REMARKS ON THE ALEXANDER POLYNOMIAL

by

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Date for about 63/64
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Introduction:

Guided by the results of Artin and Tate applied to the calculation of the Grothendieck Cohomology Groups of the schemes;

$$\text{Spec}\ (\mathbb{Z}/p_{\mathbb{Z}}) \subset \text{Spec}\ \mathbb{Z} \quad .$$

Mumford has suggested a most elegant model as a geometric interpretation of the above situation : Spec $(\mathbb{Z}/p_{\mathbb{Z}})$ is like a one-dimensional knot in Spec $\mathbb{Z}$ which is like a simply connected three-manifold.

This analogy cannot fail to strike the imagination of a Topologist. Moreover, it gives one the impetus to review classical knot theory with an eye towards making the Arithmetic connections more explicit. The object of this paper is to study the Alexander Polynomial with these connections in mind.

The rules of the game, then, are to give definitions and statements of results which admit ready translation into Arithmetic. This is carried out in §4, §5, for much of the classical theory of the Alexander Polynomial.

In §4, the main theorem (Thm.3) suggests a new definition of the Alexander Polynomial of a knot. It is merely the determinant of a certain module (suitably interpreted). (It is somewhat related to Milnor's definition [13]). In §1, §2, §3, the algebraic theory of determinants [14] is studied and prepared for eventual application.

Perhaps, before embarking on our project, it might be useful to show explicitly the number-theoretic analogical interpretations we have in mind.

Let p be a prime number, and ζ_{p^ν} a primitive p^{ν} th root of 1 and consider the Dedekind Domain $D_\nu = \mathbb{Z}[\zeta_{p^{\nu+1}}]$ Then Spec D_ν is a cyclic covering of Spec $\mathbb{Z}$ ramified exactly at the prime p if $\nu \geq 0$. The D_ν's play the role of the branched cyclic coverings of a knot.

One is thus led to compare the deep Arithmetic theory of Iwasawa, regarding the p-primary component, X_ν, of the ideal class group of D_ν, with the classical studies of the first homology group of branched cyclic coverings of a knot (of course the latter theory is quite elementary).

Iwasawa considers

$$X = \varprojlim_{\nu} X_\nu$$

where the X_ν are related by norm mappings, and takes it as a module over $\hat{\mathbb{Z}}_p[\Gamma]$, the p-adic integral group ring of

$$\Gamma = G(K_\infty / K_0)$$

where K_ν is the quotient field of D_ν and $K_\infty = \bigcup_\nu K_\nu$.

One of the theorems crucial to the theory of Iwasawa is

Theorem: X is a torsion module of finite type over $\hat{\mathbb{Z}}_p[\Gamma]$. This is in perfect analogy with Theorem 3 & 4 (except for the projective dimension 1 statement). Note that Theorem 3 is exactly what is necessary in order to define the Alexander Polynomial of a knot.

Similarly the above theorem allows us to form

$$\det_{\hat{\mathbb{Z}}_p[\Gamma]}(X) \in \hat{\mathbb{Z}}_p[\Gamma]^0 / U(\hat{\mathbb{Z}}_p[\Gamma])$$

which we may interpret as a power series in $T = (\gamma - 1)$ since:

$$\hat{\mathbb{Z}}_p[\Gamma] \approx \hat{\mathbb{Z}}_p[[T]] .$$

This we may call $\Delta(p,t)$. "the Alexander Polynomial" of the prime p. One may use the structure theorem of Iwasawa to compute $\Delta(p;t)$ since it is a mod p invariant (See [17], [11]). One obtains

$$\Delta(p;t) = p^m \cdot P_\gamma(t)$$

where m is the numerical invariant of X defined by Iwasawa and P_γ is the characteristic polynomial of the fundamental transformation $\gamma \in \Gamma$ acting on the finite dimensional vector space over $\hat{Q}_p$: $X \otimes_{\hat{\mathbb{Z}}_p} \hat{Q}_p$.

Iwasawa conjectures that $m = 0$.

It is interesting to consider the duality relation satisfied by polynomials which are the Alexander Polynomials of knots. Namely:

$$\Delta(t) = \pm t^n \Delta(1/t)$$

for some even n. (See [13])

One is led to ask whether the characteristic polynomial P_γ also satisfies the above functional equation.

1. The Determinant

We refer to the forthcoming book of D. Mumford[14] for a treatment of the determinant (or the first chern class of a coherent sheaf over a noetherian scheme. This notion was introduced by Serre and developed by Auslander and Mumford (an elaboration of that excellent notion due to Cayley !).

Given a noetherian ring Λ and a module M of finite type over Λ satisfying (i), (ii) below, the above theory assigns to M a non-zero principal ideal in Λ, called its determinant. More generally, and in the language of schemes, (see (7), (6)) if X is a noetherian scheme and $\mathcal{F}$ a coherent sheaf over X (satisfying (i), (ii) below) one may associate to $\mathcal{F}$ an effective Cartier Divisor of X, called $\mathrm{Div}(\mathcal{F})$. For the theory and definition of Cartier Divisors, see (8), (14). X will be assumed noetherian throughout.

An effective Cartier Divisor D over X may be identified with a coherent subsheaf $\mathcal{O}_X(D) \subset \mathcal{O}_X$ which is an invertible $\mathcal{O}_X$-module. (This may be taken as definition). Then for each $x \in X$, $\mathcal{O}_X(D)_x \subset \mathcal{O}_{X,x}$ is a principal ideal. A generator of this ideal is called a local equation for D at x.

There are only a finite number of points $x \in X$ of depth zero (i.e. such that $\mathcal{O}_{x,X}$ is of depth zero, which means that any non-unit of $\mathcal{O}_{x,X}$ is a zero-divisor). Two Cartier Divisors on X are equal if they possess equal 'local equations' at all points $x \in X$ of depth 1.

Let $\mathcal{F}$ be a coherent sheaf on X such that:

(i) Supp $(\mathcal{F})$ contains no points of depth zero.

(ii) $\mathcal{F}$ is an $\mathcal{O}_{X,\bar{X}}$ module of finite cohomological dimension (i.e. possesses a finite projective resolution) for all $x \in X$.

If $\mathcal{E}$ is a free $\mathcal{O}_X$-module of rank r, $\wedge\mathcal{E}$ will denote the r-fold exterior product sheaf, which is, consequently, free of rank 1. There is an isomorphism

$$\Sigma : \wedge\mathcal{E} \xrightarrow{\sim} \mathcal{O}_X$$

uniquely determined up to a unit.

