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The topological s-cobordism theorem fails in dimension 4 or 5

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Although examples have been known since 1969 [Si], which reveal a quite similar failure of the differentiable and piecewise-linear s-cobordism theorems, these did not immediately suggest any topological failure.

THEOREM. *There exists a compact topological* ($=$ *TOP*) *s-cobordism* $(W;\ V,\ V')$ *rel boundary that is not a product cobordism, i.e. not homeomorphic to* $(V \times I;\ V \times 0,\ V \times 1)$. *Indeed there exists such an s-cobordism with* V, V' *two copies of* $I \times P^2$ *or of* $I \times P^2 \times S^1$ *(we do not know which). Here* P^2 *is the real projective plane,* $I = [0, 1]$ *is the unit interval, and* $S^1 = I/\partial I$ *is the circle.*

Recall that a triad of compact topological manifolds $(W^w;\ V,\ V')$ is a *cobordism* when V, V' are disjoint $(w-1)$-submanifolds of the manifold boundary ∂W; it is *rel boundary* when further $\partial W - \text{int}\,(V \cup V')$ is homeomorphic to $(\partial V) \times I$; it is an *s-cobordism* when moreover the inclusions $V \to W$ and $V' \to W$ are *simple homotopy equivalences* (see [KS, III] and references there). For fundamental group 0 or Z_2, every homotopy equivalence of compact manifolds is simple [Hi]. The topological s-cobordism theorem asserts that every topological s-cobordism $(W^w;\ V,\ V')$ rel boundary as just defined is a product cobordism. Smale's handlebody theory led to a proof valid for $w \geqslant 6$, see [KS; III, §3·4] the cases $w = 3, 4, 5$ are hitherto undecided.

Some familiarity with *TOP* and *PL* ($=$ piecewise linear) surgery [Wa] (see also [KS] for *TOP*), is required from this point on.

LEMMA. *There exists a compact TOP manifold* X^5 *and a simple homotopy equivalence* $f: X^5 \to I^2 \times P^2 \times S^1$ *giving a homeomorphism of boundaries such that the normal invariant* $N(f) \in [I^2 \times P^2 \times S^1/\partial,\ G/TOP]$ *comes via projection to* $I^2 \times P^2/\partial$ *from a normal invariant* x *in* $[I^2 \times P^2/\partial,\ G/TOP]$ *that is not in the image of* $[I^2 \times P^2/\partial,\ G/PL]$.

Proof of lemma.

Clearly $I^2 \times P/\partial$ is the double suspension $\Sigma^2 P$. Now

$$[\Sigma^2 P, G/TOP] = H^4(\Sigma^2 P; Z) \oplus H^2(\Sigma^2 P; Z_2) = Z_2 \oplus Z_2,$$

and the image of $[\Sigma^2 P, G/PL]$ is the second Z_2, see [KS; p. 328, §15]. Let x be the generator of the first Z_2.

The surgery obstruction map $\theta: [I^2 \times P/\partial,\ G/TOP] \to L_4(Z_2^-) = Z_2$ has target Z_2 by [Wa, §13A] and is given by an Arf invariant c that is zero on x because of a formula found by Sullivan, see [Wa, §13·5B]. It states in this situation that $c(g)$ for

$$g: I^2 \times P/\partial \to G/TOP$$

is given by evaluating on the Z_2 orientation class of $I^2 \times P$ the Z_2 cohomology class $w_2(I^2 \times P)g_2 + (g_2)^2$, where g_2 is the component of g on $H^2(I^2 \times P, \partial; Z_2)$.

Letting g represent x we compose g with projection to $I^2 \times P/\partial$ to get

$$g': I^2 \times P \times S^1/\partial \to G/TOP.$$

One has surgery obstruction $\theta(g') = 0$ in $L_6(Z \times Z_2^-)$ because $\theta(g) = 0$. As we are now in dimension $\geqslant 5$, a topological transversality theorem [KS, III] permits one to represent g' by a normal map over a degree 1 map $f: X^5 \to I^2 \times P \times S^1$. Since its surgery obstruction is $\theta(g') = 0$, this f can be surgered to become a simple homotopy equivalence; its normal invariant certainly comes from x as asserted.

Proof of theorem. Suppose our theorem is false. Then an s-cobordism theorem in dim 5 is available to identify X^5 topologically with the product $I \times (I \times P \times S^1) = I^2 \times P \times S^1$ in such a way that f becomes the identity on $(0 \times I \cup I \times \partial I) \times P \times S^1$, and a self-homeomorphism on $1 \times I \times P \times S^1$ fixing the boundary $1 \times \partial I \times P \times S^1$.

Now pass to the infinite cyclic covering $\tilde{f}: \tilde{X} = I^2 \times P \times R \to I^2 \times P \times R$ of f corresponding to the universal covering $R \to S^1$. For k large, $1 \times I \times P \times k$ is disjoint from $\tilde{f}(1 \times I \times P \times 0)$; thus, assuming our theorem false, the relative s-cobordism in

$$1 \times I \times P \times R,$$

between these two manifolds, is a product and so one can readily modify $\tilde{f}$, by a compact support homotopy that is an isotopy on boundary, to a map

$$\bar{f}: I^2 \times P \times R \to I^2 \times P \times R$$

that near $I^2 \times P \times 0$ is a product of the identity on R with a homotopy equivalence

$$\bar{f}_0: I^2 \times P \to I^2 \times P = I^2 \times P \times 0.$$

Now, on boundary, this $\bar{f}_0$ is a homeomorphism of PL 3-manifolds, and thus using Moise's Hauptvermutung result (see [Mo] or alternative proofs discussed in [KS, p. 248†] [Sh]) it can be replaced by a nearby (and hence isotopic [Ce] [EK]) PL homeomorphism. Hence $\bar{f}_0$ certainly has a PL normal invariant.

This is a contradiction, for it is easily seen, cf. [KS, p. 266], that the normal invariant of each of $f, \tilde{f}, \bar{f}, \bar{f}_0$, gives by restriction to $I^2 \times P \times 0 = I^2 \times P$ the element

$$x \in [I^2 \times P/\partial, G/TOP]$$

of the lemma, which is *not* PL.

REMARKS. (1) It is possible to replace $I \times P^2$ and $I \times P^2 \times S^1$ in our theorem by the *closed manifolds* $S^1 \times P^2$ and $S^1 \times P^2 \times S^1$. A suitably revised version of the lemma follows from the present one; then the present proof of the theorem applies with possibly a refinement, namely the 4-dimensional h-cobordism encountered may have non-zero torsion in $Wh(Z \times Z_2) = Wh(Z_2) \oplus \tilde{K}_0(Z_2) \oplus \mathrm{Nil}(Z_2) = \mathrm{Nil}(Z_2)$, see [FH]. These Nil type torsions at any rate perish under transfer to a large finite covering along the new circle factor; if the covering is of odd order the required behaviour of normal invariant persists and the proof goes through.

(2) We do not know whether there is a failure of the *TOP* ribbon theorem (the class group analogue of the s-cobordism theorem, see [Si]). However the first author observes that its 5-dimensional version is false if the 4-dimensional *TOP* s-cobordism theorem is true. Indeed, starting from $\tilde{f}: \tilde{X} \to I^2 \times P \times R$, one application of each of these gives a homotopy equivalence $I^2 \times P \to I^2 \times P$ that, like $\bar{f}_0$, is a homeomorphism on boundary and has normal invariant x.

(3) Is there a counter-example to the *TOP* s-cobordism theorem given by a compact C^∞-*smooth* manifold?

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