

30. On 4-Manifolds Fibered by Tori. II

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This is a sequel to our previous note [2]. We will prove a signature formula for torus fibrations (Theorem 6) by combining Novikov additivity and W. Meyer's theorem [4]. This formula seems useful, especially in the study of singular fibers. Some computations will be presented. Also we will give a necessary condition for the existence of good torus fibrations in the sense of [3].

§ 5. The signature formula. Throughout the note, all manifolds will be compact, oriented and smooth. $\text{Sign}(M)$ will denote the signature of the homological intersection form $H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$, where M is a connected 4-manifold with or without boundary.

Let F_i be a singular fiber of a torus fibration $f_i: M_i^4 \rightarrow B_i^2$, for each $i=1, 2$. Let $\{p_i\} = f_i(F_i)$.

Definition. F_1 and F_2 are said to be *topologically equivalent* if there exist neighborhoods U_1, U_2 of p_1, p_2 in $\text{Int}(B_1^2), \text{Int}(B_2^2)$, respectively, and orientation preserving homeomorphisms $h: U_1 \rightarrow U_2$ and $H: f_1^{-1}(U_1) \rightarrow f_2^{-1}(U_2)$, so that (i) $h(p_1) = p_2$ and (ii) $h \circ f_1 = f_2 \circ H$.

Let S denote the totality of topological equivalence classes of singular fibers. Let $(1/3)\mathbb{Z} = \{m/3 \mid m \in \mathbb{Z}\} \subset \mathbb{Q}$.

Theorem 6. *There exists a (practically computable) function $\sigma: S \rightarrow (1/3)\mathbb{Z}$ with the following property: If $\{F_1, \dots, F_r\}$ is the set of all the singular fibers of a given torus fibration $f: M^4 \rightarrow B^2$ with M^4 closed, then $\text{Sign}(M) = \sum_{i=1}^r \sigma(F_i)$ holds.*

Consider a situation in which a singular fiber F_0 splits into several singular fibers $F'_1, \dots, F'_r$ through a certain deformation process (cf. [2], § 3). In that case, we have the following:

Corollary 6.1. $\sigma(F_0) = \sum_{i=1}^r \sigma(F'_i)$.

Remark. As we see above, each singular fiber behaves as if it has 'fractional signature'.

Proof of Theorem 6. Let $\omega: E \rightarrow X$ be any torus bundle over a connected surface X . Let $\partial X = C_1 \cup \dots \cup C_r$. Let $\alpha_i \in \text{SL}(2, \mathbb{Z})$ be a monodromy matrix of the restriction $E|_{C_i}$ of E to C_i , where C_i is oriented so that the orientation is concordant with that of X . The conjugacy class of α_i is uniquely determined.

Let $\phi: \text{SL}(2, \mathbb{Z}) \rightarrow (1/3)\mathbb{Z}$ be Meyer's class function whose explicit formula (containing the Dedekind sum) is found in [4], § 5.

Theorem 7 (Meyer [4], Satz 5). $\text{Sign}(E) = \sum_{i=1}^r \phi(\alpha_i)$.

Remark. $\tau(E, \partial E)$ in [4], Satz 5, is equal to $-\text{Sign}(E)$. See the equation (6) in [4].

Now let $f: M \rightarrow B$ be a torus fibration, M being closed. Let $\{F_1, \dots, F_r\}$ be the set of all the singular fibers of f , D_i a small disk in B centred at $p_i = f(F_i)$, $i = 1, \dots, r$. We remove the saturated neighborhoods $N_i = f^{-1}(D_i)$, $i = 1, \dots, r$, from M . Then we get a torus bundle $f|E: E \rightarrow X$, where $E = M - \bigcup_{i=1}^r \text{Int}(N_i)$, $X = B - \bigcup_{i=1}^r \text{Int}(D_i)$. Let β_i be a monodromy matrix of a singular fiber F_i . Here we notice our convention that when we speak of a monodromy matrix of a singular fiber F_i , it is always computed w. r. t. the orientation of ∂D_i which is concordant with that of D_i (see [3], § 4). Therefore β_i is conjugate to the inverse of α_i defined above, and we have $\phi(\beta_i) = -\phi(\alpha_i)$ (see the formula (42) in [4]).

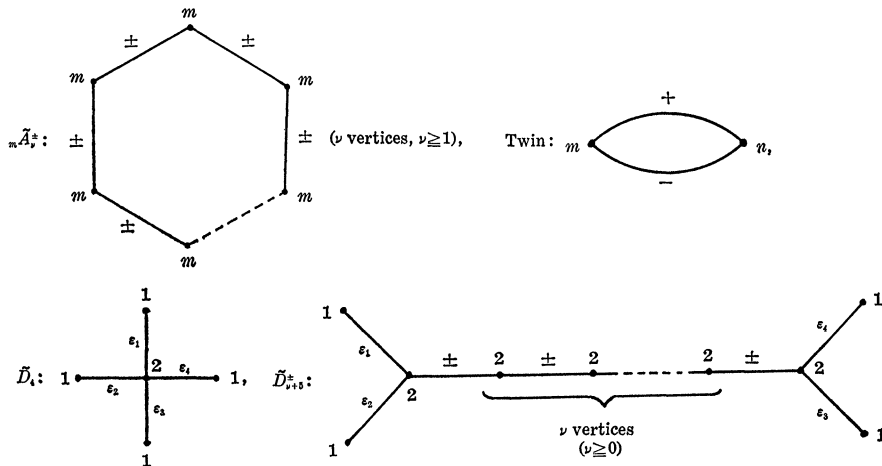
By Novikov additivity and Theorem 7, it follows that

