Topology Vol. 10, pp. 111—118 Pergamon Press, 1971. Printed in Great Britain.

ON THE PONTRJAGIN CLASSES OF HOMOLOGY MANIFOLDS

C. R. F. MAUNDER

(*Received* 27 *April* 1970)

§1. INTRODUCTION

IN [3], MILNOR has given a definition of the Pontrjagin classes of orientable (polyhedral) homology manifolds. It is the object of this paper to extend the definition of Pontrjagin classes to orientable homology cobordism bundles, in the sense of [2], in such a way that the familiar properties for vector bundles are preserved, and that the Pontrjagin classes of a homology manifold coincide with those of its tangent bundle. We give two applications, the first to cobordism of homology manifolds, the second being an example of a Poincaré complex that is not homotopy-equivalent to a homology manifold.

Since much of the detailed work will involve Whitney sums of homology cobordism bundles, these are described first in Section 2. Pontrjagin classes of bundles are the subject of Section 3, and the applications follow in Section 4.

Throughout the rest of this paper, "manifold" will mean "polyhedral homology manifold" (possibly with boundary) unless otherwise stated.

§2. WHITNEY SUMS OF HOMOLOGY COBORDISM BUNDLES

Let K be an ordered simplicial complex, taken to be the homology cell complex with the individual simplexes as cells, and let E, F be homology cobordism bundles over K with fibres D^m, D^n respectively (see [2] for definitions).

Definition 2.1. The Whitney sum $E \oplus F$ is defined as follows. Write $E \times F$ for the D^{m+n}-bundle over the cell complex $K \times K$, defined by

$$(E \times F)(\sigma \times \tau) = E(\sigma) \times F(\tau).$$

Now by Theorem 4.5 of [2], there exists a bundle G over $(K \times K)'$, unique up to isomorphism, such that the amalgamation $\mathscr{A}(G)$ is isomorphic to $E \times F$ (where $(K \times K)'$ denotes the "simplicial subdivision" of the cell complex $K \times K$, into its individual simplexes). The diagonal map $\Delta : K \to (K \times K)'$ is a simplicial embedding; define $E \oplus F = \Delta^*(G)$, a D^{m+n}-bundle over K.

THEOREM 2.2. *The Whitney sum is invariant under isomorphism, commutative natural and associative.*

Proof. That $\oplus$ is invariant under isomorphism is obvious: if $E_1 \cong E_2$ and $F_1 \cong F_2$, then $E_1 \times F_1 \cong E_2 \times F_2$.

To show that $\oplus$ is commutative, let $t:(K \times K)' \to (K \times K)'$ be the simplicial homeomorphism that exchanges the two factors. Now t^*G is the bundle defined by $t^*G(\sigma) = G(t\sigma)$, so that $\mathscr{A}t^*G \cong F \times E$. Hence $F \oplus E = \Delta^* t^* G \cong (\Delta t)^* G = \Delta^* G = E \oplus F$.

To show that $\oplus$ is natural, let $f, g : L \to K$ be continuous maps, which we may as well assume to be simplicial (it is easy to see that $\oplus$ is invariant under amalgamation and hence subdivision). Now a representative for f^*E, for example, may be constructed by extending E, as in the proof of Theorem 3.5 of [2], over the mapping cylinder M_f of f and restricting this extension to L: this is because the inclusion of K in M is a homotopy equivalence. Thus if we extend E and F over M_f, M_g respectively, we obtain an extension of $E \times F$ over $M_f \times M_g$; and if we subdivide over $(M_f \times M_g)'$, we get a bundle whose restriction to $(L \times L)'$ is $(f \times g)^*G$: hence $\mathscr{A}(f \times g)^*G \cong f^*E \times g^*F$. It follows in particular that

$$\begin{aligned} f^*(E \oplus F) &\cong (f\Delta)^*G \\ &\cong \Delta^*(f \times f)^*G \\ &= f^*E \oplus f^*F. \end{aligned}$$

Finally, to show that $\oplus$ is associative, observe first that if ε^0 denotes the trivial D^0-bundle over K, we have

$$\begin{aligned} E \oplus \varepsilon^0 &= \Delta^*H, \text{ where } \mathscr{A}(H) \cong E \times \varepsilon^0 \\ &\cong \Delta^*(1 \times f)^*H, \text{ where } f: K \to K \text{ is the constant map to a point} \\ &\cong i^*H, \text{ where } i: K \to K \times K \text{ is the inclusion as } K \times \text{point} \\ &\cong E. \end{aligned}$$

