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ZEEMAN'S FILTRATION OF HOMOLOGY

BY

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ABSTRACT. Geometric interpretations of Zeeman's filtrations of the homology and cohomology of a triangulable space are given, using an analysis of his spectral sequence for Poincaré duality.

The failure of a space to satisfy Poincaré duality is reflected by a topologically invariant filtration of its homology groups. In some sense, the filtration of a homology class measures the "degree of freedom" of its cycles. This filtration was first studied by Zeeman, who defined it using a spectral sequence for Poincaré duality. (This spectral sequence was also discovered independently by Cartan and by Fáry.) Zeeman started a geometric investigation of the spectral sequence for spaces such as singular algebraic varieties, for which it reveals a wealth of information by relating their local and global homology [**30**]. I was introduced to Zeeman's spectral sequence by Sullivan (cf. [**25**, p. 202] and [**26**]).

In this paper I show that the Zeeman filtration of a homology class α in a triangulable space is equal to the largest integer q such that α is represented by a cycle in the complement of any closed subspace of dimension less than q (Theorem 8.3). If $\mathcal{S}$ is a stratification of the space X, then the homology class α of X has filtration $\geqslant q$ if and only if α is represented by a cycle whose intersection with each stratum S of $\mathcal{S}$ has codimension $\geqslant q$ in S (Theorem 8.4). This new geometric interpretation of the Zeeman filtration has been generalized recently by M. Goresky and R. MacPherson to define "intersection homology theories" for stratified spaces [**9**].

Zeeman also defined a filtration of the cohomology of a space. Here I prove his conjecture that the filtration of a cohomology class β of a triangulable space is the smallest integer q such that β is represented by a cocycle whose support has dimension q (Theorem 8.8). (An incomplete proof of Zeeman's conjecture is given in [**7**].) In another paper [**21**] I will show that there is a simple relation between the Zeeman filtration and the Deligne weight filtration of the rational cohomology of a complex projective variety.

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My proofs use triangulations in an essential way, to define a "coskeleton" filtration whose associated spectral sequence is isomorphic with Zeeman's (Theorem 6.1). This isomorphism is constructed using a remarkable formula for cap product. (After discovering this formula, I found that it was known to W. Flexner in 1940 [5].)

A certain amount of technical background on Zeeman's spectral sequence is necessary to prove my geometric characterization of the associated filtration. §1 contains a brief description of the spectral sequence. The second section is devoted to a proof of its topological invariance which is somewhat simpler than Zeeman's. To give some feeling for the spectral sequence, some of its applications are outlined in §3. (These applications are not new, but it is interesting that in each case, analysis of the coskeleton spectral sequence leads to an elementary proof without spectral sequences.) §§4 through 6 are concerned with the coskeleton spectral sequence and the cap product. §7 is about the dual cohomology spectral sequence. The geometry of these algebraic structures is revealed in §8, which contains the main theorems. The relation of the homology filtration to stratifications is described, and the cohomology filtration is explained using cycles in a Euclidean neighborhood. Several examples are given in §9. (I advise the geometrically minded reader to start with these last two sections.)

I adopt the following conventions. All simplicial complexes are assumed to be locally finite and finite dimensional. An *oriented* simplicial complex is one in which each simplex has been oriented arbitrarily. Then simplicial homology is defined using oriented simplexes as in [**10**, Chapter 2]. Homology is with integer coefficients unless stated otherwise, but any coefficient ring would do.

This paper is a revised version of part of my doctoral thesis at Brandeis University [17]. I thank my advisor Jerome Levine, David Stone, and Dennis Sullivan for all they have taught me. Some helpful criticism of this paper was given to me by Kari Vilonen.

1. Zeeman's spectral sequence. In his thesis (1954) Zeeman developed a theory of double complexes (dihomology) with which he investigated a relation between homology and cohomology generalizing the Poincaré duality for manifolds [**30**]. In this section his Poincaré duality spectral sequence for finite simplicial complexes is extended to pairs of locally finite complexes.

1.1. Let K be an oriented simplicial complex and let L be a full subcomplex of K. Let $C.(K)$ be the oriented simplicial chain complex of K, with boundary map ∂. If $x \in C.(K)$ the *support* of x, denoted $\bar{x}$, is the smallest subcomplex of K containing all the simplexes which occur with nonzero coefficient in x. Let $C^{\cdot}(K) = \mathrm{Hom}(C.(K), \mathbf{Z})$ be the cochain complex of K, with coboundary

δ. If J is a subcomplex of K and $y \in C^{\cdot}(K)$ then $y|J \in C^{\cdot}(J)$ denotes the *restriction* of y to J.

Define two differentials d' and d'' on the group $C_{\cdot}(K) \otimes C^{\cdot}(K)$ as follows. If $x \in C_p(K)$ and $y \in C^q(K)$, then $d'(x \otimes y) = \partial x \otimes y$ and $d''(x \otimes y) = (-1)^{p-q} x \otimes \delta y$. Let $D = D(K, L)$ be the quotient of $C_{\cdot}(K) \otimes C^{\cdot}(K)$ by the subgroup generated by all elements $x \otimes y$ such that $y|\bar{x} = 0$ or $y|L = 0$. Let $D_{p,q}$ be the image of $C_p(K) \otimes C^q(K)$ in D. The differentials d' and d'' induce differentials on D. Since $d'd'' + d''d' = 0$, the bigraded group $D = \{D_{p,q}\}$ together with the two families of homomorphisms

$$d'\colon D_{p,q} \to D_{p-1,q}, \qquad d''\colon D_{p,q} \to D_{p,q+1}$$

is a double complex [**6**, p. 31]. A total grading on D is defined by $D_s = \Sigma_{p-q=s} D_{p,q}$. If $d = d' + d''$ then $d\colon D_s \to D_{s-1}$ and $d \circ d = 0$.

Let $'E$ and $''E$ be the first and second spectral sequences of the double complex D [**6**, p. 86]. The first comes from the p-filtration of D and the second comes from the q-filtration. Both spectral sequences converge to the homology $H(D)$ with respect to tbe total differential d. We will see that the first spectral sequence $'E$ collapses. The *Zeeman spectral sequence* of the pair (K, L) is the second spectral sequence $E = ''E$.

1.2. Let $N(L)$ be the open stellar neighborhood of L in K. A simplex σ of K is in $N(L)$ if and only if some face of σ is in L. Thus $K \setminus N(L)$ is a subcomplex of K. Let $\bar{\sigma}$ denote the subcomplex of K consisting of the faces of σ (including σ). Since L is a full subcomplex of K, if $\sigma \in N(L)$ then $\bar{\sigma} \cap L = \bar{\tau}$ for some face τ of σ.

PROPOSITION. *The first spectral sequence $'E$ collapses to an isomorphism*

$$H_s(K, K \setminus N(L)) \cong H_s(D).$$

PROOF. By [**6**, p. 87], $'E^2_{p,q} = {}'H_p({}''H_q D)$ where $'H$ is homology with respect to d' and $''H$ is homology with respect to d''. Let (σ) be the subgroup of $C_p(K)$ generated by the oriented p-simplex σ. If D''_σ is the image in D of the subgroup $(\sigma) \otimes C^{\cdot}(K)$ of $C_{\cdot}(K) \otimes C^{\cdot}(K)$, then $D''_\sigma = (\sigma) \otimes C^{\cdot}(\bar{\sigma} \cap L)$. Since $\bar{\sigma} \cap L = \bar{\tau}$ for some $\tau \leqslant \sigma$, it follows that ${}''H_{p,q}(D) = C_p(K)/C_p(K \setminus N(L))$ if $q = 0$, and ${}''H_{p,q}(D) = 0$ if $q \neq 0$. Thus $'E^2_{p,q} = H_p(K, K \setminus N(L))$ if $q = 0$ and $'E^2_{p,q} = 0$ if $q \neq 0$, so $'E$ collapses as desired.

