STABLE ALGEBRAIC TOPOLOGY, 1945–1966

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Contents

Stable algebraic topology is one of the most theoretically deep and calculationally powerful branches of mathematics. It is very largely a creation of the second half of the twentieth century. Roughly speaking, a phenomenon in algebraic topology is said to be "stable" if it occurs, at least for large dimensions, in a manner independent of dimension. While there are important precursors of the understanding of stable phenomena, for example in Hopf's introduction of the Hopf invariant [Hopf35, FS], Hurewicz's introduction of homotopy groups [Hur35], and Borsuk's introduction of cohomotopy groups [Bor36], the first manifestation of stability in algebraic topology appeared in Freudenthal's extraordinarily prescient 1937 paper [Fr37, Est], in which he proved that the homotopy groups of spheres are stable in a range of dimensions.

Probably more should be said about precursors, but I will skip ahead and begin with the foundational work that started during World War II but first reached print in 1945. Aside from the gradual development of homology theory, which of course dates back at least to Poincaré, some of the fundamental precursors are treated elsewhere in this volume [Ma, BG, Mc, We]. However, another reason for not attempting such background is that I am not a historian of mathematics, not even as a hobby. I am a working mathematician who is bemused by the extraordinarily rapid, and perhaps therefore jagged, development of my branch of the subject. I am less interested in who did what when than in how that correlated with the progression of ideas.

My theme is the transition from classical algebraic topology to stable algebraic topology, with emphasis on the emergence of cobordism, K-theory, generalized homology and cohomology, the stable homotopy category, and modern calculational techniques. The history is surprising, not at all as I imagined it. For one example, we shall see that the introduction of spectra was quite independent of the introduction of generalized cohomology theories. While some key strands developed in isolation, we shall see that there was a sudden coalescence around 1960: this was when the subject as we know it today began to take shape, although in far from its final form: I doubt that we are there yet even now.

Younger readers are urged to remember the difficulty of communication in those days. Even in 1964, when I wrote my thesis, the only way to make copies was to type using carbon paper: mimeographing was inconvenient and the xerox machine had not been invented, let alone fax or e-mail. Moreover, English had not yet become the lingua franca. Many relevant papers are in French or German (which I read) and some are in Russian, Spanish, or Japanese (which I do not read); further, the Iron Curtain hindered communication, and translation from the Russian was spotty. On the other hand, the number of people working in topology was quite small: most of them knew each other from conferences, and correspondence was regular. Moreover, the time between submission and publication of papers was shorter than it is today, usually no more than a year.

I have profited from a perusal of all of Steenrod's very helpful compendium [StMR] of Mathematical Reviews in algebraic and differential topology published between 1940 and 1967. Relatively few papers before the mid 1950's concern stable algebraic topology, whereas an extraordinary stream of fundamental papers was published in the succeeding decade. That stream has since become a torrent. I will focus on the period covered in [StMR], especially the years 1950 through 1966, which is an arbitrary but convenient cut-off date. For the later part of that period, I have switched focus a little, trying to give a fairly complete indication of the actual mathematical content of all of the most important relevant papers of the period. I shall also point out various more recent directions that can be seen in embryonic form during the period covered, but I shall not give references to the modern literature except in cases of direct follow up and completion of earlier work. I plan to try to bring the story up to date in a later paper, but lack of time has prevented me from attempting that now.

References to mathematical contributions give the year of publication, the only exception being that books based on lecture notes are dated by the year the lectures were given. References to historical writings are given without dates.

1. Setting up the foundations

A great deal of modern mathematics, by no means just algebraic topology, would quite literally be unthinkable without the language of categories, functors, and natural transformations introduced by Eilenberg and MacLane in their 1945 paper [EM45b]. It was perhaps inevitable that some such language would have appeared eventually. It was certainly not inevitable that such an early systematization would have proven so remarkably durable and appropriate; it is hard to imagine that this language will ever be supplanted.

With this language at hand, Eilenberg and Steenrod were able to formulate their axiomatization of ordinary homology and cohomology theory. The axioms were announced in 1945 [ES45], but their celebrated book "The foundations of algebraic topology" did not appear until 1952 [ES52], by which time its essential ideas were well-known to workers in the field. It should be recalled that Eilenberg had set the stage with his fundamentally important 1940 paper [Eil40], in which he defined singular homology and cohomology as we know them today.

I will say a little about the axioms shortly, but another aspect of their work deserves immediate comment. They clearly and unambiguously separated the algebra from the topology. This was part of the separation of homological algebra from algebraic topology as distinct subjects. As discussed by Weibel [We], the subject of homological algebra was set on firm foundations in the comparably fundamental book "Homological algebra" of Cartan and Eilenberg [CE56].

Two things are conspicuously missing from Eilenberg-Steenrod. We think of it today as an axiomatization of the homology and cohomology of finite CW complexes, but in fact CW complexes are nowhere mentioned. The definitive treatment of CW complexes had been published by J.H.C. Whitehead in 1948 [Whi48], but they were not yet in regular use. Many later authors continued to refer to polyhedra where we would refer to finite CW complexes, and I shall sometimes take the liberty of describing their results in terms of finite CW complexes.

Even more surprisingly, Eilenberg-Mac Lane spaces are nowhere mentioned. These spaces had been introduced in 1943 [EM43, EM45a], and the relation

$$\tilde{H}^n(X;\pi) \cong [X, K(\pi,n)] \tag{1.1}$$

was certainly known to Eilenberg and Steenrod. It seems that they did not believe it to be important. Nowadays, the proof of this relation is seen as one the most immediate and natural applications of the axiomatization.

However, there was something missing for the derivation of this relation. Despite their elementary nature, the theory of cofiber sequences and the dual theory of fiber sequences were surprisingly late to be formulated explicitly. They were implicit, at least, in Barratt's papers on "track groups" [Ba55], but they were not clearly articulated until the papers of Puppe [Pu58] and Nomura [Nom60]. The concomitant principle of Eckmann-Hilton duality also dates from the late 1950's [Eck57, EH58] (see also [Hil65]). The language of fiber and cofiber sequences pervades modern homotopy theory, and its late development contrasts vividly with the earlier introduction of categorical language. Probably not coincidentally, the key categorical notion of an adjoint functor was also only introduced in the late 1950's, by Kan [Kan58].

Although a little peripheral to the present subject, a third fundamental text of the early 1950's, Steenrod's "The topology of fiber bundles" [St51] nevertheless must be mentioned. In the first flowering of stable algebraic topology, with the introduction of cobordism and K-theory, the solidly established theory of fiber bundles was absolutely central to the translation of problems in geometric topology to problems in stable algebraic topology.

2. The Eilenberg-Steenrod axioms

The functoriality, naturality of connecting homomorphism, exactness, and homotopy axioms need no comment now, although their economy and clarity would not have been predicted from earlier work in the subject. Remember that these are axioms on the homology or cohomology of pairs of spaces. The crucial and subtle axiom is excision. A triad $(X; A, B)$ is *excisive* if X is the union of the interiors of A and B. In homology, the excision map $H_*(B, A \cap B) \longrightarrow H_*(X, A)$ must be an isomorphism. One subtlety is that I have stated the axiom in the form that Eilenberg and Steenrod verify it for singular homology. With a view towards other theories, they state the axiom under the stronger hypothesis that B is closed in X.

Conveniently for later developments, the dimension axiom was stated last. The fundamental theorem is that homology and cohomology with a given coefficient group is unique on triangulable pairs or, more generally, on finite CW pairs.

Several important extensions of the axioms came later. First, one wants an axiom that characterizes ordinary homology and cohomology on general CW pairs. For that Milnor [Mil62] added the additivity axiom. It asserts that homology converts disjoint unions to direct sums and cohomology converts disjoint unions to direct products. It implies that the homology of a CW complex X is the colimit of the homologies of its skeleta X^n. In cohomology, it implies $\lim^1$ exact sequences

$$0 \longrightarrow \lim{}^1 H^{q-1}(X^n) \longrightarrow H^q(X) \longrightarrow \lim H^q(X^n) \longrightarrow 0. \tag{2.1}$$

This allows the extension of the uniqueness theorem to infinite CW pairs.

One next wants an axiom that distinguishes singular theories from other theories on general pairs of spaces. I do not know who first formulated it; it appears in [Swi75] and may be due to Adams. This is the weak equivalence axiom. It asserts that a weak equivalence of pairs induces an isomorphism on homology and cohomology. Any space is weakly equivalent to a CW complex, any pair of spaces is weakly equivalent to a CW pair, and any excisive triad is weakly equivalent to a triad that consists of a CW complex X and a pair of subcomplexes A and B. Here $B/A \cap B \cong X/A$ as CW complexes, which neatly explains the excision axiom. The weak equivalence axiom reduces computation of the homology and cohomology of general pairs to their computation on CW pairs. Thus it implies the uniqueness theorem for homology and cohomology on general pairs.

Finally, one wants an axiom system for the reduced homology and cohomology of based spaces. The earliest published account is in the 1958 paper [DT58] of Dold and Thom, who ascribe it to Puppe. They use it to prove that the homotopy groups of the infinite symmetric products $SP^\infty X$ of based spaces X can be computed as the reduced integral homology groups of X. There are several slightly later papers [Ke59, BP60, Hu60] devoted to single space axioms for the homology and cohomology of both based spaces and, curiously, unbased spaces.

For the reduced homology of nondegenerately based spaces, the axioms just require functors $\tilde{k}_q$ together with natural suspension isomorphisms

$$\Sigma_* : \tilde{k}_q(X) \cong \tilde{k}_{q+1}(\Sigma X) \tag{2.2}$$

that satisfy the exactness, wedge, and weak equivalence axioms. Here the exactness axiom requires the sequences

$$\tilde{k}_q(X) \xrightarrow{f_*} \tilde{k}_q(Y) \longrightarrow \tilde{k}_q(Cf) \tag{2.3}$$

to be exact for a map $f : X \longrightarrow Y$ with cofiber $Cf = Y \cup_f CX$. The wedge axiom requires the functors $\tilde{k}_q$ to carry wedges (1-point unions) to direct sums. The weak equivalence axiom requires a weak equivalence to induce isomorphisms on all homology groups. Given such a reduced homology theory, one obtains an unreduced homology theory by setting $k_q(X) = \tilde{k}_q(X_+)$, where X_+ is the union of X and a disjoint basepoint, and $k_q(X, A) = \tilde{k}(Cf)$, where $f : A \longrightarrow X$ is the inclusion. For an unreduced homology theory k_*, one obtains a reduced homology theory by setting $\tilde{k}_q(X) = k_q(X, *)$. Thus reduced and unreduced homology theories are equivalent notions. The same is true for cohomology theories. The summary in this paragraph makes no reference to the dimension axiom and applies in general.

In view of (2.2), all of ordinary homology and cohomology theory is actually part of stable algebraic topology. As an informal rule of thumb, when thinking in terms of classical algebraic topology, one uses unreduced theories. When thinking in terms of stable algebraic topology, one wants the suspension axiom to hold without qualification in all degrees and one therefore works with reduced theories. In fact, in a great deal of recent work, it is an accepted convention that k_* *means* reduced homology, and one writes $k_*(X_+)$ for unreduced homology. I shall not take that point of view here, however.

This summary of the axioms is skewed towards singular homology and cohomology. The viewpoint of someone working in, say, algebraic geometry would be quite different. However, there are two footnotes to the axioms that are not well-known and may be worth mentioning. To characterize Čech cohomology on compact Hausdorff spaces, Eilenberg and Steenrod add the continuity axiom. Keesee [Kee51] observed that this axiom implies the homotopy axiom.

More substantively, let us go back to (1.1) above. If X has the homotopy type of a CW complex, then the square brackets denote homotopy classes of based maps. Huber [Hu61] proved that if X is a paracompact Hausdorff space, then the *Čech* cohomology group $\check{H}^n(X; \pi)$ is isomorphic to the set of homotopy classes of maps $X \longrightarrow K(\pi, n)$. In contrast, for the general representation of *singular* cohomology in the form (1.1), we must understand $[X, K(\pi, n)]$ to be the set of maps in the category that is obtained from the homotopy category of based spaces by adjoining formal inverses to the weak equivalences; equivalently, we must replace X by a CW complex weakly equivalent to it before taking homotopy classes of maps.

3. Stable and unstable homotopy groups

Another important precursor of stable algebraic topology was a substantial increase in the understanding of the relationship between stable and unstable homotopy groups and of certain fundamental exact sequences relating homotopy groups in different dimensions. I am here thinking of what was achieved by bare hands work, in the early to mid 1950's, using CW complexes and homotopical methods rather than the contemporaneous and overlapping progress that came with the introduction of spectral sequences.

We have seen that the critical axiom for homology is excision. In the early 1950's, Blakers and Massey [BM51, BM52, BM53] made a systematic study of excision in homotopy theory, proving that homotopy groups satisfy the excision axiom in a range of dimensions. This gave a new proof of the Freudenthal suspension theorem and considerably clarified the conceptual relationship between homology and homotopy. The proofs were quite difficult, and it soon became fashionable to prove versions of their results using homology and spectral sequences. However, Boardman later came up with a quite accessible direct homotopical proof, which is presented in [Swi75], for example. It is worth emphasizing that the homotopical proof gives a stronger result than can be obtained by homological methods.

The Freudenthal suspension theorem establishes the stable range for homotopy groups, roughly twice the connectivity of a space. It was shown by G.W. Whitehead [Wh53] that there is a metastable range for the homotopy groups of spheres. The suspension homomorphism E fits into the EHP exact sequence

$$\cdots \longrightarrow \pi_q(S^n) \xrightarrow{E} \pi_{q+1}(S^{n+1}) \xrightarrow{H} \pi_{q+1}(S^{2n+1}) \xrightarrow{P} \pi_{q-1}(S^n) \xrightarrow{E} \pi_q(S^{n+1}) \longrightarrow \cdots$$

when $q \leq 3n-2$. Here H is a (generalized) Hopf invariant that Whitehead had introduced earlier [Wh50] and P is the (J.H.C.) Whitehead product. There were many extensions and refinements of these results. For example, Hilton [Hil51] gave a definition of the Hopf invariant in the next range of dimensions, $q < 4n$, in the sequence above. The extrapolation of calculations and understanding in stable homotopy theory to calculations and understanding in the metastable range, and further, has been a major theme ever since.

James [Ja55, Ja56a, Ja56b, Ja57] and Toda [To62a] went much further with this. James proved that, on 2-primary components, there is an EHP exact sequence that is valid for all values of q, and Toda proved an appropriate analogue for odd primes. James introduced the James construction JX for the purpose. Here JX is the free topological monoid generated by a based space X. For a connected CW complex X, James proved that JX is homotopy equivalent to $\mathcal{M}\Sigma X$. The space JX comes with a natural filtration, and its simple combinatorial structure allows direct construction of suitable Hopf invariant maps. Milnor [Mil56b] proved that ΣJX splits up to homotopy as the wedge of the suspensions of its filtration quotients. These arguments were the prototypes for a great deal of later work in which combinatorial approximations to the n-fold loop spaces $\mathcal{M}^n\Sigma^n X$ have been used to obtain stable decompositions of such spaces, leading to a great deal of new calculational information in stable homotopy theory. However, this goes beyond the present story.

The power and limitations of such direct homotopical methods of calculation are well illustrated in Toda's series of papers [To58a, To58b, To58c, To59] and monograph [To62b]; while cohomology operations, spectral sequences, and the method of killing homotopy groups are used extensively, most of the work in these calculations of the groups $\pi_{n+k}(S^n)$ for small k consists of direct elementwise inductive arguments in the EHP sequence. Later work of this sort gave these groups for a few more values of k, but it was apparent that this was not the route towards major progress in the determination of the homotopy groups of spheres.

4. Spectral sequences and calculations in homology and homotopy

Although the credit for the invention of spectral sequences belongs to Leray [Le49, Mc], for algebraic topology the decisive introduction of spectral sequences is due to Serre [Se51]. For a fibration $p : E \longrightarrow B$ with connected base space B and fiber F, the Serre spectral sequence in homology has $E^2_{p,q} = H_p(B; H_q(F;\pi))$, where local coefficients are understood, and it converges in total degree $p+q$ to $H_*(E;\pi)$. The analogous cohomology spectral sequence with coefficients in a commutative ring π is a spectral sequence of differential algebras, and it converges to the associated graded algebra of $H^*(E;\pi)$ with respect to a suitable filtration.

With the Serre spectral sequence, algebraic topology emerged as a subject in which substantial calculations could be made. While its applications go far beyond our purview, many of the calculations that it made possible and ideas to which it led were essential prerequisites to the emergence of stable algebraic topology.

Work of Borel [Bo53a, Bo53b] and others gave a systematic understanding of the homology and cohomology of the classical Lie groups and of their classifying spaces and homogeneous spaces. The basic characteristic classes had all been defined earlier, but the precise detailed analysis of the various cohomology algebras and their induced maps was vital to future progress.

Serre's introduction of class theory [Se53a], and his use of the spectral sequence to prove the finiteness of the homotopy groups of spheres, save for $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$, were to change the way people thought about algebraic topology. Earlier calculations had generally had as their goal the understanding of homology and cohomology with integer or with real coefficients. In the years since, calculations have largely focused on mod p homology and cohomology, especially in stable algebraic topology where the rational theory is essentially trivial. Moreover, this change in point of view led ultimately to the study of all of homotopy theory in terms of localized and completed spaces.

The method of killing homotopy groups introduced by Cartan and Serre [CS52a, CS52b] was also profoundly influential. It provided the first systematic route to the computations of homotopy groups. The idea is easy enough. Let X be a simple space. Inductively, by killing homotopy groups and passing to homotopy fibers, one can construct a sequence of fibrations

$$p_n : X[n+1,\infty) \longrightarrow X[n,\infty)$$

with fibre $K(\pi_n(X), n-1)$, where $X[n,\infty)$ is $(n-1)$-connected and its higher homotopy groups are those of X. The initial map p_1 is just the universal covering of X. Assuming that one knows the first n homotopy groups of X, one should have enough inductive control on the space $X[n,\infty)$ to use the Serre spectral sequence to compute $H_{n+1}(X[n+1,\infty))$, which by the Hurewicz isomorphism is $\pi_{n+1}(X)$. This is closely related to Postnikov systems [Pos51a, Pos51b, Pos51c], which were not yet available to Cartan and Serre and so were implicitly reinvented by them. If $i_n : X \longrightarrow X_n$ is the nth term of the Postnikov tower of X, then i_n induces an isomorphism on π_q for $q \leq n$ and the higher homotopy groups of X_n are zero; $X[n+1,\infty)$ is the homotopy fiber of i_n.

An interesting companion to this method was given in Moore's study [Mo54] of the homotopy groups of spaces with a single non-vanishing homology group, which are now called Moore spaces. This work led later to the introduction of the mod p homotopy groups of spaces. Cohomotopy groups with coefficients were introduced and studied earlier, by Peterson [Pe56a, Pe56b]. Moore also gave a functorial, semi-simplicial, construction of Postnikov systems, in [Ca54-55] and [Mo58], which are sometimes called Moore-Postnikov systems as a result. This and related work of Moore in [Ca54-55], Heller in [He55], and especially Kan in [Kan55] and many later papers (see [May67]), began the modern systematic use of simplicial methods in algebraic topology.

5. Steenrod operations, $K(\pi, n)$'s, and characteristic classes

For the method of killing homotopy groups to be useful, one must know something about the cohomology of Eilenberg-Mac Lane spaces. The problem of calculating these cohomology groups was intensively studied by Eilenberg and Mac Lane, notably in [EM50], and was solved a few years later by Cartan [Ca54-55], using methods of homological algebra. However, Cartan's original answer was not in the form we know it today. In fact, in mod p cohomology for odd primes p, it is still not obvious how to correlate Cartan's calculations with the definitive calculations in terms of Steenrod operations.

I will not say anything about the invention and development of the basic properties of the Steenrod operations [St47, St52, St53a, St53b, St57, ST57] since that is interestingly discussed in [Ma] and [Wh1]. Steenrod and Epstein [SE62] published a systematic account of the results. Epstein [Ep66] later showed how to construct Steenrod operations in a general context of homological algebra. In fact, simply by separating the algebra from the topology, Steenrod's original definition can be adapted to a variety of situations in both topology and algebra [May70].