Hence

$$\begin{aligned} E \oplus (F \oplus G) &\cong \Delta^*H \oplus \Delta^*M, \text{ where } \mathscr{A}(M) \cong F \times G \\ &\cong \Delta^*(H \oplus M) \\ &\cong \Delta^*\Delta^*(N), \text{ where } \mathscr{A}(N) \cong H \times M \text{ over } (K \times K)' \times (K \times K)'. \end{aligned}$$

But, over $K \times K \times K \times K$, $\mathscr{A}(N) \cong E \times \varepsilon^0 \times F \times G$. By symmetry, therefore, $E \oplus (F \oplus G) \cong (E \oplus F) \oplus G$.

As a consequence of Theorem 2.2, the set $K_H(X)$ of stable isomorphism classes of disc bundles over a polyhedron X defines a contravariant functor from the category of polyhedra and continuous maps to the category of monoids and homomorphisms. It is easy to see (using Section 4 of [2]), that K_H is a homotopy functor in the sense of [1], so that there exists a classifying space B, such that

$$K_H(X) \cong [X, B].$$

It follows from [6], therefore, that $K_H(X)$ is actually an abelian group, and so in particular every disc bundle has a stable inverse.

Sometimes it is convenient to have a rather different definition of the Whitney sum, along the lines of Milnor's "composite" construction (see [4]).

Definition 2.3. Let K be a homology cell complex, and let E be a D^m-bundle over K, that contains K as a zero cross-section (see the note after Proposition 3.3 of [2]). Let F be a D^n-bundle over a triangulation of E: the *composite* $E \circ F$ is defined by

$$(E \circ F)(C) = F(E(C)),$$

where C is a cell of K.

Observe that, since each $E(C)$ is contractible, we have

$$\begin{aligned} F(E(C)) &\cong E(C) \times D^n, \\ &\cong C \times D^m \times D^n, \end{aligned}$$

so that $E \circ F$ is a D^{m+n}-bundle over K.

Let $i : K \to E$ be the inclusion map of K as the zero cross-section of E; we shall see that $E \circ F \cong E \oplus i^*F$ over K, if K is an (ordered) simplicial complex. In order to prove this, a lemma is necessary.

LEMMA 2.4. *With the notation of Definition* 2.3, *let K' be the "simplicial subdivision" of K, and let P be a bundle over K' such that $\mathscr{A}(P) \cong E$, and P contains K' as a zero cross-section. Then there exists a bundle Q over P, such that $\mathscr{A}(P \circ Q) \cong E \circ F$.*

Proof. Let $G : \mathscr{A}(P) \cong E$ be the isomorphism, regarded as a bundle with zero cross-section over $K \times I$. Now the inclusion of E in G is the same, up to homotopy equivalence, as the inclusion of $K \times 1$ in $K \times I$, and so is a homotopy equivalence. Let $\rho : G \to E$ be a simplicial deformation retraction, obtained without subdividing E, and let H be ρ^*F, which may be assumed to extend F over E, since it is easy to see that $M_\rho \searrow E \times I$.

Finally, set $Q = H \mid G(K \times 0)$ (subdivided if necessary): then $G \circ H$ provides an isomorphism between $\mathscr{A}(P \circ Q)$ and $E \circ F$.

THEOREM 2.5. *Let K be an ordered simplicial complex, and let E and F be as in Definition* 2.3; *let $i : K \to E$ be the inclusion map. Then $E \circ F \cong E \oplus i^*F$ over K.*

Proof. Over $K \times E$, $E \times F$ is the composite $(E \times E) \circ (E \times F)$ (we regard E and F as D^0-bundles over themselves). By Lemma 2.4, this is isomorphic to the amalgamation of $P \circ Q$, where P over $(K \times E)'$ satisfies $\mathscr{A}(P) \cong E \times E$, and Q is a bundle over P. Now restrict this composite to the image of the diagonal map $\Delta : K \to (K \times E)'$: we have

$$\begin{aligned} E \oplus i^*F &\cong \Delta^*(P \circ Q) \\ &\cong \Delta^*P \circ Q \mid (\Delta^*P). \end{aligned}$$

By the proof of Theorem 2.2, we have $G : \Delta^*P \cong E$. Hence, as in Lemma 2.4, $E \oplus i^*F \cong E \circ R$, where $R = j^*\rho^*(Q \mid \Delta^*P)$, and $j : E \to G$, $\rho : G \to \Delta^*P$ are the inclusion and deformation retraction, respectively.