If $0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to 0$ is an exact sequence of locally free $\mathcal{O}_X$-modules of finite rank,

$$\text{rank } (\mathcal{E}_i) = r_i$$

we may obtain a canonical isomorphism

$$\eta : \mathcal{O}_X \xrightarrow{\sim} \bigotimes_{i=0}^{n} (\wedge \mathcal{E}_i)^{(-1)^i}$$

(the "collation isomorphism"). (For the definition of "collation", see (14). One reduces the problem of defining the collation to sequences of length 2 (see lemma 5.6 of (14))

$$0 \to \mathcal{E}_2 \xrightarrow{\varphi_2} \mathcal{E}_1 \xrightarrow{\varphi_1} \mathcal{E}_0 \to 0$$

and for such sequences we produce an explicit isomorphism $\Psi : \wedge \mathcal{E}_2 \otimes \wedge \mathcal{E}_0 \to \wedge \mathcal{E}_1$ as follows: If $r_0 = 0$, take $\Psi = \varphi_2$. Otherwise,

for any open set U, over which the E_i are free, if

$$x \in \Gamma(\Lambda^{r_2} E_2; U) = \Lambda^{r_2}(E_2(U))$$

$$y \in \Gamma(\Lambda^{r_0} E_0; U) = \Lambda^{r_0}(E_0(U))$$

choose some $z \in \Lambda^{r_1}(E_1(U))$ such that

$$\varphi_1(z) = y$$

and set $\varphi(X \otimes y) = \varphi_2(X) \wedge z$

which is in:

$$\Lambda^{r_2}(E_1(U)) \wedge \Lambda^{r_0}(E_1(U)) = \Lambda E_1(U).$$

From the exact sequence:

$$\Lambda^{r_2} E_2 \xrightarrow{\varphi_1} \Lambda^{r_0} E_1 \xrightarrow{\varphi_1} \Lambda^{r_0} E_0 \to 0$$

one sees that another choice z' above would differ only by:

$$z' = z + \varphi_2(w)$$

and consequently

$$\varphi_2(x) \wedge z - \varphi_2(x) \wedge z' = \varphi_2(x) \wedge \varphi_2(w) = \varphi_2(x \wedge w) = 0$$

since $x \wedge w \in \Lambda^{r_2 + r_0}(E_2) = 0$.

Existence and Uniqueness

Theorem[14] If $\mathcal{T}$ is a coherent sheaf on $\underline{X}$ such that:

(i) $\mathcal{T}$ is a torsion sheaf. That is, Supp $\mathcal{T}$ contains no points of depth 0.

(ii) $\mathcal{T}_X$ is an $\mathcal{O}_{X,X}$-module of finite projective dimension for all $x \in X$,

then there is an effective Cartier Divisor Div $\mathcal{T}$ on X (which plays the role of the first chern Class of $\mathcal{T}$) uniquely characterized as follows:

Given any open U, and free resolution

$$0 \to E_n \to \dots \to E_1 \to E_0 \to \mathcal{F} \to 0$$

over U, set $U' = U -$ Supp $\mathcal{F}$, and consider the exact sequence of free sheaves:

$$0 \to E_n \to \dots \to E_0 \to 0$$

on U'. Then there is a local equation $f \in \Gamma(U, \mathcal{O}_X)$ for Div $\mathcal{F}$ such that

$$\begin{array}{ccc} \mathcal{O}_X/U' & \xrightarrow{f/U'} & \mathcal{O}_X/U' \\ & \searrow^{\eta} \quad \nearrow_{\varepsilon} & \\ & \bigotimes (\Lambda \Sigma_i/U_i)^{(-1)^i} & \end{array}$$

is commutative, where η is the collation coming from (*) and $\varepsilon = \bigotimes \varepsilon_i^{(-1)^i}$ is built from the canonical isomorphisms $\varepsilon_i : \Lambda \mathcal{O}_X \to \mathcal{E}_X$.

The assignment $\mathcal{F} \xrightarrow{D}$ (Div $\mathcal{F}$) satisfies the following properties:

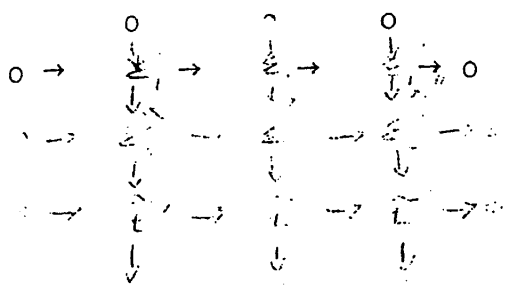
(I) It is multiplicative. That is, suppose given an exact sequence of coherent sheaves which satisfy (i), (ii) above:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

Then Div $\mathcal{F}$ = Div $\mathcal{F}'$ · Div $\mathcal{F}''$

(writing the group of Cartier Divisors multiplicatively).

Proof: The above formula, being a formula for Cartier Divisors, need only be checked at points of depth 1, where by the Auslander - Buchsbaum Theorem (1) we may find a free resolution:



But $\det \alpha = \det \alpha' \det \alpha''$, which gives (I).

(II) The support of $\mathrm{Div}(\mathcal{L})$ is contained in the support of $\mathcal{L}$. If X is normal, the supports coincide at points of depth 1.

Proof: The first statement follows from the local character of D and the fact that $\mathrm{Div}(\{\mathcal{O}\})$ is the identity element in the group of Cartier Divisors. If $x \in X$ is of depth 1 and X is normal, then $\mathcal{O}_x$ is a discrete valuation ring. We may again find a free resolution

$$0 \to E_{1,x} \xrightarrow{\alpha} E_{0,x} \to \mathcal{L} \to 0$$

(by the Auslander-Buchsbaum theorem).

where the $E_{j,x}$ are of finite and equal rank. Since $\mathcal{O}_x$ is principal we have that $\mathcal{L} = 0$ if and only if $\det \alpha$ is a unit. Bbki. Alg. Ch. VII §4, No.5, prop.4.

(III) Let $\mathcal{L}$ be a coherent sheaf on X and $f: Y \to X$ a morphism of noetherian schemes such that

$$\mathrm{Tor}_g^{\mathcal{O}_x}(f_*\mathcal{O}_y, \mathcal{L}_x) = 0$$

for all $g > 0$, $y \in Y$, $x = f(y)$.

Then: $\mathrm{Div}(f^*\mathcal{L}) = f^* \mathrm{Div}(\mathcal{L})$.

assuming both $\mathcal{L}$ and $f^*\mathcal{L}$ satisfy (i), (ii) above. (The above condition holds, e.g. if f is a flat morphism).

Proof: Straightforward vanishing of the above Tor_g's is precisely what is necessary to ensure that the inverse image of a projective resolution of $\mathcal{L}$ over $\mathcal{O}_x$ is a projective resolution of $f^*\mathcal{L}$ over $\mathcal{O}_y$. This allows us to compute local equations for $f^*\mathcal{L}$.

(IV) Let D be a Cartier Divisor and form the exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Then Div $(\mathcal{O}_D) = D$.

Proof: immediate.

(V) Let $f : X \to Y$ be a faithfully flat, finite morphism of neetherian schemes. Suppose $\mathcal{F}$ coherent on X and satisfies (i), (ii) above. Then

$$f_* \text{ Div } (\mathcal{F}) = \text{Div } (f_* \mathcal{F}).$$

Proof: Check it again at a point x of depth 1. Take a free resolution

$$0 \to \mathcal{E}_{1,x} \xrightarrow{\alpha} \mathcal{E}_{0,x} \to \mathcal{F}_x \to 0.$$

Let $y = f(x)$.

Since f is finite and faithfully flat, the natural homomorphism $f_x = \mathcal{O}_y \to \mathcal{O}_x$ is an inclusion (SGA) IV, proposition 2.6), and $\mathcal{O}_x$ is a free $\mathcal{O}_y$-module of finite rank.

