

83. On 4-Manifolds Fibered by Tori

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§ 1. Definitions. The class of elliptic surfaces plays an important role in the theory of complex surfaces, [2]. In this note, we consider an analogous structure on smooth 4-manifolds, which we call *torus fibration*, and announce some results. Before giving the definition of torus fibration, we need slightly extend the notion of fibered link in the 3-sphere.

Definition. A smooth map $g: S^3 \rightarrow C$ is called a *multiple fibered link* if it satisfies the following:

- (i) $g^{-1}(0) \neq \emptyset$;
- (ii) the map $\varphi(x) = g(x)/|g(x)|: S^3 - g^{-1}(0) \rightarrow S^1$ is a submersion;
- (iii) at each $x_0 \in g^{-1}(0)$, there exist local coordinates u_1, u_2, u_3 in S^3 so that

$$g(x) = (u_2(x) + \sqrt{-1}u_3(x))^m$$

holds for all x near x_0 , m being a certain positive integer (called the *multiplicity at x_0*).

Definition. A map $f: R^4 \rightarrow C$ is a *cone-extension* of a smooth map $g: S^3 \rightarrow C$, if it is given as follows:

$$f(x) = \begin{cases} \|x\|^d g(x/\|x\|) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where d is an integer > 0 depending on f .

Clearly f is smooth outside of the origin $0 \in R^4$. Let $f_i: M_i^m \rightarrow N_i^k$ be a map, $p_i \in M_i^m$ a point, for $i=1, 2$, where M_i^m and N_i^k are oriented smooth manifolds. We say that the germ (f_1, p_1) is *smoothly (+)-equivalent* to the germ (f_2, p_2) if they are equivalent through orientation preserving local diffeomorphisms around p_i and $f_i(p_i)$.

Now we define the torus fibration. Let M and B be oriented smooth manifolds of dimensions 4 and 2, respectively. In this note, we assume that M and B are closed for the sake of convenience.

Definition. A *torus fibration of M with base space B* is an onto map $f: M \rightarrow B$ with the following properties:

- (i) at each point $p \in M$, the germ (f, p) is smoothly (+)-equivalent to the germ at 0 of a cone-extension of a multiple fibered link;
- (ii) the inverse image $C_u = f^{-1}(u)$ of any *general* point $u \in B$ is diffeomorphic to the 2-torus T^2 .

Note that the projection map f is smooth outside a finite set of

points. A special type of torus fibration has been studied by Moishezon [4] and Harer [7].

Torus fibration is an underlying structure of elliptic surfaces :

Proposition 1. *Suppose that $\Phi: M \rightarrow B$ is an analytic fiber space of elliptic curves [2], then there exist a torus fibration $f: M \rightarrow B$ and an orientation preserving homeomorphism $h: M \rightarrow M$ which is a diffeomorphism outside a finite set of points, so that the following diagram commutes :*

$$\begin{array}{ccc} M & \xrightarrow{h} & M \\ & \searrow \phi & \swarrow f \\ & B & \end{array}$$

Given a torus fibration, we can define (multiple or simple) singular fibers, divisors, the monodromy around a singular fiber, etc. in the same way as in the case of complex surfaces.

§ 2. An existence theorem. There are several necessary conditions for a 4-manifold to admit a torus fibration.

Proposition 2. *If $f: M \rightarrow B$ is a torus fibration, then the Euler number $\chi(M)$ is non-negative.*

It is shown that the fundamental group of a singular fiber is either $\{1\}$, $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. Using this fact, Koichi Yano proved the following theorem (he remarks that this theorem also follows from a result of Gromov [1, § 3.1]):

Theorem 3 (Yano, Gromov). *If $f: M \rightarrow B$ is a torus fibration, then the Gromov invariant of M vanishes.*

We have the following existence theorem :

Theorem 4. *Suppose that M has a handle-body decomposition of the form $M = H^0 \cup \mu H^2 \cup \nu H^3 \cup H^4$ ($\nu \leq 1$). Then there exists a torus fibration $f: M \rightarrow S^2$.*