However, R is the pull-back of $E \times F$ over $E \times E$, by a map that covers (up to homotopy) $\Delta : K \to K \times E$, and so is homotopic to the diagonal map $\Delta: E \to E \times E$. Hence $R \cong F$, so that $E \oplus i^*F \cong E \circ F$.

COROLLARY 2.6. *Let $r : E \to K$ be a homotopy inverse to $i : K \to E$, and let F be another bundle over K. Then $E \oplus F \cong E \circ r^*F$.*

Proof. $E \oplus F \cong E \oplus i^*r^*F \cong E \circ r^*F$.

§3. PONTRJAGIN CLASSES

(Throughout this section, we shall assume unless otherwise stated that all manifolds and bundles are oriented.)

We start by defining the total Pontrjagin class of a disc bundle over a manifold.

Definition 3.1. Let ξ be an (oriented) D^n-bundle over an (oriented) manifold M (possibly with boundary). Then E, the total space of ξ, is an oriented $(m + n)$-manifold, and we can define the total Pontrjagin class of ξ, $p(\xi) \in H^*(M; Q)$, by

$$i^*p(E) = p(M) \cdot p(\xi),$$

where $p(M)$, for example, is the total Pontrjagin class of M, and $i : M \to E$ is the inclusion map (since $p(M)$ and $p(E)$ are polynomials with constant term 1, this defines a unique polynomial $p(\xi)$).

Observe that, by the Milnor compatibility theorem [3], this definition of $p(\xi)$ coincides with the usual one if ξ is a vector bundle and M is a differentiable manifold.

We check now that $p(\xi)$ has the expected properties, and the first is invariance under isomorphism. This follows from

THEOREM 3.2. *Let W be an orientation-preserving H-cobordism between n-manifolds M, N, where $i : M \to W$, $j : N \to W$ are the inclusion maps.*
Then

$$p(M) = i^*(j^*)^{-1}p(N).$$

Proof. Let W_o be the "sub-cobordism" between ∂M and ∂N. Consider the diagram

$$\begin{array}{ccc}
(M, \partial M) & & \\
\big\downarrow{\scriptstyle i} & & \\
(W, W_0) & \xrightarrow{f} & (S^{n-4i}, *) \\
\big\uparrow{\scriptstyle j} & & \\
(N, \partial N), & &
\end{array}$$

where f is a PL-map and $n \geq 8i + 2$. For almost all $x \in S^{n-4i}$, $f^{-1}(x)$ is a cobordism (in $W - W_o$) between $(fi)^{-1}x$ and $(fj)^{-1}x$: hence $I(fi) = I(fj)$ in the notation of [3]. It follows that there is a commutative diagram

$$\begin{array}{ccccc}
 & & \pi^*(W, W_0) \otimes Q & & \\
 & \swarrow{\scriptstyle i_*}\ \cong & & \cong\ \searrow{\scriptstyle j_*} & \\
\pi^*(M, \partial M) \otimes Q & \xrightarrow[I]{} & Q & \xleftarrow[I]{} & \pi^*(N, \partial N) \otimes Q,
\end{array}$$

which is sufficient to prove that $(i^*)^{-1}p(M) = (j^*)^{-1}p(N)$ (if $n < 8i + 2$, replace W by $W \times S^m$, where m is large).

COROLLARY 3.3. *If $\xi \cong \eta$ over M, then $p(\xi) = p(\eta)$.*

Proof. The total spaces of ξ and η are H-cobordant.

Since it is clear that $p(\xi) = p(\mathscr{A}(\xi))$, the Pontrjagin classes are uniquely defined for isomorphism classes of disc (or sphere) bundles over the *polyhedron* M.