A local equation for f_* Div $(\mathcal{F})$ at x may be given by

$$N_{\mathcal{O}_x/\mathcal{O}_y}(\det \alpha_x).$$

where, if $\in \mathcal{O}_x$, $N_{\mathcal{O}_x/\mathcal{O}_y}(\in)$ is defined to be the determinant of the matrix (with coefficients in $\mathcal{O}_y$) which describes multiplication by $\in$ in $\mathcal{O}_x$. (Here one uses that $\mathcal{O}_x$ is a free $\mathcal{O}_y$ - module of finite rank)

Since $(f_* \mathcal{F})_x$ is just $\mathcal{F}_x$, regarded as an $\mathcal{O}_y$-module, take the same resolution for $(f_* \mathcal{F})_x$ as above, only regarded

as an $\mathcal{O}_y$ resolution. A local equation for $\mathrm{Div}(f_* \tilde{\iota})$ at y may then be given by

$$\det_{\mathcal{O}_y}(\alpha).$$

We are reduced to demonstrating an identity in the theory of determinants.

Proposition $\{B, A\}$:

Let B be a commutative finite dimensional A-algebra (free as a module over A). Let $\tilde{T}$ be any B-endomorphism of some free B-module E of rank g. Then:

$$N_{B/A} \circ \det_B(T) = \det_A(T).$$

The determinants here refer to the expression of T as a square matrix over B or over A upon choice of a free B or A basis of E respectively.

Proof: First a series of reductions. It suffices to prove $\{B,A\}$ for

1. A an integral domain
2. A an algebraically closed field
3. and where B possesses a unique maximal ideal.

Proof of 1. Let A' be an integral domain and $\tau: A \to A$ a surjective homomorphism, (i.e. take $A' = \mathbb{Z}[\{t_a; a \in A\}]$ and $\tau(t_a) = a$.) Express B via its "structural constants"

$$B = A(\bar{X}_1, \ldots, \bar{X}_\gamma) \Big/ (f_{i,j}) \quad i,j = 1;\ldots,\gamma$$

where $f_{ij} = \bar{X}_i \bar{X}_j - \sum_k g_{ij}^k \bar{X}_k \in A(\bar{X}_1,..,\bar{X}_\gamma)$

Choose liftings $\gamma_{i,j}^k \in A'$ of the g_{ij}^k and define

$$B' = A'(\bar{X}_1,\ldots\bar{X}_\gamma) \Big/ (f'_{i,j}) \qquad i,j = 1;\ldots,\gamma$$

where $f'_{ij} = \bar{X}_i\bar{X}_j - \sum_k \gamma_{ij}^k \bar{X}_k$

B is a free module over A generated by the images of $\{\bar{X}_j\}$ and $B = B' \otimes_{A'} A$.

By the universal nature of the determinant formula we have:

Lemma 1

(i) $\sigma \det_{B'}(T') = \det_B(\sigma T')$

(ii) $\sigma \det_{A'}(T') = \det_A(\sigma T')$

(iii) $\sigma N_{B'/A'}(b') = N_{B/A}(\sigma b')$

for T' any square matrix over B' and $b' \in B'$.

We have therefore shown $\{B', A'\}$ implies $\{B, A\}$, and consequently (1).

Proof of 2: Imbed the integral domain A in an algebraically closed field K. Then $\{B \otimes_A K, K\}$ implies $\{B, A\}$.

Proof of 3: Suppose B decomposes over A, $B = B_1 \oplus B_2$, corresponding to the idempotent, decomposition $1 = \xi_1 \oplus \xi_2$.

Lemma 2:

(i) $\det_B(T) = \det_{B_1}(\xi_1 T) + \det_{B_2}(\xi_2 T)$

(ii) $\det_A(T) = \det_A(\xi_1 T) \cdot \det_A(\xi_2 T)$

where $\xi_j T$ is regarded as a matrix over B_j and:

(iii) $N_{B/A}(b) = N_{B_1/A}(\xi_1 b) \cdot N_{B_2/A}(\xi_2 b)$

Proof: (i) e.g. Use the formula for the determinant.

(ii) $T = (\xi_1 T + \xi_2 I) \circ (\xi_1 I + \xi_2 T)$

and $\det_A (\xi_1 T \otimes \xi_2 I) = \det_A (\xi, T)$.

The best basis (to see this last fact) is one which animates the splitting $E = \xi_1 E + \xi_2 E$.

(iii) The "matrix" $b \in B$ over A is the diagonal matrix of blocks

$$\begin{pmatrix} \xi_1 b & 0 \\ 0 & \xi_2 b \end{pmatrix}$$

Lemma 2 shows that $\{B_1, A\}$ and $\{B_2, A\}$ imply $\{B, A\}$. Since A is an algebraically closed field, we are led to the local ring situation, q.e.d.

Now let $m \subset B$ be the unique maximal ideal, $B \xrightarrow{\varphi} B/m \simeq A$, and choose a basis

$$\{1 = b_0, b_1, \ldots, b_r\} \subset B$$

as vector space over A, such that reading from right to left they form bases for the successive powers $m^j \subset B$ in order of decreasing j.

Represent the free module E as $V \otimes_A B$ where V is a vector space over A of dimension q, and consider the sequence of subspaces $E_j = b_j V \otimes b_{j+1} V \otimes \ldots \otimes b_r V$ forming a decreasing sequence of vector spaces over A.

$$\{0\} = E_{r+1} \subset \ldots \subset E_0 = E.$$

The B-homomorphism $T: E \to E$ will leave the above flag invariant and if we convene that

$$\varphi : B \to B/m \simeq A \subset B$$

be defined on matrices co-ordinate-wise, we have that T induces

the matrix $\varphi(T)$ on any E_j/E_{j+1} $(j=0, \ldots, r)$.

Thus we have:

$$\text{(i)} \quad \det_A (T) = \left\{ \det_A \varphi(T) \right\}^{r+1}$$

On the other hand

$$\text{(ii)} \quad N_{B/A} (b) = \varphi(b)^{r+1}$$

since a matrix for b over A may be given as triangular, with $\varphi(b)$ along the diagonal,

$$\text{(iii)} \quad \varphi \det_B (T) = \det_A (\varphi T)$$

since φ is a ring homomorphism.

Proposition $\{B,A\}$ therefore follows and consequently $\underline{V}$.

Converting terminology from noetherian schemes to noetherian rings, A we consider modules M of finite type over A satisfying:

(i) M is a <u>torsion</u> module over A (i.e. its associated sheaf $\widetilde{M}$ over Spec A is torsion). This condition says that $M \otimes_A q(A) = \{0\}$ where q (A) is the total quotient ring of A (See [A-S], Vol.I, Ch.I, 19, Theorem 17)

(ii) M_p is of finite projective dimension over A_p for all prime ideals p of A.

A Cartier divisor of X = Spec A (i.e. a non-vanishing section in $\Gamma(X, \mathcal{O}_X/\mathcal{O}_X^*)$) may be identified with an element of A^0 (a non-zero element of A) modulo the group g units of A, $U(A)$.

Thus, if M satisfies (i), (ii) above, define

$$\det_A (M) \quad \in \quad A^0/U(A)$$

to be the Cartier Divisor, Div $(\widetilde{M})$.

We may, also, if we wish, interpret $\det_A (M)$ as a non-zero principal ideal of A.

Definition: The module M has a determinant (over A) if M is of finite type over A and satisfies (i), (ii) above.

For convenience, we restate an affine corollary of property V.