An essential point is that the Steenrod operations are stable, in the sense that the following diagrams commute, where $\mathbb{Z}_2$ is the field $\mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd} \tilde{H}^q(X, \mathbb{Z}_2) \arrow[r, "Sq^i"] \arrow[d, "\Sigma^*"'] & \tilde{H}^{q+i}(X; \mathbb{Z}_2) \arrow[d, "\Sigma^*"] \\ \tilde{H}^{q+1}(\Sigma X; \mathbb{Z}_2) \arrow[r, "Sq^i"'] & \tilde{H}^{q+1+i}(\Sigma X; \mathbb{Z}_2). \end{tikzcd} \tag{5.1}$$

The analogous diagram commutes for odd primes, where P^i has degree $2i(p-1)$.

Serre [Se53b] computed $H^*(K(\pi_2, n); \mathbb{Z}_2)$, where π_2 is cyclic of order 2, in modern terms: it is the free commutative algebra on suitable composites of Steenrod operations acting on the fundamental class $\iota_n \in H^n(K(\pi_2, n); \mathbb{Z}_2)$. The analogue for odd primes was worked out by Cartan in [Ca54-55], in later exposés that are in fact independent of his original calculations published in the same place.

Formulas for the iteration of the Steenrod operations were first proven by Adem [Adem52] at the prime 2 and by Adem and Cartan [Adem53, Adem57, Ca55], independently, at odd primes. However, it was Cartan who first defined the Steenrod algebra A_p and determined its basis of admissible monomials.

In the paper [Se53b], Serre also formulated the modern viewpoint on cohomology operations. A cohomology operation ϕ of degree i is a natural transformation $k^q \longrightarrow \ell^{q+i}$ for some fixed q, where k^* and ℓ^* are any cohomology theories. When k^* is ordinary cohomology with coefficients in π and ℓ^* is ordinary cohomology with coefficients in ρ, ϕ is determined by naturality by the element $\phi(\iota_q) \in H^{q+i}(K(\pi, q); \rho)$. Observe that, by (1.1), this element may be viewed as a homotopy class of maps $K(\pi, q) \longrightarrow K(\rho, q+i)$.

A crucial point quickly understood was the calculation of the Steenrod operations in the cohomologies of Lie groups and their classifying spaces and homogeneous spaces. In particular, already in 1950 [Wu50a, Wu53], Wu proved his basic formula for the calculation of the Steenrod operations on the Stiefel-Whitney classes:

$$Sq^r(w_s) = \sum_t \binom{s-r+t-1}{t} w_{r-t}w_{s+t} \quad \text{for} \quad s > r \geq 0. \tag{5.2}$$

Borel and Serre made a systematic study shortly afterwards [BS51, BS53].

Also in 1950 [Wu50b], Wu proved his formula giving an algorithm for the calculation of the Stiefel-Whitney classes of the tangent bundle of a manifold directly in terms of its cup products; see Section 12. Wu was a close collaborator of Thom, and his work was dependent on work of Thom, announced in part in 1950 [Thom50a, Thom50b] and published in 1952 [Thom52]. In that paper, Thom proved the Thom isomorphism theorem and used it to give the now familiar description of Stiefel-Whitney classes in terms of Steenrod operations. Since [Thom52] was later overshadowed by Thom's great work on cobordism, it is well worth describing some of its original contributions.

Thom considered locally trivial fiber bundles $p : E \longrightarrow B$ with fiber S^{k-1}, with no assumptions about the group of the bundle. Working sheaf theoretically and resolutely avoiding the use of spectral sequences, which were available to him, Thom proved the Thom isomorphism

$$\phi : H^q(B) \longrightarrow H^{q+k}(Mp, E), \tag{5.3}$$

where Mp is the mapping cylinder of p. He worked with twisted integer coefficients, thus allowing for non-oriented fibrations, before studying the mod 2 case. Observe that, in the motivating example of the unit sphere bundle $E = S(E')$ of a k-dimensional vector bundle $p' : E' \longrightarrow B$ with a Riemannian metric, the quotient space Mp/E is homeomorphic to the quotient space $D(E')/S(E')$, where $D(E')$ is the unit disk bundle of E'. This quotient space is called the Thom space of p' and now usually denoted Tp' or $T(E')$.

Using mod 2 coefficients in the Thom isomorphism, Thom defined the Stiefel-Whitney classes of E by

$$w_i = \phi^{-1} Sq^i \phi(1), \tag{5.4}$$

and he proved that, in the case of vector bundles, these are the classical Stiefel-Whitney classes of E. He rederived the properties of Stiefel-Whitney classes, in particular the Whitney duality theorem, from the new definition. This gave an elegant new proof of Whitney's result [Whit41] that the Stiefel-Whitney classes of the normal bundle of an immersion f are invariants of the induced map f^* on mod 2 cohomology. In particular, they are independent of the choice of the differentiable structures on the manifolds in question. It is worth emphasizing that Whitney's foundational work in [Whit41] and other papers, for example on embeddings and immersions of smooth manifolds, was an essential prerequisite to virtually all of the later applications of algebraic topology to geometric topology.

Thom then generalized to obtain results of this form for purely topological immersions, with no hypothesis of differentiability. It should be remembered that this paper appeared four years before Milnor's discovery of exotic differential structures on spheres [Mil56a]. For an embedding f, he went further and showed that the homotopy type of a tubular neighborhood of f is independent of the differentiable structure on the ambient manifold. He then introduced the notion of fiber homotopy equivalence and proved that the fiber homotopy type of the tangent bundle of a manifold is independent of its differentiable structure. He observed that the

Stiefel-Whitney classes are invariant under fiber homotopy equivalence, and asked what other such classes there might be. The determination of all characteristic classes for spherical fibrations evolved over the following two decades. That is a long story, intertwined with the theory of iterated loop spaces, and is well beyond our present scope.

6. The introduction of cobordism

In the last chapter of [Thom52], Thom set up the modern theory of Poincaré duality for manifolds with boundary and explained the now familiar necessary Euler characteristic and index conditions for a differentiable manifold to be the boundary of a compact differentiable manifold. The emphasis he placed on the index was a precursor of things to come. He also recalled Pontryagin's fundamental observation [Pon42, Pon47] that, for M to be such a boundary, it is necessary that all of its characteristic numbers be zero. He went on to observe that the vanishing of Stiefel-Whitney numbers is still a necessary condition when M is not assumed to be differentiable. He observed that "la recherche de conditions suffisantes est un problème beaucoup plus difficile".

Two years later, as announced in [Thom53a, Thom53b, Thom53c] and published in his wonderful 1954 paper [Thom54], he had solved this problem for smooth compact manifolds. The importance to modern topology, both geometric and algebraic, of his introduction and calculation of cobordism cannot be exaggerated. For example, Milnor's construction of exotic differentiable structures on S^7 begins with Thom's theory and in particular with Thom's result that every smooth compact 7-manifold is a boundary.

Cobordism theory was not wholly unprecedented. In 1950, Pontryagin [Pon50] showed that the stable homotopy groups of spheres, in low dimension at least, are isomorphic to the framed cobordism groups of smooth manifolds. His motivation was to obtain methods for the computation of stable homotopy groups, and he used this technique to prove that $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$, thus correcting an earlier mistake of his. While that motivation seems misguided in retrospect, it was an imaginative attack on the problem. Pontryagin's paper was in Russian, never translated, and it is not quoted by Thom. However, Thom does quote earlier papers of Pontryagin [Pon42, Pon47] in which the idea of pulling back the zero-section in Grassmannians along a smooth approximation to a classifying map plays a prominent role.

Thom's paper [Thom54] reads a little surprisingly today. Its main focus is not cobordism, which does not appear until the last chapter, but rather the realization of homology classes of manifolds by submanifolds. It seems that it was this that first motivated Thom to a detailed analysis of the cohomology and homotopy of Thom complexes, not just in the stable range relevant to corbordism but also in the unstable range. Moreover, the existence of a stable range for the homotopy groups of $TSO(k)$ and $TO(k)$ is proven by direct methods of algebraic topology, rather than as a consequence of the isomorphism between homotopy groups and cobordism groups.

For a closed subgroup G of $O(k)$, Thom lets $T(G)$ be the Thom space of the universal bundle $E_G \longrightarrow B_G$ with fiber S^{k-1}. He considers a compact oriented manifold V^n and asks when a homology class $x \in H_{n-k}(V)$ is realizable as the image of the fundamental class of submanifold W^{n-k} of codimension k. He dualizes the question as follows. For any space X, say that a class $y \in H^k(X)$ is G-realizable if there is a map $f : X \longrightarrow T(G)$ such that $f^*(\mu) = y$, where $\mu \in H^k(T(G))$ is the Thom class. Let $y \in H^k(V)$ be Poincaré dual to x. Then "le théorème fondamental" asserts that x is realizable by a submanifold W such that the structure group of the normal bundle of W in V can be reduced to G if and only if y is G-realizable. Of course, the analogue with mod 2 coefficients does not need orientability. As we shall see in Section 16, Atiyah explained this result conceptually almost a decade later.

Taking G to be the trivial group, it follows from a result of Serre [Se53a] that x is realizable if k is odd or if $n < 2k$ and that Nx is realizable for some integer N that depends only on k and n. However, the main focus is on $G = O(k)$ and $G = SO(k)$. Here Thom shows directly that $\pi_{k+i}(TO(k))$ is independent of k when $i < k$, and similarly for $TSO(k)$. Moreover, crucially, he proves that $TO(k)$ has the same $2k$-type as a precisely specified product of Eilenberg-Mac Lane spaces $K(\mathbb{Z}_2, k+i)$. The Wu formula (5.2) is the key to the calculation. He goes on to study $H^*(TO(k);\mathbb{Z}_2)$ in low dimensions beyond the stable range for $k \leq 3$. For the realizability problem, he deduces that $x \in H_i(V^n;\mathbb{Z}_2)$ is realizable for $i < [n/2]$, with further information in low codegrees $n - i$.

The problem for $TSO(k)$ is much harder, and $\pi_{k+i}(TSO(k))$ is only determined completely for $i \leq 7$; more detailed information is obtained for $k \leq 4$. However, Thom shows that $TSO(k)$ has the rational cohomology type of an explicitly specified product of Eilenberg-Mac Lane spaces $K(\mathbb{Z}, k+i)$. For the realizability problem, he deduces that some integer multiple of any $x \in H_i(V^n;\mathbb{Z})$ is realizable, and that any x is realizable if $i \leq 5$ or $n \leq 8$.

Before turning to cobordism, Thom studies the problem posed by Steenrod of determining which homology classes $x \in H_r(K)$ of a finite polyhedron K are realizable as $f_*(z)$, where z is the fundamental class of a compact manifold M^r and $f : M^r \longrightarrow K$ is a map. By embedding K as a retract of a manifold with boundary M and taking the double V of M to obtain a manifold without boundary, Thom reduces this question to the realizability question already studied. He thereby proves that, in mod 2 homology, every class x is realizable. In retrospect, of course, this presages unoriented bordism and its relationship to ordinary mod 2 homology. Similarly, he proves that, in integral homology, some integer multiple of every class x is realizable. Remarkably, he then proves that every class x is realizable if $r \leq 6$, but that there are unrealizable classes in all dimensions $r \geq 7$.

Only after all of this does he prove the cobordism theorems. Let $\mathcal{N}_n$ be the set of cobordism classes of smooth compact n-manifolds, where two n-manifolds are cobordant if their disjoint union is the boundary of a smooth compact $(n+1)$-manifold with boundary. Define $\Updownarrow ega_n$ similarly for oriented n-manifolds. Under disjoint union, $\mathcal{N}_n$ is a $\mathbb{Z}_2$-vector space and $\mathcal{M}_n$ is an Abelian group; any boundary is the zero element. Under cartesian product, $\mathcal{N}_*$ and $\mathcal{M}_*$ are graded rings. Moreover, the index defines a ring homomorphism $I : \mathcal{M}_* \longrightarrow \mathbb{Z}$. The fundamental geometric theorem is the Thom isomorphism: $\mathcal{N}_n$ is isomorphic to the stable homotopy group $\pi_{k+n}(TO(k))$ and $\mathcal{M}_n$ is isomorphic to the stable homotopy group $\pi_{k+n}(TSO(k))$.

While modern proofs are easier reading than Thom's, the basic ideas are the same. In slightly modernized terms, an isomorphism $\phi : \mathcal{N}_n \longrightarrow \pi_{k+n}(TO(k))$ is constructed as follows. Embed a given n-manifold M in $\mathbb{R}^{k+n}$ for k large, let ν be the normal bundle of the embedding, and construct a tubular neighborhood V of M in $\mathbb{R}^{k+n}$. Define a map f from S^{k+n} to the Thom space $T(\nu)$ by identifying V with the total space of ν and mapping points not in V to the basepoint. This is the

Pontryagin-Thom construction. Classify ν and compose f with the induced map of Thom spaces $T(\nu) \longrightarrow TO(k)$ to obtain $\phi(M)$, checking that the homotopy class of the composite is independent of the choice of M in its cobordism class and of the embedding. To construct an inverse isomorphism ψ to ϕ, view the classifying space $BO(k)$ as a Grassmannian manifold of sufficiently high dimension. Up to homotopy, any map $g : S^{k+n} \longrightarrow TO(k)$ can be smoothly approximated by a map that is transverse to the zero section. Define $\psi(g)$ to be the cobordism class of the inverse image of the zero section, checking that this class is independent of the homotopy class of g. Transversality is the crux of the proof, and Thom was the first to develop this notion.

From here, the earlier calculations in the paper immediately identify the groups $\mathcal{N}_n$. Using this identification, Thom proves that two manifolds are cobordant if and only if they have the same Stiefel-Whitney numbers. By calculating the Stiefel-Whitney numbers of products, this allows him to determine the ring structure of $\mathcal{N}_*$: it is a polynomial algebra on one generator of dimension n for each $n \geq 2$ not of the form $2^j - 1$. The even dimensional generators can be chosen to be the cobordism classes of the real projective spaces $\mathbb{RP}^{2n}$.

Similarly, the groups $\mathcal{M}_n$ are identified modulo torsion by the earlier calculations. Using this, Thom proves that if all Pontryagin numbers of an oriented manifold are zero, then the disjoint union of some number of copies of that manifold is a boundary. This allows determination of the ring $\mathcal{M}_* \otimes \mathbb{Q}$: it is a polynomial algebra on generators of dimension $4n$ for $n \geq 1$. The generators can be chosen to be the cobordism classes of the complex projective spaces $\mathbb{C}P^{2n}$.

Dold [Dold56a] soon after identified odd dimensional generators of $\mathcal{N}_*$. The Wu formula for the computation of Stiefel-Whitney classes of manifolds give restrictions on which collections of Steifel-Whitney numbers actually correspond to the cobordism class of a manifold, and Dold [Dold56b] proved that these relations are complete: a collection of Stiefel-Whitney numbers that satisfies the Wu relations corresponds to a manifold. In modern invariant terms, the Stiefel-Whitney numbers of manifolds define a monomomorphism $\mathcal{N}_* \longrightarrow \mathrm{Hom}(H^*(BO; \mathbb{Z}_2), \mathbb{Z}_2)$, and its image consists of those homomorphisms that annihilate the subgroup generated by the Wu relations.

7. The route from cobordism towards K-theory

Hirzebruch [Hirz53] had already introduced multiplicative sequences of characteristic classes before Thom's paper. However, cobordism theory provided exactly the right framework for their study and allowed him to prove the index theorem [Hirz56]: the index of a smooth oriented $4n$-manifold M is the characteristic number $\langle L(\tau), [M] \rangle$, where L is the L-genus and τ is the tangent bundle of M. Here $L(\tau)$ is a polynomial in the Pontryagin classes of M determined in Hirzebruch's formalism by the power series $L(x) = x/\tanh(x)$. Using Thom's observation that the index defines a ring homomorphism $\mathcal{M}_* \longrightarrow \mathbb{Z}$, Hirzebruch's formalism shows that the index formula must hold for some power series L, and $L(x)$ is the only power series that gives the correct answer on complex projective spaces.

The purpose of Hirzebruch's monograph [Hirz56] was to prove the Riemann-Roch theorem for algebraic varieties of arbitrary dimension. It would take us too far afield to say much about this, and a quite detailed summary may be found in Dieudonné [Dieu, pp. 580-595]. Suffice it to say that Hirzebruch's essential strategy was to reduce the Riemann-Roch theorem to the index theorem. One key ingredient in the reduction should be mentioned, namely a method for splitting vector bundles that led later to the splitting principle in K-theory.

Another nice discussion of [Hirz56] may be found in Bott's review [Bott61] of the second part of Borel and Hirzebruch's deeply influential work [BH58, BH59, BH60]. The Riemann-Roch theorem showed that the characteristic number $\langle T(\tau_c), [M]\rangle$ of any projective non-singular variety M is an integer, namely the arithmetic genus of M; here τ_c is the complex tangent bundle of M and T is the Todd genus, which is determined by the power series $T(x) = x/1 - e^{-x}$. Borel and Hirzebruch sought and proved an analogous integrality theorem for arbitrary differentiable manifolds. The $\hat{A}$-genus is related to the Todd genus by the formula $T(x) = e^{x/2}\hat{A}(x)$, and it satisfies $\hat{A}(x) = \hat{A}(-x)$. As Bott explains clearly, this makes it plausible that the $\hat{A}$ genus should satisfy a similar integrality relation on arbitrary compact manifolds, as Borel and Hirzebruch prove. More precisely, they prove it up to a factor of 2 that was later eliminated by Milnor's proof (implicit in [Mil60]) that the Todd genus of an almost complex manifold is an integer.

Milnor and Kervaire [Mil58b, KM60] gave an important application of the integrality of the $\hat{A}$-genus. In 1943, G.W. Whitehead introduced the stable J-homomorphism $J : \pi_q(SO(n)) \longrightarrow \pi_{q+n}(S^n)$, n large. Writing $\pi_q^s = \pi_{q+n}(S^n)$ for the qth stable homotopy group of spheres and letting n go to infinity, this can be written $J : \pi_q(SO) \longrightarrow \pi_q^s$. Milnor and Kervaire used the integrality theorem to prove that, when $q = 4k-1$, the order j_n of the image of J is divisible by the denominator of $B_k/4k$, where B_k is the kth Bernoulli number. This result gave the first sign of regularity in the stable homotopy groups of spheres, and their proof showed that the J-homomorphism is of considerable relevance to geometric topology. In fact, although this is a result in stable homotopy theory, they derive it from a generalization of a theorem of Rohlin in differential topology. Rohlin's theorem [Ro51, Ro52] states that the Pontrjagin number $p_1(M)$ of a compact oriented smooth 4-manifold M with $w_2(M) = 0$ is divisible by 48. Milnor and Kervaire mimic his arguments to prove that the Pontrjagin number $p_n(M)$ of an almost parallelizable smooth $4n$-manifold is divisible by $(2n-1)!j_n a_n$, where a_n is 2 if n is even and 1 if n is odd, with equality for at least one such manifold M.

For the historical story, one striking feature of the work of Borel and Hirzebruch is its systematic use of multiplicative functions $F_{\mathbb{C}}(X) \longrightarrow H^{**}(X;\mathbb{R})$ and $F_{\mathbb{R}}(X) \longrightarrow H^{**}(X;\mathbb{R})$, where $F_{\mathbb{R}}(X)$ and $F_{\mathbb{C}}(X)$ are the semi-groups of equivalence classes of complex and real vector bundles over X and $H^{**}(X;\mathbb{R})$ is the direct product of the real cohomology groups of X. A multiplicative function is one that converts sums to products. The authors are tantalizingly close to K-theory. Two things are missing: the Grothendieck construction and Bott periodicity.

The first was introduced by Grothendieck [BS58], who needed it to formulate his generalized, relative, version of the Riemann-Roch theorem in algebraic geometry. Grothendieck is the inventor of the general subject of K-theory, and his ideas played a centrally important role in the introduction of topological K-theory.

As to the second, as Bott notes in his review, the work of Borel and Hirzebruch led them to an exact sequence

$$0 \longrightarrow \mathbb{Z}_{n!} \longrightarrow \pi_{2n}(U(n)) \longrightarrow \pi_{2n}(U(n+1)) \longrightarrow 0. \tag{7.1}$$

More precisely, they proved the sequence to be exact modulo 2-torsion. As Bott writes: "The exact sequence conflicted, at the time of its discovery, with computations of homotopy theorists and led to a spirited controversy. At present it is known the sequence is exact even with regard to the prime 2." What he neglects to say is that the sequence also follows from Bott periodicity, and the conflict for some time held up publication of that result.