The next property to check is that, if ξ, η are bundles over manifolds M, N respectively, then $p(\xi \times \eta) = p(\xi) \times p(\eta)$. Once again we first establish a similar property for the Pontrjagin classes of a product of two manifolds.

THEOREM 3.4. *If M and N are manifolds of dimensions m, n respectively, $p(M \times N) = p(M) \times p(N) \in H^*(M \times N; Q)$.*

Proof. Given PL-maps $f : (M, \partial M) \to (S^{m-4i}, *)$, $g : (N, \partial N) \to (S^{n-4j}, *)$ (where $m \geq 8i + 2$, $n \geq 8j + 2$), consider the composite

$$(M \times N, \partial(M \times N)) \xrightarrow{f \times g} (S^{m-4i} \times S^{n-4j}, S^{m-4i} \vee S^{n-4j}) \xrightarrow{q} (S^{m+n-4(i+j)}, *),$$

where q is the standard identification map onto the reduced product. For almost all $x \in S^{m+n-4(i+j)}$, $q^{-1}(x)$ is a point (y, z) in $S^{m-4i} \times S^{n-4j} - S^{m-4i} \vee S^{n-4j}$, and so $[q(f \times g)]^{-1}x = f^{-1}y \times g^{-1}z$. Hence $I(q(f \times g)) = I(f) \times I(g)$, and there is a commutative diagram

$$\begin{tikzcd}[column sep=small]
\sum \pi^*(M, \partial M) \otimes Q \otimes \pi^*(N, \partial N) \otimes Q \arrow[rr, "m"] \arrow[dr, "I \otimes I"'] & & \pi^*(M \times N, \partial(M \times N)) \otimes Q \arrow[dl, "I"] \\
& Q, &
\end{tikzcd}$$

where m is the "cup-product". It follows that $p(M \times N) = p(M) \times p(N)$ (if the restriction $m \geq 8i + 2$, $n \geq 8j + 2$ is not satisfied, we just take the products of M and N with spheres of large dimensions).

COROLLARY 3.5. *If ξ is a trivial D^n-bundle over a manifold M, then $p(\xi) = 1$.*

Proof. The total space E of ξ is H-cobordant to $M \times D^n$, so that

$$p(E) = p(M \times D^n) = p(M) \times p(D^n) = p(M).$$

COROLLARY 3.6. *Given bundles ξ, η over manifolds M, N respectively, $p(\xi \times \eta) = p(\xi) \times p(\eta) \in H^*(M \times N; Q)$.*

COROLLARY 3.7. *If ξ is a bundle over a manifold M, $p(\xi)$ depends only on the stable isomorphism class of ξ.*

Proof. Let ε^1 denote the trivial D^1-bundle over M. By Corollary 2.6, the total space of $\xi \oplus \varepsilon^1$ may be taken to be $E \times I$, where E is the total space of ξ: hence $p(\xi \oplus \varepsilon^1) = p(\xi)$.

The proof that the Pontrjagin classes are natural (for induced bundles) is a little more complicated, and proceeds via a number of steps.

THEOREM 3.8. *Let M be a manifold, and N be a submanifold of codimension zero, where $\partial M \cap \partial N$ is a submanifold of ∂M of codimension zero. Then if $i : N \to M$ is the inclusion map, $i^*p(M) = p(N)$.*

Proof. Let $P = \mathrm{cl}[M - N]$. There is a commutative diagram

$$\begin{tikzcd} & \pi^*(M, P \cup (\partial M \cap \partial N)) \otimes Q \arrow[dl, "i^*"', "\cong"] \arrow[d, "I"] \arrow[dr, "j^*"] & \\ \pi^*(N, \partial N) \otimes Q \arrow[r, "I"'] & Q & \pi^*(M, \partial M) \otimes Q, \arrow[l, "I"] \end{tikzcd}$$

where i^* is an excision isomorphism. By duality, therefore, $i^*p(M) = p(N)$.

COROLLARY 3.9. *If ξ is a bundle over M, $i^*p(\xi) = p(i^*\xi)$.*

COROLLARY 3.10. *Let τ be the tangent bundle of M. Then $p(\tau) = p(M)$.*

Proof. Let T be the total space of τ. We have inclusions

$$M \xrightarrow{i} T \xrightarrow{j} M \times M,$$

and by definition $i^*p(T) = p(M) \cdot p(\tau)$. But by Theorem 3.8 we have $p(T) = j^*p(M \times M) = j^*[p(M) \times p(M)]$. Since ji is the diagonal map Δ, it follows that $i^*p(T) = p(M) \cdot p(M)$, so that $p(\tau) = p(M)$.