1.3. The simplicial complex K can be regarded as a category with objects its simplexes σ and morphisms the face operators $\sigma \geqslant \tau$. A *stack* (or *system of coefficients*) S on K is a contravariant functor from K to the category of abelian groups. A cochain on K with coefficients in S is a function c which assigns an element of $S(\sigma)$ to each simplex σ of K. The cochain c has finite support if $c(\sigma) = 0$ for all but a finite number of simplexes σ. If $\sigma \geqslant \tau$ let $S_{\sigma\tau}$: $S(\tau) \to S(\sigma)$ be the corresponding morphism, and let $[\sigma, \tau] = \pm 1$ be the

incidence number of σ and τ (the coefficient of τ in the chain $\partial\sigma$). If c is a cochain with finite support and coefficients in S let δc be the cochain defined by $\delta c(\sigma) = \Sigma[\sigma, \tau]S_{\sigma\tau}(c(\tau))$. Let $\bar{C}^{\cdot}(K; S)$ be the resulting chain complex, with homology $\bar{H}^{\cdot}(K; S)$.

If J is a subcomplex of K and S is a stack on K, then $\bar{H}^{\cdot}(J; S)$ is defined by restricting S to J.

The pth *local homology stack* $h_p = h_p(K)$ is defined as follows. If $\tau \in K$ then $h_p(\tau) = H_p(K, K \setminus \mathrm{St}(\tau))$, where $\mathrm{St}(\tau) = \{\sigma \in K | \sigma \geqslant \tau\}$ is the *open star* of τ in K. If $\sigma \geqslant \tau$ then $\mathrm{St}(\sigma) \subset \mathrm{St}(\tau)$ and $(h_p)_{\sigma\tau}\colon h_p(\tau) \to h_p(\sigma)$ is induced by the inclusion $K \setminus \mathrm{St}(\tau) \subset K \setminus \mathrm{St}(\sigma)$.

PROPOSITION. *The second spectral sequence has*

$$''E^2_{p,q} = \bar{H}^q(L; h_p(K)).$$

PROOF. Again $''E^2_{p,q} = ''H_q('H_pD)$. If τ is an oriented simplex of K (which is identified with a generator of $C^{\cdot}(K)$) let D'_τ be the image in D of the subgroup $C_{\cdot}(K) \otimes (\tau)$ of $C_{\cdot}(K) \otimes C^{\cdot}(K)$. Then $D'_\tau = C_{\cdot}(K, K \setminus \mathrm{St}(\tau)) \otimes (\tau)$ if $\tau \in L$, and $D'_\tau = 0$ if $\tau \notin L$. Therefore $'H_{p,q}(D) = \bar{C}^q(L; h_p(K))$ and the proposition follows.

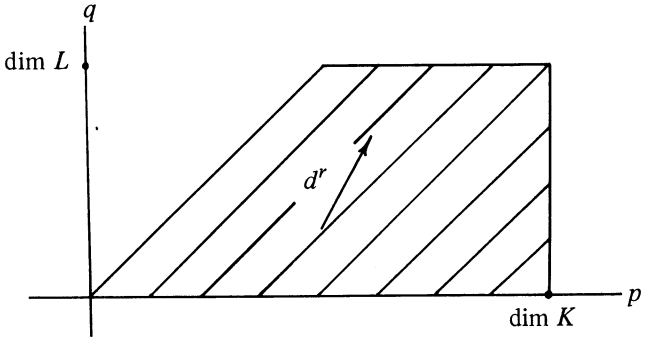

FIGURE 1

1.4. In summary the *Zeeman spectral sequence* $E = E(K, L)$ runs

$$E^2_{p,q} = \bar{H}^q(L; h_p(K)) \overset{q}{\Rightarrow} H_{p-q}(K, K \setminus N(L)),$$
$$E^r_{p,q} \neq 0 \quad \text{only if } 0 \leqslant q \leqslant \dim(L) \text{ and } q \leqslant p \leqslant \dim(K),$$
$$d^r\colon E^r_{p,q} \to E^r_{p+r-1,q+r}.$$

(See Figure 1.)

2. Topological invariance. We will give a topological description of Zeeman's spectral sequence using standard constructions of sheaf theory. In essentially this form it was first defined by Cartan in 1951 [3] to prove

Poincaré duality for manifolds (cf. 3.1 below). A general reference for sheaf theory is [6].

2.1. Let X be a topological space, with A a closed subspace. Let $\Delta.$ be the differential sheaf of germs of singular chains on X; that is, the sheaf generated by the presheaf $U \mapsto C.(X, X \setminus U)$. Let $D = D(X, A) = \check{C}_c^{\cdot}(A; \Delta.)$ be the bigraded group of Čech cochains of A with compact supports, and coefficients in the restriction of $\Delta.$ to A (cf. [6, II§5]). Let d': $\check{C}_c^q(A; \Delta_p) \to \check{C}_c^q(A; \Delta_{p-1})$ be induced by the differential $\Delta_p \to \Delta_{p-1}$, and let d'': $\check{C}_c^q(A; \Delta_p) \to \check{C}_c^{q+1}(A; \Delta_p)$ be $(-1)^{p-q}$ times the differential of the complex $\check{C}_c^{\cdot}(A; \Delta_p)$. Then $d'd'' + d''d' = 0$, so D is a double complex with total differential $d = d' + d''$. Let $'E$ and $''E$ be the first and second spectral sequences of D. The spectral sequence $'E$ collapses to an isomorphism of the singular homology group of $(X, X \setminus A)$ with the homology of D with respect to d. The *Zeeman spectral sequence* of the pair (X, A) is the spectral sequence $E = ''E$.

2.2. Let $h_p(X)$ be the pth *local homology sheaf*, or the sheaf generated by the presheaf $U \to H_p(X, X \setminus U)$ (singular homology). Let $\check{H}_c^q(A; h_p(X))$ be the qth Čech cohomology group with compact supports and coefficients in the restriction $h_p(X)|A$.

If either (i) A has finite covering dimension a or (ii) $h_p(X)|A = 0$ for $p > n$, then the Zeeman spectral sequence $E = E(X, A)$ converges, with

$$E_{p,q}^2 = \check{H}_c^q\left(A; h_p(X)\right) \overset{q}{\Rightarrow} H_{p-q}(X, X \setminus A),$$

$$E_{p,q}^r \neq 0 \quad \text{only if } 0 \leqslant q \leqslant a \text{ and } 0 \leqslant p \leqslant n.$$

(Compare [6, p. 178], [2, IV 2.9, V 8.6].)

2.3. Let $(|K|, |L|)$ be the geometric realization of the simplicial pair (K, L), with L full in K.

PROPOSITION. *There is an isomorphism of spectral sequences*

$$E(K, L) \cong E(|K|, |L|).$$

PROOF. Let $(X, A) = (|K|, |L|)$. We shall define a morphism ϕ: $D(K, L) \to D(X, A)$ of bigraded differential groups which induces an isomorphism ϕ^2: $E^2(K, L) \cong E^2(X, A)$, so ϕ induces an isomorphism of spectral sequences.

By definition $D(K, L) = \bar{C}^q(L; c_p(K))$, where $c_p(K)$ is the stack of local p-chains of K, $c_p(\tau) = C_p(K, K \setminus \mathrm{St}(\tau))$. Similarly $D(X, A) = \check{C}_c^q(A; \Delta_p)$.