V affine: Suppose $i:A \to B$ is a ring homomorphism where B is a free A-module of finite type. Then the norm $N_{B/A} : B \to A$ is defined, and we have the following relation:

$$\det_A (M) \quad = \quad N_{B/A} \quad \det_B \quad (M).$$

where it is understood that if the right-hand side is defined it follows that the left is also defined and the above equality results.

2, Modules over F [π].

F will denote either the integers, $\mathbb{Z}$, or the integers localized at a fixed prime p,

$$\mathbb{Z}_{(p)} = \{ a/b \mid a, b \in \mathbb{Z}, \ (p, b) = 1 \}$$

Set $A = F[\pi] = F[t, t^{-1}]$. Then F is principal and has the property that for any module X of finite type over F, Tor (X) is finite. The natural inclusion $\Lambda \subset A$ is flat. A module M over A may be regarded as a module over F with a given automorphism γ: M → M which describes the operation of $t \in F[\pi]$ on M. γ is called the fundamental automorphism of the module M.

Since Spec A is non-singular its local rings are all regular and we have:

Proposition 1: Any A-module M of finite type satisfies condition (ii) of §1. Thus M has a determinant if and only if M is a torsion module over A.

Proof: This follows from the criterion of Serre for regularity of a local ring (Thm. 3, § 5, (16).

We will need an elementary consequence of the noetherian decomposition theorem.

Lemma 3: Any ideal $\mathfrak{a} \subset \Lambda$ may be (uniquely) expressed as

$$\mathfrak{a} = (f) \cap \mathfrak{q}_0$$

where (f) is the smallest principal ideal containing $\mathfrak{a}$ and

and Supp $(A/\mathfrak{a}_0)$ contains only points in Spec A of depth 2.

The neotherian decomposition theorem will factor $\mathfrak{a}$ into primary ideals associated to primes of depth 1 and 2. Let $\mathfrak{a}_0$ denote the intersection of all the primary ideals which occur in the decomposition and are associated to primes of depth 2. Recall that a prime ideal of depth 1 is minimal, hence principal, since A is a unique factorization domain (2.6), Ch IV §14 top page 238). Consequently all primary ideals associated to primes of depth 1 are principal (loc. cit. Ch III, §9, Note of p.155).

We may then write

$$\mathfrak{a} = (f_1^{n_1}) \cap \ldots \cap (f_s^{n_s}) \cap \mathfrak{a}_0$$

where the f_i are non-associated irreducible elements. Since A is a Unique Factorization Domain:

$$(f_1^{n_1}) \cap \ldots \cap (f_s^{n_s}) = (f_1^{n_1} \ldots f_s^{n_s}) = (f).$$

Finally, suppose $\mathfrak{a} \subset (g)$ for some principal ideal g. Then the associated primes to (g) in its noetherian decomposition must be among the (f_i) i = 1, ..., s.

Thus

$$(g) = \bigcap_{i=1}^{s} (f_i^{m_i})$$

Suppose some exponent, say m_1 is greater than n_1. Then

$$\mathfrak{a} = (f_1^{m_1}) \cap (f_2^{n_2}) \cap \ldots \cap (f_s^{n_s}) \cap \mathfrak{a}$$

contradicting uniqueness of the primary decomposition of $\mathfrak{a}$. Therefore $m_i \leq n_i$. We conclude that $(f) \subset (g)$. q.e.d.

Proposition 2: Let $M = A/\mathfrak{a}$ have a determinant $\Delta(t)$. Then $(\Delta(t)$ is the smallest principal ideal containing $\mathfrak{a}$.

Proof: By Lemma 3, decompose

$$\mathfrak{a} = (f)_n \mathfrak{a}_0.$$

Then proposition 2 will follow from the assertion:

$$\det (A/(f)_n \mathfrak{a}_0) = \det (A/(f)).$$

But to check this it suffices to check it only at points x of depth 1. At any such point x, however, $\mathfrak{a}_{0,x} = A_x$.

Let $\mathcal{F}$ be the collection of finite A-modules.

Proposition 3: Let M have a determinant over A. Then $\det_A (M)$ is a unit if and only if $M \in \mathcal{F}$. ($\det_A$ is therefore a mod $\mathcal{F}$ invariant of modules)

Proof: Spec A is normal. By property (II), of §1, $\det_A (M)$ is a unit if and only if supp M consists exactly in points of depth 2. Proposition 3 then follows from theorems 1, 2 of Bourbaki, Alg.Comm. Ch IV §1, no.4, after one observes that any prime of depth 2 in A has finite residue field.

3. Monic Modules

Again let F denote either $\mathbb{Z}$, or the integers localized at p, $\mathbb{Z}_{(p)}$.

Let $A = F[\pi] = F[t, t^{-1}]A_\gamma = A/(1-t^\gamma)$ and let M be an A-module possessing a determinant. The determinant of M may then be interpreted as a polynomial

$$\Delta(t) \in F(\pi)^0 / \{U(F) \cdot t^n\}$$

$n \in \mathbb{Z}$. This may be normalized (by multiplication with a suitable power of t, to achieve a non-zero constant term and no negative exponents).

Definition: The A-module M is monic if $\Delta(t)$ is a monic polynonical in F[t] whose constant term is a unit.
(This amounts to the requirement that Δ is monic when expressed both in t and t^{-1})

Theorem 1: If M is a monic A-module, then M is of finite type over F.

Proof: By induction on the number of generators of M. Suppose M monogenic, $M = F[\pi]/\mathcal{A}$. Then $\mathcal{A} \subset (\det M)$ by , and we have the exact sequence:

$$0 \to (\Delta(t)) / \mathcal{A} \to M \to A/(\Delta(t)) \to 0$$

Lemma 4: Consider an exact sequence of A-modules

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

If M_1 is a torsion module of finite type, so are M_0 and M_2.

Proof: The finite type statement follows trivially for M_2 and for M_o by noetherian-ness of A. The torsion statement follows because q(A) is A-flat (A being an integral domain).

Consequently, after Proposition 1, §2 $(\Delta(t))/\sigma$ has a determinant and:

$$\det M = \det (A/(\Delta(t)) \cdot \det (\Delta(t))/\sigma).$$

by property I of §1. By Proposition 2, §2, $\det (A/(\Delta(t)) = \Delta(t)$ and we have $\det (\Delta(t)/\sigma) = 1$; i.e. $\Delta(t)/\sigma$ is finite after Proposition 3, §2. Therefore, in the light of the exact sequence (*), to show A/σ of finite type over F it suffices to show:

Lemma 5: $\left\{\right.$ $A/(\Delta(t))$ is of finite type over F.

Let S = deg $\Delta(t)$ and consider the sub-module $M_o \subset A/(\Delta(t))$ over F, generated by the basis $\{1, t, \ldots, t^{s-1}\}$. Since $\Delta(t)$ is monic, after multiplication by a unit we may write it as

$$\Delta(t) = t^s - \sum_{i=o}^{s-1} \alpha_i t^i$$

Now assume by induction that

$t^j \in M_o$ for $0 \leq j \leq N-1$. Since

$$t^N = \sum_{i=o}^{S-1} \alpha_i t^{(N-S)+i}$$

in $A/(\Delta(t))$, we get $t^N \in M_o$ for all $N \geq 0$. Using monicity of $\Delta(t^{-1})$ we get $t^N \in M_o$ for $N \leq 0$. Consequently, $M_o = A/(\Delta(t))$ q.e.d.