8. Bott periodicity and K-theory

One version of the Bott periodicity theorem asserts that there is a homotopy equivalence $BU \longrightarrow \mathcal{M}SU$. The periodicity is clearer in the equivalent reformulation $BU \times \mathbb{Z} \simeq \mathcal{M}^2(BU \times \mathbb{Z})$. The real analogue gives $BO \times \mathbb{Z} \simeq \mathcal{M}^8(BO \times \mathbb{Z})$. Bott's original proof of these beautiful results is based on the use of Morse theory. Before proving the periodicity theorem, Bott had clearly demonstrated the power of Morse theory by using it to prove that there is no torsion in the integral homology of $\mathcal{M}G$ for any simply connected compact Lie group G [Bott56]. Bott announced the periodicity theorem in [Bott57], and he gave two somewhat different proofs, both based on Morse theory, in [Bott58, Bott59a].

It immediately became a challenge to reprove the periodicity theorems using the standard methods of algebraic topology. In the complex case, a proof was given by Toda [To62b], together with a rederivation of the Borel-Hirzebruch exact sequence (7.1), but his proof did not show that BU and $\mathcal{M}SU$ have the same homotopy type. The space BU is an H-space under Whitney sum, and Bott's proofs led to simple and explicit H-maps that give the equivalences. In the real case, there are actually six maps that must be proven to be equivalences. These explicit maps were exploited by Dyer and Lashof [DL61] and Moore (written up by Cartan [Ca54-55]) to give direct calculational proofs. Actually, there is a curious simplification to be made: comparison of the proofs in [DL61] and [Ca54-55] shows that each finds particular difficulty in proving one of the required equivalences, but they find difficulty with different maps: combining the best of both proofs gives a quite tractable argument.

Finally, in their announcement [AH59], submitted in May, 1959, Atiyah and Hirzebruch introduce the functor $K(X)$ for a finite CW complex X: it is the Grothendieck construction on the semi-group $F_{\mathbb{C}}(X)$, and it is a ring with multiplication induced by the tensor product of vector bundles. They define $KO(X)$ similarly. They noticed a striking reinterpretation of Bott periodicity: tensor product of bundles induces a natural isomorphism β that fits into the commutative diagram

$$\begin{tikzcd}
K(X) \otimes K(S^0) \arrow[r, "\beta"] \arrow[d, "ch"'] & K(X \times S^2) \arrow[d, "ch"] \\
H^{**}(X;\mathbb{Q}) \otimes H^{**}(S^2);\mathbb{Q}) \arrow[r, "\alpha"'] & H^{**}(X \times S^2;\mathbb{Q}),
\end{tikzcd}$$

where ch is the Chern character and α is the cup product isomorphism.

They observe that, for connected X, the kernel $\tilde{K}(X)$ of the dimension map $\varepsilon : K(X) \longrightarrow \mathbb{Z}$ can be identified with the set of homotopy classes of maps $X \longrightarrow BU$. In principle, modulo a $\lim^1$ argument not yet available, this leads to a homotopy equivalence from BU to the basepoint component of $\mathcal{M}^2BU$. However, their reinterpretation of Bott periodicity was by no means an obvious one. In

[Bott58], Bott related his explicit maps to tensor products of bundles and so proved that his original version of the periodicity theorem really did imply the version noticed by Atiyah and Hirzebruch. Moreover, he gave the analogous reinterpretation in the real case, where a direct proof of the new version was less simple.

Jumping ahead to 1963 for a moment, Atiyah and Bott together [AB64] then found a direct and elementary analytic proof of the complex case of the periodicity isomorphism in its tensor product formulation, using clutching functions to describe bundles over $X \times S^2$ explicitly. Their proof actually gives a more general result, namely a Thom isomorphism, and important refinements and generalizations are given in their lecture notes [At64] and [Bott63]. The analytic proof is relevant to the Atiyah-Singer index theorem, which was already announced in 1963 [AS63] and which generalizes Hirzebruch's index theorem. The first published proof appeared in 1965 [Pa65], based on seminars in 1963-64.

In their 1959 announcement [AH59] and also in [Hirz59], Atiyah and Hirzebruch give a Riemann-Roch theorem relative to a suitable map $f : M \longrightarrow N$ of differential manifolds; see Section 12 for the statement. They observe that their theorem can be rewritten for holomorphic maps between complex manifolds in the same form as Grothendieck's version of the Reimann-Roch theorem. Their results imply a new proof of the integrality of the $\hat{A}$-genus, together with a sharpening in the case of *Spin*-manifolds of dimension congruent to 4 mod 8 that had been conjectured by Borel and Hirzebruch. They also rederive and give a conceptual sharpening of Milnor's result on the J-homomorphism.

In [AH59], nothing is said about $K(X)$ being part of a generalized cohomology theory. Moreover, it is clear that the authors as yet have no hint of K-homology and Poincaré duality: their statement of the Riemann-Roch theorem involves a pushforward map $f_!$, as it must, but that map was not well understood. They remark that "It is probable that $f_!$ is actually induced by a functorial homomorphism $K(Y) \longrightarrow K(X)$".

Rather than proceed directly to 1960 and the first published account of K-theory as a generalized cohomology theory, I shall interpolate a discussion of several quite different lines of work that were going on in the late 1950's.

As preamble, Milnor [BM58, Mil58c] saw immediately, in February, 1958, that Bott's results led to the solution of two longstanding problems; [BM58] is a pair of letters between Milnor and Bott on this subject, and [Mil58c] fills in the details. The relevant result of Bott is that the image of the Hurewicz homomorphism $\pi_{2n}(BU) \longrightarrow H_{2n}(BU)$ is divisible by exactly $(n-1)!$. This is closely related to the exact sequence (7.1). What Milnor deduces from this is:

1. The vector space $\mathbb{R}^n$ possesses a bilinear product without zero divisors only for n equal to 1, 2, 4, or 8.
2. The sphere S^{n-1} is parallelizable only for $n-1$ equal to 1, 3, or 7.

The latter result was also proven at about the same time by Kervaire [Ker58].

9. The Adams spectral sequence and Hopf invariant one

Milnor's results just cited are also among the many implications of Adams' celebrated theorem that $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant one if and only if n is 1, 2, 4, or 8 [Ad60]. This result was announced in [Ad58b], which was submitted in April, 1958. This work was a sequel to and completion of work begun in [Ad58a], submitted in June, 1957, in which Adams first attacked the Hopf invariant one problem and introduced the Adams spectral sequence.

Fix a prime p, let A be the mod p Steenrod algebra, and let X be a space. In its original form in [Ad58a], the Adams spectral sequence satisfies

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X), \mathbb{Z}_p),$$

where s is the homological degree, t is the internal degree, and $t-s$ is the total degree, so that $E_2^{s,t} = 0$ if $s < 0$ or $t < s$. The differentials are of the form

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}.$$

There is a filtration of the stable homotopy groups $\pi_n^s(X)$ such that

$$E_\infty^{s,n+s} = F^s\pi_n^s(X)/F^{s+1,n+s+1}\pi_n^s(X).$$

The intersection of the filtrations consists of the elements of $\pi_n^s(X)$ that are of finite order prime to p. When $X = S^0$, $\{E_r^{*,*}\}$ is a spectral sequence of differential $\mathbb{Z}_p$-algebras and converges as an algebra to the associated graded algebra of the ring of stable homotopy groups of spheres under the composition product.

The Adams spectral sequence can be thought of in several ways: it is a sophisticated reformulation and generalization of the Cartan-Serre method of killing homotopy groups, and it is an extension and systematization of the method of studying homotopy groups by considering higher order cohomology operations.

The idea of higher order operations first appeared with Steenrod's introduction of functional cohomology operations [St49]. Let $f : Y \longrightarrow X$ be a map. Steenrod showed how to construct an element $x \cup_f x'$ in $H^*(Y)$ from a pair of elements x, x' in $H^*(X)$ such that $x \cup x' = 0$ and $f^*(x') = 0$. He defined functional mod 2 Steenrod operations similarly. These operations are defined on a subspace of $H^*(X)$, and they are well-defined up to indeterminacy. Adem [Adem56] made a systematic study of functional cohomology operations associated to stable cohomology operations, and Peterson [Pe57] gave a presentation in terms of Postnikov systems with stable k-invariants. Although a few low dimensional examples had appeared earlier, Adem [Adem58] gave the first systematic study of secondary cohomology operations, building on his earlier proof of the Adem relations for the iterated Steenrod operations. He related secondary and functional cohomology operations in [Adem59]. Peterson and Stein [PS59] then gave a treatment of secondary and functional operations in terms of two-stage Postnikov systems.

It was this kind of treatment that Adams had in mind. Secondary and higher operations come from relations between relations, and homological algebra is the natural tool for the study of relations between relations. The essential idea of the construction of the Adams spectral sequence is to construct a realization of a free resolution of the A-module $H^*(X)$ (in a range of dimensions) by means of a resolution of the space X. This gives a kind of exact couple of spaces that leads to an exact couple giving the desired spectral sequence on passage to homotopy groups. Implicitly, as became much clearer with a later reformulation in terms of the homology of spectra rather than the cohomology of spaces, the fundamental points are the representation (1.1) of cohomology and the calculation of the cohomology of Eilenberg-Mac Lane spaces in terms of Steenrod operations.

The relationship to the Hopf invariant one problem comes about as follows. There is an element of Hopf invariant one in $\pi_{2n-1}(S^n)$ if and only if there is a (stable) two-cell complex such that the Steenrod operation Sq^n connects the bottom cell to the top cell in mod 2 cohomology. If n is not a power of two, then Sq^n is decomposable as a linear combination of iterated Steenrod operations, by the Adem relations, and no such two-cell complex is possible. Now, for any connected $\mathbb{Z}_p$-algebra A, $\mathrm{Ext}_A^{1,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ is isomorphic to the dual of the vector space of degree t indecomposable elements of A. Take A to be the mod 2 Steenrod algebra and consider the Adams spectral sequence for $X = S^0$. Then we have elements $h_i \in E_2^{1,2^i}$ dual to the Steenrod operations Sq^{2^i}. It is direct from the construction of the spectral sequence that there is an element of Hopf invariant one detected by Sq^{2^i} if and only if h_i is a permanent cycle in the spectral sequence.

The element h_0 corresponds to the Bockstein $Sq^1 = \beta$, and multiplication by h_0 in the spectral sequence detects multiplication by 2 in the stable homotopy groups of spheres. Adams computes enough of $E_2^{s,*}$, $s = 2$ and $s = 3$, to see that the elements $h_0h_i^2$ are non-zero in E_2 for $i \geq 3$. The only way that $h_0h_i^2$ can be a boundary is if $d_2(h_{i+1}) = h_0h_i^2$. If $i \geq 3$ and both h_i and h_{i+1} are permanent cycles, we conclude that h_i represents an odd dimensional homotopy class x_i such that $2x_i^2$ is non-zero. This is impossible since π_*^s is a graded commutative ring. This implies the main theorem of [Ad58a]: if both $\pi_{2n-1}(S^n)$ and $\pi_{4n-1}(S^{2n})$ contain elements of Hopf invariant one, then $n \leq 4$, which was tantalizingly close to the expected answer.

This line of argument doesn't work to solve the problem. However, the method of proof implies that Sq^{16}, although indecomposable in A, admits a decomposition in terms of composites of primary and secondary operations, taking into account the relevant domains of definition and indeterminacy. In [Ad60], Adams constructs such a decomposition of Sq^{2^i} for all $i \geq 4$. While the argument makes no use of the Adams spectral sequence, it implies the differential $d_2(h_{i+1}) = h_0h_i^2$ for $i \geq 3$.

The arguments in [Ad60] are very long, and I won't attempt a complete summary. They require a more thorough exposition of the foundations of graded homological algebra than was needed in [Ad58a], and this work has been used ever since. They also require an axiomatization and construction of secondary cohomology operations in terms of universal examples, together with a detailed study of how to relate the homological algebra to the analysis of the operations. Finally, particular operations relevant to the problem at hand are constructed, a putative decomposition formula for Sq^{2^n} is proven formally by means of the general theory, and the coefficient of Sq^{2^n} in the decomposition is proven to be non-zero by explicit calculation in a specific example.

There are two crucially important ingredients in the work that must be singled out. First, the work of Milnor and Moore [MM65] on graded Hopf algebras plays a key role in the relevant homological algebra. Although [MM65] was not published until 1965, a mimeographed version was distributed much earlier and was an essential prerequisite to the higher level of algebraic sophistication that Adams introduced into algebraic topology.

Second, Adams needed to make some calculations of E_2 beyond those of [Ad58a], and for this purpose he made substantial use of Milnor's remarkable analysis of the structure of the Steenrod algebra [Mil58a]. This analysis has played a central role in a great many later calculations in stable algebraic topology. The Steenrod algebra A is a Hopf algebra. Its coproduct is determined by the Cartan formula and is cocommutative. Therefore the dual Hopf algebra, denoted A_*, is commutative as an algebra. Milnor proved that it is a free commutative algebra in the graded sense.

Explicitly, for an odd prime p, it can be written as a tensor product

$$A_* = E\{\tau_i | i \geq 0\} \otimes P\{\xi_i | i \geq 1\} \tag{9.1}$$

of an exterior algebra on odd degree generators τ_i and a polynomial algebra on even degree generators ξ_i. Moreover, the coproduct on the generators admits a simple explicit formula, in principle equivalent to the Adem relations but far more algebraically tractable. The dual B of $P\{\xi_i | i \geq 1\}$ can be identified both with the subalgebra of A generated by the Steenrod operations P^i and with the quotient of A by the two-sided ideal generated by the Bockstein β. Note that, in quotient form, B also makes sense when $p = 2$. We shall come back to it later.

Shortly after Adams' work, the techniques he developed were adapted to solve the analogue of the Hopf invariant one problem at odd primes p, showing that there can be a two-cell complex with P^n connecting the bottom cell to the top cell in mod p cohomology if and only if $n = 1$. This work was done independently by Liulevicius [Liu62a] and by Shimada and Yamanoshita [SY61].

Using the structure theory for mod p Hopf algebras of Milnor and Moore and Milnor's analysis of the Steenrod algebra, I later developed tools in homological algebra that allowed the use of the Adams spectral sequence for explicit computation of the stable homotopy groups of spheres in a range of dimensions considerably greater than had been known previously [May65, May65, May66]. Correspondence initiated in the course of this work led Adams and myself to a long friendship, and I have given a brief account of all of Adams' work in [May2] and a eulogy and personal reminiscences in [May1].

10. S-duality and the introduction of spectra

Setting up the Adams spectral sequence as Adams did it originally is a tedious business, the reason being that one is trying to do stable work with unstable objects: one should be using "spectra" rather than spaces. Similarly, the representability of ordinary cohomology and the introduction of cobordism and K-theory must eventually have forced the introduction of spectra, which appear naturally as sequences of Eilenberg-Mac Lane spaces, as sequences of Thom spaces, and as sequences of spaces featuring in the Bott periodicity theorem.

Nevertheless, the fact is that the introduction of spectra had nothing whatever to do with these lines of work. Rather, it grew out of the work on S-duality of Spanier and Whitehead. I will be brief about this since it is also treated in [BG] in this volume.

In 1949, Spanier [Sp50] reconsidered Borsuk's cohomotopy groups [Bor36]. For a (nice) compact pair (X, A), where dim X $< 2n - 1$, Spanier defined $\pi^n(X, A)$ to be the set of homotopy classes of maps $(X, A) \longrightarrow (S^n, *)$. As in Borsuk [Bor36], these are abelian groups, and Spanier showed that these cohomotopy groups satisfy *all* of the Eilenberg-Steenrod axioms for a cohomology theory, except that they are only defined in a range of non-negative degrees depending on the dimension of X. He also showed that the cohomotopy groups map naturally to the integral Čech cohomology groups and that, for a CW complex X with subcomplex A, $\pi^n(X^m \cup A, X^{m-1} \cup A)$ is isomorphic to the cellular cochain group $C^m(X, A; \pi_m(S^n))$. These were puzzling results. The real explanation, that these cohomotopy groups are the terms in a positive range of dimensions of a cohomology theory whose coefficients are nonzero in negative dimensions, would come later. With hindsight, the cellular cochain isomorphism just mentioned is the first hint of the Atiyah-Hirzebruch spectral sequence for stable cohomotopy theory. Spanier also observed that the Hurewicz isomorphism theorem for $[S^n, X]$ and the Hopf classification theorem for $[X, S^n]$ are dual to one another.

To make a home for such duality phenomena in all dimensions, Spanier and Whitehead devised the S-category in [SW53, SW57]. Its objects are based spaces, and the set $\{X, Y\}$ of S-maps $X \longrightarrow Y$ is

$$\{X, Y\} = \operatorname{colim}_{n\geq 0}[\Sigma^n X, \Sigma^n Y].$$

That is, homotopy classes of based maps $f : \Sigma^n X \longrightarrow \Sigma^n Y$ and $g : \Sigma^q X \longrightarrow \Sigma^q Y$ define the same S-map if $\Sigma^k f$ and $\Sigma^{n-q+k} g$ are homotopic for some $k \geq 0$. The S-category is additive, and $\Sigma : \{X, Y\} \longrightarrow \{\Sigma X, \Sigma Y\}$ is a bijection.

Although obscured by their language of "carriers", in retrospect a most unfortunate choice of technical details, Spanier and Whitehead introduce graded morphisms by setting $\{X, Y\}_q = \{\Sigma^q X, Y\}$ if $q \geq 0$ and $\{X, \Sigma^{-q} Y\}$ if $q < 0$. They prove that, for CW complexes X and Y with X finite, the abelian groups $\{X, Y\}_q$ satisfy all except the dimension axiom of the Eilenberg-Steenrod axioms for a homology theory in Y when X is fixed and for a cohomology theory in X when Y is fixed. They even set up the Atiyah-Hirzebruch spectral sequences for stable homotopy and stable cohomotopy.

However, they do not take the step of describing their results in a language of homology and cohomology theories, and none of their later papers return to this point of view. With their definitions, the wedge axiom would not be satisfied in cohomology for infinite X, and only homology and cohomology theories represented by suspension spectra of spaces would be obtained. Thus this would not have been the right way to set up generalized homology and cohomology theories, and that was far from their intention. The useful version of the Spanier-Whitehead category is its full subcategory of *finite* CW complexes. This category is far too small to form a satisfactory foundation for stable homotopy theory, but it is appropriate for the study of duality between finite CW complexes, which is the main point of the papers [SW55, SW58] and the expository notes [Whi56, Sp56, Sp58].

The 1956 note [Sp56] of Spanier, reviewed by Hilton, gives a nice description of dual theorems in algebraic topology and seems to have been a forerunner of Eckmann-Hilton duality. The 1956 survey of Whitehead [Whi56] looks more towards the past, based as it was on Whitehead's presidential address to the London Mathematical Society. Prior to this point, it had been common practice to discuss duality in ordinary homology and cohomology in terms of Pontryagin duality of groups. Whitehead gives an interesting exposition of this point of view on duality, the role of colimits in understanding singular homology and Čech cohomology, and various other aspects of duality theory in algebraic topology. At that stage in our story, it is not very surprising that Whitehead understands the Eilenberg-Steenrod axioms solely in terms of ordinary homology and cohomology theories.

In retrospect, it is more surprising that Spanier in his 1959 paper [Sp59b] still understands the axioms this way. In a footnote, he refers to the Eilenberg-Steenrod axioms to specify what he means by homology and cohomology, and of course he means all of the axioms. There is no hint of generalized homology and cohomology theories in the paper, although one of its main points is the convenience and importance of spectra in the study of duality theory. Nevertheless, the work of

Spanier and Whitehead, especially the work in [Sp59b], was soon to lead to duality theorems in generalized homology and cohomology.

Before saying more about [Sp59b], I should mention the interesting paper [Sp59a] that Spanier wrote a year earlier. In it, he returns to the Dold-Thom description [DT58] of integral homology as the homotopy groups of the infinite symmetric product, and he shows how this can be related to the S-category and Spanier-Whitehead duality. Function spaces are used heavily in the comparison, and it seems that their use may have led to the idea of spectra.

In any case, Spanier's student Lima introduced spectra in his 1958 thesis, published in [Lima59]. In Lima's work, a spectrum is a sequence of based finite CW complexes L_i and S-maps $\lambda_i : \Sigma L_i \longrightarrow L_{i+1}$. Lima also considers inverse spectra, with structure maps reversed. He uses spectra to give an extension of the S-category and an extension of Spanier-Whitehead duality from polyhedra embedded in spheres to general compact subspaces of spheres. In a sequel, Lima [Lima60] develops Postnikov systems in his category of spectra. He also gives a curious dual theory whose dual Postnikov invariants lie in homology groups with coefficients in cohomotopy groups.

