Poincaré Duality in Spaces with Singularities

A Dissertation Presented to

The Faculty of the Graduate School of Arts and Sciences

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Department of Mathematics

In Partial Fulfillment

of the Requirements of the Degree

Doctor of Philosophy

by

Clinton G. McCrory

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Abstract

Poincaré Duality in Spaces with Singularities

A Dissertation Presented to the Faculty of the Graduate School of Arts and Sciences of Brandeis University, Waltham, Massachusetts.

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If X is a compact topological n-dimensional manifold without boundary, the Poincaré duality is an isomorphism from the $q^{\underline{th}}$ cohomology $H^q(X)$ to the $(n-q)^{\underline{th}}$ homology $H_{n-q}(X)$ for each integer q. In 1954, E. C. Zeeman introduced a spectral sequence $E(X)$ associated to any topological space X, which collapses to the Poincaré duality isomorphism when X is a manifold.

Using this spectral sequence, we study the failure of Poincaré duality for spaces which are not manifolds; for example, singular algebraic varieties.

We give a geometrical characterization of the filtration associated to $E(X)$ of a homology class $\alpha \in H_s(X)$. If X is triangulable, this filtration equals the "degrees of freedom" of α in X - the maximum integer q such that α is represented by a (singular) cycle in the complement of any $(q-1)$-dimensional subspace of X. If X is stratified by piecewise-linear manifolds, α has $\geq q$ degrees of freedom if and only if α is represented by a cycle whose support intersects each stratum in codimension $\geq q$.

For example, a normal complex projective algebraic variety V satisfies Poincaré duality (with respect to its canonical orientation class) if and only if every homology class $\alpha \in H_s(V)$ has $n-s$ degrees of freedom, where n is the (real) dimension of V.

We also show that the filtration of a cohomology class in the dual spectral sequence $\hat{E}(X)$ equals its geometric "codimension" in X (when X is triangulable), as conjectured by Zeeman.

Our analysis rests on a simple combinatorial formula for cap product, which provides an isomorphism from E (or $\hat{E}$) to a spectral sequence of G. Whitehead.

One application of the spectral sequences is to the study of "characteristic homology classes" of varieties. One such class is the diagonal $\Delta_V \in H_n(V \times V)$ (V as above). We show that Δ_V always has n degrees of freedom with field coefficients, and we study the "Thom classes" $U_V \in H^n(V \times V)$ geometrically dual to Δ_V. The principal result of the discussion is that V is a homology manifold if and only if V has a Thom class U_V which vanishes off the diagonal in $V \times V$. (Actually, our results hold for arbitrary topological n-circuits, or "geometric n-cycles.")

Poincaré Duality in Spaces with Singularities - Errata

Clint McCrory, September 1972.

Page	Line	
7	-7	Omit $[\tau, \sigma] = [\sigma, \tau]$. Let $[\sigma, \tau] = 0$ whenever σ is not a top dimensional proper face of τ.
18	-5	For an arbitrary topological space X, the filtrations (1.5) and (1.6) do not have length (n-s), but length n, the covering dimension of X.
21	-7	Replace $b \mid \text{support } b \neq 0$ by $b \mid X\text{-support } b = 0$.
35	6	The first term in $\partial(\sigma \cap \tau)$ is $\sum_{\gamma} \eta(\gamma) \langle \underline{\omega}_1, \ldots, \underline{\omega}_{p-q-1}, \underline{\tau} \rangle$
32		The example illustrated by the diagram indicates that there is a non-standard sign in my definition of cap product. In fact, the geometric definition of the duality chain map $\mathcal{D}$ (p.25) agrees with the algebraic definition of cap product (so $D = s \circ (+\mathcal{D})$, page 30). However, the cap product (2.3) satisfies $1 \cap x = (-1)^{1+\cdots+p}\, s(x)$, where $x \in C_p(K)$ and $s\colon C_p(K) \to C_p(K')$ is subdivision. This sign problem pops up for the product (2.6) in the following places:

Page	Line	
41	7	$\epsilon(y \cap x) = \pm (ty)(x)$
42	-6	(2.6) satisfies 0) with a sign .

A simple (but fundamental) change of notation removes these sign difficulties. Namely, let the "canonical" orientation of a simplex of the first derived complex K' be $\langle \underline{\sigma}_o, \ldots, \underline{\sigma}_i \rangle$, where $\sigma_o > \ldots > \sigma_i$ (instead of $\sigma_o < \ldots < \sigma_i$). This necessitates writing homology before cohomology in the cap product, and (2.3) (p. 32) becomes

$$\gamma_p \cap \sigma^q = \Sigma\, \eta(\tau, \omega_{p-q-1}, \ldots, \omega_1, \sigma) \left\langle \underline{\tau}, \underline{\omega}_{p-q-1}, \ldots, \underline{\omega}_1, \underline{\sigma} \right\rangle.$$

The analogous change is made in (2.6) (p.38). We then have $1 \cap x = s(x)$ for (2.3), and the sign for (2.6) on p.41 is also positive.