We have thus proved the theorem for monogenic monic modules M.

for

Now assume it proved monic modules generated over Λ by $(\gamma-1)$ elements or less. Let M be generated over Λ by γ elements and let $M_o \subset M$ be the sub-module generated by all but the last.

We have an exact sequence:

(**) $$0 \to M_o \to M \to M/M_o \to 0, \text{ with } M/M_o \text{ monogenic.}$$

By lemma 4, all three modules have determinants, and

$$\det M = \det M_o \cdot \det (M/M_o)$$

so they are all monic. Both M_o and M/M_o are of finite type over F; M_o by our inductive assumptions; M/M_o since it is monic and monogenic. It follows from (**) that M is of finite type over F.

Theorem 2. If M is monic and of projective dimension 1, then M is free over F of finite rank $\delta = \deg \Delta(t)$. Its fundamental automorphism γ may then be identified (up to equivalence over F) with a $\delta \times \delta$ matrix over F, whose characteristic polynominal may be identified with $\Delta(t)$.

Proof: The resolution:

$$0 \to \Lambda \to \Lambda \to \Lambda_\gamma \to 0$$

yields: $$0 \to \operatorname{Tor}_1^{\Lambda}(\Lambda_\gamma, M) \to M \xrightarrow{1-\gamma} M \to M \otimes_{\Lambda} \Lambda_\gamma \to 0$$

which allows us to identify:

$$\operatorname{Tor}_1^{\Lambda}(\Lambda_\gamma, M) = M^{\pi_1}.$$

Since M is of projective dimension 1, we also have a resolution $$0 \to P_1 \to P_0 \to M \to 0$$

yielding: $$0 \to \mathrm{Tor}_1^{A}(M, A_\sim) \to P_1 \otimes A_\sim$$

But $P_1 \otimes A_\sim$ is torsion-free over F. Using commutatively of Tor_1 we get that M^{π_ν} is torsion free over F. Since M is of finite type, its torsion module Tor (M)$\subset$ M is finite. Moreover, the fundamental automorphism γ must preserve Tor (M). Thus γ determines an automorphism $\bar{\gamma}$ of Tor (M). But Tor (M) being a finite group, its group G of automorphisms is again finite of order, say, g. Thus $\bar{\gamma}^g = 1$. Taking $\nu = g$, we see that γ^ν is the identity on Tor (M). Therefore:

$$\mathrm{Tor}\,(M) \subset M^{\pi^\nu}.$$

Since M^{π^ν} is torsion-free, it follows that

$$\mathrm{Tor}\,(M) = 0.$$

Since F is principal, M is therefore free over F of finite rank.

Consider the following sequence

$$0 \to M \otimes_F F[\pi] \xrightarrow{\psi} M \otimes_F F[\pi] \xrightarrow{\varphi} M \to 0$$

where ψ, φ are the $F[\pi]$ homomorphisms determined as follows:

$$\begin{cases} \psi & (m \otimes t^n) = \gamma^n(m) \\ \varphi & (m \otimes t^n) = m \otimes t^{n+1} - \gamma(m) \otimes t^n. \end{cases}$$

The sequence is exact since ker φ is generated as a $F[\pi]$-module by elements of form $(m \otimes t - \gamma(m) \otimes 1)$. It is a free resolution of finite type since M is free over F (of finite type).

Thus we may compute $\Delta(t)$ from this resolution, and the full statement of the theorem follows.

Note that we have also:

Proposition 4: Let M be of projective dimension 1, and possess a determinant over A.

Then:

$$\mathrm{Tor}_1^A(M, A_\nu) \simeq M^{\pi_\nu}$$

is a sub-module of a free module over A_ν.

§4. The Alexander Polynomial

Let $K \subset S^3$ be a polygonal one-sphere knot and consider M the bounded complementary manifold. We may take M as a connected finite simplicial complex, hence a connected finite CW-complex. In the usual manner, after a series of elementary contractions of M into its 2-skeleton, and collapsing a maximal subtree to a single vertex, we may represent M (up to homotopy type) by a 2-dimensional finite CW-complex F, possessing a single 0-cell.

Denote the infinite cyclic group, $H_1(M, \mathbb{Z})$ by π, and choose a generator $t \in \pi$. If $\tilde{M}$ is the maximal connected abelian covering bundle of M, π acts freely on M and $M = \tilde{M}/\pi$.

Let us replace $\tilde{M}$ by the maximal connected abelian covering bundle of F. $\tilde{F}$ inherits a CW-structure from F, and the group π permutes the cells of F (principally). Let C, $\tilde{C}$ denote the chain complexes (integer coefficients) associated to the CW-structures $F, \tilde{F}$. The action of the group π allows us to consider $\tilde{C}$ as a chain complex of free $\mathbb{Z}[\pi]$-modules, of finite rank. Consequently, $H_1(\tilde{M}) = H_1(\tilde{C})$ is a module of finite type over $\Lambda = \mathbb{Z}[\pi]$.

Evidently $H_1(\tilde{M})$, as an abelian group is the abelianization of the commutator subgroup of the knot group $\pi_1(M)$.

The main theorem concerning the structure of $H_1(\tilde{M})$ as a Λ-module is the following:

Theorem 3: (i) $H_1(\tilde{M})$ is a torsion module of finite type over Λ.

(ii) It is of projective dimension 1.

(iii) Its determinant $\Delta(t)$ is the classical Alexander Polynomial of the knot K.

(iv) The natural map $p_r : \tilde{M} \to S_r$ (where S_r is the branched cyclic covering of the knot K of order r; See §5) induces an isomorphism,
$H_1(\tilde{M}) \otimes_{\Lambda} \Lambda_r \xrightarrow{\simeq} H_1(S_r)$
as Λ_r-modules, p_r^* where $\Lambda_r = \Lambda/(1-t^r)$.

Remarks: (1) Assertion (iv) will be proved in §5, (It is exactly proposition 5), where branched cyclic coverings are discussed.

(2) Theorem 3 suggests the following definitions (in the light of §3).

The norm of a knot K is the constant term of its Alexander Polynomial $\Delta(t)$.

$$\text{Norm}(K) = \Delta(0) \in \mathbb{Z}/\{\pm 1\}$$

A knot is monic if its norm is a unit.

Theorem 4: Let K be a knot and p a prime which does not divide the norm of K. Then $H_1(\tilde{M}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a free module of finite rank over $\mathbb{Z}_{(p)}$. Its rank is equal to the degree $\delta = 2g$ of the Alexander Polynomial, $\Delta(t)$ of K. $\Delta(t)$ is the characteristic polynomial of the fundamental automorphism of $H_1(\tilde{M})_{(p)} = H_1(\tilde{M}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Theorem 5: If K is monic, then $H_1(\widetilde{M})$ is free abelian of rank equal to $\delta = 2g$. The fundamental automorphism γ may then be regarded (up to integral equivalence) as a matrix:

$$\gamma \in SL(\delta, \mathbb{Z})$$

whose characteristic polynomial is the Alexander Polynomial of K.