COROLLARY 3.11. *If M is stably parallelizable, $p(M) = 1$.*

PROPOSITION 3.12. *Given bundles ξ, $\bar{\xi}$, η, $\bar{\eta}$ over a manifold M, such that $\xi \oplus \eta \cong \bar{\xi} \oplus \bar{\eta}$, then $p(\xi) \,.\, p(\eta) = p(\bar{\xi}) \,.\, p(\bar{\eta})$.*

Proof. Consider the diagram

$$\begin{tikzcd} F \arrow[r] & E \,|\, T \arrow[r, "g"] & E \\ M \arrow[u] \arrow[r, "i"'] & T \arrow[u, "f"] \arrow[r, "j"'] & (M \times M)', \arrow[u] \end{tikzcd}$$

where F is the total space of $\xi \oplus \eta$, and $\mathscr{A}(E) \cong \xi \times \eta$. Since i is a homotopy equivalence, and $\xi \oplus \eta \cong \bar{\xi} \oplus \bar{\eta}$, we have $E \,|\, T \cong \bar{E} \,|\, T$, where $\mathscr{A}(\bar{E}) \cong \bar{\xi} \times \bar{\eta}$: hence $f^*g^*p(E) = \bar{f}^*\bar{g}^*p(\bar{E})$ since $E \,|\, T$ is a submanifold of E of codimension zero. It follows that $\Delta^*p(\xi \times \eta) = \Delta^*p(\bar{\xi} \times \bar{\eta})$, or $p(\xi) \cdot p(\eta) = p(\bar{\xi}) \cdot p(\bar{\eta})$.

PROPOSITION 3.13. *Given a bundle ξ over M, with total space E, and a bundle η over E, we have $i^*p(\eta) = p(i^*\eta)$, where $i : M \to E$ is the inclusion map.*

Proof. Let F be the total space of η, and consider the commutative diagram

$$\begin{tikzcd} F \,|\, M \arrow[r] & F \\ M \arrow[u, "k"] \arrow[r, "i"'] & E. \arrow[u, "j"'] \end{tikzcd}$$

Now $j^*p(F) = p(E) \cdot p(\eta)$ and $k^*p(F|M) = p(M) \cdot p(i^*\eta)$. But consider the bundles $(i^*\eta \oplus \xi) \times \varepsilon^0$, $i^*\eta \times \xi$ over $M \times M$, which by Theorem 2.2 become isomorphic after applying Δ^*. By Theorem 2.5, the total space of $i^*\eta \oplus \xi$ may be taken to be F, and so by Proposition 3.12 we have

$$i^*j^*p(F) \cdot p(M) = k^*p(F|M) \cdot i^*p(E).$$

That is, $i^*p(E) \cdot i^*p(\eta) \cdot p(M) = p(M) \cdot p(i^*\eta) \cdot i^*p(E)$, whence $i^*p(\eta) = p(i^*\eta)$.

COROLLARY 3.14. *Let ξ be a bundle over a manifold M, and let N be a proper submanifold of M. Then $i^*p(\xi) = p(i^*\xi)$, where $i : N \to M$ is the inclusion map.*

Proof. Let E be the total space of the normal (disc) bundle, so that we have inclusions

$$N \xrightarrow{j} E \xrightarrow{k} M,$$

where $kj = i$. Then

$$\begin{aligned} i^*p(\xi) &= j^*k^*p(\xi) \\ &= j^*p(k^*\xi), \text{ by Corollary 3.9,} \\ &= p(j^*k^*\xi), \text{ by Proposition 3.13,} \\ &= p(i^*\xi). \end{aligned}$$

COROLLARY 3.15. *Let M, N be manifolds, let ξ be a bundle over N, and let $f: M \to N$ be any map. Then $f^*p(\xi) = p(f^*\xi)$.*

Proof. If M and N are both closed, this follows from Corollary 3.14 on replacing f by the embedding of M as the graph of f in $M \times N$. If M is closed and N has a boundary, replace N by its double $2N (= N \cup \partial N \times I \cup N)$, and use also Corollary 3.9. If M has a boundary and N is closed, let $g : M \to I$ be a continuous map such that

$$g(x)\begin{cases} = 0, & x \in \partial M \\ > 0, & x \notin \partial M; \end{cases}$$

then $(f, g) : M \to N \times [0, 2]$ is a proper map, and its graph is a proper submanifold. Finally, if both M and N have boundaries, first replace N by $2N$ and then proceed as if N were closed.