Let c_p' be the presheaf on X defined by $c_p' = (X, X \setminus U)$ if $U \cap A \neq \varnothing$ and $c_p(U) = 0$ if $U \cap A = \varnothing$. The sheaf generated by c_p' is $c_p(X)_A$, the unique sheaf on X whose restriction to A equals $c_p(X)|A$ and whose restriction to $X \setminus A$ is zero.

If $\mathcal{U}$ is an open covering of X let $N\mathcal{U}$ be the *nerve* of $\mathcal{U}$. If $\mathcal{U}(K)$ is the covering of X by the open sets $|\mathrm{St}(v)|$ where v is a vertex of K, then $N\mathcal{U}(K) = K$.

The morphism ϕ is the composition

$$\bar{C}^q(L; c_p(K)) \xrightarrow{f} \bar{C}^q(N\mathcal{U}(K); c_p') \xrightarrow{g} C_c^q(N\mathcal{U}(K); c_p(X)_A) \xrightarrow{h} \check{C}_c^q(A; c_p(X)),$$

where f is induced by the canonical coefficient map $C_p(K, K \setminus \mathrm{St}(\tau)) \to C_p(X, X \setminus |\mathrm{St}(\tau)|)$, g is induced by the coefficient map $c_p' \to c_p(X)_A$, and h comes from the fact that $\check{C}_c^q(A; c_p(X)) = \lim C_c^q(N\mathcal{U}; c_p(X)_A)$, the direct limit over all open coverings $\mathcal{U}$ of X.

Let h_p' be the local homology sheaf corresponding to c_p'. The morphism ϕ^2 of E^2 is the composition

$$\bar{H}^q(L; h_p(K)) \xrightarrow{f^2} \bar{H}^q(N\mathcal{U}(K); h_p') \xrightarrow{g^2} H_c^q(N\mathcal{U}(K); h_p(X)_A) \xrightarrow{h^2} \check{H}_c^q(A; h_p(X)).$$

It is easy to see that f^2 and g^2 are isomorphisms. It remains to see that h^2 is an isomorphism.

Let J be a simplicial subdivision of K. If τ is a q-simplex of K and $x \in |\tau|$ (the "interior" of τ) then the restriction $H_p(X, X \setminus \mathrm{St}(\tau)) \to H_p(X, X \setminus \{x\})$ is an isomorphism. Thus if ρ is a q-simplex of J such that $|\rho| \subset |\tau|$, then the restriction $H_p(X, X \setminus |\mathrm{St}(\tau, K)|) \to H_p(X, X \setminus |\mathrm{St}(\rho, J)|)$ is an isomorphism. A standard spectral sequence argument shows that the subdivision map $\bar{H}^q(N\mathcal{U}(K); h_p') \to \bar{H}^q(N\mathcal{U}(J); h_p')$ is an isomorphism (cf. [**30**, Lemma 10]). Since the collection of open coverings of X of the form $\mathcal{U}(J)$ for J a subdivision of K is cofinal in the directed system of open coverings of X, the map h^2 is an isomorphism.

2.4. REMARK. Here are two variations of the Zeeman spectral sequence:

(1) For a triple $X \supset A \supset B$ with A and B closed, there is a spectral sequence

$$\check{H}_c^q(A, B; h_p(X)) \overset{q}{\Rightarrow} H_{p-q}(X \setminus B, X \setminus A).$$

(2) For a map $f\colon X \to Y$ there is a spectral sequence

$$\check{H}_c^q(Y; fh_p) \overset{q}{\Rightarrow} H_{p-q}(X).$$

The latter version is done by Zeeman [**30**, p. 171].

3. Applications. Generalizations of the classical Poincaré duality theorem can be proved using the spectral sequence E. The following representative

theorems were all first proved (for triangulable spaces) without spectral sequences.

3.1. *Duality for manifolds.* Let X be a topological n-manifold without boundary. That is, every point of X has a neighborhood homeomorphic with Euclidean n-space. The *orientation sheaf* of X is the local singular homology sheaf h_n. (The manifold X is orientable if and only if h_n is the constant sheaf $\mathbf{Z}$.) Consider the spectral sequence $E(X, A)$ where A is a closed subspace of X. Since $h_p = 0$ for $p \neq n$, it follows that $E^2_{p,q} = \check{H}^q_c(A; h_p) = 0$ for $p \neq n$, so E collapses to the Alexander-Lefschetz duality isomorphism

$$\check{H}^q_c(A; h_p) \cong H_{n-q}(X, X \setminus A).$$

If X is compact and oriented and $A = X$ we obtain the Poincaré duality isomorphism

$$\check{H}^q_c(X) \cong H_{n-q}(X)$$

(cf. [**3**, exp. 20, p. 3]). A more general duality theorem can be obtained by using the spectral sequence 2.4(1) above.

3.2. *Partial Poincaré duality.* For simplicity let us consider finite simplicial complexes. Let K be a finite n-dimensional complex with $h_n(K) = \mathbf{Z}$. The complex K is *s-regular* if $C^q(K; h_p) = 0$ for all p and q such that $p - q = s$ and $p \neq n$. Thus in the spectral sequence $E(K)$ we have that for such p and q the terms $E^r_{p,q}$ are zero for $1 \leqslant r \leqslant \infty$, so the edge morphism $H^{n-s}(K) \to H_s(K)$ is *surjective*. Furthermore $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p+r-1,p+q}$ must be zero for $p - q = s$, so $E^2_{p,q} = E^\infty_{p,q}$ for $p + q = s - 1$. Therefore the edge morphism $H^{n-s+1}(K) \to H_{s-1}(K)$ is *injective*. So if K is both s-regular and $(s-1)$-regular then $H^{n-s}(K) \cong H_s(K)$. This result is due to Čech [**4**, p. 693] and Wylie [**29**, p. 187]. (Their result is actually a dual statement which is proved using the dual of E, defined in §7 below.) Therefore if K is s-regular for all $s \leqslant k$ then $H^{n-s}(K) \cong H_s(K)$ for $s < k$ and $H^{n-k}(K) \to H_k(K)$ is surjective. This result was recently rediscovered and applied to complex projective varieties by Kaup [**13**], [**14**] and Kato [**11**], [**12**].

3.3. Let K be a finite simplicial complex and let L be a full subcomplex of dimension l. Suppose that $h_p(K)|L = 0$ for $p < l$. Then in the spectral sequence $E(K, L)$ we have $E^2_{p,q} = 0$ for $p = q \neq l$, so $H^l(L; h_l(K)) \cong H_0(K, K \setminus N(L))$, or

$$H^q(L; h_l(K)) \cong H_{l-q}(K, K \setminus N(L)) \quad \text{for } q \geqslant l,$$

since both groups are zero for $q > l$. This isomorphism was discovered recently by J. Munkres.

4. The coskeleton filtration. Dual cells were used by Poincaré in 1899 [**22**, §VII] to prove his duality theorem. The dual skeleton construction for combinatorial manifolds generalizes to arbitrary simplicial complexes, provi-

ding a framework for the geometric study of duality.