It remains to check the boundary formulas for these new definitions (Lemma 1A, p.34, and Lemma 1B, p.39). A short calculation shows that (2.3) satisfies

$$\partial(x_p \cap y^q) = \partial x \cap y + (-1)^{p-q}\, x \cap \delta y$$

(This is the same sign as [Sp], p.253.) (2.6) satisfies

$$\partial(x_p \cap y^q) = (-1)^q\, \partial x \cap y + (-1)^{q+1}\, x \cap \delta y$$

(This is the same sign as [HW], p.154.) These formulas enable all subsequent arguments to go through.

Page	Line	
56	7	e_X is not defined for an arbitrary n-dimensional space X. (The local singular homology of X may be non-zero in dimensions greater than n if X is locally pathological.) A convenient class of spaces for which e_X is defined is <u>locally triangulable</u> spaces.
62	2	For any locally triangulable n-dimensional space X ...

<u>Remark</u>. The geometric characterization of the filtration of a homology (or cohomology) class in Zeeman's spectral sequence (section 5) should hold for locally triangulable spaces. (This was suggested by J. Munkres.)

Page	Line	
72		Clearly c_3 is a boundary! Thus $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, $H^1(X; H_3) \cong \mathbb{Z} \oplus \mathbb{Z}$, and e_X is still an isomorphism, whereas $\cdot \cap [X]$ and θ aren't surjective, as desired.
73	-2	$L_1, \ldots, L_k$ are the closures of the components of link v - S(link v)
81	3	$c_{\sigma_i} = \sum_{\omega_n > \sigma} \omega_n$
90	2, 12	See erratum, p.179 .
91	1	See erratum, p.179 .
94	5	$D_q \circ \delta = \partial \circ D_{q-1}$

Page	Line	
98		This proposition was proved by Akin [Ak] (by a different argument), without the condition that the isotopy can be made arbitrarily small.
101	-4	See erratum, p.179 .
115	11	See erratum, p.179 .
125	7	$f^{-1}(\lvert L_q\rvert)$ should be $X-f^{-1}(\lvert L_{q-1}\rvert)$
125	9	The argument given here for (ii) is wrong, but it can be modified (cf. erratum, p.179). The argument given is valid if A is a subpolyhedron of Y.
174	-6	If X has intersection pairings φ_F on $H_*(X; F)$ <u>satisfying (4.5)</u>...

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Introduction

The topology of a space is reflected in the structure of its homology groups. The homogeneity of a manifold manifests itself in the Poincaré duality, which has its most precise geometrical expression in Lefschetz' intersection theory. Cycles in a manifold can be intersected because they have enough freedom of movement to be made transverse to each other. In this thesis, I will discuss how the singularities of a space (for example, an algebraic variety) disturb Poincaré duality by restricting the "freedom" of its cycles.

My main tool is a spectral sequence due to Zeeman [Ze 1]. It provides a framework for the geometrical study of Poincaré duality (as undertaken classically by Čech and Wylie - cf. [Wy]). The most striking feature of this spectral sequence is the topologically invariant filtration it induces on the homology groups of a space. The principal result of chapter I is that the filtration of a homology class in a triangulable space X is equal to its "degrees of freedom" - the maximum integer q such that the class is represented by a cycle in the complement of any $(q-1)$-dimensional subspace. As a corollary, a "normal" geometric cycle X (e.g. a normal algebraic variety) satisfies Poincaré duality if and only if each homology class in

X has degree of freedom equal to its formal codimension. I also show that the filtration of a cohomology class in the "dual" spectral sequence is equal to its geometric "codimension," as conjectured by Zeeman.

If X has a p.l. structure, the degrees of freedom of a homology class has a more precise description. Given a p.l. stratification of X (for example, a p.l. triangulation, or a Whitney stratification of an algebraic variety), a homology class has $\geq q$ degrees of freedom if and only if it is represented by a cycle whose support intersects each stratum in codimension $\geq q$.

The passage from algebra to geometry is effected by a simple combinatorial formula for the cap product. If σ^q and τ^p are (oriented) simplexes of the complex K, and $\sigma < \tau$, let

$$\sigma \cap \tau = \Sigma \pm \langle \underline{\sigma}, \underline{\omega}_1, \ldots, \underline{\omega}_{p-q-1}, \underline{\tau} \rangle$$

where the sum is over all $(p-q)$-simplexes of the first derived complex K' consisting of barycenters $\underline{\omega}$ of simplexes ω with $\sigma \leq \omega \leq \tau$. It follows from this formula that a homology class has filtration q if and only if it is represented by a (simplicial) cycle in the "q-coskeleton" of K. (The q-coskeleton is a canonical codimension q subcomplex of K', equal to the $(n-q)$-skeleton of the classical dual cell complex when K is a combinatorial n-manifold.) The q-coskeleton has q degrees of freedom in $|K|$ since it intersects each simplex of K in codimension q.