In Spanier's paper [Sp59a], he redefines spectra X to be sequences of based spaces T_i and based maps, not S-maps, $\sigma_i : \Sigma T_i \longrightarrow T_{i+1}$ that satisfy certain connectivity and convergence conditions. These conditions have the effect of giving his spectra a stable range analogous to the one implied for the suspension spectrum $\{\Sigma^i X\}$ of a based space X by the generalized Freudenthal suspension theorem, which was first proven in [SW57]. His intent is to recast Spanier-Whitehead duality in terms of smash products $X \wedge Y$ and function spectra $\mathbb{F}(X,Y)$, where X and Y are based spaces and $\mathbb{F}(X,Y)$ has ith space the function space $F(X,\Sigma^i Y)$. Curiously, he does not define general function spectra $\mathbb{F}(X,T)$. He writes $\mathbb{F}(X)$ for $\mathbb{F}(X,S^0)$ and calls it the functional dual of X, and he observes that $H^{-q}(X) \cong H_q(\mathbb{F}(X))$. He defines stable maps $\{X,T\}$ from a space to a spectrum and shows that there are canonical duality isomorphisms

$$\{X,\mathbb{F}(Y,S^n)\} \cong \{X \wedge Y, S^n\} \cong \{Y,\mathbb{F}(X,S^n)\}.$$

(Actually, his statement of this has $\mathbb{F}(-,S^n)$ replaced with the n-fold suspension of the functional dual, but his definition of suspension disagrees with the modern one.) While the asymmetry between spaces and spectra is clearly unsatisfactory, this was a step from the S-category towards the true stable homotopy category.

He then redefines what it means for spaces X and Y to be n-dual to one another. Let $i_n \in \tilde{H}^n(S^n)$ be the fundamental class. A map $\varepsilon : Y \wedge X \longrightarrow S^n$ is said to be an *n-duality map* if the homomorphism $f_\varepsilon : \tilde{H}_q(Y) \longrightarrow \tilde{H}^{n-q}(X)$ defined by $f_\varepsilon(y) = \varepsilon^*(i_n)/y$ is an isomorphism, where / is the slant product. He proves that ε determines and is determined by a weak equivalence ξ from the suspension spectrum of Y to $\mathbb{F}(X,S^n)$ such that the following diagram of spaces commutes in the S-category:

$$\begin{tikzcd} Y \wedge X \arrow[rr, "\xi_0 \wedge \mathrm{id}"] \arrow[dr, "\varepsilon"'] & & F(X,S^n) \wedge X \arrow[dl, "\varepsilon"] \\ & S^n. & \end{tikzcd}$$

This gives an intrinsic characterization of the n-dual of X that leads to all of the properties proven in the earlier work of Spanier and Whitehead [SW55]. The earlier work shows that if X is embedded in S^{n+1} and Y is embedded in the complement of X in such a way that the inclusion $Y \longrightarrow S^{n+1} - X$ induces an isomorphism of all homology groups, then there is a duality map $\varepsilon : Y \wedge X \longrightarrow S^n$. This unfortunately means that Spanier's new notion of an n-duality is what in the earlier work was called an $(n+1)$-duality. The new notion relegates the role of the embeddings to the verification of a more conceptual defining property and makes it much simpler to determine when spaces X and Y are n-dual to one another. It is equivalent to the modern homotopical definition of a duality map in the stable homotopy category.

All of this work of Spanier and Whitehead was independent of the work on cobordism, integrality theorems, and K-theory that was going on at the same time. In [MS60], submitted a month after [Sp59a], Milnor and Spanier show that if a smooth compact n-manifold M is embedded in the pair $\mathbb{R}^{n+k}$ with normal bundle ν, then the Thom space $T(\nu)$ is $(n+k)$-dual (new style) to M_+. Moreover, they show that if k is sufficiently large, then ν is fiber homotopy trivial if and only if there is an S-map $S^n \longrightarrow M$ of degree one. They also make the nice observation that Adams' solution to the Hopf invariant one problem implies that the tangent bundle of a homotopy n-sphere is fiber homotopy trivial if and only if n is 1, 3, or 7.

A year later, in [At61c], Atiyah made a systematic study of the relationship between Thom complexes and S-duality. In particular, he proved the Atiyah duality theorem, which identifies the $(n+k)$-dual of the cofibration sequence $\partial M_+ \to M_+ \to M/\partial M$ of a smooth compact n-manifold M with boundary ∂M as the cofibration sequence

$$T(\nu(\partial M)) \longrightarrow T(\nu(M)) \to T(\nu(M))/T(\nu(\partial M))$$

associated to the normal bundles of a proper embedding of the pair $(M, \partial M)$ in $(\mathbb{R}^{n+k-1} \times [0, \infty), \mathbb{R}^{n+k-1} \times \{0\})$. He also proved that, for any bundle ξ over a smooth compact manifold M without boundary, the Thom complex $T(\xi)$ is S-dual to the Thom complex $T(\nu \surd lus\xi^{\perp})$, where $\xi \surd lus\xi^{\perp}$ is trivial. We will return to this paper when we discuss the J-homomorphism.

11. Oriented cobordism and complex cobordism

With the aid of the Adams spectral sequence, the work of Thom on the oriented cobordism ring could be completed. Although slightly ahistorical, the language of spectra will clarify how this came about. Using the structural maps $\sigma : \Sigma T_n \longrightarrow T_{n+1}$, the homotopy, homology, and cohomology of a spectrum $T = \{T_n\}$ can be defined as follows:

$$\pi_q(T) = \operatorname{colim} \pi_{n+q}(T_n) \tag{11.1}$$

$$H_q(T) = \operatorname{colim} \tilde{H}_{n+q}(T_n) \tag{11.2}$$

and

$$H^q(T) = \lim \tilde{H}^{n+q}(T_n), \tag{11.3}$$

where the last definition is only correct when $\lim^1 \tilde{H}^{n+q-1}(T_n) = 0$. As Adams noted in 1959 [Ad59], the Adams spectral sequence generalizes readily to a spectral sequence for the computation of $\pi_*(T)$ in terms of the mod p cohomology $H^*(T)$, regarded as a module over the Steenrod algebra A. The E_2-term is given by

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(H^*(T), \mathbb{Z}_p),$$

and everything said earlier applies, with simpler proofs, in this more general setting.

For each of the familiar sequences of classical groups $G(n)$, namely $G = O$, SO, U, SU, Sp, and $Spin$, the Thom spaces $TG(n)$ of the universal bundles give a Thom spectrum MG. A uniform method of attack on the problem of computing $\pi_*(MG)$ is to first compute the mod p cohomology of MG for each prime p and then compute the mod p Adams spectral sequence.

A key reason that Thom was able to compute $\mathcal{N}_*$ completely was that the mod 2 cohomology $H^*(MO)$ is a free module over the mod 2 Steenrod algebra A. A quick direct proof of this fact, using Hopf algebra techniques, was given by Liulevicius [Liu62b] in 1962.

For an abelian group π, the sequence of spaces $K(\pi, n)$ gives a spectrum $H\pi$ such that $\pi_0(H\pi) = \pi$ and the remaining homotopy groups of $H\pi$ are zero. The mod p cohomology of $H\mathbb{Z}_p$ is the mod p Steenrod algebra, as Cartan had implicitly shown [Ca55]. The representation of cohomology (1.1) generalizes to spectra. Representing generators of $H^*(MO)$ as maps from MO to suspensions of $H\mathbb{Z}_2$, one obtains a map from MO to a product of suspensions of $H\mathbb{Z}_2$ that induces an isomorphism on mod 2 cohomology. Since one knows that $\pi_*(MO)$ is a $\mathbb{Z}_2$-vector space, one readily deduces that this map is an equivalence of spectra, allowing one to read off $\pi_*(MO)$. However, a good homotopy category of spectra in which to make such a deduction only appeared later.

Using spectra and the Adams spectral sequence, Milnor [Mil60] in 1959 proved that $\mathcal{M}_* = \pi_*(MSO)$ has no odd torsion. This was proven independently by Averbuh [Av59] and, a little later, Novikov [Nov60]. These are announcements. Averbuh's proofs never appeared and Novikov's proofs [Nov62] seem never to have been translated from the Russian.

Also in 1959 [Wall60], but without using spectra or the Adams spectral sequence, Wall determined the 2-torsion in $\mathcal{M}_*$. In particular, he proved that $\mathcal{M}_*$ has no elements of order 4 and that two oriented manifolds are cobordant if and only if they have the same Stiefel-Whitney and Pontryagin numbers. These results were both conjectured by Thom [Thom54]. A nice deduction from the explicit form of the generators Wall found is that the square of any manifold is cobordant to an oriented manifold, and he remarked the desirability of a direct geometric proof; we shall return to this in Sections 16 and 17.

After calculating the 2-torsion in $\mathcal{M}_*$ by other means, Wall used this calculation to prove that the mod 2 cohomology $H^*(MSO)$ is the direct sum of suspensions of copies of A and of A/ASq^1. He remarks "It seems that a direct proof ... would be extremely difficult", but he found such a direct proof not long afterwards [Wall62]. That allows a more direct calculation of $\mathcal{M}_*$. In fact, the mod 2 cohomology of $H\mathbb{Z}$ is A/ASq^1. As Browder, Liulevicius, and Peterson observed later [BLP66], it follows that there is a map f from the spectrum MSO to a product of suspensions of copies of $H\mathbb{Z}$ and $H\mathbb{Z}_2$ that induces an isomorphism on mod 2 cohomology. In a good homotopy category of spectra, one readily deduces that f is a 2-local equivalence. Of course, the foundations for such an argument only came later, but the calculation of homotopy groups is easily made by use of the Adams spectral sequence.

Milnor [Mil60] and Novikov [Nov60, Nov62] also introduced and calculated complex cobordism $\pi_*(MU)$. Although the geometric interpretation was not included in Milnor [Mil60], this is the cobordism theory of weakly almost complex manifolds, namely manifolds with a complex structure on their stable normal bundles. The explicit calculation, carried out one prime at a time and then collated algebraically, showed that $\pi_*(MU)$ is a polynomial ring on one generator of degree $2i$ for each $i \geq 1$. Interestingly, there is no known geometric reason why the complex cobordism ring should be concentrated in even degrees. The analogue for symplectic cobordism is false. The cited papers of Milnor and Novikov raise the question of determining $\pi_*(MG)$ for other classical groups G and give some information. We will return to this in Sections 16 and 17.

12. K-theory, cohomology, and characteristic classes

In their 1960 paper [AH61a], Atiyah and Hirzebruch explicitly introduce K-theory as a generalized cohomology theory. Whether or not the idea of taking a generalized cohomology theory seriously occurred to anyone before, this paper is the first published account. They restrict attention to finite CW complexes X for convenience, but they are fully aware of both represented K-theory and inverse limit K-theory, namely the inverse limit of $K^*(X^n)$ as X^n runs over the skeleta of X. Using Bott periodicity, they prove that $\mathbb{Z}$-graded K-theory satisfies all of the Eilenberg-Steenrod axioms except the dimension axiom and they introduce $\mathbb{Z}_2$-graded K-theory. Regarding ordinary rational cohomology as $\mathbb{Z}_2$-graded by sums of even and odd degree elements, they prove that the Chern character extends to a multiplicative map of cohomology theories $ch : K^*(X) \longrightarrow H^{**}(X;\mathbb{Q})$ which becomes an isomorphism when the domain is tensored with $\mathbb{Q}$.

They also introduce what is now called the Atiyah-Hirzebruch spectral sequence. It satisfies

$$E_2^{p,q} = H^p(X; K^q(pt)),$$

and it converges to $K^*(X)$. Since it is compatible with Bott periodicity, it may be regraded so as to eliminate the grading q. It collapses, $E_2 = E_\infty$, if $H^*(X;\mathbb{Z})$ is concentrated in even degrees or, using the Chern character, if $H^*(X;\mathbb{Z})$ has no torsion. They state without proof that d_3 can be identified with the integral operation Sq^3, and they give partial information about the product structure. They also state without proof that the spectral sequence generalizes to a Serre type spectral sequence for the K-theory of fibre bundles.

The Riemann-Roch theorem of their earlier paper [AH59] is generalized to the cohomology theory K^*, but still with no hint of K-homology and a genuine push-forward map in K-theory. The theorem states that if $f : M \longrightarrow N$ is a continuous map betweeen compact oriented differentiable manifolds and if there is a given element $c_1(f) \in H^2(M;\mathbb{Z})$ such that $c_1(f) \equiv w_2(M) - f^*w_2(N)$ mod 2, then, for $x \in K^*(M)$,

$$f_!(ch(x)e^{c_1(f)/2} \cdot \hat{A}(M)) = ch(f_!(x)) \cdot \hat{A}(N) \tag{12.1}$$

in $H^*(N;\mathbb{Q})$. On the left $f_!$ is the pushforward in rational cohomology determined by Poincaré duality and f_*; a posteriori, $f_!$ is defined similarly in K-theory.

Using both the Riemann-Roch theorem and the spectral sequence, they study the K-theory of certain differentiable fiber bundles and compute $K^*(G/H)$ explicitly when H is a closed connected subgroup of maximal rank in a compact connected Lie group G. Moreover, when $H^*(G;\mathbb{Z})$ has no torsion, they prove that the natural map $R(H) \longrightarrow K(G/H)$ is surjective. Calculations with the maximal rank condition dropped came much later.

Taking $\mathcal{K}(BG)$ to be the inverse limit K-theory of BG, they define a homomorphism $\alpha : R(G)^{\wedge}_{I} \longrightarrow \mathcal{K}(BG)$ and prove that it is an isomorphism when G is a compact *connected* Lie group. They also prove that $\mathcal{K}^1(BG) = 0$ for such G. The proof is by direct calculation when T is a torus and by comparison with the result for a maximal torus in general. They conjecture that this result remains true for any compact Lie group G.

In [At61b], which appeared in 1961, Atiyah proves the same result for *finite* groups G. The proof is by direct calculation when G is cyclic, by induction up a composition series when G is solvable, and by application of the Brauer induction theorem to pass from solvable groups to general finite groups. The second step depends on a Hochschild-Serre type spectral sequence that satisfies $E_2^{p,q} = H^p(G/N; \mathcal{K}^q(BN))$ and converges to $\mathcal{K}^*(BG)$, where N is a normal subgroup of G. The last step depends on the transfer homomorphisms in K-theory associated to finite covers. Atiyah claims in a footnote that the result does remain true for general compact Lie groups. However, a proof did not appear until the 1969 paper [AS69] of Atiyah and Segal, which is based on the use of equivariant K-theory. This was developed in lectures at Harvard and Oxford in 1965, but the first published accounts appeared later [At66a, Seg68].

In 1961 [AH61c], Atiyah and Hirzebruch make use of real K-theory KO to obtain a number of interesting results on characteristic classes in ordinary mod p cohomology. These are less well-known than they ought to be, perhaps because [AH61c] is written in German; some of its results were later reworked by Dyer [Dyer69]. Atiyah and Hirzebruch greatly extend and clarify observations Hirzebruch had already made in 1953 [Hirz53], and they improve results in the expository paper [AH61c], also in German, which was written a bit earlier and contains a nice general overview of the authors' results on K-theory, including some that I will not discuss here.

In [AH61c], using Milnor's analysis of the Steenrod algebra, Atiyah and Hirzebruch first determine the group of natural ring isomorphisms $\lambda : H^{**}(X) \longrightarrow H^{**}(X)$. The obvious examples are $\lambda = Sq \equiv \sum Sq^r$ if $p = 2$ and $\lambda = P \equiv \sum P^r$ if $p > 2$. For a $\mathbb{Z}_p$-oriented vector bundle ξ with Thom isomorphism ϕ, they define $\underline{\lambda}(\xi) = \phi^{-1}\lambda\phi(1)$. Thus $\underline{Sq}$ is the total Stiefel-Whitney class and $\underline{P}$ is the total Wu class. They observe that, for a finite CW complex X, $\underline{\lambda}$ extends to a natural homomorphism from $KO(X)$ to the group $G^{**}(X)$ of elements of $H^{**}(X)$ with zeroth component 1 and, if $p > 2$, odd components zero, where the multiplication in $G^{**}(X)$ is given by the cup product. Write $Wu(\lambda, \xi) = \lambda^{-1}\underline{\lambda}(\xi)$. Then, when $p = 2$,

$$Wu(Sq, \xi) = \sum_{i \geq 0} 2^i T_i(w_1(\xi), \cdots, w_i(\xi)),$$

where the T_i are the Todd polynomials. Here the right side makes sense since $2^i T_i$ is a rational polynomial with denominator prime to 2. When $p > 2$, let $f = p^{1/p-1}$ and let P_i be the ith Pontryagin class. Then

$$Wu(P, \xi) = \sum_{i \geq 0} f^{2i} L_i(P_1(\xi), \cdots, P_i(\xi)) = \sum_{i \geq 0} f^{2i} \hat{A}_i(P_1(\xi), \cdots, P_i(\xi)).$$

In both cases, there is an implied analogue for complex bundles, with Chern classes appearing on the right-hand sides of the equations.

These formulas suggest a relationship between the differential Riemann-Roch theorem and Wu's formulas for the characteristic classes of manifolds. Let f :

$M \longrightarrow N$ be a continuous map between differentiable manifolds M and N. Atiyah and Hirzebruch prove that, for any $x \in H^*(M)$,

$$f_!(\lambda(x) \cdot Wu(\lambda^{-1}, \tau_M)) = \lambda(f_!(x)) \cdot Wu(\lambda^{-1}, \tau_N), \tag{12.2}$$

where $f_!$ is the pushforward map determined by Poincaré duality and f_*. When N is a point, this reduces to

$$\langle \lambda(y), [M] \rangle = \langle (y \cdot (Wu(\lambda, \tau_M), [M] \rangle.$$

Taking $\lambda = Sq$ if $p = 2$ or $\lambda = P$ if $p > 2$, this is Wu's formula for the determination of the Stiefel-Whitney or L-classes of M in terms of Steenrod operations and cup products in $H^*(M)$.

It should be remarked at this point that Adams [Ad61b] proved the Wu relations for not necessarily differentiable manifolds in 1961. In 1960 [Ad61a], he proved an integrality theorem for the Chern character. Atiyah and Hirzebruch [AH61c] observe that (12.2) is an analogue of the differentiable Riemann-Roch theorem (12.1), and they show that this is more than just an analogy by using Adams' integrality theorem to derive important cases of (12.2) from (12.1). In a noteworthy remark, they point out that one can ask for such a Riemann-Roch type theorem whenever one has a natural transformation from one generalized cohomology theory to another, provided that both theories satisfy an analogue of Poincaré duality that allows pushforwards to be defined. This still precedes Poincaré duality in K-theory.

Even without K-homology, Atiyah in 1962 [At62] found an ingenious and influential proof of a Künneth theorem for K-theory, obtaining a short exact sequence of the expected form

$$0 \longrightarrow K^*(X) \otimes K^*Y) \xrightarrow{\alpha} K^*(X \times Y) \xrightarrow{\beta} \mathrm{Tor}(K^*(X), K^*(Y)) \longrightarrow 0.$$

13. Generalized homology and cohomology theories

The work of G.W. Whitehead [Wh60, Wh62a] and Brown [Br63, Br65] defined and characterized represented generalized homology and cohomology theories in close to their modern forms. We have seen that K-homology is nowhere mentioned in the work of Atiyah and Hirzebruch. However, Whitehead's announcement [Wh60] of his definition of represented homology was already submitted in February, 1960, and appeared that year, although the full paper [Wh62a] was not submitted until May, 1961, and appeared in 1962. More surprisingly, [Wh62a] makes no mention of either K-theory or bordism and contains no references to Atiyah and Hirzebruch, although the Bott spectrum is mentioned briefly. There seems to have been little mutual influence.

It seems that the main influence on Whitehead was his own earlier work on the homotopy groups of smash products of spaces [Wh56] and the work on duality of Spanier and J.H.C. Whitehead [SW55] and its further development by Spanier [Sp59b]. Whitehead defines a spectrum E to be a sequence of spaces E_i and maps $\sigma_i : \Sigma E_i \longrightarrow E_{i+1}$, dropping the convergence conditions that Spanier imposed. He says that E is an $\mathcal{M}$-spectrum if the adjoint maps $\tilde{\sigma} : E_i \longrightarrow \mathcal{M}E_{i+1}$ are homotopy equivalences. Actually, he insists on spaces E_i for all integers i, rather than for $i \geq 0$ as is now more usual. He defines a map $f : E \longrightarrow E'$ to be a sequence of maps $f_i : E_i \longrightarrow E'_i$ such that the diagrams

$$\begin{tikzcd} \Sigma E_i \arrow[r, "\sigma_i"] \arrow[d, "\Sigma f_i"'] & E_{i+1} \arrow[d, "f_{i+1}"] \\ \Sigma E'_i \arrow[r, "\sigma'_i"] & E'_{i+1} \end{tikzcd} \tag{13.1}$$

commute *up to homotopy*, and he says that two maps f and g are homotopic if $f_i \simeq g_i$ for all i.