4.1. Let K' be the barycentric subdivision of the simplicial complex K. The vertices of K' are the barycenters $b\sigma$ of the simplexes σ of K, and the simplexes of K' are the sets $\langle b\sigma_0, \ldots, b\sigma_s\rangle$ with $\sigma_0 > \cdots > \sigma_s$ in K. (We will always use the *decreasing* order of simplexes.) Define

$$K_p = \{\langle b\sigma_0, \ldots, b\sigma_s\rangle \in K' | \dim \sigma_0 \leqslant p\},$$

$$K^q = \{\langle b\sigma_0, \ldots, b\sigma_s\rangle \in K' | \dim \sigma_s \geqslant q\}.$$

The subcomplex K_p of K' is just the barycentric subdivision of the pth skeleton of K. The subcomplex K^q of K' is the qth *coskeleton* of K. Notice that K^q has *codimension* q in K. Clearly

$$\dim(K_p \cap K^q) = p - q,$$

$$K_p = K' \setminus N(K^{p+1}), \qquad K^q = K' \setminus N(K_{q-1}).$$

4.2. If L is a subcomplex of K let $C = C(K', L')$ be the simplicial chain complex $C.(K', K' \setminus N(L'))$, and let $F^iC = C.(K^i, K^i \setminus N(L^i))$. (Note that $K^i \cap L' = L^i$.) The filtration F^iC of C gives rise to a spectral sequence $E' = E'(K, L)$ (cf. [**6**, p. 77]). This *coskeleton spectral sequence* is a special case of a spectral sequence of G. Whitehead (the spectral sequence $^*E$ of [**27**, p. 275]). *A priori* it runs

$$E'(K, L)\colon (E')^2_{i,j} \overset{i}{\Rightarrow} H_{i+j}(K', K' \setminus N(L')),$$

$$(E')^r_{i,j} \neq 0 \quad \text{only if } 0 \leqslant i \leqslant \dim(K) \text{ and } 0 \leqslant i + j \leqslant \dim(K),$$

$$(d')^r\colon (E')^r_{i,j} \to (E')^r_{i+r,j-r-1}.$$

In §6 it will be shown that $E'(K, L)$, suitably reindexed, is isomorphic with Zeeman's spectral sequence $E(K, L)$ (if L is full in K).

5. Cap product. Čech and Whitney [**28**] observed that the Poincaré duality isomorphism can be defined using the cap product pairing. Comparison of this cap product isomorphism with the classical dual cell isomorphism leads to a remarkable definition of cap product.

5.1. Let K be an oriented simplicial complex. If $S = \langle b\omega_0, \ldots, b\omega_s\rangle$ is a simplex of the barycentric subdivision K', let $\varepsilon(S) = [\omega_0, \omega_1][\omega_1, \omega_2] \cdots [\omega_{s-1}, \omega_s]$. (Thus $\varepsilon(S) \neq 0$ only if $\dim \omega_i = \dim \omega_0 - i$ for $i = 1, \ldots, s$.) If $\sigma \geqslant \tau$ are simplexes of K, let $D(\sigma, \tau) = \{\langle b\omega_0, \ldots, b\omega_s\rangle \in K' | \sigma \geqslant \omega_0$ and $\omega_s \geqslant \tau\}$, the *dual cell* to τ in σ. If σ is a p-simplex and τ is a q-simplex, define a chain $c'(\sigma \otimes \tau) \in C_{p-q}(K')$ by

$$c'(\sigma \otimes \tau) = \sum_{\substack{S \in D(\sigma,\tau) \\ \dim S = p-q}} \varepsilon(S)S.$$

This determines a *cap product* homomorphism

$$c'\colon C_p(K) \otimes C^q(K) \to C_{p-q}(K').$$

This definition is due to W. Flexner [**5**].

PROPOSITION. *If* $x \in C_p(K)$ *and* $y \in C^q(K)$, *then*

$$\partial c'(x \otimes y) = c'(\partial x \otimes y) + (-1)^{p-q} c'(x \otimes \delta y).$$

PROOF. This is a straightforward calculation, using the identity $\Sigma[\rho, \sigma][\sigma, \tau] = 0$ (i.e. $\partial^2 = 0$).

It follows that c' induces a homomorphism

$$C'\colon H_p(K) \otimes H^q(K) \to H_{p-q}(K').$$

5.2. Now let $s\colon H_{p-q}(K) \to H_{p-q}(K')$ be the subdivision isomorphism, and let $C\colon H_p(K) \otimes H^q(K) \to H_{p-q}(K)$ be the classical Whitney cap product $C(\alpha \otimes \beta) = \alpha \cap \beta$ defined using the Alexander-Whitney diagonal approximation, with sign convention as in [**23**, p. 254].

PROPOSITION. *Flexner's cap product is homologous with the subdivision of Whitney's, i.e.* $C' = s \circ C$.

PROOF. Let $z \in C_p(K)$ be a cycle and let $A.$ be the chain complex $A_i = C^{p-i}(K)$ for $i \geqslant 0$ with differential δ, and $A_i = 0$ for $i < 0$. Define an augmentation $\varepsilon_z\colon A. \to \mathbf{Z}$ by $\varepsilon_z(y) = y(z)$ if $y \in C^p(K)$, $\varepsilon_z(y) = 0$ otherwise (or $\varepsilon_z(y) = \langle z, y \rangle$). Let $\varepsilon\colon C.(K') \to \mathbf{Z}$ be the standard augmentation.

Define an acyclic carrier Γ (cf. [**10**, 3.4]) from $A.$ to $C.(K')$ as follows. The carrier Γ assigns to the generator τ the subcomplex $\Gamma(\tau) = \{S \in K' |$ for each vertex $b\omega$ of S, either $\omega \geqslant \tau$ or $\tau \geqslant \omega\}$, the closed star of $b\tau$ in K'.

Define two homomorphisms c'_z, $c_z\colon A. \to C.(K')$ by $c'_z(y) = c'(z \otimes y)$ and $c_z(y) = c(z \otimes y)$, where c is the chain-level Whitney cap product. By Proposition 5.1, c'_z is a chain map. By [**23**, p. 253], c_z is a chain map. Both c'_z and c_z are augmentation preserving and carried by Γ. Therefore c'_z and c_z are chain homotopic, so $c'(z \otimes y)$ is homologous with $c(z \otimes y)$ for all cocycles y.

5.3. REMARKS. (1) Alternatively, the cap product can be characterized by several axioms (listed in [**24**, p. 959]), which can easily be checked for the Flexner and Whitney products.

(2) It follows from Flexner's definition that the cap product can be defined as a homomorphism

$$H_p(K) \otimes H^q(L, M) \to H_{p-q}(K' \setminus N(M'), K' \setminus N(L'))$$

(cf. [**27**, p. 265]).

6. An isomorphism of spectral sequences. Cap product defines an isomorphism from Zeeman's spectral sequence to the coskeleton spectral

sequence of G. Whitehead. This gives a powerful geometric interpretation of Zeeman's spectral sequence and implies the topological invariance of the coskeleton spectral sequence.