Note: For monic knots, then, one might try to delve into the second commutation group by considering $\breve{M}$, the maximal abelian unramified covering bundle over $\widetilde{M}$. Then $H_1(\breve{M})$ is a module of finite type over the integral group ring of $H_1(\widetilde{M})$ which is just:

$$\Lambda^\delta = \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_\delta, x_\delta^{-1}].$$

For knots-of-rotation, $H_1(\breve{M})$ is free of rank (δ_2) over this ring. Can one expect a similar rank (over the generic point of Spec (Λ^δ)) for all monic knots?

Proof of Theorems 4 and 5: Letting F stand for either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}$, in either circumstance we have that $H_1(\widetilde{M}) \otimes_{\mathbb{Z}} F$ is monic, as a module over $F[\pi]$. This follows because:

(a) The inclusion $\mathbb{Z}[\pi] \hookrightarrow F[\pi]$ is flat. Consequently property III of §1 applies, yielding

$$\det_{F[\pi]}(H_1(\widetilde{M}) \otimes_{\mathbb{Z}} F) = i\,\Delta(t).$$

(b) $\Delta(t)$ is a symmetric polynomial. Consequently since $i\,\Delta(0)$ is a unit in F, so is the leading coefficient.

Theorem 3 gives us that it is of projective dimension 1, hence Theorem 2 of §3 applies, q.e.d.

A special case of a monic knot is one for which the zeroes of the Alexander Polynomial are of absolute value 1. (One may call them cyclotomic knots). It will follow from Theorem 2 and duality that the fundamental automorphism is periodic in this case i.e. $\gamma^r = 1$ for some r). This "animates" the phenomenon of periodicity tabulated in (4).

Proof of Theorem 3: We first prove (i):

Let $\beta : \mathbb{Z}[\pi] \to Q(t)$ denote the imbedding of $\mathbb{Z}[\pi]$ in its field of fractions. The assertion is that

$$H_1(\widetilde{M}) \underset{\mathbb{Z}[\pi]}{\otimes} Q(t) = 0$$

Lemma 6: $\widetilde{C} \underset{\mathbb{Z}[\pi]}{\otimes} Q(t)$ is acyclic

Proof: See ((13), Lemma 4 and page 145). Theorem 3 will then follow from the general assertion:

Lemma 7: Let $\widetilde{C}$ be a free chain complex over $\Lambda = \mathbb{Z}[\pi]$, such that $H_0(\widetilde{C}) \simeq \mathbb{Z}$ (i.e. $\widetilde{C}$ is connected). Let $\gamma : \Lambda \to B$ be a ring homomorphism.

Then the natural map

$$H_1(\widetilde{C}) \underset{\Lambda}{\otimes} B \xrightarrow{\iota} H_1(\widetilde{C} \underset{\Lambda}{\otimes} B)$$

is injective.

Proof: Since $\widetilde{C}$ is free, we may use the spectral sequence: (See theorem 5.5.1 page 102, (6))

$$E^2_{p,q} = \mathrm{Tor}^{\Lambda}_p (H_q(\tilde{C}), B) \Rightarrow H_{p+q}(\tilde{C} \otimes_{\Lambda} B).$$

The natural map i factors:

$$\begin{array}{ccc} E^2_{2,0} & & \\ \downarrow d^2 & & \\ E^2_{0,1} & \to & H_1(\tilde{C}) \otimes_{\Lambda} B \\ \downarrow & & \downarrow i \\ E^3_{0,1} & & \\ \downarrow & & \\ E^\infty_{0,1} & \subseteq & H_1(\tilde{C} \otimes_{\Lambda} B) \end{array}$$

and the vertical sequence is exact at the point $E^2_{0,1}$. Lemma 7 will follow if $E^2_{2,0} = 0$.

But:- $E^2_{2,0} = \mathrm{Tor}^{\Lambda}_2 (H_0(\tilde{C}), B) \simeq \mathrm{Tor}^{\Lambda}_2(\mathbb{Z}, B) = 0$,

the middle isomorphism comes by hypothesis and the last since $\mathbb{Z}$ has projective dimension 1 over $\Lambda = \mathbb{Z}[t]$. q.e.d.

Proof of (ii) and (iii):

Let us denote the cells of F of dimension q by C^q_j $(j = 1, \ldots \rho_q)$. Suppose $\rho_2 = n$. Then since $\rho_0 = 1$ and the euler characteristic of F is zero, we have: $\rho_1 = n + 1$. Take e^1_1 to be such that $\partial e^1_1 = (t-1) \cdot C_0$ (Compare (13)) and regard the basis:

$$-e^2_1, \ldots e^2_n, e^1_1, e^1_{n+1}$$

as free bases of the Λ-modules $\tilde{C}_2$ and $\tilde{C}_1$ respectively. In terms of these bases, the boundary homomorphism

$$\partial : \tilde{C}_2 \to \tilde{C}_1$$

may be written as an $n \times n+1$ matrix with coefficients in Λ. (The Alexander Matrix of this presentation $\tilde{F}$). $||\alpha_{ij}(t)||$.

Consider the following commutative diagram:

$$0 \to e_1^1 \cdot \Lambda \to \tilde{C}_1 \xrightarrow{\pi} \tilde{C}_1^* \to 0$$
$$\qquad\qquad\quad C \uparrow \quad \nearrow f \quad \uparrow \alpha$$
$$\tilde{Z}_1 \leftarrow \tilde{C}_2 \leftarrow 0$$
$$\qquad\quad \tilde{\partial}_2$$

Hence $e_1^1 \Lambda$ is the submodule of $\tilde{C}_1$ generated by e_1^1, $\tilde{C}_1^*$ is the free Λ-module generated by: $e_2^1, \ldots, e_{n+1}$, π the natural projection and Z_1 is the submodule of 1-cycles.

Lemma 8: p is an isomorphism. Consequently $H_1(\tilde{M})$ admits the free Λ-resolution:

$$* \qquad 0 \to \tilde{C}_2 \xrightarrow{\alpha} \tilde{C}_1^* \to H_1(\tilde{M}) \to 0.$$

Lemma 8 will then prove (ii). Also (iii) will follow, since the determinant of $H_1(\tilde{M})$ may then be computed from the Λ-resolution (*):

$$\det_\Lambda \{H_1(\tilde{M})\} = \det \alpha = \det \| \alpha_{i,j}(t) \|_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}}$$

But the latter determinant is the classical Alexander Polynomial. (Compare (13)).

Proof of Lemma 8: First

$$\tilde{Z}_1 \cap \{ e_1^1 \Lambda \} = \{ 0 \}$$

because $\partial(e_1^1 \lambda) = \lambda(t-1) \cdot e_1^0$, which is zero only if $\lambda = 0$. Therefore p is injective. Now let $C^* = \sum_{i=2}^{n+1} \lambda_i \cdot e_2^1 \in \tilde{C}_1^*$. Since $\tilde{\partial}_1(C^*) \in (t-1) \cdot \tilde{C}_0$, we have $\tilde{\partial}_1(C^*) = (t-1) \cdot \lambda e_1^0$ for some $\lambda \in \Lambda$. Let $c = c^* - \lambda e_1^1$. Then $c \in \tilde{Z}_1$ and $p(c) = c^*$. Therefore p is surjective.