Taking the obvious steps beyond Spanier [Sp59b], Whitehead defines the function spectrum $\mathbb{F}(X, E)$ and the smash products $E \wedge X \cong X \wedge E$ between a based space X and a spectrum E. As an unfortunate choice, he restricts X to be compact in these definitions, and his homology and cohomology theories are only defined on finite CW complexes. Remember that the additivity axiom came a bit later. In particular, these definitions give $\mathcal{M}E = \mathbb{F}(S^1, E)$ and $\Sigma E = E \wedge S^1$ (except that he writes the suspension coordinate on the left). Defining the homotopy groups of spectra as in (11.1), he proves that suspension gives an isomorphism $\Sigma_* : \pi_q(E) \longrightarrow \pi_{q+1}(\Sigma E)$.

For finite based CW complexes X and a spectrum E, Whitehead defines

$$\tilde{H}_q(X; E) = \pi_q(E \wedge X). \tag{13.2}$$

This is suggested by the more obvious cohomological analogue

$$\tilde{H}^q(X; E) = \pi_{-q}(F(X, E)). \tag{13.3}$$

In retrospect, this definition of homology is correct for general CW complexes X, but this definition of cohomology is only correct for general CW complexes X when E is an $\mathcal{M}$-spectrum.

Much of [Wh62a] is concerned with products in generalized homology and cohomology theories. These are induced by pairings $(D, E) \longrightarrow F$ of spectra, which are specified by maps

$$D_m \wedge E_n \longrightarrow F_{m+n}$$

that are suitably compatible up to homotopy with the structure maps σ of D, E, and F. Starting from such pairings of spectra, Whitehead defines and studies the properties of external products

$$\tilde{H}_m(X; D) \otimes \tilde{H}_n(Y; E) \longrightarrow \tilde{H}_{m+n}(X \wedge Y; F)$$
$$\tilde{H}^m(X; D) \otimes \tilde{H}^n(Y; E) \longrightarrow \tilde{H}^{m+n}(X \wedge Y; F)$$

and slant products

$$\backslash : \tilde{H}_n(X \wedge Y; D) \otimes \tilde{H}^m(X; E) \longrightarrow \tilde{H}_{n-m}(Y; F)$$
$$/ : \tilde{H}_n(X \wedge Y; D) \otimes \tilde{H}_m(Y; E) \longrightarrow \tilde{H}^{n-m}(X; F).$$

He obtains cup and cap products by pulling back along diagonal maps. By now, all of this is familiar standard practice.

Similarly, the familiar duality theorems are proven. Whitehead defines a ring spectrum E in terms of a product $(E, E) \longrightarrow E$ and unit $S \longrightarrow E$, where S is the sphere spectrum, namely the suspension spectrum of S^0. He defines an E-orientation of a compact connected n-manifold M in terms of a fundamental class in $\tilde{H}_n(M; E)$, and he proves a version of Alexander duality for dual pairs embedded in M. This specializes to give Poincaré duality for M. Taking $M = S^{n+1}$, which is E-oriented for any E, it specializes to give Spanier-Whitehead duality in any theory.

When [Wh62a] was written, Brown [Br63] had already proven his celebrated representation theorem. That paper also gave an incorrect first approximation to Milnor's additivity axiom [Mil62]. In fact, James and Whitehead [JW58] had exhibited homology theories that fail to satisfy the additivity axiom and whose existence contradicted one of Brown's results. The correction of [Br63] noted this and pointed out simpler axioms for the representability theorem. Brown later published the improved version in a general categorical setting [Br65]. That version is one of the foundation stones of modern abstract homotopy theory.

Let k be a contravariant set-valued homotopy functor defined on based CW complexes. The functor k is said to satisfy the Mayer-Vietoris axiom if, for a pair of subcomplexes A and B of a CW complex X with union X and intersection C, the natural map from $k(X)$ to the pullback of the pair of maps $k(A) \longrightarrow k(C)$ and $K(B) \longrightarrow k(C)$ is surjective; k is said to satisfy the wedge axiom if it converts wedges to products. Brown in [Br65] proves that $k(X)$ is then naturally isomorphic to $[X, Y]$ for some CW complex Y. If k is only defined on finite CW complexes, Brown reaches the same conclusion but with a countability assumption on the $k(S^q)$. Adams [Ad71a] later showed that the countability assumption can be removed when the functor k is group-valued.

Applied to the term $\tilde{k}^n(-)$ of a (reduced) generalized cohomology theory $\tilde{k}^*$, Brown's theorem gives a CW complex E_n such that $\tilde{k}^n(X) \cong [X, E_n]$ for all CW complexes X. The suspension axiom on the theory leads to homotopy equivalences $E_n \longrightarrow \mathcal{M}E_{n+1}$. Thus a cohomology theory $\tilde{k}^*$ gives rise to an $\mathcal{M}$-spectrum E. Whitehead [Wh62a] followed up by using Spanier's version [Sp59b] of duality theory to show that a homology theory gives rise to a cohomology theory on finite CW complexes. Applying Brown's theorem for finite CW complexes (and using Adams' variant to avoid countability hypotheses), it follows that a homology theory on finite CW complexes is also representable by a spectrum.

Since the Brown representation is natural, a map of cohomology theories gives rise to a map of $\mathcal{M}$-spectra. Defining the category of cohomology theories on spaces in the evident way, we see that it is equivalent to the homotopy category of $\mathcal{M}$-spectra E whose spaces E_n are homotopy equivalent to CW complexes. We call this the Whitehead category of $\mathcal{M}$-spectra. Milnor's basic result [Mil59] that the loop space of a space of the homotopy type of a CW complex has the homotopy type of a CW complex is relevant here.

Via the suspension spectrum functor and a functor that converts spectra to $\mathcal{M}$-spectra, one can check that the S-category of finite CW complexes embeds as a full subcategory of the Whitehead category. Thus the Whitehead category is an approximation to stable homotopy theory that substantially improves on the S-category by providing the proper home for cohomology theories on spaces. However, as we shall see in Section 21, this is not yet the genuine stable homotopy category.

In the summer of 1962, there was an International Congress in Stockholm, preceded by a colloquium on algebraic topology at Aarhus. The proceedings of the latter contain brief expositions of generalized cohomology by Dold [Dold62], Dyer [Dyer62], and Whitehead [Wh62b]. Dold was the first to make the important observation that rational cohomology theories are products of ordinary cohomology theories, and he gave the first general exposition of the Atiyah-Hirzebruch spectral sequence. Making systematic use of Brown's representability theorem, his later book [Dold66], in German, gave a complete treatment of these matters and much else. Dyer was the first to write down a general treatment of the Riemann-Roch theorem, although already in 1962 he described the result as a folk theorem known to Adams, Atiyah, Hirzebruch and others. His later book [Dyer69] gave a complete treatment, along with an exposition of much of the work of Atiyah and Hirzebruch described in the previous section. He still avoids use of K_*, but this appears implicitly in the form of Atiyah duality, which allows an appropriate definition of pushforward maps.

Not everything in cohomology theory was to be done using its represented form. For example, working directly from the axioms, Araki and Toda [AT65] made a systematic study of products in mod q cohomology theories and of Bockstein spectral sequences in generalized cohomology. Nevertheless, most work was to be simplified and clarified by working with represented theories.

14. Vector fields on spheres and $J(X)$

In the proceedings of the 1962 Aarhus and Stockholm conferences, Adams [Ad62d] described his solution of the vector fields on spheres problem [Ad62b, Ad62c] and outlined his work on the groups $J(X)$, which appeared gradually in [Ad63, Ad65a, Ad65b, Ad66a]. I summarized these papers in [May2], emphasizing their impact on later work and the reformulations that became possible with later technology. These applications of K-theory have been of central importance to the development of stable algebraic topology.

The key new idea was the introduction of the Adams operations ψ^k in real and complex K-theory. These play a role in K-theory that is of comparable importance to the role played by Steenrod operations in ordinary mod p cohomology. It was clear from Grothendieck's work [Gro57] how to extend the exterior power operations λ^k from vector bundles to K-theory. The "Newton polynomials" Q_k that express the power operations $x_1^k+\cdots+x_n^k$ in a polynomial ring $\mathbb{Z}[x_1,\ldots,x_n]$ as polynomials in the elementary symmetric polynomials σ_k were familiar to topologists from their role in the study of characteristic classes. Adams' ingenious idea was to define

$$\psi^k(x) = Q_k(\lambda^1(x),\ldots,\lambda^n(x)).$$

Here X is a finite CW complex, $x \in K(X)$, and n is large.

Either by a representation theoretical argument, as in [Ad62c], or by use of the splitting principle and reduction to the case of line bundles, one finds that the ψ^k are natural ring homomorphisms that commute with each other. They are easily evaluated on line bundles and on the K-theory of spheres, and their relationship to the Chern character and the Bott isomorphism are easily determined. They greatly enhance the calculational power of K-theory.

Adams discovered these operations after first trying to solve the vector fields on spheres problem by use of secondary and higher operations in ordinary cohomology in [Ad62a], a paper that was obsolete by the time it appeared. The idea that a problem that required higher order operations in ordinary cohomology could be solved using primary operations in K-theory had a strong impact on the directions taken by stable algebraic topology.

The vector fields on spheres problem asks how many linearly independent vector fields there are on S^{n-1}. The answer is $\rho(n)-1$. Here $\rho(n) = 2^c + 8d$, where $n = (2a+1)2^b$ and $b = c + 4d$, $0 \le c \le 3$. It had long been known [Eck42] that there exist $\rho(n)-1$ such fields. Adams proved that there are no more. Work of James [Ja58a, Ja58b, Ja59] had reduced the problem to a question about the reducibility of a certain complex. Up to suspension, Atiyah [At61c] identified the S-dual of that complex with a stunted projective space. This reduced the problem to the question of the coreducibility of $X = \mathbb{RP}^{m+\rho(n)}/\mathbb{RP}^{m-1}$ for a suitable m. Here coreducibility means that there is a map $f : X \longrightarrow S^m$ that has degree 1 when restricted to the bottom cell S^m of X. Adams proves that X is not coreducible, thus solving the problem.

For the proof, Adams starts with the calculation of $K(\mathbb{CP}^n)$ and $K(\mathbb{CP}^n/\mathbb{CP}^m)$, which was first carried out by Atiyah and Todd [AT60]. He next calculates $K(\mathbb{RP}^n)$ and $K(\mathbb{RP}^n/\mathbb{RP}^m)$. Finally he calculates $KO(\mathbb{RP}^n)$ and $KO(\mathbb{RP}^n/\mathbb{RP}^m)$. In each case, he obtains complete information on the ring structure and the Adams operations. The main tools are just the Atiyah-Hirzebruch spectral sequence and the Chern character. For X as above, the existence of a coreduction f and the naturality relation $f^*\psi^k = \psi^k f^*$ lead to a contradiction.

For a connected finite CW complex X, define $J(X)$ to be $\mathbb{Z}\surd lus\tilde{J}(X)$, where $\tilde{J}(X)$ is the quotient of $\tilde{K}(X)$ obtained by identifying two stable equivalence classes of vector bundles if they are stably fiber homotopy equivalent. Let $J : K(X) \longrightarrow J(X)$ be the evident quotient map. Atiyah in [At61b] (where $J(X)$ means what we and Adams call $\tilde{J}(X)$) proved that the bundle $O(n)/O(n-k) \longrightarrow S^{n-1}$, $n \geq 2k$, admits a section if and only if n is a multiple of the order of $J(1-\xi)$, where ξ is the canonical line bundle over $\mathbb{RP}^{k-1}$. Thus the vector fields problem can be viewed as a special case of the problem of determining $J(X)$. In fact, as Bott first observed [Bott62, Bott63], Adams' calculations in [Ad62c] imply that $KO(\mathbb{RP}^n) \cong J(\mathbb{RP}^n)$. While Adams was aware of the relationship between the vector fields problem and the study of J, he chose not to discuss this in [Ad62c]; he published a proof of the cited isomorphism in [Ad65a].

The results just discussed have complex analogues, using $U(n)/U(n-k)$ and $\mathbb{CP}^{k-1}$. The bundle $\pi_{n,k} : U(n)/U(n-k) \longrightarrow S^{2n-1}$ admits a section if and only if n is divisible by a certain number M_k. The necessity was proven first, by Atiyah and Todd [AT60], and the sufficiency was then proven by Adams and Walker [AW64]. For the proof, they compute $KO(\mathbb{CP}^n)$ and $KO(\mathbb{CP}^n/\mathbb{CP}^m)$, use the methods and results of [Ad63, Ad65a] to study $J : KO(\mathbb{CP}^n) \longrightarrow J(\mathbb{CP}^n)$, and deduce that the order of $J(1-\xi)$ is M_k, where ξ is the canonical line bundle over $\mathbb{CP}^{k-1}$.

Many of the results of Atiyah [At61b] and Adams [Ad62c] on stunted projective spaces have analogues for stunted lens spaces, and these were worked out by Kambe, Matsunaga, and Toda [Ka66, KMT66].

The papers [Ad63, Ad65a, Ad65b, Ad66a] carry out the general study of $J(X)$ for a connected finite CW complex X. The overall plan is to define two further, more computable, quotients $J'(X)$ and $J''(X)$ of $K(X)$ such that the quotient homomorphisms from $K(X)$ factor to give epimorphisms $J''(X) \longrightarrow J(X) \longrightarrow J'(X)$ and then to prove that $J'(X) = J''(X)$. Thus $J'(X)$ is a lower bound and $J''(X)$ an upper bound for $J(X)$, and these two bounds coincide.

That $J''(X)$ really is an upper bound depends on the Adams conjecture: "If k is an integer, X is a finite CW complex and $y \in KO(X)$, then there exists a non-negative integer $e = e(k,y)$ such that $k^e(\psi^k - 1)y$ maps to zero in $J(X)$." Adams [Ad63] proved this when y is a linear combination of $O(1)$ or $O(2)$ bundles and when $X = S^{2n}$ and y is a complex bundle. His proof is based on the "Dold theorem mod k", which asserts that if $f : \eta \longrightarrow \xi$ is a fiberwise map of sphere bundles of degree $\pm k$ on each fiber, then $k^e\eta$ and $k^e\xi$ are fiber homotopy equivalent for some $e > 0$. For $k = 1$, this is a result of Dold [Dold63].

The groups $J'(X)$ and $J''(X)$ are defined and calculated in favorable cases in [Ad65a]. In particular, the image of J in π^s_{4k-1} is shown to be either the denominator of $B_k/4k$, as expected, or twice it; the expected answer would follow from the Adams conjecture. The group $J''(X)$ is $KO(X)/W(X)$, where $W(X)$ is the subgroup generated by all elements $k^{e(k)}(\psi^k - 1)y$ for a suitable function e. The content of the Adams conjecture is that $J''(X)$ is indeed an upper bound for $J(X)$.

To define $J'(X)$, Adams needs certain operations ρ^k which he calls "cannibalistic classes". They are related to the ψ^k as the Stiefel-Whitney classes are related to the Steenrod operations. That is, $\rho^k = \phi^{-1}\psi^k\phi(1)$ where ϕ is the KO-theory Thom isomorphism. This definition and calculations based on it require good control on KO-orientations of vector bundles. While Adams developed some of this himself, the published version of [Ad65a] relies on the paper [ABS64] of Atiyah, Bott, and Shapiro, and I shall say more about that in the next section. This definition only works for Spin$(8n)$-bundles, in which case the operations ρ^k were introduced by Atiyah (unpublished) and Bott [Bott62, Bott63], who denoted them θ_k. Adams shows that the operations can be extended to all of $KO(X)$ if one localizes the target groups away from k. If sphere bundles η and ξ are fiber homotopy equivalent, then $\rho^k(\xi) = \rho^k(\eta)[\psi^k(1+y)/(1+y)]$ for some $y \in \tilde{K}O(X)$, independent of k. The group $J'(X)$ is $KO(X)/V(X)$, where $V(X)$ is the subgroup of these elements x such that $\rho^k(x) = \psi^k(1+y)/(1+y)$ in $KO(X) \otimes \mathbb{Z}[1/k]$ for all $k \neq 0$ and some $y \in \tilde{K}O(X)$.

Adams gives the proof that $J'(X) = J''(X)$ in [Ad65b]. This entails a good deal of representation theory, some of it involving the extension to the real case of arguments used by Atiyah and Hirzebruch [AH61a] in their comparison between $R(G)^\wedge_I$ and $K(BG)$ for a compact connected Lie group G. This is used to construct a certain diagram between K-groups, the motivation for which is the heuristic idea that $1 + y = \rho^k x$ is a solution of the equation $\rho^\ell(\psi^k - 1)x = \psi^\ell(1+y)/(1+y)$. This diagram is then proven to be a weak pullback by calculational analysis. To get a more precise hold on $J'(X)$, Adams proves that the ψ^k are periodic in the sense that, for any positive integer m, there is an exponent e, depending only on X, such that, for any $x \in KO(X)$, $\psi^k(x) \equiv \psi^\ell(x) \bmod m$ if $k \equiv \ell \bmod m^e$. He uses this to characterize which elements $(\nu_k) \in \prod_{k\neq 0}(1 + \tilde{K}O(X)[1/k])$ are of the form $\nu_k = \rho^k(x)\psi^k[(1+y)/(1+y)]$ for some $x \in \tilde{K}Spin(X)$ and $y \in \tilde{K}O(X)$.

Modulo the Adams conjecture, Adams proves in [Ad66a] that $J(S^n)$ is a direct summand of π^s_n. He does this by studying invariants d and e that are associated to maps $f : S^{q+r} \longrightarrow S^q$; there are two variants, real and complex. The real invariant $d_{\mathbb{R}}(f)$ is just the induced homomorphism f^* on $\tilde{K}O$, and it is zero unless $r \equiv 1$ or 2 mod 8, when it detects certain well-known direct summands $\mathbb{Z}_2$ of π^S_*. When $d_{\mathbb{R}}(f) = 0$ and $d_{\mathbb{R}}(\Sigma f) = 0$, the cofiber sequence $S^q \longrightarrow Cf \longrightarrow S^{q+r+1}$ gives a short exact sequence on application of $\tilde{K}O$, and $e_{\mathbb{R}}(f)$ is the resulting element of the appropriate Ext^1 group of extensions. Here Ext^1 is taken with respect to an abelian category of abelian groups with Adams operations that commute with each other and satisfy the periodicity relations. Building in that much structure allows direct computation of the relevant Ext^1 group, which in the cases of interest is an explicitly determined subgroup of $\mathbb{Q}/\mathbb{Z}$. Adams' algebraic formalism leads to an analysis of how $e_{\mathbb{R}}$ relates Toda brackets in homotopy theory to Massey products in Ext groups, and these relations are the key to many of Adams' detailed calculations.

The real e-invariant is essential to the proof of the splitting of π_*^s. The complex e-invariant $e_{\mathbb{C}}$ admits a more elementary description in terms of the Chern character and was introduced and studied independently by Dyer [Dyer63] and Toda [To63]. Adams, Dyer, and Toda all show that $e_{\mathbb{C}}$ can be used to reprove the Hopf invariant one theorem, at all primes p. Adams [Ad66a] also uses $e_{\mathbb{C}}$ to prove that if Y is the mod p^f Moore space, p odd, with bottom cell in a suitable odd dimension, and if $r = 2(p-1)p^{f-1}$, then there is a map $A : \Sigma^r Y \longrightarrow Y$ that induces an isomorphism on $\tilde{K}$. Iterating A s times, by use of suspensions, and first including the bottom cell and then projecting on the top cell, there result elements $\alpha_s \in \pi_{rs-1}^s$, and Adams uses $e_{\mathbb{C}}$ to prove that these maps are all essential. This generalized and clarified a construction of Toda [To58a] and was a forerunner of a great deal of recent work on periodicity phenomena in stable homotopy theory. When $f = 1$, Toda himself [To63] showed how to use $e_{\mathbb{C}}$ to detect these elements as Toda brackets.