6.1. Let L be a full subcomplex of K. Consider the cap product homomorphism c': $C_\cdot(K) \otimes C^\cdot(K) \to C_\cdot(K')$ (5.1). If $\sigma \not\geqslant \tau$ then $c'(\sigma \otimes \tau) = 0$, and if $\tau \notin L$ then $c'(\sigma \otimes \tau) \subset K' \setminus N(L')$ for all σ. Therefore c' induces a homomorphism from $D(K, L)$ (1.1) to $C(K', L')$ (4.2). Proposition 5.1 says that $\partial c' = c'd$, where d is the total differential on the double complex $D(K, L)$. Furthermore c' preserves the total degree and maps the second filtration of D to the filtration of C, i.e. $c'(D_{p,q}) \subset F^q(C_{p-q})$. Therefore c' induces a map of spectral sequences

$$c': E(K, L) \to E'(K, L),$$
$$(c')^r: E^r_{p,q} \to (E')^r_{q,p-2q}.$$

THEOREM. *The map c' is an isomorphism of spectral sequences. In fact $(c')^r$ is an isomorphism for all $r \geqslant 1$.*

PROOF. It is enough to show that $(c')^1$ is an isomorphism. This follows from local simplicial geometry. Recall that if $\tau \in K$ then $\mathrm{St}(\tau) = \{\sigma \in K | \sigma \geqslant \tau\}$ is the *open star* of τ in K. Let $\overline{\mathrm{St}}(\tau) = \{\omega \in K | \omega \leqslant \sigma \text{ for some } \sigma \in \mathrm{St}(\tau)\}$ be the *closed star* of τ in K. Let $\partial\overline{\mathrm{St}}(\tau) = \overline{\mathrm{St}}(\tau) \setminus \mathrm{St}(\tau)$. Let $\mathrm{Dl}(\tau) = \{\langle b\omega_0, \ldots, b\omega_s\rangle \in K' | \omega_s \geqslant \tau\}$ be the *dual cone* of τ in K', and let $\mathrm{Lk}(\tau) = \{\langle b\omega_0, \ldots, b\omega_s\rangle \in K' | \omega_s > \tau\}$ be the *link* of τ in K'. The subcomplex $\mathrm{Dl}(\tau)$ of K' is the simplicial cone with base $\mathrm{Lk}(\tau)$ and apex $b\tau$.

By the proof of Proposition 1.2, the group $E^1_{p,q}$ is the direct sum over all q-simplexes $\tau \in L$ of the groups $H_p(K, K \setminus \mathrm{St}(\tau))$, which equal $H_p(\overline{\mathrm{St}}(\tau), \partial\overline{\mathrm{St}}(\tau))$ by excision. On the other hand $(E')^1_{q,p-2q}$ equals $H_{p-q}(K^q, (K \setminus N(L^q)) \cup K^{q+1})$ by definition (cf. [6, p. 77]). Now K^q is the union of $\mathrm{Dl}(\tau)$ for all q-simplexes $\tau \in K$, so $K^q \setminus N(L^q)$ is the union of $\mathrm{Dl}(\tau)$ for all q-simplexes $\tau \in K \setminus L$, and K^{q+1} is the union of $\mathrm{Lk}(\tau)$ for all q-simplexes $\tau \in K$. Thus $(E')^1_{p,p-2q}$ is the direct sum of the groups $H_{p-q}(\mathrm{Dl}(\tau), \mathrm{Lk}(\tau))$ over all q-simplexes τ of L.

The homomorphism $(c')^1$: $E^1 \to (E')^1$ is induced by the sum of the cap product maps

$$(c')^\tau: C_p\left(\overline{\mathrm{St}}(\tau), \partial\overline{\mathrm{St}}(\tau)\right) \to C_{p-q}(\mathrm{Dl}(\tau), \mathrm{Lk}(\tau)),$$
$$(c')^\tau(x) = c'(x \otimes \tau).$$

It remains to show that $(c')^\tau$ induces an isomorphism in homology.

Recall the subcomplex $D(\sigma, \tau)$ (5.1), the *dual cell* to τ in σ. Let $L(\sigma, \tau) = \{\langle b\omega_0, \ldots, b\omega_s\rangle \in D(\sigma, \tau) | \omega_s \neq \tau\}$, the *link cell* of τ in σ. The geometric realizations $|D(\sigma, \tau)|$ and $|L(\sigma, \tau)|$ are closed piecewise-linear cells.

($D(\sigma, \tau)$ is the cone from $b\tau$ on $L(\sigma, \tau)$, and $L(\sigma, \tau)$ is simplicially isomorphic with $(\bar{\rho})'$, where ρ is the simplex spanned by the vertices of σ not in τ.) The collection of these cells for all $\sigma \in \mathrm{St}(\tau)$ is a cell subdivision of $(|\mathrm{Dl}(\tau)|, |\mathrm{Lk}(\tau)|)$. If σ is a p-simplex and τ is a q-simplex, then the $(p-q)$-chain $c'(\sigma \otimes \tau)$ defines an orientation for the $(p-q)$-cell $|D(\sigma, \tau)|$, since the boundary of $c'(\sigma \otimes \tau)$ lies in the boundary of $|D(\sigma, \tau)|$ by Proposition 5.1. Let $C.(|\mathrm{Dl}(\tau)|, |\mathrm{Lk}(\tau)|)$ be the resulting oriented cellular chain complex. There is a commutative triangle of homomorphisms

$$\begin{array}{ccc} C_p(\mathrm{St}(\tau), \partial\,\mathrm{St}(\tau)) & \xrightarrow{(c')^\tau} & C_{p-q}(\mathrm{Dl}(\tau), \mathrm{Lk}(\tau)) \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & C_{p-q}(|\mathrm{Dl}(\tau)|, |\mathrm{Lk}(\tau)|) & \end{array}$$

where g sends the generator σ to the generator $|D(\sigma, \tau)|$ and h is the subdivision map. Since g is a chain isomorphism and h induces an isomorphism in homology, $(c')^\tau$ induces an isomorphism in homology.

6.3. As a corollary of this theorem, the edge morphisms of the spectral sequence $E(X, A)$ can be described using cap product. If $h_p(X)|A = 0$ for $p > n$ and $h_n(X)|A \neq 0$ there are edge morphisms

$$e_q\colon \check{H}_c^q(A; h_n(X)) \to H_{n-q}(X, X \setminus A).$$

For simplicity we will discuss the case $A = X$.

A *normal n-circuit* is a triangulable space X together with a class $[X] \in H_n(X)$ with the following properties. First, the sheaf $h_n(X)$ is constant. It follows that X is n-dimensional and the stalk $h_n(X)_x = H_n(X, X \setminus \{x\})$ is infinite cyclic for all $x \in X$. Second, the restriction of $[X]$ to $H_n(X, X \setminus \{x\})$ is a generator for all $x \in X$. The class $[X]$ is the *orientation class* of X. For example, a normal complex projective variety of pure complex dimension d is a normal $2d$-circuit.

PROPOSITION. *If X is a normal n-circuit, then $e_q(\beta) = [X] \cap \beta$ for all $\beta \in H^q(X)$.*

This follows immediately from the preceding theorem. It can be generalized using cap product with sheaf coefficients (cf. [**2**, V 10]) or by using a topological normalization process ([**17**], [**13**]).

7. The dual spectral sequence. The companion spectral sequence which converges to cohomology rather than homology was emphasized by Zeeman. The topological version of this spectral sequence is technically harder to set up since it involves inverse rather than direct limits.

7.1. Let K be a simplicial complex. The pth *local cohomology costack* $h^p(K)$

is defined by $\tau \mapsto H^p(K, K \setminus \mathrm{St}(\tau))$. Let L be a full subcomplex of K, and let $\overline{H}_q(L; h^p(K))$ be the homology group of L with coefficients in $h^p(K)|L$, defined using possibly infinite sums of simplexes. The Zeeman spectral sequence $\hat{E}(K, L)$ runs

$$\hat{E}^2_{p,q} = \overline{H}_q(L; h^p(K)) \overset{q}{\Rightarrow} H^{p-q}(K, K \setminus N(L)).$$

It is defined dually to $E(K, L)$. Briefly, let $\hat{D}$ be the subgroup of $C^\cdot(K) \otimes C_\cdot(K)$ generated by the elements $\sigma \otimes \tau$ such that $\sigma \geqslant \tau$ and $\tau \in L$. If $x \in C^p(K)$ and $y \in C_q(K)$, let $d'(x \otimes y) = \delta x \otimes y$ and $d''(x \otimes y) = (-1)^{p-q} x \otimes \partial y$. These differentials restrict to $\hat{D}$, making it a double complex. Its second spectral sequence is $E(K, L)$.