$$\begin{aligned} \text{Sign}(M) &= \text{Sign}(E) + \sum_{i=1}^r \text{Sign}(N_i) \\ &= \sum_{i=1}^r \{-\phi(\beta_i) + \text{Sign}(N_i)\}. \end{aligned}$$

Defining $\sigma(F_i)$ to be $-\phi(\beta_i) + \text{Sign}(N_i)$, we have the desired formula.

§ 6. Computations in good torus fibrations. A good torus fibration (GTF) is a torus fibration whose singular fibers are of normal type in the sense of [2], § 4. Such singular fibers without removable linear branches are classified in [2], § 4, [3], Thm. 3.1, and their monodromy matrices are known, [3], Thm. 4.1. Thus their σ -numbers can be computed.

Before stating the results of computation we introduce some refined notation for the classes of singular fibers,



$$\begin{aligned}\tilde{E}_6^\pm &: \tilde{E}_6 \text{ with } \varepsilon_3 m_3 \equiv \mp 1 \pmod{3}, \\ \tilde{E}_7^\pm &: \tilde{E}_7 \text{ with } \varepsilon_3 m_3 \equiv \mp 1 \pmod{4}, \\ \tilde{E}_8^\pm &: \tilde{E}_8 \text{ with } \varepsilon_3 m_3 \equiv \mp 1 \pmod{6},\end{aligned}$$

where in the last three classes, ε_3 (or m_3) denotes the sign of the edge (or the multiplicity of the vertex) in the (p_3) -branch which is adjacent to the m -vertex. Here we are referring to the graph on p. 300 of [2]. (Cf. [3], Thm. 4.1.)

Theorem 8. *We have the following table (in which $\Sigma\varepsilon$ stands for the sum of the signs of all the edges):*

class of F		$\sigma(F)$	euler number $\chi(F)$
${}_m I_0$		0	0
$\tilde{A}$	${}_m \tilde{A}_\nu^\pm$	$\mp(2/3)\nu$	ν
	Twin	0	2
$\tilde{D}$	$\tilde{D}_4$	$-(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$	6
	$\tilde{D}_{\nu+5}^\pm$	$\mp(2/3)(\nu+1) - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$	$\nu+7$
$\tilde{E}_6^\pm$		$\pm(2/3) - \Sigma\varepsilon$	(number of vertices) + 1
$\tilde{E}_7^\pm$		$\pm 1 - \Sigma\varepsilon$	
$\tilde{E}_8^\pm$		$\pm(4/3) - \Sigma\varepsilon$	

Although they are not necessarily of normal type, we can make similar computations for the singular fibers of complex elliptic surfaces, either by using the Table I in [1], p. 604 or by blowing down exceptional curves in the singular fibers of normal type:

Corollary 8.1 (for the symbols ${}_m I_b$, ${}_m I_b^*$, II, etc., see [1]).

F	${}_m I_0$	${}_m I_b$	${}_m I_b^*$	II	II*	III	III*	IV	IV*
$\sigma(F)$	0	$-(2/3)b$	$-(2/3)(b+6)$	$-4/3$	$-20/3$	-2	-6	$-8/3$	$-16/3$

Comparing the table with the values of euler numbers, we obtain $\sigma(F) = -(2/3)\chi(F)$. This is considered as a 'local form' of a known relation for each elliptic surface M which contains no exceptional curves in its fiber: $\text{Sign}(M) = -(2/3)\chi(M)$.

Returning to our torus fibration, we have the following theorem by a closer analysis of singular fibers:

Theorem 9. *Let $F = \Sigma m_i \Theta_i$ be a singular fiber of normal type. If the self-intersection number $\Theta_i \cdot \Theta_i$ is an even integer for each Θ_i , then $|\sigma(F)| \leq (2/3)\chi(F)$.*

Corollary 9.1. *Let $f: M \rightarrow B$ be a GTF. Suppose that M is closed and $w_2(M) = 0$, then $|\text{Sign}(M)| \leq (2/3)\chi(M)$.*

For example, a connected sum of $K3$ surfaces $M' = K3 \# K3$ does not admit any GTF, because $w_2(M') = 0$, $\text{Sign}(M') = -32$ and $\chi(M') = 46$.

On the other hand M' has a general torus fibration [2], Thm. 4. Thus the torus fibration cannot be deformed into GTF. This means that the 'program' stated in [3], § 1 must be amended slightly.

References

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