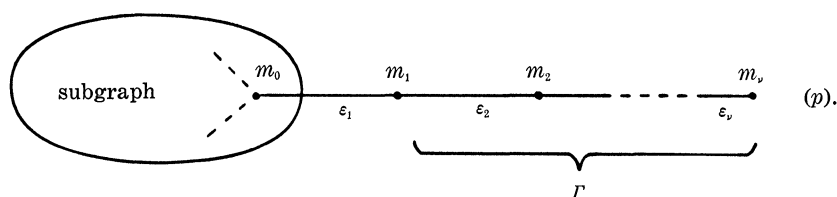
According to Mandelbaum [3], every nonsingular complete intersection of k distinct hypersurfaces in the complex projective space $\mathbb{C}P^{k+2}$ has a handle-body decomposition without 1 and 3 handles. Thus it admits a torus fibration.

§ 3. A torus fibration of S^4 . The 4-sphere S^4 has a torus fibration. The author's original construction was inspired by Montesinos' work [6] and made use of the Heegaard diagrams of 4-manifolds [5]. Afterwards, K. Fukaya gave the following nice construction: Let $H: S^3 \rightarrow S^2$ be the Hopf fibration, $\Sigma H: S^4 \rightarrow S^3$ its suspension. Then the composition $H \circ \Sigma H: S^4 \rightarrow S^2$ is a torus fibration. If S^3 is identified with ΣS^2 so that a fiber of H goes through the two suspension vertices, then the singular fiber consists of two S^2 's which intersect transversely in two points with opposite signs (a *twin* in the sense of [6]). As we

change the identification of S^3 with ΣS^2 , the torus fibration deforms, and the twin singular fiber splits into two simpler singular fibers.

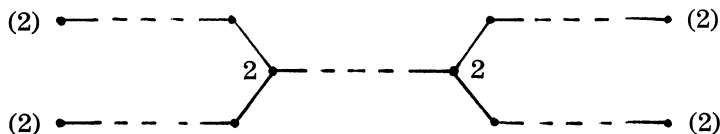
§ 4. Singular fibers with normal crossings. To obtain somewhat deeper results, certain restrictions on the type of singular fibers would be desirable. A singular fiber is said to be of *normal type*, if it consists of smoothly embedded S^2 's or T^2 which intersect transversely. Such a fiber is represented by a dual weighted graph, in which each vertex stands for an embedded 2-sphere. Two vertices are joined by an edge (labelled with sign $\varepsilon = \pm 1$) if and only if the corresponding 2-spheres transversely intersect in a point with the sign ε . The weight of a vertex represents the multiplicity.

Consider the linear branch Γ

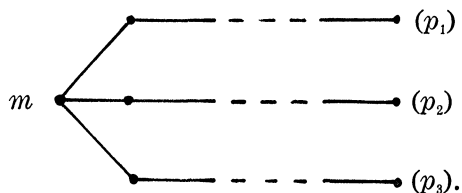


It is shown that $\gcd(m_0, m_1) = \gcd(m_1, m_2) = \cdots = \gcd(m_{v-1}, m_v) = m_v$. The number $p = p(\Gamma)$ in the parentheses is given by $p = m_0/m_v$. A *removable linear branch* (RLB) is a linear branch Γ for which $p = 1$ holds. The neighbourhood boundary of an RLB is S^3 , and if we 'remove' an RLB from a singular fiber, the monodromy remains unaffected.

Theorem 5. *Singular fibers of normal type without RLB are classified into the following six classes: (i) class mI_0 (multiple tori), (ii) class $\tilde{A}$ in which the graphs are cyclic (iii) class $\tilde{D}$ in which the graphs are of the form*



(iv) class $\tilde{E}_6$ with $(m=3, p_1=p_2=p_3=3)$, (v) class $\tilde{E}_7$ with $(m=4, p_1=2, p_2=p_3=4)$, (vi) class $\tilde{E}_8$ with $(m=6, p_1=2, p_2=3, p_3=6)$, where in the last three classes the graphs have the form



When all the signs of intersections are $+1$, these singular fibers reduce to Kodaira's singular fibers [2] or their blown ups. For example, the class $\tilde{E}_s$ reduces to $\{II, II^*\}$.

The detailed proofs will appear elsewhere.

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