7.2. The coskeleton spectral sequence $\hat{E}'(K, L)$ is the standard cohomology spectral sequence arising from the filtration of K' by its coskeletons, with

$$(\hat{E}')^2_{i,j} \overset{i}{\Rightarrow} H^{i+j}(K', K' \setminus N(L')).$$

7.3 THEOREM. *There is an isomorphism of spectral sequences*

$$\hat{E}(K, L) \cong \hat{E}'(K, L).$$

PROOF. Dualize the proof of Theorem 6.1.

7.4. Now let X be a space and A a closed subspace. The topological Zeeman spectral sequence $\hat{E}(X, A)$ can be defined for coefficients in a *field* Λ merely by dualizing the definition of $E(X, A)$. The resulting spectral sequence is strictly dual to $E(X, A)$, i.e. $\hat{E}^r = \mathrm{Hom}(E^r, \Lambda)$ and $\hat{d}^r = \mathrm{Hom}(d^r, \Lambda)$ for all r. It runs

$$\hat{E}^2_{p,q} = \check{H}_q(A; h^p(X)) \overset{q}{\Rightarrow} H^{p-q}(X, X \setminus A),$$

where $\check{H}_q(A; L^p(X))$ is the qth Čech homology group of A with arbitrary supports and coefficients in the restriction of the pth local cohomology cosheaf of X. With field coefficients an isomorphism $\hat{E}(K, L) \cong \hat{E}(|K|, |L|)$ is obtained by dualizing Proposition 2.3.

7.5. Zeeman defined $\hat{E}(X)$ for arbitrary coefficients as a "semispectral sequence", the inverse limit of spectral sequences defined on the nerves of open coverings, and he proved that $\hat{E}(K) \cong \hat{E}(|K|)$ [**30**, p. 171]. It follows that if X is triangulable then $\hat{E}(X)$ is a spectral sequence, and that the spectral sequence of a complex is a topological invariant. Zeeman's definition and topological invariance theorem can be extended to pairs (X, A).

8. Topology of the filtrations. Associated to the spectral sequences $E(X)$ and $\hat{E}(X)$ there are filtrations of the homology and cohomology of X which reflect the interaction of cycles and cocycles with the singularities of X. We will give geometric interpretations of these filtrations for a triangulable space.

Our description of the cohomology filtration was conjectured by Zeeman [**30**, p. 178]. Our description of the homology filtration is new.

Throughout this section we will assume that X is homeomorphic with the geometric realization of an n-dimensional simplicial complex K.

8.1. *The homology filtration.* The spectral sequence $E(X)$ converges to a topologically invariant filtration

$$H_s(X) = F^0H_s(X) \supset \cdots \supset F^qH_s(X) \supset F^{q+1}H_s(X) \supset \cdots$$

of the singular homology of X by subgroups. There are isomorphisms

$$E^\infty_{p,q} \cong F^qH_{p-q}(X)/F^{q+1}H_{p-q}(X)$$

for all p and q. Since X is finite dimensional it follows that $F^qH_s(X) = 0$ for q sufficiently large. If $\alpha \in H_s(X)$ is nonzero, the *Zeeman filtration* $f\alpha$ is the largest integer q such that $\alpha \in F^qH_s(X)$.

8.2. Of the following properties of the filtration, (1), (2), and (3) are due to Zeeman. Recall that the n-dimensional triangulable space X is a *homology n-manifold* if $h_p(X) = 0$ for $p < n$ and $h_n(X)$ is locally constant.

PROPOSITION. *Let X be an n-dimensional triangulable space, and let $\alpha \in H_s(X)$ be nonzero.*

(1) *$f\alpha \leqslant n - s$.*

(2) *If X is a homology n-manifold then $f\alpha = n - s$.*

(3) *If $\beta \in H^q(X)$ then $f(\alpha \cap \beta) \geqslant q$.*

(4) *If X is a normal n-circuit then $f\alpha = n - s$ if and only if there exists $\beta \in H^{n-s}(X)$ with $[\dot{X}] \cap \beta = \alpha$.*

(5) *If X is a normal n-circuit then the map*

$$[X] \cap \cdot \ : H^{n-s}(X) \to H_s(X)$$

is an isomorphism for all s if and only if $\alpha \in H_s(X)$ implies $f\alpha = n - s$ for all s.

PROOF. (1) follows from the fact that $E^2_{p,q} = 0$ for $p > n$, so $s = p - q \leqslant n - q$, i.e. $q \leqslant n - s$. (2) holds since $E^2_{p,q} = 0$ for $p \neq n$. (3) is true because $\alpha \cap \beta$ is represented by a cycle in the q-coskeleton $|K^q|$ by Flexner's definition of cap product (5.1), and the filtration of $H_s(X)$ is induced by the coskeleton filtration of X by Theorem 6.1. One direction of (4) follows from (1), (3), and Proposition 6.3. The other direction follows easily from Flexner's definition. (For details, see [**18**, p. 289].) (5) follows from (4) and a universal coefficient argument (see [**19**, Lemma 8]).

8.3. DEFINITION. The class $\alpha \in H_s(X)$ has *geometric codimension* $\geqslant q$ if $\alpha \in \text{Image}[H_s(X \setminus A) \to H_s(X)]$ for every closed subspace A of X with covering dimension less than q.

For example if X is an annulus in the plane and $\alpha \in H_1(X)$ is nonzero

then α has geometric codimension one. But if a line segment connecting the two boundary components of X is collapsed to a point, then the image of α in the resulting *pinched annulus* has geometric codimension zero.

THEOREM. *In a triangulable space the Zeeman filtration of a homology class equals its geometric codimension.*

PROOF. Suppose that $\alpha \in H_s(X)$ has geometric codimension $\geqslant q$. Then

$$\alpha \in \operatorname{Im}\left[H_s(X \setminus |K_{q-1}|) \to H_s(X)\right] = \operatorname{Im}\left[H_s(X \setminus |K^q|) \to H_s(X)\right] = F^q H_s(X)$$

by Theorem 6.1. On the other hand suppose that $f\alpha \geqslant q$. Let A be a closed subspace of X of dimension $< q$. We shall show that $r\alpha = 0$ where r: $H_s(X) \to H_s(X, X \setminus A)$ is the restriction map. The restriction map $\check{C}_c^q(X; \Delta_p) \to \check{C}_c^q(A; \Delta_p|A)$ of double complexes (2.1) induces a map $E(X) \to E(X, A)$ of spectral sequences. Therefore $f(r\alpha) \geqslant f\alpha = q$, where $f(r\alpha)$ is the filtration of $r\alpha$ with respect to $E(X, A)$. But since $\dim A < q$, this implies $r\alpha = 0$.

REMARK. The proof shows that even if X is not triangulable the geometric codimension of a homology class is greater than or equal to its Zeeman filtration.

8.4. The choice of a triangulation $|K| = X$ determines a piecewise-linear structure on X. A *piecewise-linear cycle* $\mathbf{a}$ of X is a simplicial cycle in some subdivision of K. The support $|\mathbf{a}|$ is the union of all the closed simplexes which occur with nonzero coefficient in $\mathbf{a}$.