Now let X be any compact polyhedron, and let ξ be an (oriented) bundle over X. Let $f: X \to M$ be a homotopy equivalence onto a manifold, and let η be a bundle over M such that $f^*\eta \cong \xi$. Define the total Pontrjagin class of ξ by

$$p(\xi) = f^*p(\eta):$$

by Corollary 3.15 this is unambiguous, and $p(\xi)$ has all the properties one would expect, including naturality for induced bundles and $p(\xi \oplus \eta) = p(\xi) \cdot p(\eta)$.

In particular, let γ be the universal D^n-bundle over $BSH(n)$ (see [2], Section 4). For any finite skeleton X of $BSH(n)$, we have already defined $p(i^*\gamma)$ (where $i : X \to BSH(n)$ is the inclusion map), and since $H^*(BSH(n); Q) = \varprojlim H^*(X; Q)$, this defines

$$p(\gamma) \in H^*(BSH(n); Q),$$

the *universal* total Pontrjagin class.

§4. APPLICATIONS

The first is to cobordism of homology manifolds.

THEOREM 4.1. *Let M be a closed orientable homology manifold. Then the Pontrjagin numbers of M are (oriented) cobordism invariants.*

Proof. Suppose that M is a boundary, say $M = \partial W$. By replacing W by $W \cup \partial W \times I$ if necessary, we may assume that M has a neighbourhood $M \times I$ in W. Thus if $i : M \to W$ is the inclusion map, and $\tau(W)$ is the tangent bundle of W, the total space of $i^*\tau(W)$ is $\tau(M) \times I$. It follows that $i^*p(\tau(W)) = p(\tau(M))$, or $i^*p(W) = p(M)$ by Corollary 3.10. As in [3], this is sufficient to show that the Pontrjagin numbers of M all vanish.

COROLLARY 4.2. *If $\Omega_H{}^*$ denotes oriented cobordism classes of homology manifolds, the "inclusion" $\Omega^* \to \Omega_H{}^*$ is a monomorphism.*

Our other application is an example of a Poincaré complex that is not homotopy-equivalent to a homology manifold. This needs

THEOREM 4.3. *Let M be an oriented homology $4k$-manifold, and $f : \tilde{M} \to M$ be an oriented n-fold covering. Then $I(\tilde{M}) = nI(M)$, where $I(M)$ is the index (signature) of M.*

Proof. This follows from the fact that $f^*\tau(M) = \tau(\tilde{M})$, Corollary 3.10, and the fact that the Pontrjagin classes of homology manifolds are defined so that the Hirzebruch index theorem holds.

COROLLARY 4.4. *There exists a compact Poincaré complex X, of dimension 4, such that X is not of the homotopy type of a homology manifold.*

Proof. Wall [5] has constructed a 4-dimensional Poincaré complex X with $I(X) = 4$, $I(\tilde{X}) = 0$, where $\tilde{X}$ is a double cover. If X were of the homotopy type of a homology manifold, this would contradict Theorem 4.3.

REFERENCES

1. E. H. BROWN: Abstract homotopy theory. *Trans. Am. Math. Soc.* **119** (1965), 79–85.
2. N. MARTIN and C. R. F. MAUNDER: Homology cobordism bundles *Topology* **10** (1971), 93–110.
3. J. W. MILNOR: Lectures on characteristic classes, mimeographed notes, Princeton (1957).
4. J. W. MILNOR: Microbundles—I. *Topology* **3**, Supplement 1 (1964), 53–80.
5. C. T. C. WALL: Poincaré complexes—I. *Ann. Math.* **86** (1967), 213–245.
6. J. W. WEST: A short proof of the existence of additive inverses of topological microbundles. *Proc. Cambridge Phil. Soc.* **63** (1967), 587–589.

Christ's College
Cambridge