Once the Adams conjecture was proven, various classifying spaces not available to Adams were constructed, and the theories of localization and completion were developed, the proof that $J'(X) = J''(X)$ could later be carried out in a more conceptual homotopy theoretic way. The speculative last section of [Ad65b] anticipated much of this. Adams showed that, once appropriate foundations were in place, one would be able to deduce that, for any KO-oriented spherical fibration ξ of dimension $8n$, the sequence $\rho^k(\xi) = \phi^{-1}\psi^k\phi(1)$ would be of the form cited above. This would imply that, for any x in the group $\tilde{K}(F; KO)(X)$ of KO-oriented stable spherical fibrations, there is an element $x' \in \tilde{K}Spin(X)$ such that $\rho^k(x) = \rho^k(x')$ for all k. In retrospect, this was headed towards localized splittings of the classifying space for KO-oriented spherical fibrations, with one factor being $BSpin$ and the other a space $BCokerJ$ whose homotopy groups are essentially the cokernel of $J : \pi_*(BSpin) \longrightarrow \pi_*^s$.

Adams asked, among other things, whether or not the $J(X)$ specify a natural direct summand of some other functor of X, and he observed that, since the $J(X)$ do not give a term in a cohomology theory on X, they cannot be direct summands of a term of a cohomology theory. We now fully understand the answers to his questions. The process of reaching that understanding was to have major impact on geometric topology and algebraic K-theory, as well as on many topics within algebraic topology.

15. Further applications and refinements of K-theory

The need for K and KO orientations of suitable vector bundles was apparent from the moment K-theory was introduced. Such orientations were essential to the work of Adams just discussed and were first studied in detail by Bott [Bott62, Bott63]. However, the definitive treatment was given in the beautiful paper [ABS64] of Atiyah, Bott, and Shapiro, which was written by the first two authors after Shapiro's untimely death.

The authors first give a comprehensive algebraic treatment of Clifford algebras and their relationship to spinor groups. Let C_k be the Clifford algebra of the standard negative definite quadratic form $-\sum x_i^2$ on $\mathbb{R}^k$ and let $M(C_k)$ be the free abelian group generated by the irreducible $\mathbb{Z}_2$-graded C_k-modules. The inclusion of C_k in C_{k+1} induces a homomorphism $M(C_{k+1}) \longrightarrow M(C_k)$. Let A_k be its cokernel.

Then the groups A_k are periodic of period 8 and are isomorphic to the homotopy groups $\pi_k(BO)$. Their complex analogues A_k^c are isomorphic to the homotopy groups of BU. Under tensor product, the A_k and A_k^c form graded rings isomorphic to the positive dimensional homotopy groups of KO and KU. These facts are far too striking to be mere coincidences.

They next give an account of relative K-theory in bundle theoretic terms, proving that, for any n, a suitably defined set $L_n(X,Y)$ of equivalence classes of sequences of vector bundles over X, exact over Y and of length any fixed $n \geq 1$, maps isomorphically to $K(X,Y)$ under an Euler characteristic they construct. The proof depends on a difference bundle construction that is important in many applications.

Combining ideas, they view the algebraic theory as a theory of bundles over a point and generalize it to a theory of bundles over X. Starting from a fixed Euclidean vector bundle V over X, they construct an associated Clifford bundle $C(V)$ over X whose fiber over x is the Clifford algebra $C(V_x)$. They define $M(V)$ to be the Grothendieck group of $\mathbb{Z}_2$-graded $C(V)$-modules over X and define $A(V)$ to be the cokernel of the homomorphism $M(V \surd lus 1) \longrightarrow M(V)$. Using their explicit description of relative K-theory, an elementary construction gives a natural homomorphism

$$\chi_V : A(V) \longrightarrow \tilde{K}O(B(V), S(V)) \cong \tilde{K}O(TV).$$

It is multiplicative on external sums of bundles in the sense that

$$\chi_V(E) \cdot \chi_W(F) = \chi_{V \surd lus W}(E \otimes F).$$

If V is the associated bundle $V = P \times_{Spin(k)} \mathbb{R}^k$ of a principal $Spin(k)$-bundle P and M is a C_k-module, then $E = P \times_{Spin(k)} M$ is a $C(V)$-module. This gives a homomorphism $\beta_P : A_k \longrightarrow A(V)$ and thus a composite homomorphism $\alpha_P = \xi_V \beta_P : A_k \longrightarrow \tilde{K}O(TV)$. Taking X to be a point and P to be trivial, there results a homomorphism of rings

$$\alpha : A_* \longrightarrow \sum_{k \geq 0} KO^{-k}(pt).$$

The beautiful theorem now is that α and its complex analogue are isomorphisms of rings. This suggests that a proof of Bott periodicity based on the use of Clifford algebras should be possible. Using Banach algebras, Wood [Wood65] and Karoubi [Kar66, Kar68] later found such proofs..

Now consider a $Spin$-bundle $V \cong P \times_{Spin(n)} \mathbb{R}^n$, where $n = 8k$. Define $\mu_V = \alpha_P(\lambda^k) \in \tilde{K}O(TV)$. Then μ_V restricts on fibers to the canonical generator of the free $KO^*(pt)$-module $KO^*(S^n)$. That is, it is an orientation of V, and so it induces a Thom isomorphism $\phi : KO^*(X) \longrightarrow \tilde{K}O^*(TV)$. It follows that a $Spin(8k)$-bundle V is KO-orientable if and only $w_1(V) = 0$ and $w_2(V) = 0$. The orientation is multiplicative in the sense that $\mu_{V \surd lus W} = \mu_V \cdot \mu_W$. The authors prove that the orientation they construct coincides with that constructed earlier by Bott [Bott62, Bott63]. Similarly, they obtain an orientation $\mu_V^c \in \tilde{K}U(TV)$ for a $Spin^c$-bundle of dimension $n = 2k$. They state that the agreement of their orientations with Bott's gives additional good properties, but they do not say what these properties are.

In [Ad65a], Adams explained some of these properties, since he needed them for computation. Note first that, since $U(k) \longrightarrow SO(2k)$ lifts canonically to $Spin^c(2k)$, the orientations of $Spin^c$-bundles give orientations of complex bundles. The complexification of the orientation of a $Spin$-bundle V is the orientation of $V \otimes \mathbb{C}$. According to Adams, the Todd and $\hat{A}$ classes are given in terms of the K-theory and rational cohomology Thom isomorphisms by the formulas

$$e^{c_1(V)}T^{-1}(\xi) = \phi^{-1} ch \mu_V^c$$

for a complex bundle V and

$$\hat{A}^{-1}(V) = \phi^{-1} ch \mu_{V\otimes\mathbb{C}}^c$$

for a $Spin$-bundle V. According to Adams "It is well known that this is the way $\hat{A}$ enters the theory of characteristic classes". That is, $\hat{A}(M) \equiv \hat{A}(\tau) = \phi^{-1} ch \mu_{\nu\otimes\mathbb{C}}^c$, where τ is the tangent bundle of a manifold M with normal bundle ν.

We have noted the analogy between Adams operations and Steenrod operations. In the 1966 paper [At66a], Atiyah went further and showed that this analogy could be made into a precise mathematical relationship, at least for complex K-theory. He redefined the Adams operations by constructing a homomorphism of rings

$$j : R_* = \sum_k \mathrm{Hom}_{\mathbb{Z}}(R(\Sigma_k), \mathbb{Z}) \longrightarrow \mathrm{Op}(K).$$

Here Σ_k is the kth symmetric group, $R(\Sigma_k)$ its character ring, and $\mathrm{Op}(K)$ is the ring of natural transformations from the functor K to itself. This makes essential use of equivariant K-theory and the isomorphism $K_G(X) \cong K(X) \otimes R(G)$ for a finite group G and a space X regarded as a G-space with trivial action. The kth tensor power of a vector bundle over X is a Σ_k-bundle over X, and this gives a kth power map $K(X) \longrightarrow K(X) \otimes R(\Sigma_k)$; composing with homomorphisms $R(\Sigma_k) \longrightarrow \mathbb{Z}$, we obtain the kth component of j. As a matter of algebra, there is a copy of the polynomial algebra generated by certain elements that deserve to be denoted ψ^k sitting inside R_*, and the images of the ψ^k under j are the Adams operations.

Making essential use of the construction of relative K-theory in [ABS64], this allows Atiyah to relate the Adams operations to Steenrod operations by a direct comparison of definitions. The K-theory of a CW complex X is filtered by $K_q(X) = \mathrm{Ker}(K(X) \longrightarrow K(X^q))$ with associated graded group $E_0^*K(X)$. Suppose that $H_*(X)$ has no torsion and let p be a prime. The Atiyah-Hirzebruch spectral sequence implies an isomorphism $H^{2q}(X;\mathbb{Z}_p) \cong E_0^{2q}K(X) \otimes \mathbb{Z}_p$. Atiyah proves that, for $x \in K_{2q}(X)$, there are elements $x_i \in K_{2q+2i(p-1)}(X)$ such that $\psi^p(x) = \sum_{i=0}^q p^{q-i}x_i$. Writing $\bar{x}$ for the mod p reduction of x and letting $P^i = Sq^{2i}$ when $p = 2$, he then proves the remarkable formula $P^i(\bar{x}) = \bar{x}_i$. The idea of introducing Steenrod operations into generalized homology theories along the lines that Atiyah worked out in the case of K-theory has had many subsequent applications.

In another influential 1966 paper, Atiyah [At66b] introduced Real K-theory KR, which must not be confused with real K-theory KO. In the paper, real vector bundles mean one thing over "real spaces" and another thing over "spaces", which has bedeviled readers ever since: we distinguish Real from real, never starting a sentence with either. A Real space is just a space with a $\mathbb{Z}_2$-action, or involution, denoted $x \to \bar{x}$. A Real vector bundle $p : E \longrightarrow X$ is a complex vector bundle E with involution such that $\sqsubseteq erline cy = \sqsubseteq erline c \sqsubseteq erline y$ and $\sqsubseteq erline p(y) = p(\sqsubseteq erline y)$ for $c \in \mathbb{C}$ and $y \in E$. There is a Grothendieck ring $KR(X)$ of Real vector bundles over a compact Real space X.

Atiyah shows that the elementary proof of the periodicity theorem in complex K-theory that he and Bott gave in [AB64] transcribes directly to give a periodicity theorem in KR-theory. The wonderful thing is that this general theorem specializes and combines with information on coefficient groups deduced from Clifford algebras to give a new proof of the periodicity theorem for real K-theory. An essential point is to introduce a bigraded version of KR-theory, as was first done by Karoubi [Kar66] in a more general context. In more modern terms, KR is a theory graded on the real representation ring $RO(\mathbb{Z}_2)$, and it is the first example of an $RO(G)$-graded cohomology theory. Such theories now play a central role in equivariant algebraic topology.

In Atiyah's notation, define groups

$$KR^{p,q}(X,A) = KR(X \times B^{p,q}, X \times S^{p,q} \cup A \times B^{p,q}),$$

where $B^{p,q}$ and $S^{p,q}$ are the unit disk and sphere in $\mathbb{R}^q \surd lusi\mathbb{R}^p$. In the absolute case, these are the components of a bigraded ring. There is a Bott element $\beta \in KR^{1,1}(B^{1,1}, S^{1,1})$, and multiplication by β is an isomorphism. Setting $KR^p(X,A) = KR^{p,0}(X,A)$, it follows that $KR^{p,q}(X,A) \cong KR^{p-q}(X,A)$, and it turns out that this is periodic of period 8. When the involution on X is trivial, $KR(X) \cong KO(X)$, and this gives real Bott periodicity. Complex K-theory K and self-conjugate K-theory KSC, which is defined in terms of complex bundles E with an isomorphism from E to its conjugate, are also obtained from KR-theory by suitable specialization. This leads to long exact sequences relating real, complex, and self-conjugate K-theory that have been of considerable use ever since. The self-conjugate theory had been introduced by Green [Gr64] and Anderson [An64], who first discovered these exact sequences. The ideas in [At66b] have found a variety of recent applications. This is the paper of which Adams wrote in his review: "This is a paper of 19 pages that cannot adequately be summarized in less than 20".

In contrast, we come now to the definitive proof by K-theory of the Hopf invariant one theorem, for all primes p, that was given in the paper [AA66] of Adams and Atiyah. They give a complete proof of the Hopf invariant one theorem for $p = 2$ in just over a page (see also [May1]). The essential idea is to apply the relation $\psi^2\psi^3 = \psi^3\psi^2$ in the K-theory of a two-cell complex $S^n \cup_f e^{2n}$, n even. If the Hopf invariant of f is one, then a simple calculation shows that this relation leads to a contradiction unless n is 2, 4, or 8. The proof at odd primes takes only a little longer.

16. Bordism and cobordism theories

We now back up and return to the story of cobordism. Immediately after the introduction of K-theory, in 1960, Atiyah [At61a] introduced the oriented bordism and cobordism theories, denoted $MSO_*(X)$ and $MSO^*(X)$, for finite CW complexes X. Just as K^* was the first explicitly specified generalized cohomology theory, MSO_* was the first explicitly specified generalized homology theory.

For a finite CW pair (X,A) and any integer q, Atiyah defines

$$MSO^q(X,A) = \operatorname{colim}[\Sigma^{n-q}X/A, TSO(n)] \tag{16.1}$$

and verifies that these groups satisfy all of the Eilenberg-Steenrod axioms except the dimension axiom. This is the theory represented by the spectrum MSO, but Atiyah's work precedes Whitehead's paper [Wh62a], and that language was not yet available.

He defines oriented bordism groups geometrically. He proceeds a little more generally than is currently fashionable, but with good motivation. He considers the category $\mathcal{B}$ of pairs (X,α), where X is a finite CW complex (say) and α is a principal $\mathbb{Z}_2$-bundle over X, that is, a not necessarily connected double cover. Maps and homotopies of maps in $\mathcal{B}$ are bundle maps and bundle homotopies. For a smooth manifold M (with boundary), let γ denote the orientation bundle of M. Then $MSO_q(X,\alpha)$ is defined to be the set of "bordism classes" of maps $f:(M,\gamma)\longrightarrow(X,\alpha)$, where M is a q-dimensional closed manifold. Here f is bordant to $f':(M',\gamma)\longrightarrow(X,\alpha)$ if there is a manifold W such that $\partial W = M\amalg M'$ together with a map $g:(W,\gamma)\longrightarrow(X,\alpha)$ that restricts to f on M and to f' on M'. When α is trivial, f is just a map $M\longrightarrow X$, where M is an oriented q-manifold, and Atiyah writes $MSO_q(X)$ for the resulting oriented bordism group. He observes that $MG_q(X)$ can be defined similarly for the other classical groups G.

One virtue of the more general definition is the observation that, for large n,

$$MSO_q(\mathbb{RP}^n,\xi)\cong\mathcal{N}_q, \tag{16.2}$$

where $\xi: S^n\longrightarrow\mathbb{RP}^n$ is the canonical double cover. More deeply, Atiyah proves that, for an n-manifold M without boundary $MSO_q(M,\gamma)$ is isomorphic in the stable range $2q<n$ to a certain group $L_q(M)$ introduced by Thom[Thom54] and used in the proof of his "théorème fondamental". This allows Atiyah to show that Thom's theorem directly implies Poincaré duality: for a finite CW pair (X,A) such that $X-A$ is a closed oriented n-manifold

$$MSO^q(X,A)\cong MSO_{n-q}(X-Y,\gamma). \tag{16.3}$$

Taking Y to be empty and X to be oriented, this specializes to

$$MSO^q(X)\cong MSO_{n-q}(X).$$

Although he doesn't go into detail, Atiyah was aware of the expected interpretation in terms of cup and cap products induced from the maps

$$TSO(m)\wedge TSO(n)\longrightarrow TSO(m+n).$$

For n large and even, so that $\gamma=\xi$, (16.2) and (16.3) imply that

$$\mathcal{N}_q\cong MSO^{2n-q}(\mathbb{RP}^{2n}). \tag{16.4}$$

One of Atiyah's main motivations was to understand certain exact sequences relating oriented and unoriented cobordism groups, in particular the exact sequence

$$\mathcal{M}_n\xrightarrow{2}\mathcal{M}_n\longrightarrow\mathcal{N}_n, \tag{16.5}$$

due originally to Rohlin [Ro53, Ro58] and also proven by Dold [Dold60]. These exact sequences play a central role in Wall's computation of $\mathcal{M}_*$. Using (16.4), Atiyah shows that they are just long exact sequences obtained by applying the theory MSO^* to pairs of projective spaces.

Conner and Floyd [CF64a] followed up Atiyah's work with a thorough exposition and many interesting applications of the theories MO_* and MSO_*. Atiyah did not give a geometric definition of the relative groups $MSO_*(X,A)$. Conner and Floyd do so carefully, and they prove that $MSO_*(X,A)$ so defined satisfies

$$MSO_q(X,A)\cong\pi_{n+q}(X/A\wedge TSO(n)) \text{ if } n\geq q+2.$$

This shows that the geometrically defined theory agrees with the theory given by Whitehead's prescription. They construct the bordism Atiyah-Hirzebruch spectral sequence converging from $H_*(X, A; \mathcal{M}_*)$ to $MSO_*(X, A)$. For the unoriented theory, they show that

$$MO_*(X, A) \cong H_*(X, A; \mathbb{Z}_2) \otimes \mathcal{N}_*, \tag{16.6}$$

as we see from the splitting of MO as a product of Eilenberg-Mac Lane spectra. Similarly, they show that, modulo the Serre class of odd order abelian groups,

$$MSO_*(X, A) \cong H_*(X, A; \mathcal{M}_*).$$

Using this, they reinterpret and generalize Thom's work on the Steenrod representation problem. For example, they show that the natural map $MSO_*(X, A) \longrightarrow H_*(X, A; \mathbb{Z})$ is an epimorphism if and only if the oriented bordism spectral sequence for (X, A) collapses and that this holds if $H_*(X, A; \mathbb{Z})$ has no odd torsion. They also generalize (16.5) to an exact sequence

$$MSO_n(X, A) \xrightarrow{2} MSO_n(X, A) \longrightarrow MO_n(X, A).$$

However, the main point of Conner and Floyd's monograph [CF64a] was the use of cobordism for the study of transformation groups of manifolds. The cohomological study of group actions was initiated in the remarkable early work of P.A. Smith [Sm38]. The use of cohomological methods in the study of transformation groups was systematized in the seminar [Bo60] of Borel and others, including Floyd. In its introduction, Borel had pointed out the desirability of making more effective use of differentiability assumptions than had been possible previously. Conner and Floyd introduced equivariant cobordism as a follow up, and they found many very interesting applications of it to the study of fixed point spaces of differentiable group actions. I shall only indicate a little of what they do.

They define oriented and unoriented geometric equivariant cobordism groups for any finite group G with respect to group actions on manifolds with isotropy groups constrained to lie in any set of subgroups of G closed under conjugacy. Write $\mathcal{N}_*^G$ and $\mathcal{M}_*^G$ for these groups when all subgroups are allowed as isotropy groups. Conner and Floyd focus on the case of free actions (trivial isotropy group). Here the geometric description of bordism theory directly implies that the cobordism groups of smooth compact manifolds with free G-actions are isomorphic to the bordism groups $MO_*(BG)$. Restricting to oriented manifolds and orientation preserving actions, the resulting cobordism groups are isomorphic to the bordism groups $MSO_*(BG)$. This opens the way to calculations. As in Atiyah's work on $K^*(BG)$, transfer homomorphisms play a significant role.

In the unoriented case, $MO_*(BG)$ is calculated in terms of $H_*(G; \mathbb{Z}_2)$ by (16.6). As an elementary application, Conner and Floyd give a geometric proof of Wall's observation that the square of a manifold is cobordant to an oriented manifold. However, the main applications concern the fixed point space F of a non-trivial smooth involution on a closed n-manifold M, which for clarity we assume to be connected. Let F^m be the union of the components of F of dimension m. If the Stiefel-Whitney classes of the normal bundle of F^m in M are trivial for $0 \leq m < n$, then F^m is a boundary for $0 \leq m < n$. This is a substantial generalization of the fact that F cannot have exactly one fixed point, a fact that, with its odd primary analogue, motivated their entire study.

Remarkably, although they did not have a description of $\mathcal{N}_*^{\mathbb{Z}_2}$ as the homotopy groups of a space, Conner and Floyd were able to compute these cobordism groups in terms of bordism groups; precisely, they obtained a split short exact sequence

$$0 \longrightarrow \mathcal{N}_n^{\mathbb{Z}_2} \longrightarrow \sum_{m=0}^{n} MO_m(BO(n-m)) \longrightarrow MO_{n-1}(B\mathbb{Z}_2) \longrightarrow 0.$$

For an odd prime p, Conner and Floyd calculate the bordism groups $MSO_*(B\mathbb{Z}_p)$ completely and give partial information on $MSO_*(B\mathbb{Z}_{p^k})$ for $k > 1$. They also study $MSO_*((B(\mathbb{Z}_p)^k))$, ending with a conjecture on annihilator ideals that was only proven much later. In this connection, they obtained partial information on a Künneth theorem for the computation of $MSO_*(X \times Y)$. Landweber [Lan66] later gave the complete result, along with the easier analogue for MU_*. Conner and Floyd went on to study the equivariant complex bordism groups $MU_*(BG)$ for free G-actions in [CF64b]. This work has been very influential in the development of both equivariant geometric topology and equivariant stable algebraic topology, which recently has become a major subject in its own right.

