_1^T \bar{B}_1 + (\bar{a}^T B - a^T \bar{A}B - I_1^T)(B^*a - B^*A\bar{a} - I_1) \\ &= B_1^T \bar{B}_1 + \bar{a}^T B_1 - \bar{a}^T(I - A\bar{A})A\bar{a} - \bar{a}^T B_1 \\ &\quad - a^T \bar{A}B_1 + a^T \bar{A}(I - A\bar{A})A\bar{a} + a^T \bar{A}B_1 \\ &\quad - \bar{B}_1^T a + \bar{B}_1^T A\bar{a} + 1 \\ &= B_1^T \bar{B}_1 - \bar{a}^T A\bar{B}B^T \bar{a} + a^T(I - \bar{B}B^T)\bar{B}B^T \bar{a} \\ &\quad - \bar{B}_1^T a + \bar{B}_1^T A\bar{a} + 1 \\ &= B_1^T \bar{B}_1 - (BB^*a)^T \bar{B}_1 + 1 = 1. \end{aligned}$$

Let  $B = (B_1 : B_2)$ , let  $x_{11}$  be the first element of  $X_1$  and let  $X_1^T = (x_{11} : X_2^T)$ . Now  $A$  is changed to  $\begin{bmatrix} A & B_1 \\ B_1^T & x_{11} \end{bmatrix}$  and  $B$  is changed to  $\begin{bmatrix} B_2 \\ X_2^T \end{bmatrix}$ . So (2) becomes

$$(8) \quad \begin{bmatrix} A & B_1 \\ B_1^T & x_{11} \end{bmatrix} \begin{bmatrix} A & B_1 \\ B_1^T & x_{11} \end{bmatrix}^* + \begin{bmatrix} B_2 \\ X_2^T \end{bmatrix} \begin{bmatrix} B_2 \\ X_2^T \end{bmatrix}^* = I.$$

Equation (8) follows directly from (2), (6) and (7).

Thus the desired column matrix  $X_1$  can be constructed. The first row of  $X$  is  $X_1^T$ , and the first column of  $X$  is  $X_1$ . This process can be repeated until the complete matrix  $X$  is found. If the first column of  $B$  is interchanged with some other column, it is possible to begin constructing  $X$  with this column. It is obvious from the nature of the problem and the proof that  $X$  need not be unique.

*Remark.* In some cases the computation of  $X_1$  is easier and more direct than in the proof of the theorem. If  $B_1 = 0_1$ , then  $X_1 = I_1$  is a solution. If there exists

a column matrix  $b$  such that  $B^*b = I_1$ , let  $X_1 = -B^T \bar{A}b$ . It can be verified that this  $X_1$  satisfies (6) and (7).

If there exists a column matrix  $c$  such that  $B^*Bc = I_1$ , let  $b = Bc$ . If  $B^*B$  is nonsingular, it is always possible to solve the equation  $B^*Bc = I_1$ . Consequently, if  $B^*B$  is nonsingular,  $X_1 = -B^T \bar{A}B(B^*B)^{-1}I_1$  satisfies (6) and (7).

## REFERENCES

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