5. Branched Cyclic Coverings

Let M_ν be the ν^{th} cyclic covering bundle over M (i.e. the bundle associated to the subgroup of $\pi_1(M)$ which is the universe image of $\pi^\nu \subset \pi$ under the natural map

$$0 \to [\pi_1(M), \pi_1(M)] \to \pi_1(M) \to \overline{\pi} \to 0$$

Here $\pi^\nu = (t^{\lambda\nu} \in \pi \mid \lambda \in \mathbb{Z})$.

Then M_ν may be "completed" to a 'covering space with singularities' S_ν, over S^3, branched exactly at the knot K. S_ν is called the branched cyclic covering of K, of degree ν. See (3). The Mayer-Vietoris sequence for homology yields the following exact sequence:

$$H_1(S^1)_{\alpha_\nu} \to H_1(M_\nu) \xrightarrow{i} H_1(S_\nu) \to 0$$

where α represents a standard loop in M_ν, "transversal to the knot K".

The CW-structure F, representing the homotopy type of M will lift to a CW-structure F_ν yielding a free chain complex C_ν over the ring $\Lambda_\nu = \Lambda/(1-t^\nu)$. One has the relation

$$C_\nu = \widetilde{C} \otimes_\Lambda \Lambda_\nu$$

and the natural maps $p_{\mu,\nu} : M_{\mu\nu} \to M$ give rise to commutative diagrams:

Lemma 9:

a)
$$\begin{array}{ccccc} H_1(M_{\mu\nu}) & = & H_1(C_{\mu\nu}) & \simeq & H_1(\widetilde{C} \otimes_\Lambda \Lambda_{\mu\nu}) \\ \downarrow p_{\mu,\nu} & & & & \downarrow \{\otimes_\nu \Lambda_\nu\}_* \\ H_1(M_\nu) & = & H_1(C_\nu) & \simeq & H_1(\widetilde{C} \otimes_\Lambda \Lambda_\nu) \end{array}$$

(b)
$$\begin{array}{ccc} H_1(\tilde{M}) & = & H_1(\tilde{C}) \\ \downarrow p_\nu & & \downarrow \{\alpha_\Lambda \wedge \nu\}_* \\ H_1(M_\nu) & \simeq & H_1(C \otimes_\Lambda \Lambda_\nu) \end{array}$$

Using the spectral sequence,

$$\mathrm{Tor}_p^\Lambda (H_q(\tilde{C}), \Lambda_\nu) \Rightarrow H_{p+q}(\tilde{C} \otimes_\Lambda \Lambda_\nu)$$

again, as in §4, lemma 7, and that $E_{2,0} = \mathrm{Tor}_2^\Lambda (\mathbb{Z}, \Lambda_\nu) = 0$, we get an exact sequence:

$$0 \to H_1(\tilde{M}) \otimes_\Lambda \Lambda_\nu \to H_1(\tilde{C} \otimes_\Lambda \Lambda_\nu) \to \mathrm{Tor}_1^\Lambda (H_0(\tilde{C}), \Lambda_\nu) \to 0$$

Using the resolution

$$\Lambda: \quad 0 \to \Lambda \xrightarrow{(1-t)} \Lambda \to \Lambda_\nu \to 0$$

one sees that $\mathrm{Tor}_1^\Lambda (H_0(\tilde{C}), \Lambda_\nu) \subset H_0(\tilde{C}) \otimes \Lambda$.

is the infinite cyclic subgroup generated by

$$(\epsilon_1^0) \otimes \left(\sum_{j=0}^{\nu-1} t^j\right) \in H_0(\tilde{C}) \otimes \Lambda$$

where (ϵ_1^0) is the homology class of C_1^0.

Combining exact sequences, we get:

$$\begin{array}{ccccccccc} & & & & H_1(S^1) & & & & \\ & & & & \downarrow \alpha_\nu & \searrow \Psi_\nu & & & \\ \bullet \to & H_1(\tilde{M}) \otimes_\Lambda \Lambda_\nu & \to & H_1(\tilde{C} \otimes \Lambda_\nu) & \xrightarrow{\varphi} & \mathrm{Tor}_1^\Lambda (H_0(\tilde{C}), \Lambda_\nu) & \to & 0 & \\ & \searrow \varphi_\nu & & \downarrow & & & & & \\ & & & H_1(S_\nu) & & & & & \\ & & & \downarrow & & & & & \\ & & & 0 & & & & & \end{array}$$

where the diagonal arrows are the composites.

Proposition 5: Ψ_ν (hence also φ_ν) is an isomorphism (which proves assertion (iv) of Theorem 3, §4.

By diagram-chasing, Ψ_ν is an isomorphism if ψ_ν is. Since both $H_1(S^1)$ and $\mathrm{Tor}_1^{\Lambda}(H_0(\tilde{C}),\Lambda_\nu)$ are infinite cyclic, it suffices to show that a generator of $H_1(S^1)$ is sent by ψ_ν to a generator of $\mathrm{Tor}_1^{\Lambda}(H_0(\tilde{C}),\Lambda_\nu)$. By choice of e_1^1, $\varkappa_\nu$ sends a generator of $H_1(S^1)$ to

$$\sum_{j=0}^{\nu-1} t_j \, e_1^1 \in \tilde{C}_\nu = \tilde{C} \otimes_{\Lambda} \Lambda_\nu .$$

The homomorphism $\mathcal{E}$ is the natural "step-ladder" map of the double complex $\tilde{C} \otimes \Lambda$. That is, if $h \in H_1(\tilde{C} \otimes \Lambda_\nu)$ is a class, $\mathcal{E}(h)$ may be obtained by choosing a cycle $z \in \tilde{C}_1 \otimes \Lambda_\nu$ which represents h, and ascending the following staircase:

$$\begin{array}{ccccccc}
 & & & & \mathrm{Tor}_1(H_0,\Lambda_\nu) & & \\
 & & \tilde{C}_0 \otimes \Lambda & \to & H_0 \otimes \Lambda & \to & 0 \\
\tilde{C}_1 \otimes \Lambda & \xrightarrow{\partial} & \tilde{C}_0 \otimes_{\Lambda} \Lambda & \to & H_0 \otimes_{\Lambda} \Lambda & \to & 0 \\
\tilde{C}_1 \otimes \Lambda_\nu & \to & \tilde{C}_0 \otimes_{\Lambda} \Lambda_\nu & \to & H_0 \otimes_{\Lambda} \Lambda_\nu & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

(Here $H_0 = H_0(\tilde{C})$)

Thus, if $\bar{z} \in \tilde{C}_1 \otimes_{\Lambda} \Lambda$ is a class mapping onto z we may represent $\mathcal{E}(h)$ by the following class in $H_0 \otimes_{\Lambda} \Lambda$:

$$[1/(1-t) \cdot \partial \bar{z}] \in H_0 \otimes_{\Lambda} \Lambda .$$

Thus for $z = \sum_0^{\nu-1} t^j \, e_1^1$, take

$$\bar{z} = \sum_0^{\nu-1} t^j \, e_1^1 \in \tilde{C}_1 \otimes \Lambda, \text{ and}$$

$$\partial \bar{z} = (t-1) \cdot \left(\sum^{\nu-1} t^j\right) \cdot e_1^0 \in \tilde{C}_0 \otimes \Lambda$$

which gives us:

$$[1/(1-t) \, \partial \bar{z}] = \left(\sum^{\nu-1} t^j\right) \cdot [e_1^0] .$$

q.e.d.