17. Further work on cobordism and its relation to K-theory

We have seen that Milnor [Mil60, Mil62] and Novikov [Nov60, Nov65] raised the problem of determining the cobordism groups $\mathcal{M}_*^G \cong \pi_*(MG)$ of G-manifolds for $G = SU$, Sp and $Spin$. They were aware that only the question of 2-torsion was at issue. Liulevicius [Liu64] described $H^*(MG, \mathbb{Z}_2)$ as a coalgebra over the Steenrod algebra for various G and began the study of the relevant mod 2 Adams spectral sequences. In particular, he calculated E_2 and showed that $E_2 \neq E_\infty$ for MSU and MSp. He also computed $\pi_*(MSp)$ in low dimensions. The calculation of the 2-torsion in $\pi_*(MSp)$ has been studied extensively over the last 30 years, and a complete answer is still out of sight. I shall say no more about that here. However, the remaining cases were all completely understood by the end of 1966. The literature in this area burgeoned in the mid 1960's, and I will mention only some of the main developments. Stong [Sto68], unfortunately out of print, gives an excellent and thorough survey of results through 1967, with a complete bibliography. Foundationally, he starts from the systematic treatment of the geometric interpretation of $\pi_*(MG)$ that was given by Lashof in 1963 [Las63].

As a preamble to explicit calculations, Milnor [Mil65] and others gave some attractive conceptual results concerning the squares of manifolds. As a consequence of their work on fixed points of involutions in [CF64b], Conner and Floyd had observed that if $V_{\mathbb{R}}$ is the real form of a complex algebraic variety $V_{\mathbb{C}}$ and both are non-singular, then $V_{\mathbb{C}}$ is unoriented cobordant to $V_{\mathbb{R}} \times V_{\mathbb{R}}$. Milnor [Mil65] showed that this implies that an unoriented cobordism class contains a complex manifold if and only if it contains a square. He also explained in terms of Stiefel-Whitney numbers when a manifold is unoriented cobordant to a complex manifold. Further, he conjectured and proved in low dimensions that the square of an orientable manifold is unoriented cobordant to a $Spin$-manifold. P.G. Anderson [And66] proved that the square of a torsion element of $\mathcal{M}_*$ is unoriented cobordant to an SU-manifold, and he deduced Milnor's conjecture from that. Stong [Sto66b] later gave a simpler proof.

In their monograph [CF66a], Conner and Floyd worked out the analogue of their development of geometric and represented oriented cobordism theory in the complex case, together with its SU variant. Although the details are a good deal more complicated, they follow the methods used by Wall [Wall60] and Atiyah [At61a] in the case of oriented cobordism to determine the additive structure of the SU-cobordism ring $\mathcal{M}_*^{SU}$. The essential point is to determine the torsion, and they prove that the torsion subgroup of $\mathcal{M}_q^{SU}$ is zero unless $q = 8n + 1$ or $q = 8n + 2$, in which cases it is a $\mathbb{Z}_2$-vector space whose dimension is the number of partions of n. Wall [Wall66] later completed the determination of the multiplicative structure of $\mathcal{M}_*^{SU}$.

In concurrent work, Anderson, Brown, and Peterson [ABP66a] calculated the mod 2 Adams spectral sequence for $\pi_*(MSU)$. They use a result of Conner and Floyd [CF66a] to determine the differential d_2, and they deduce that $E_3 = E_\infty$. This is a more sophisticated application of the Adams spectral sequence than had appeared in earlier work, and it was the first significant example in which the Adams spectral sequence was determined completely despite the presence of non-trivial differentials. Morever, they prove that an SU-manifold is a boundary if and only if all of its Chern numbers and certain of its (normal) KO-characteristic numbers are zero.

To define KO-characteristic numbers, they make one of the first explicit uses of Poincaré duality in KO-theory, relying on the Atiyah-Bott-Shapiro orientation to obtain canonical KO-fundamental classes of SU-manifolds. Another interesting feature of their work is the complete determination of the image of the framed cobordism groups $\mathcal{M}_*^{fr}$, that is the stable homotopy groups of spheres, in $\mathcal{M}_*^{SU}$. This allows them to connect up their calculations with the Kervaire surgery invariant and the realization of Poincaré duality spaces as SU-manifolds up to homotopy equivalence.

Soon afterwards, Anderson, Brown, and Peterson [ABP66b] followed up their work on $\mathcal{M}_*^{SU}$ with a calculation of $\mathcal{M}_*^{Spin}$, which is a good deal harder. Let $bo\langle n\rangle$ denote the spectrum obtained from the real Bott spectrum by killing its homotopy groups in dimensions less than n. They construct a map f from $MSpin$ to an appropriate product of copies of spectra $bo\langle 2n\rangle$ and suspensions of $H\mathbb{Z}_2$ and prove that f induces an isomorphism on mod 2 cohomology. A posteriori, f is a 2-local equivalence.

The essential input that makes this calculation possible is Stong's calculation [Sto63] of the mod 2 cohomology of the $bo\langle 2n\rangle$ as modules over the Steenrod algebra. These modules are of the form $A/A(Sq^1, Sq^2)$ or A/ASq^3, and this allows calculation of the relevant Adams spectral sequences. However, a good deal of work, most of it dealing with the algebra of modules over the Steenrod algebra, is needed to go from this input to the final conclusion. Incidentally, working on the space level, Adams had earlier calculated the mod p cohomologies of the $bu\langle 2n\rangle$ for all primes p [Ad61a].

Similarly to the case of $\mathcal{M}_*^{SU}$, a $Spin$-manifold is a boundary if and only if all of its Stiefel-Whitney numbers and certain of its KO-characteristic numbers are zero. Moreover, a manifold is cobordant to a $Spin$-manifold if and only all of its Stiefel-Whitney numbers involving w_1 or w_2 are zero. The image of $\mathcal{M}_*^{fr}$ in $\mathcal{M}_*^{Spin}$ is determined by comparison with the case of $\mathcal{M}_*^{SU}$. A result of Stong [Sto66a] determines the ring structure on the torsion free part of $\mathcal{M}_*^{Spin}$.

In their monograph [CF66b], Conner and Floyd give a general exposition of the relationship between K-theory and cobordism, starting from a variant of the orientation theory of Atiyah, Bott, and Shapiro. They construct compatible natural transformations of multiplicative cohomology theories

$$\mu_c : \tilde{M}U^*(X) \longrightarrow \tilde{K}^*(X)$$

and

$$\mu_r : \tilde{M}SU^*(X) \longrightarrow \tilde{K}O^*(X)$$

on finite CW complexes X. Thinking of an element of $\tilde{M}U^n(X)$ as a homotopy class of maps $f : S^{2k-n} \wedge X \longrightarrow TU(k)$, k large, they obtain $\mu_c(f)$ by transporting the Thom class along the composite

$$\tilde{K}(TU(k)) \xrightarrow{f^*} \tilde{K}(S^{2k-n} \wedge X) \cong \tilde{K}^n(X).$$

Up to sign, $\mu_c : \mathcal{M}^U_* \longrightarrow \mathbb{Z}$ gives the Todd genus $T[M]$ of U-manifolds. Since μ_c is a ring homomorphism, it gives $\mathbb{Z}$ a structure of MU^*-module, where $MU^n = \mathcal{M}^U_{-n}$. Conner and Floyd prove the remarkable facts that complex cobordism determines complex K-theory and symplectic cobordism determines real K-theory. Precisely, the maps μ_c and μ_r induce isomorphisms

$$MU^*(X, A) \otimes_{MU^*} K^*(pt) \cong K^*(X, A)$$

and

$$MSp^*(X, A) \otimes_{MSp^*} KO^*(pt) \cong KO^*(X, A)$$

on finite CW pairs (X, A). The reason that MSp comes in is clear from the proof, which makes heavy use of the Atiyah-Hirzebruch spectral sequence and relies on the fact that $H^*(BSp)$ is concentrated in even degrees. Along the way, considerable information about characteristic classes in cobordism theories is obtained.

There is a last part of [CF66b] that deserves to be better known than it is. In slightly modernized terms, Conner and Floyd consider the cofiber MU/S of the unit $S \longrightarrow MU$. They give a cobordism interpretation of $\pi_*(MU/S)$ in terms of U-manifolds with stably framed boundary, or (U, fr)-manifolds. The cofiber sequence gives rise to a short exact sequence

$$0 \longrightarrow \mathcal{M}^U_{2n} \longrightarrow \mathcal{M}^{U,fr}_{2n} \longrightarrow \mathcal{M}^{fr}_{2n-1} \longrightarrow 0$$

for each $n > 0$. The Todd genus defines a homomorphism $T : \mathcal{M}^{U,fr}_* \longrightarrow \mathbb{Q}$, and it turns out that there is a closed U-manifold with the same Chern numbers as a given (U, fr)-manifold M if and only if $T(M)$ is an integer. Therefore T induces a homomorphism $\pi_{2n-1}(S) \cong \mathcal{M}^{fr}_{2n-1} \longrightarrow \mathbb{Q}/\mathbb{Z}$. Conner and Floyd show that this homomorphism coincides with Adams' complex e-invariant. This allows the use of Adams' complete determination of the behavior of $e_{\mathbb{C}}$ to obtain geometric information. Using SU in place of U, they obtain a similar interpretation of Adams' real e-invariant $\pi_{8n+3}(S) \longrightarrow \mathbb{Q}/\mathbb{Z}$, and they use explicit manifold constructions modelled on Toda brackets appearing in Adams' work to reprove the result of Anderson, Brown, and Peterson on the image of $\mathcal{M}^{fr}_*$ in $\mathcal{M}^{SU}_*$.

The work of Conner and Floyd uses a basic theorem of Hattori [Ha66] and Stong [Sto66b]. The tangential characteristic numbers of a U-manifold M^{2n} determine a homomorphism $H^{2n}(BU; \mathbb{Q}) \longrightarrow \mathbb{Q}$. Let I^{2n} be the subgroup of $H^{2n}(BU; \mathbb{Q})$ consisting of all elements that are mapped into $\mathbb{Z}$ by all such homomorphisms. The Riemann-Roch theorem of Atiyah and Hirzebruch shows that the $2n$ component of $ch(x)T$ is in I^{2n} for all $x \in K(BU)$, where T is the universal Todd class. Atiyah and Hirzebruch [AH61c] conjectured that these Riemann-Roch integrality relations are complete, in the sense that every element of I^{2n} is of this form.

This is the theorem of Hattori and Stong. It can be rephrased in several ways. Stong uses methods of cobordism to show that all homomorphisms $\mathcal{M}_n^U \longrightarrow \mathbb{Z}$ are integral linear combinations of certain homomorphisms given by K-theory characteristic numbers. Hattori shows that the conjecture is equivalent to the assertion that, for large k, the homomorphism

$$\alpha : \tilde{K}(TU(k)) \longrightarrow \mathrm{Hom}(\pi_{2n+2k}(TU(k)), \tilde{K}(S^{2n+2k}))$$

given by $\alpha(y)(x) = x^*(y)$ is an epimorphism. He proves that the K-theory Hurewicz homomorphism

$$\pi_{2n+2k}(TU(k)) \longrightarrow \tilde{K}_{2n+2k}(TU(k)),$$

which is induced by the unit $S \longrightarrow K$ of the K-theory spectrum, is a split monomorphism. He then deduces the required epimorphism property by use of Poincaré duality in K-theory. Adams and Liulevicius [AL72] later gave a spectrum level reinterpretation and proof of Hattori's theorem, viewing it as a result about the connective K-theory Hurewicz homomorphism of MU.

18. High dimensional geometric topology

The period that I have been discussing was of course also a period of great developments in high dimensional geometric topology. There was a closer interaction between algebraic and geometric topology throughout the period than there is today, and some of the most important work in both fields was done by the same people. Cobordism itself is intrinsically one such area of interaction. It would be out of place to discuss such related topics as h-cobordism and s-cobordism here. However, some geometric work was so closely intertwined with the main story or was to be so important to later developments that it really must be mentioned, if only very briefly.

First, there is the work of Kervaire and Milnor [KM63] on groups of homotopy spheres. This gives one of the most striking reductions of a problem in geometric topology to a problem in stable homotopy theory, albeit in this case to the essentially unsolvable one of computing the cokernel of the J-homomorphism.

As we have already mentioned, the starting point of modern differential topology was Milnor's discovery [Mil56b] of exotic differentiable structures on S^7. Kervaire and Milnor classify the differentiable structures on spheres in terms of the stable homotopy groups of spheres and the J-homomorphism. Let Θ_n be the group of h-cobordism classes of homotopy n-spheres under connected sum. By Smale's h-cobordism theorem [Sm62], Θ_n is the set of diffeomorphism classes of differentiable structures on S^n when $n \neq 3$ or 4. Kervaire and Milnor show that every homotopy sphere is stably parallelizable. The proof uses Adams' result [Ad65a] that $J : \pi_*(SO) \longrightarrow \pi_*^s$ maps the torsion classes monomorphically. They then show that the homotopy spheres that bound a parallelizable manifold form a subgroup bP_{n+1} of Θ_n such that Θ_n/bP_{n+1} embeds as a subgroup of $\pi_n^s/J(\pi_n(SO))$. This embedding is an isomorphism if $n = 4k + 1$.

In the 1960's, geometric topologists began to take seriously the classification of piecewise linear and topological manifolds, and the appropriate theories of bundles and classifying spaces were developed. A few of the important relevant papers are those of Hirsch [Hir61], Milnor [Mil64], Kister [Ki64], Lashof and Rothenberg [LR65], and Haefliger and Wall [HW65]. We point out one conclusion that is particularly relevant to our theme, namely a theorem of Hirsch and Mazur that is explained in [LR65] and that is closely related to the work of Kervaire and Milnor just discussed. If M is a smoothable combinatorial manifold, then the set of concordance classes of smoothings of M is in bijective correspondence with the set of homotopy classes of maps $M \longrightarrow PL/O$. Part of the explanation is that $PL(n)/O(n) \longrightarrow PL/O$ induces an isomorphism of homotopy groups through dimension n, which converts an unstable problem into a stable one. Williamson [Wi66] proved the analogue of Thom's theorem for PL-manifolds, showing that the cobordism ring $\mathcal{M}_*^{PL}$ of oriented PL-manifolds is isomorphic to $\pi_*(MSPL)$, and similarly in the unoriented case.

These results raised the question of computing characteristic classes for PL and topological bundles and of computing the PL-cobordism groups. These calculational questions, which turn out to be closely related to the questions raised by Adams in [Ad65b], would later motivate a substantial amount of work in stable algebraic topology. A 1965 paper of Hsiang and Wall [HsW65] discussed the orientability of non-smooth manifolds with respect to generalized cohomology theories. A year or two later, Sullivan discovered [Sull70] that PL-bundles admit canonical KO-orientations (away from 2). That fact has played an important role in answering such questions.

Although almost nothing was known about these questions in 1966, a useful conceptual guide to later calculations was published that year by Browder, Liulevicius, and Peterson [BLP66]. By then, classifying spaces $BF(n)$ for spherical fibrations were also on hand, by work of Stasheff [Sta63c] and later Dold [Dold66]. The authors consider a system of spaces $BG(n)$, where $G(n)$ may have no a priori meaning, and maps $BO(n) \longrightarrow BG(n) \longrightarrow BF(n)$, $BG(n) \longrightarrow BG(n+1)$ and $BG(m) \times BG(n) \longrightarrow BG(m+n)$ satisfying some evident compatibility conditions. They define the Thom space $TG(n)$ by use of the pullback of the universal spherical fibration over $BF(n)$ and have a Thom spectrum MG. They have a Thom isomorphism $H^*(BG) \longrightarrow H^*(MG)$ in mod 2 cohomology, and they define Stiefel-Whitney classes as usual.

With this set up, they observe that theorems of Milnor and Moore [MM65] imply that $H^*(MG)$ is a free A-module and there is a Hopf algebra $C(G)$ over A such that

$$H^*(BG) \cong H^*(BO) \otimes C(G)$$

as Hopf algebras over A and

$$\pi_*(MG) \cong \pi_*(MO) \otimes C(G)^*$$

as algebras. Letting BSG be the 2-fold cover of BG determined by w_1, they also observe that $H^*(MSG)$ is the direct sum of a free A-module and suspensions of copies of A/ASq^1, so that, a posteriori, MSG splits 2-locally as a product of corresponding Eilenberg-Mac Lane spectra, just as MSO does. Moreover

$$H^*(BSG) \cong H^*(BSO) \otimes C(G)$$

as Hopf algebras over A, and at least the additive structure of $\pi_*(MSO)$, modulo odd torsion, is determined by $C(G)$ and the Bockstein spectral sequence of $H^*(BSG)$.

Intuitively, this means that the mod 2 characteristic classes for "G-bundles" completely determine the unoriented G-cobordism ring and the 2-local part of the oriented G-cobordism ring. The proofs require no geometry, but when one has a manifold interpretation of $\pi_*(MG)$, for example when $G = PL$, it follows directly that a G-manifold is a boundary if and only if all characteristic numbers defined in terms of $H^*(BG)$ are zero.

For an odd prime p, they prove that $H^*(BSG; \mathbb{Z}_p)$ is a free B-module, where B is the sub Hopf algebra of A generated by the P^i, but the calculation of the odd torsion in $\pi_*(MG)$ requires use of the Adams spectral sequence and is thus of a quite different character than the determination of the 2-torsion.

19. Iterated loop space theory

So far I have focused on the mainstream of developments through 1966, but there are some other directions of work that were later to become important to stable algebraic topology. This section describes one stream of work that was later to merge with the mainstream. Although the connection was not yet visible in 1966 and won't be made visible here, the relevant later work was to provide key tools for the calculations called for in the previous section.

Let X be an H-space. One can ask whether or not X has a classifying space Y, so that $X \simeq \mathcal{M}Y$. If so, one can ask whether Y is an H-space. If so, one can ask whether Y has a classifying space. Iterating, one can ask whether X is an n-fold loop space, or even an infinite loop space. One wants the answers to be in terms of internal structure on the space X. The answers are closely related to an understanding of the spaces $\mathcal{M}^n\Sigma^n X$, which play a role roughly dual to the role of Eilenberg-Mac Lane spaces in ordinary homotopy theory. Such questions were later to be a major part of stable algebraic topology, but some important precursors were on hand by 1966.

Recall that a topological monoid X, that is an associative H-space with unit, has a classifying space BX and $X \simeq \mathcal{M}BX$ if $\pi_0(X)$ is a group under the induced multiplication. This result has a fairly long history, which would be out of place here.

In 1957, Sugawara [Su57a] gave a fibration-theoretic necessary and sufficient condition for a space X to be an H-space, or to be a homotopy associative H-space. In the same year [Su57b], he followed up by giving necessary and sufficient conditions for X to have a classifying space. Obviously, X must be homotopy associative, but that is not sufficient. Sugawara described an infinite sequence of higher homotopies that must be present on loop spaces and showed that the existence of such homotopies is sufficient. Three years later, he took the next step and displayed an infinite sequence of higher commutativity homotopies such that a loop space $\mathcal{M}Y$ has such homotopies if and only if Y is an H-space. Stasheff [Sta63a, Sta63b] later reformulated Sugawara's higher associativity homotopies in a much more accessible fashion, introducing A_n and A_∞-spaces. The latter are Sugawara's H-spaces with all higher associativity homotopies, and Stasheff reproved the result that such an H-space has a classifying space.

Systematic computations of $H_*(\mathcal{M}^n\Sigma^n X; \mathbb{Z}_p)$ began in 1956 with the work of Kudo and Araki [AK56]. Using higher commutativity homotopies, they mimic Steenrod's original construction of the Steenrod squares in mod 2 cohomology in terms of $\cup_i$-products to obtain mod 2 homology operations for n-fold loop spaces. They use these operations to compute $H_*(\mathcal{M}^n S^{n+k}; \mathbb{Z}_2)$. To compute $H_*(\mathcal{M}^n\Sigma^n X; \mathbb{Z}_2)$ for general spaces X, bracket operations of two variables are needed. These were introduced by Browder [Br60]. He reproved the results of Kudo and Araki by mimicking Steenrod's construction of Steenrod operations in terms of the homology of the cyclic group $\mathbb{Z}_2$, and he computed $H_*(\mathcal{M}^n\Sigma^n X;\mathbb{Z}_2)$ as a functor of $H_*(X;\mathbb{Z}_2)$. The functoriality is a calculational fact, not something true for general theories. For example, $K(\mathcal{M}^n\Sigma^n X)$ is not a functor of $K(X)$. It is related to Dold's earlier, but non-calculational, result [Dold58b] that the homologies of the symmetric products of X are determined by the homology of X.