THEOREM. *Let $\mathcal{S}$ be a peicewise-linear stratification of X. The class $\alpha \in H_s(X)$ has geometric codimension $\geqslant q$ if and only if α is represented by a piecewise-linear cycle $\mathbf{a}$ such that $|\mathbf{a}| \cap S$ has codimension $\geqslant q$ in S for all strata S of $\mathcal{S}$.*

PROOF. (Compare [18].) If α has geometric codimension $\geqslant q$ then

$$\alpha \in \operatorname{Im}\left[H_s(X \setminus |K_{q-1}|) \to H_s(X)\right] = \operatorname{Im}\left[H_s(|K^q|) \to H_s(X)\right].$$

Let $\mathbf{a}$ be any simplicial cycle in K^q representing α. Since $\dim(K_p \cap K^q) = p - q$ for all p and q (4.1) and $S \subset |K_p|$ if $\dim S = p$, it follows that $\dim(|\mathbf{a}| \cap S) \leqslant p - q$. Conversely, suppose that α is represented by a cycle $\mathbf{a}$ such that $|\mathbf{a}| \cap S$ has codimension $\geqslant q$ in S for all strata $S \in \mathcal{S}$. By Akin's stratified general position theorem [1, Theorem 6, p. 471], [20] there is an isotopy of X which moves $|\mathbf{a}|$ off the $(q-1)$-skeleton of K, so

$$\alpha \in \operatorname{Im}\left[H_s(X \setminus |K_{q-1}|) \to H_s(X)\right] = \operatorname{Im}\left[H_s(|K^q|) \to H_s(X)\right] = F^q H_s(X),$$

so α has geometric codimension $\geqslant q$ by Theorem 8.3.

8.5. *The cohomology filtration.* The spectral sequence $\hat{E}(X)$ converges to a topologically invariant filtration

$$H^s(X) \supset \cdots \supset F_q H^s(X) \supset F_{q-1} H^s(X) \supset \cdots \supset F_0 H^s(X) \supset 0$$

of the singular cohomology of X by subgroups. There are isomorphisms

$$\hat{E}_{p,q}^{\infty} \cong F_q H^{p-q}(X)/F_{q-1} H^{p-q}(X)$$

for all p and q. Since X is finite dimensional it follows that $F_q H^s(X) = H^s(X)$ for q sufficiently large. If $\beta \in H^s(X)$ the *Zeeman filtration* $f\beta$ is the smallest integer q such that $\beta \in F_q h^s(X)$.

8.6. Of the following properties of the filtration, (1) and (2) are due to Zeeman.

PROPOSITION. *Let X be an n-dimensional triangulable space, and let $\beta \in H^s(X)$ be nonzero.*

(1) *$f\beta \leqslant n - s$.*

(2) *If X is a homology n-manifold then $f\beta = n - s$.*

(3) *If $f\beta \leqslant q$ then $\alpha \cap \beta = 0$ for all $\alpha \in H_p(X)$ with $p - q > s$.*

(4) *If X is a normal n-circuit then $f\beta < n - s$ if and only if $[X] \cap \beta = 0$.*

(5) *If $f\beta \leqslant q$ then $f(\beta \cup \gamma) \leqslant q$ for all $\gamma \in H^{\cdot}(X)$.*

In other words, $F_q H^{\cdot}(X)$ is an ideal of the ring $H^{\cdot}(X)$.

PROOF. The proofs of (1) through (4) are dual to their counterparts in Proposition 8.2. And (5) follows from the fact that $f\beta \leqslant q$ if and only if $\beta|(X \setminus |K_q|) = 0$.

8.7. PROPOSITION. *Let Λ be a field. If $\beta \in H^s(X; \Lambda)$ then $f\beta \leqslant q$ if and only if $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in H_s(X; \Lambda)$ with $f\alpha > q$.*

PROOF. The filtration $f\beta \leqslant q$ if and only if $\beta|\,|K^{q+1}| = 0$ if and only if $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \mathrm{Im}[H_s(|K^{q+1}|) \to H_s(X)]$, i.e. $f\alpha > q$.

8.8. DEFINITION. The class $\beta \in H^s(X)$ has *geometric dimension* $\leqslant q$ if $\beta \in \mathrm{Ker}[H^s(X) \to H^s(X \setminus A)]$ for some closed subspace A of X with dimension less than or equal to q.

In other words β is represented by a cocycle whose support has dimension $\leqslant q$. (Zeeman calls the geometric dimension of β the "codimension" of β.) For example if X is an annulus in the plane and $\beta \in H^1(X)$ is nonzero then β has geometric dimension one. But β is the pull-back of a class in the pinched annulus with geometric dimension zero.

THEOREM. *In a triangulable space the Zeeman filtration of a cohomology class equals its geometric dimension.*

PROOF. If $\beta \in H^s(X)$ and $f\beta \leqslant q$ then by Theorem 7.3,

$$\beta \in \mathrm{Ker}\left[H^s(X) \to H^s\left(|K^{q+1}|\right)\right] = \mathrm{Ker}\left[H^s(X) \to H^s\left(X \setminus |K_q|\right)\right],$$

so β has geometric dimension $\leqslant q$. For coefficients in a field, the proof that β has geometric dimension $\leqslant q$ implies $f\beta \leqslant q$ is dual to the second part of the proof of Theorem 8.3. For arbitrary coefficients, there are technical problems if X cannot be triangulated so that A is a subcomplex. These difficulties are handled by Zeeman in [**30**, Theorem 4].

REMARK. Zeeman proves filtration $\leqslant$ geometric dimension in the cohomology of any compact Hausdorff space.

8.9. The following result was motivated by the work of G. L. Gordon [**7**], [**8**]. Again fix a piecewise-linear structure on X.

THEOREM. *Let X be a compact subpolyhedron of R^k. The class $\beta \in H^s(X)$ has geometric dimension $\leqslant q$ if and only if the Alexander-Lefschetz dual class $\alpha \in H_{k-s}(R^k, R^k \setminus X)$ is represented by a piecewise-linear cycle* **a** *such that* $\dim(|\mathbf{a}| \cap X) \leqslant q$.

PROOF. Let J be a triangulation of R^k by linear simplexes such that X is triangulated by a full subcomplex K. Let N be the closed stellar neighborhood of X in R^k with respect to the barycentric subdivision J', let ∂N be the boundary of N, and let $\pi\colon N \to X$ be the canonical simplicial retraction. (If $\sigma \in J$ with $|\bar{\sigma}| \cap X \neq \varnothing$ then $|\bar{\sigma}| \cap X = |\bar{\tau}|$ for some $\tau \in K$. Then $\pi(b\sigma) = b\tau$, and π is linear on the simplexes of J' in N.)

Now β has geometric dimension $\leqslant q$ if and only if $\beta \in \operatorname{Ker}[H^s(X) \to H^s|K^{q+1}|] = \operatorname{Im}[H^s(X, |K^{q+1}|) \to H^s(X)]$. Consider the following commutative diagram of isomorphisms:

$$\begin{array}{ccccc} H^s(X, |K^{q+1}|) & \cong & H^s(N, \pi^{-1}|K^{q+1}|) & \cong & H_{n-s}(N \setminus \pi^{-1}|K^{q+1}|, (N \setminus \pi^{-1}|K^{q+1}|) \cap \partial N) \\ & & \downarrow & & \downarrow \\ H^s(X) & \cong & H^s(N) & \cong & H_{n-s}(N, \partial N) \end{array}$$

Since

$$H_{n-s}(N \setminus \pi^{-1}|K^{q+1}|, (N \setminus \pi^{-1}|K^{q+1}|) \cap \partial N) \cong H_{n-s}(\pi^{-1}|K_q|, \pi^{-1}|K_q| \cap \partial N)$$

it follows that β has geometric codimension $\leqslant q$ if and only if

$$\alpha \in \operatorname{Im}\left[H_{n-s}(\pi^{-1}|K_q|, \pi^{-1}|K_q| \cap \partial N) \to H_{n-s}(N, \partial N)\right].$$

Since $\pi^{-1}|K_q| \cap X = |K_q|$, this means that α is represented by a cycle **a** such that $|\mathbf{a}| \cap X \subset |K_q|$.