As a complement to Proposition 5, we have:

Proposition 6: For all integers μ, ν the diagrams:

$$\begin{array}{ccc} H_1(\tilde{M}) \otimes_\Lambda \Lambda_{\mu,\nu} & \xrightarrow{\mathcal{G}_{\mu,\nu}} & H_1(S_{\mu,\nu}) \\ \downarrow & & \downarrow \\ H_1(\tilde{M}) \otimes_\Lambda \Lambda_\nu & \xrightarrow{\mathcal{G}_\nu} & H_1(S_\nu) \end{array}$$

are commutative, where the vertical homomorphisms are induced from the natural maps $\rho_{\mu,\nu} : S_{\mu,\nu} \longrightarrow S_\nu$, $\lambda_{\mu,\nu} : \Lambda_{\mu,\nu} \longrightarrow \Lambda_\nu$.

Proof: The $\mathcal{G}$'s are the composites

$$\begin{array}{ccccccc} H_1(\tilde{M}) \otimes \Lambda_{\mu,\nu} & \longrightarrow & H_1(\tilde{C} \otimes_\Lambda \Lambda_{\mu,\nu}) & \cong & H_1(M_{\mu,\nu}) & \longrightarrow & H_1(S_{\mu,\nu}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_1(\tilde{M}) \otimes \Lambda_\nu & \longrightarrow & H_1(\tilde{C} \otimes_\Lambda \Lambda_\nu) & \cong & H_1(M_\nu) & \longrightarrow & H_1(S_\nu) \end{array}$$

The right-hand square is commutative since it is induced by a commutative square of continuous maps. The left-hand square is commutative by naturality of the Spectral Sequence from which it comes, and the middle square is commutative by lemma 9.

Corollary 1: The induced homomorphism

$$\rho_{\mu,\nu} : H_1(S_{\mu,\nu}) \longrightarrow H_1(S_\nu)$$

is always surjective (i.e. for all μ, ν). (Compare Th.4 of (4)).

Corollary 2: $H_1(\tilde{M}) \otimes_\Lambda \mathbb{Z} = \{0\}$

Proof: $H_1(\tilde{M}) \otimes_\Lambda \mathbb{Z} \simeq H_1(S_1) = 0$

since S_1 is the 3-sphere.

Let $\Lambda \xrightarrow{\varepsilon_\nu} \Lambda_\nu$ denote the surjective map and set $\zeta_\nu = \varepsilon_\nu(t)$.

Corollary 3:

If $H_1(\tilde{M})\otimes_\Lambda \Lambda_\nu$ is a torsion module over Λ_ν, then

(i) $\mathrm{Tor}_1^\Lambda(H_1(\tilde{M}), \Lambda_\nu) \simeq H_1(\tilde{M})^{\pi_\nu} = 0$

(ii) $\det_{\Lambda_\nu}(H_1(\tilde{M})\otimes_\Lambda \Lambda_\nu) = \Delta(\zeta_\nu)$

Proof: (i) Proposition 4 of §3 shows that we may identify the above two modules and they are both contained in free modules over Λ_ν. The sequence

$$0 \to \mathrm{Tor}_1^\Lambda(H_1(\tilde{M}), \Lambda_\nu) \to P_1\otimes\Lambda_\nu \to P_0\otimes\Lambda_\nu \to H_1(\tilde{M})\otimes_\Lambda\Lambda_\nu \to 0$$

where $P_1\otimes\Lambda_\nu$, $P_0\otimes\Lambda_\nu$ are both free over Λ_ν of equal (and finite) rank, shows that if $H_1(\tilde{M})\otimes_\Lambda\Lambda_\nu$ is a torsion module, so is $\mathrm{Tor}_1^\Lambda(H_1(\tilde{M}), \Lambda_\nu)$.

Lemma 10: Consider X, a module over a commutative ring P with unit which is both a torsion module and contained in a free module over P. Then $X = \{0\}$.

Lemma 10 will conclude the demonstration of (i), which gives us that $H_1(\tilde{M})\otimes_\Lambda\Lambda_\nu$ is of projective dimension 1 (making use of (i) plus the above exact sequence) so that $\det_\Lambda(H_1(\tilde{M})\otimes_\Lambda\Lambda_\nu)$ is defined and property III of §1, then applies to give us (ii).

To prove Lemma 10: Suppose $X \subset F$ and F free over P. Then $F \subset F\otimes_P q(P)$, and the natural diagram

$$\begin{array}{ccc} F & \subset & F\otimes_P q(P) \\ \cup & & \uparrow \\ X & \xrightarrow{i} & X\otimes_P q(P) \end{array}$$

shows that i is an inclusion q.e.d.

Corollary 4: If $H_1(\tilde{M})\otimes_\Lambda \Lambda_r$ is a finite group, its order k_r is given by:

$$k_r = [H_1(\tilde{M})\otimes_\Lambda \Lambda_r \quad 1] = \det_{\mathbb{Z}} \{H_1(\tilde{M})\otimes_\Lambda \Lambda_r\}$$
$$= \left| \prod_{\zeta = r^{th} \text{ roots of } 1} \Delta(\zeta) \right|$$

(Remark: This is (6.3) of (5))

Clearly, for finite abelian groups, G, $[G:1] = \det_{\mathbb{Z}}(G)$. If $H_1(\tilde{M})\otimes\Lambda_r$ is a finite group it is a torsion Λ_1-module and Cor.6 (i) applies. Using property 5 of we have:

$$\det_{\mathbb{Z}} \{H_1(\tilde{M})\otimes_\Lambda \Lambda_r\} = N_{\Lambda_r/\mathbb{Z}} \{\Delta(\zeta_r)\}$$
$$= \prod_{j=0}^{r-1} (\Delta(\zeta_r^j))$$

Denote by $\bar{\zeta}_r$ a primitive r^{th} root of 1 and consider the commutative diagram

$$\Lambda_r \xrightarrow{\eta} \mathbb{Z}[\zeta, \bar{\zeta}_r]$$
$$\mathbb{Z} \xrightarrow{\iota} \Lambda_r$$

where η is the surjective homomorphism sending ζ_r to $\bar{\zeta}_r$

Since $\prod_{j=0}^{r-1} \Delta(\zeta_r^j)$ is in $\mathbb{Z}$ we have:

$$\eta \prod_{j=0}^{r-1} \Delta(\zeta_r^j) = \prod_{j=0}^{r-1} \Delta(\bar{\zeta}_r^j)$$

But -

$$\eta \prod_{j=0}^{r-1} \Delta(\zeta_r^j) = \prod_{j=0}^{r-1} \Delta(\bar{\zeta}_r^j)$$
$$= \prod_{\zeta: r^{th} \text{ roots of } 1} \Delta(\zeta)$$

q.e.d.

Remarks: (1) Reading Corollary 4 for $r = 1$, and using Cor.2 we obtain the well-known fact

$$\Delta(1) = \pm 1$$

(2) Therefore we may write

$$k_r = \prod_{\zeta \neq 1;\ r^{th} \text{ roots of } 1} |\Delta(\zeta)|$$

It is interesting to compare this with the classical analytical formula for the class number times the regulator of a finite abelian extension of the rational number field. (Hasse (9), page 7, (4). Compare (12)).

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