Dyer and Lashof [DL62] studied homology operations for n-fold loop spaces at odd primes p, mimicking Steenrod's definition of Steenrod operations in terms of the homology of the symmetric group Σ_p. These operations are now generally called Dyer-Lashof operations. This method of construction does not give enough operations to compute $H_*(\mathcal{M}^n\Sigma^n X;\mathbb{Z}_p)$. Dyer and Lashof define $QX = \cup\mathcal{M}^n\Sigma^n X$, where the union is taken over the inclusions $\mathcal{M}^n\Sigma^n X \longrightarrow \mathcal{M}^{n+1}\Sigma^{n+1}X$ obtained by suspending a map $S^n \longrightarrow X\wedge S^n$ to a map $S^{n+1} \longrightarrow X\wedge S^{n+1}$. They then prove that their operations, plus the Bockstein, are sufficient to compute $H_*(QX;\mathbb{Z}_p)$ as a functor of $H_*(X;\mathbb{Z}_p)$.

In 1966, Milgram [Mil66] generalized the James construction to obtain a combinatorial model J_nX for $\mathcal{M}^n\Sigma^n X$, where X is a connected CW complex. The spaces J_nX are themselves CW complexes with cellular chain complexes identified in terms of the cellular chains of X. This allows a computation of $H_*(\mathcal{M}^n\Sigma^n X)$, but Milgram's work was not connected up with homology operations until much later. This is analogous to Cartan's original computation of the homology of Eilenberg-Mac Lane spaces without use of Steenrod operations. The later theory of operads led to a simpler, but equivalent, model for $\mathcal{M}^n\Sigma^n X$ and allowed the specification of sufficiently many homology operations to compute $H_*(\mathcal{M}^n\Sigma^n X;\mathbb{Z}_p)$ as a functor of $H_*(X;\mathbb{Z}_p)$. It also made it clear that Milgram's work and Stasheff's work on A_∞-spaces are closely related, something that was not apparent at the time.

20. Algebraic K-theory and homotopical algebra

The cohomology of groups, homological algebra, algebraic K-theory, and category theory are algebraic areas of mathematics that developed simultaneously with stable algebraic topology and gradually evolved into separate subjects. All remain closely connected to stable algebraic topology. I shall mention some directions that seem to me to be of particular interest or to have been important forerunners of later developments.

I will point to just a few relevant papers concerning the cohomology of groups. Of course, with trivial coefficients, the homology of a discrete group G agrees with the homology of its classifying space BG. At about the same time that Dyer and Lashof were computing $H_*(QX)$, Nakaoka [Na60, Na61] was computing the homologies of the symmetric groups and in particular the homology of the infinite symmetric group. With $X = S^0$, it would later turn out that these were essentially the same computation.

The equivariant homology and cohomology groups of spaces that were studied by Borel and others in [Bo60] are $H_*^G(X) = H_*(EG\times_G X)$ and $H_G^*(X) = H^*(EG\times_G X)$. Swan [Sw60a] in 1960 introduced the Tate cohomology of spaces $\hat{H}_G^*(X)$. Just as in group theory, he gave a long exact sequence relating $H_*^G(X)$, $H_G^*(X)$ and $\hat{H}_G^*(X)$. It has been shown recently that one can replace ordinary homology and cohomology by the theories represented by any spectrum and still get such a long exact sequence.

Although a little off the subject, the early applications of algebraic K-theory to algebraic topology deserve brief mention. Swan [Sw60b] used his study [Sw60c] of projective modules over group rings to show that any finite group with periodic cohomology acts freely on a homotopy sphere. This led to Wall's K-theoretic finiteness obstruction [Wall65] that determines whether or not a finitely dominated CW complex is homotopy equivalent to a finite CW complex. Applications of algebraic K-theory in surgery theory also began in the 1960's, but are beyond our scope.

There are several papers in algebraic K-theory and what later became known as homotopical algebra that may be viewed as harbingers of things to come in stable algebraic topology. The feature to emphasize is the evolution from analogies between similarly defined objects in different subjects to direct mathematical connections and fruitful common generalizations. These topics were to have much direct contact with iterated loop space theory, but that could not have been visible in 1966. Their connections with the mainstream of stable algebraic topology were visible from the beginning, although the forms these connections would eventually take could not have been anticipated.

Topological K-theory grew directly out of Grothendieck's work, and the analogy with algebraic K-theory was thus visible from the outset. Swan [Sw62] gave the analogy mathematical content by proving that, for a compact space X, $K(X)$ is naturally isomorphic to the Grothendieck group of finitely generated projective modules over the ring $C(X)$ of continuous real-valued functions on X. The isomorphism sends a vector space ξ to the $C(X)$-module $\Gamma(\xi)$ of sections of ξ; $\Gamma(\xi)$ is a finitely generated projective $C(X)$-module since ξ is a summand of a trivial bundle.

As Adams wrote in his review of a paper of Bass [Bass64]: "This leads to the following programme: take definitions, constructions and theorems from bundle-theory; express them as particular cases of definitions, constructions, and statements about finitely-generated projective modules over a general ring; and finally, try to prove the statements under suitable assumptions." With this analogy clearly in mind, Bass defines and studies K^0 (following Grothendieck) and K^1 of rings in the cited paper. Higher algebraic K-groups came later, and their study would lead to substantial developments in stable algebraic topology that would be closely related to both high dimensional geometric topology and infinite loop space theory.

There are many other areas where analogies between algebra and topology have been explored. For example, starting with Eckmann-Hilton duality [Eck57, Hil58], there was considerable work in the late 1950's and early 1960's exploring the idea of a homotopy theory of modules, or, more generally, of objects in abelian categories, by analogy with the homotopy theory of spaces. I shall say nothing about that work.

Rather, I shall say a little about the analogy between stable homotopy theory and differential homological algebra. Differential homological algebra studies such objects as differential graded modules over differential graded algebras and is a natural tool in both algebraic topology and algebraic geometry. The analogy between homotopies in topology and chain homotopies in homological algebra was already clear by 1945. However, the structural analogy between stable homotopy theory and differential homological algebra goes much deeper. It later led both to an axiomatic understanding of homotopy theory in general categories and to concrete mathematical comparisons between such categories in topology and algebra, beginning with the fundamental work of Quillen [Qu67].

Dold and Puppe gave important precursors of this in the early 1960's. The first systematic exploration of the analogy was given by Dold [Dold60], in 1960. He develops cofiber sequences of chain complexes of modules over a ring, a Whitehead type theorem for such chain complexes, Postnikov systems of chain complexes, and so forth. The next year [DP61], Dold and Puppe gave a remarkable and original use of simplicial methods in algebra by defining and studying derived functors of non-additive functors between abelian categories. Unlike the additive case, these functors do not commute with suspension. This fact is analyzed by use of a bar construction defined in terms of cross-effect functors that measure the deviation from additivity.

In 1962, Puppe [Pu62], motivated by the need for a good stable homotopy category, gave an axiomatic treatment of exact triangles. That paper precedes the introduction of the derived category of chain complexes over a ring in Verdier's 1963 thesis (which was published much later [Ver71]). Verdier's axioms for exact triangles give the notion of a "triangulated category". Algebraic topologists and algebraic geometers have developed several areas of differential homological algebra independently, with different details, nomenclature, and, of course, assignment of credit. The definition of triangulated categories is a case in point.

About the same time as Stasheff's work on A_∞ spaces, and with mutual influence, Mac Lane [Mac65] in 1963 introduced coherence theory in categorical algebra. This explains what it means for a category to have a product that is associative, commutative, and unital "up to coherent natural isomorphism". The coherence isomorphisms are analogues of higher homotopies in topology. In familiar examples, like cartesian products and tensor products, the isomorphisms are so obvious that they hardly seem worth mentioning. In less obvious situations, they require serious attention. The analogy between coherence isomorphisms and higher homotopies was later to be given mathematical content via infinite loop space theory, with extensive applications to algebraic K-theory.

Also around the same time, Adams and Mac Lane collaborated in the development and study of certain algebraic categories, the "PROPs" and "PACTs" discussed briefly in [Mac65]. Their goal was to understand coherence homotopies in differential homological algebra. I have gone through a box full of correspondence between Adams and Mac Lane and can attest that this was one of the largest scale collaborations never to have reached print. When later translated into topological terms, their work was to be very influential in infinite loop space theory; the original algebraic motivation reached fruition much more recently.

21. The stable homotopy category

In our discussion of the Adams spectral sequence and of cobordism, we have indicated the need for a good stable homotopy category of spectra, and we have discussed the S-category of Spanier and J.H.C. Whitehead [SW57] and the category of G.W. Whitehead [Wh62a] as important precursors. We begin this section by discussing a very important 1966 paper for which such foundations are needed.

We have seen that the quotient $B = A/(\beta)$ of the Steenrod algebra appears naturally in the study of cobordism. For all classical groups G and for $G = PL$, $H^*(MG;\mathbb{Z}_p)$ is a free B-module for each odd prime p, where we think of B as a sub-Hopf algebra of A. In the classical group case, but not in the case of PL, $H^*(MG;\mathbb{Z})$ is torsion free. For each prime p, Brown and Peterson [BP66] construct a spectrum, now called BP, such that $H^*(BP;\mathbb{Z}_p) \cong B$ as an A-module. They then prove that any spectrum X whose mod p cohomology is a free B-module and whose integral cohomology is torsion free admits a map f into a product of suspensions of BP that induces an isomorphism on mod p cohomology. A posteriori, f is a p-local equivalence. Since Brown and Peterson compute the homotopy groups of BP, one can read off the homotopy groups of X, modulo torsion prime to p. The method of proof is to use Milnor's results on the structure of A to write down a free resolution of $A/(\beta)$ as an A-module and then to realize the resolution by an inductive construction of a generalized Postnikov system whose inverse limit is BP.

This was the first time that a spectrum with desirable properties was tailor made. The spectra studied earlier had been ones that occurred "in nature" as sequences of spaces. For the foundations of their work, Brown and Peterson write "We will make various constructions on spectra, for example, forming fibrations and Postnikov systems, just as one does with topological spaces. For the details of this see [–]". The reference they give in [–] is Whitehead [Wh62a]. However, the Whitehead category is not designed for this purpose and is not triangulated. Intuitively, one needs a category that is equivalent to the category of cohomology theories on spectra, not just spaces.

Moreover, it would later be seen that BP, like S, K, KO, and the MG is a "commutative and associative ring spectrum". To attach a satisfactory meaning to this notion, one needs a smash product in the stable homotopy category of spectra that is associative, commutative, and unital up to coherent natural isomorphism. A ring spectrum R is then a spectrum with a product $\wedge : R \wedge R \longrightarrow R$ and unit $S \longrightarrow R$ such that the appropriate diagrams commute in the stable homotopy category.

The minimal requirements of a satisfactory stable homotopy category $\mathcal{S}_h$ include the following very partial list.

1. It must have a suspension spectrum functor $\Sigma^\infty : \mathcal{C}_h \longrightarrow \mathcal{S}_h$, where $\mathcal{C}_h$ is the homotopy category of based CW complexes.
2. It must have a suspension functor $\Sigma : \mathcal{S}_h \longrightarrow \mathcal{S}_h$ such that $\Sigma\Sigma^\infty \cong \Sigma^\infty\Sigma$.
3. Σ^∞ must induce a full embedding of the S-category of finite CW complexes, so that Spanier-Whitehead duality makes sense.
4. $\mathcal{S}_h$ must represent cohomology theories: isomorphism classes of objects E of $\mathcal{S}_h$ must correspond bijectively to isomorphism classes of cohomology theories $\tilde{E}^*$ in such a way that, $\mathcal{S}_h(\Sigma^\infty X, E) \cong \tilde{E}^0(X)$ for based CW complexes X.
5. $\mathcal{S}_h$ must be triangulated; in particular, $\mathcal{S}_h$ must be an additive category and $\Sigma : \mathcal{S}_h \longrightarrow \mathcal{S}_h$ must be an equivalence of categories.
6. $\mathcal{S}_h$ must be symmetric monoidal under a suitably defined smash product.

It is not an easy matter to construct such a category, and a rigorous development of modern stable algebraic topology would not have been possible without one.

Adams made several attempts to construct such a category, first in a very brief account in 1959 [Ad59] and then in more detail in his 1961 Berkeley notes [Ad61c]. There he gave an amusing discussion of the approaches a hare and a tortoise might take. In retrospect, his decision to come down on the side of the tortoise was misguided: a more inclusive and categorically sophisticated approach was needed. In [Ad66b], Adams assumed the existence of a good stable category and sketched the development of an Adams spectral sequence based on connective K-theory. This was the first attempt at setting up an Adams spectral sequence based on a generalized cohomology theory. However, convergence was not proven and the approach was still based on cohomology rather than homology.

More fruitful approaches would come a little later, with the development of the Adams-Novikov spectral sequence [Nov67], namely the Adams spectral sequence based on MU or, more usefully, BP. This, together with Quillen's observation of the relationship between complex cobordism and formal groups [Qu69], would lead later to the realization that MU and spectra constructed from it are central to the structural analysis of the stable homotopy category.

Adams' version [Ad61c] of the stable homotopy category and the slightly later version of Puppe [Pu67], following up [Pu62], were based on the use of spectra T such that T_n is a CW complex and ΣT_n is a subcomplex of T_{n+1}. Connectivity and convergence conditions were imposed. In Adams, these had the effect that all spectra were (-1)-connected. In Puppe, they had the effect that all spectra were bounded below. The specification of maps was a little complicated. Roughly, the basic diagrams (13.1) were required to commute on the point-set level rather than only up to homotopy, as in Whitehead's category, but maps were not required to be defined on the whole spectrum, only on some cofinal part of it. Puppe's category was triangulated, and his discussion of exact triangles has been quite influential.

Kan [Kan63a, Kan63b] introduced simplicial spectra in 1963 and began the development of the stable homotopy category in terms of them. Simplicial spectra are not defined as sequences of simplicial sets and maps, but rather as generalized analogues of simplicial sets that admit infinitely many face operators in each simplicial degree.

Neither Adams nor Puppe addressed the crucial problem of constructing a smash product. Kan's original papers did not address that problem either, but Kan and Whitehead [KW65a] constructed a smash product of simplicial spectra not much later. They proved that their smash product is commutative, but they did not address its associativity. In [KW65b], they used this product to discuss ring and module spectra and to study degrees of orientability, defined in terms of higher order cohomology operations, but still without addressing the question of associativity. In particular, they defined the notion of a commutative ring spectrum without defining the notion of an associative ring spectrum. Further study of simplicial spectra was made in a series of papers by Burghelea and Deleanu [BD67, BD68, BD69]. While they proved some additional properties of the smash product, they too failed to address the question of its associativity. In fact, as far as I know, that question has never been addressed in the literature.

Although simplicial spectra have not been studied much in recent years, the simplicial approach does lend itself naturally to the study of algebraically defined functors. This was exploited in the papers [KW65a, KW65b] of Kan and Whitehead and in the paper [BCKQRS66] of Bousfield, Curtis, Kan, Quillen, Rector, and Schlesinger. That paper gave a new construction of the Adams spectral sequence in terms of the mod-p lower central series of free simplicial group spectra. For the sphere spectrum, the E_1-term given by their construction is the "Λ-algebra", which is a particularly nice differential graded algebra whose homology is the cohomology of the Steenrod algebra. It would become apparent later that the Λ-algebra is closely related to the Dyer-Lashof algebra of homology operations on infinite loop spaces.

The first satisfactory construction of the stable homotopy category was given by Boardman in 1964 [Bo64]. Although mimeographed notes were made available [Bo65, Bo69], Boardman never published his construction. An exposition was given by Vogt [Vogt70]. Boardman begins with the category $\mathcal{F}$ of based finite CW complexes. He constructs from it the category $\mathcal{F}_s$ of finite CW spectra by a categorical stabilization construction. Its homotopy category $\mathcal{F}_{sh}$ is equivalent to the category obtained from the S-category by adjoining formal desuspensions. As Boardman notes, this is the right category in which to study Spanier-Whitehead duality since here the pesky dimension n in Spanier's definition can be eliminated: a duality between finite CW spectra X and Y is specified by a suitably behaved map $\varepsilon : Y \wedge X \longrightarrow S$.

Freyd [Fre66] studied the category $\mathcal{F}_{sh}$ categorically. He observed that any additive category $\mathcal{C}$ with cofiber sequences, such as $\mathcal{F}_{sh}$, embeds as a full subcategory of an abelian category $\mathcal{A}$, namely the evident category whose objects are the morphisms of $\mathcal{C}$. Moreover, $\mathcal{A}$ has enough injective and projective objects, its injective and projective objects coincide, and the objects of $\mathcal{C}$ map to projective objects in $\mathcal{A}$. He observed further that idempotents induce splittings into wedge summands in $\mathcal{C}$ for suitable $\mathcal{C}$, such as $\mathcal{F}_{sh}$, and deduced that $\mathcal{C}$ is then the full subcategory of projective objects of $\mathcal{A}$. Although he was not in possession of $\mathcal{S}_h$, it satisfies the hypotheses he makes on $\mathcal{C}$. Focusing on $\mathcal{F}_{sh}$, he posed a provocative question, "the generating hypothesis", which asserts that a map between finite CW spectra is null homotopic if it induces the zero homomorphism of homotopy groups. Despite much work, it is still unknown whether or not this is true.

Boardman next constructs a category $\mathcal{S} = \mathcal{F}_{sw}$ of CW spectra by a categorical adjunction of colimits construction. Thus his spectra are the colimits of directed systems of inclusions of finite CW spectra. The homotopy category $\mathcal{S}_h$ is the desired stable homotopy category. The most interesting feature of his work is his construction of smash products. He constructs a category $\mathcal{S}(U)$ similarly for each countably infinite dimensional real inner product space, and he constructs an external smash product $\bar{\wedge} : \mathcal{S}(V) \times \mathcal{S}(V) \longrightarrow \mathcal{S}(U \sqrt{} lusV)$. He shows that any linear isometry $f : U \longrightarrow U'$ induces a functor $f_* : \mathcal{S}(U) \longrightarrow \mathcal{S}(U')$, and he proves that, up to canonical isomorphism, the induced functor $\mathcal{S}_h(U) \longrightarrow \mathcal{S}_h(U')$ on homotopy categories is independent of the choice of f. An internal smash product on $\mathcal{S} = \mathcal{S}(\mathbb{R}^\infty)$ is a composite $f_* \circ \bar{\wedge} : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$ for any linear isometry $f : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$. Any two such internal smash products become canonically equivalent after passage to homotopy, and this allows the proof that $\mathcal{S}_h$ is symmetric monoidal.

This was very much the hare's approach and it has greatly influenced later hares (such as myself), who have needed vastly more precise properties of a good category of spectra than would have seemed possible in 1966. In particular, for much current work of interest, it is essential to have an underlying symmetric monoidal category of spectra, before passage to homotopy categories. However, perhaps for the benefit of the tortoises, Boardman [Bo69] gave a precise comparison between his construction of $\mathcal{S}_h$ and earlier approaches, and he explained how to modify the approaches of Adams and Puppe to obtain a category equivalent to $\mathcal{S}_h$. He wrote "the complication will show why we do not adopt this as definition". Nevertheless, Adams soon after gave an exposition along these lines [Ad71b] which, in the absence of a published version of Boardman's category, has served until recently as a stopgap reference.

In other parts of our story, definitive foundations were in place by 1966. The axioms for generalized homology and cohomology theories and the understanding of the representation of homology and cohomology theories were firmly established. So were the basics of K-theory and cobordism and much of the basic machinery of computation. Of course, the calculations themselves, once in place, are fixed forever: the answers will not change. The development of the stable category seems now also to have reached such a level of full understanding, and I ask the reader's indulgence in offering the monograph [EKMM97] as evidence.

My arbitrary stopping point of 1966 has the effect both of allowing me to document the invention of a marvelous new area of mathematics and of throwing into high relief how very much has been done since. There are truly vast areas of stable algebraic topology that were barely visible over the horizon or well beneath it in 1966. But that is a story for another occasion.

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