8.10. REMARK. In fact the Zeeman cohomology spectral sequence $\hat{E}(X)$ is isomorphic with the Leray homology spectral sequence of the map $\pi\colon (N, \partial N) \to X$ (cf. [**2**, IV 6.1]). Similarly $E(X)$ is isomorphic with the Leray cohomology spectral sequence of π.

9. Examples. The following examples illustrate the spectral sequences $E(X)$ and $\hat{E}(X)$ and the Zeeman filtrations.

9.1. *The pinched torus.* Let X be obtained by collapsing a meridian on the torus to a point: $X = S^1 \times S^1/\{t\} \times S^1$, $t \in S^1$. The space X is homeomorphic with the curve $x^3 + y^3 = xyz$ in the complex projective plane, which has one singular point $x = y = 0$. The homology of X is $H_0(X) = \mathbf{Z}$, $H_1(X) = \mathbf{Z}$, $H_2(X) = \mathbf{Z}$. The sheaf h_0 is 0, and the stalk of h_1 is 0 except at the singular point, where it is $\mathbf{Z}$. Similarly the stalk of h_2 is $\mathbf{Z}$ except at the singular point, where it is $\mathbf{Z} \oplus \mathbf{Z}$. Therefore the nonzero entries in the E^2 term are $E^2_{1,0} = H^0(X; h_1) = \mathbf{Z}$, $E^2_{2,0} = H^0(X; h_2) = \mathbf{Z}$, and $E^2_{2,2} = H^2(X; h_2) = \mathbf{Z}$. (Figure 2.) It follows that $d^2 = 0$ and $E^2 = E^\infty$, so $H_0(X) = F^2H_0(X)$, $H_1(X) = F^0H_1(X)$, and $H_2(X) = F^0H_2(X)$. Thus a generator $\alpha \in H_1(X)$ has filtration 0 instead of filtration 1 as it would in a 2-manifold. This reflects the fact that any cycle representing α must pass through the singular point of X. The dual spectral sequence $\hat{E}$ shows that a generator $\beta \in H^1(X)$ has filtration 0, since β is represented by a cocycle which is supported by the singular point.

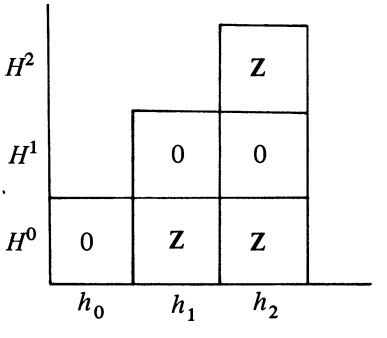

FIGURE 2

9.2. *Isolated singularities.* Zeeman describes two examples in detail: the cone on a closed manifold [**30**, p. 160] and the quadric cone in complex projective 3-space [**30**, p. 181]. The former shows that a contractible space can have a nontrivial spectral sequence, and the latter shows that E and $\hat{E}$ are not strictly dual with integer coefficients.

In general, let X be a compact n-dimensional triangulable space for which there exists a finite subset Σ such that $h_p|(X \setminus \Sigma) = 0$ for all $p < n$ and $h_n|(X \setminus \Sigma) = \mathbf{Z}$. Then the differentials and edge morphisms of $E(X)$ give a long exact sequence

$$\cdots \to H^0(X; h_{n-q+1}) \xrightarrow{d^q} H^q(X; h_n) \xrightarrow{e^q} H_{n-q}(X) \to H^0(X; h_{n-q}) \to \cdots$$

(cf. [**13**]). It follows that if the duality map e^q is an isomorphism for all q then X is in fact a homology manifold (cf. [**19**, Lemma 3]). Let K be a trian-

gulation of X with the points of Σ as vertices and with $\overline{\mathrm{St}}(v) \cap \overline{\mathrm{St}}(w) = \varnothing$ for distinct vertices v and w of Σ. Let $W = X \setminus \bigcup_{v \in \Sigma} |\mathrm{St}(v)|$. Then W is an oriented homology n-manifold with boundary, and the long exact sequence above is isomorphic with the long exact homology sequence of the pair $(W, \partial W)$.

9.3. *The Cartan umbrella* (cf. [**16**, p. 204]). Let X be the surface $wx^2 = yz^2$ in complex projective 3-space. The singular set Σ of X is the curve $x = z = 0$. Let $p = [1, 0, 0, 0]$ and $q = [0, 0, 1, 0]$. The points p and q are *pinch points* of X. Then $\{X \setminus \Sigma, \Sigma \setminus \{p, q\}, \{p\}, \{q\}\}$ is a minimal Whitney stratification of X. The local homology sheaves of X have the following stalks:

	h_0	h_1	h_2	h_3	h_4
$X \setminus \Sigma$	0	0	0	0	$\mathbf{Z}$
$\Sigma \setminus \{p, q\}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$
$\{p\}$ or $\{q\}$	0	0	$\mathbf{Z}/2$	0	$\mathbf{Z}$

The sheaf h_3 is not constant–the generators of the stalk $\mathbf{Z}$ are flipped as one runs around an essential loop in $\Sigma \setminus \{p, q\}$. The homology of X is $\mathbf{Z}$, 0, $\mathbf{Z} \oplus \mathbf{Z}$, 0, $\mathbf{Z}$. The only possible nonzero differential of E is d^2: $H^0(X; h_2) \to H^2(X; h_3)$ (Figure 3). The definition of the coskeleton sequence E'

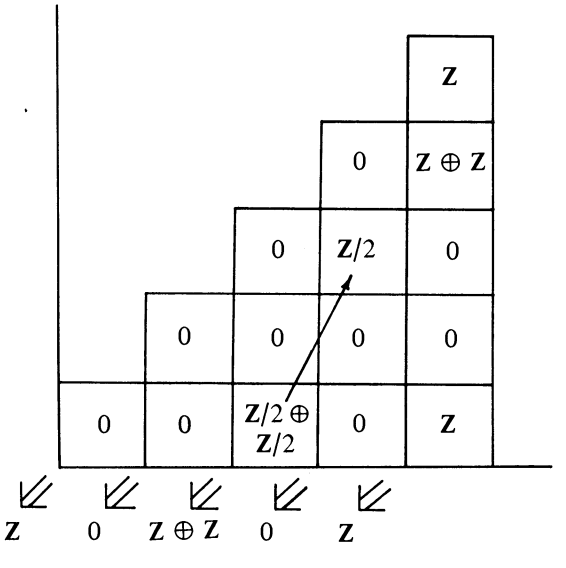

FIGURE 3

shows that $d^2(1, 0) = 1 = d^2(0, 1)$. Therefore the nonzero terms of E^∞ are $E^\infty_{4,0} = \mathbf{Z}$, $E^\infty_{4,2} = \mathbf{Z} \oplus \mathbf{Z}$, $E^\infty_{4,4} = \mathbf{Z}$, and $E^\infty_{2,0} = \mathbf{Z}/2$. Hence some classes in $H_2(X)$ have filtration zero. More precisely, let α be the image of the

orientation class of the curve $x = z = 0$ (the singular set of X), and let β be the image of the orientation class of the curve $w = y = 0$. Then each class $\gamma \in H_2(X)$ can be written uniquely as $\gamma = m\alpha + n\beta$, $m, n \in \mathbf{Z}$, and γ has filtration zero if and only if m is odd. If γ has filtration zero, then the support of any cycle representing γ must contain the points p and q. For example, the first Chern homology class $c_1(X)$ (as constructed by MacPherson [**15**]) is $5\alpha - \beta$, which has filtration zero.

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