If X is a (normal) geometric n-cycle, the homology class Δ represented by the diagonal in $X \times X$ has n degrees of freedom if and only if Δ has a "Poincaré dual" cohomology class U in $X \times X$ ($U \cap [X \times X] = \Delta$). In chapter II, I will show that such a "Thom class" U always exists with field coefficients. Furthermore, X is a duality space ($\cdot \cap [X]$ is an isomorphism) if and only if X has a Thom class U with a certain symmetry property. The main result of chapter II is that X is an integral homology manifold (i.e. its local integral homology groups are like those of Euclidean space) if and only if X has a Thom class U which vanishes off the diagonal in $X \times X$. Finally, I show that Thom classes for X are equivalent to "intersection pairings" on the homology of X. As a corollary, any geometric cycle which has a pairing satisfying certain simple axioms must be a homology manifold.

There are several exciting directions for possible applications of these results. One is the study of the Stiefel and Chern homology classes of algebraic varieties, and another is the analysis of bordism theories with singularities (see §6 of chapter I).

For many helpful conversations, I thank Phil Lynch and David Stone, who had the patience to listen to my ravings. My advisor, Professor Jerome Levine, has made quite a few clarifying suggestions, for which I am grateful. I especially thank Dennis Sullivan, who introduced me to Zeeman's spectral sequence, and who has inspired me all along the way.

I. The Poincaré duality spectral sequence

In his thesis (1954; [Ze 1] 1963) Zeeman defined a spectral sequence E for any topological space X which runs

$$H^q(X; \mathscr{H}_p) \Longrightarrow H_{p-q}(X),$$

where $\mathscr{H}_p$ is the sheaf of local homology of X. E is a topological invariant of X, but not a homotopy type invariant. If X is an n-manifold, E collapses to the Poincaré duality isomorphism $H^q(X) \xrightarrow{\approx} H_{n-q}(X)$. If X is a general space, E relates its local and global homological structures, providing subtle information about how the "singularities" of X disturb duality.

Associated with E is a filtration of the homology of X. The filtration of a cycle α in X measures how close the cycle comes to being geometrically dual to a cocycle β. (If α is dual to β, and γ is a cycle, $\beta(\gamma)$ is the "intersection" of γ with α.) A problem posed by Zeeman is to give a geometric interpretation of the filtration of a cycle -- in order to gain some insight into the failure of duality.

Zeeman gave an elegant combinatorial definition of E for a triangulable space, and defined E for arbitrary spaces as the limit of spectral sequences associated to nerves of open coverings of X. Cartan ([SC], 1951) defined on isomorphic spectral sequence, using sheaf theory, to prove duality for topological manifolds. (The modern sheaf-theoretic proofs of duality use generalizations of Cartan's spectral sequence - see [Sw] or [Br] for example.)

I will discuss the geometry of Zeeman's spectral sequence for triangulable spaces by explicit use of their combinatorial structure. This analysis will be put in a geometrical setting in section 5, where I will describe the relation between the filtration of a cycle and its interaction with a (p.l) stratification of X. (A triangulation is the simplest stratification of a space.)

In §1 I will summarize the definition and basic properties of E for a simplicial complex K ([Zel], §2). Zeeman emphasizes the algebraically dual spectral sequence $\hat{E}$, which converges to the cohomology of K, so my disucssion of E will complement his. In §2 I will give a combinatorial definition of cap product, which provides an isomorphism from Zeeman's spectral sequence to a spectral sequence of G. Whitehead (§3). This isomorphism gives rise to a geometric description of E (§§4 and 5), including an interpretation of the filtration of a homology class, and a proof of Zeeman's conjecture that the filtration (associated to $\hat{E}$) of a cohomology class equals its "codimension". In §6 I will use the spectral sequence to study the euler characteristic and the Stiefel homology classes of a variety, and I will indicate possible ways of increasing its usefulness.

1. Zeeman's spectral sequence

Let K be a finite simplicial complex. Recall the definition of the simplicial chain complex $C_*(K)$. Suppose that each simplex of K is given an arbitrary orientation. Then $C_i(K)$ is the free abelian group generated by the oriented i-simplexes of K, and $\partial : C_i(K) \to C_{i-1}(K)$ is the homomorphism induced by

$$\partial\tau = \sum_{\sigma<\tau} [\sigma, \tau]\sigma ,$$

where if σ and τ are oriented simplexes of K, $[\sigma, \tau]$ is the incidence number of σ and τ. (If τ is an i-simplex and σ is an (i-1)-face of τ, then $[\sigma, \tau] = \pm 1$, depending on whether the given orientation of τ induces $\pm$ the given orientation on σ. Let $[\tau, \sigma] = [\sigma, \tau]$, and let $[\sigma, \tau] = 0$ whenever one is not a top dimensional proper face of the other.) The fact that $\partial \circ \partial = 0$ is equivalent to the formula

$$\text{(1.1)} \qquad \sum_{\sigma \in K} [\rho, \sigma][\sigma, \tau] = 0 \qquad \rho, \tau \in K.$$

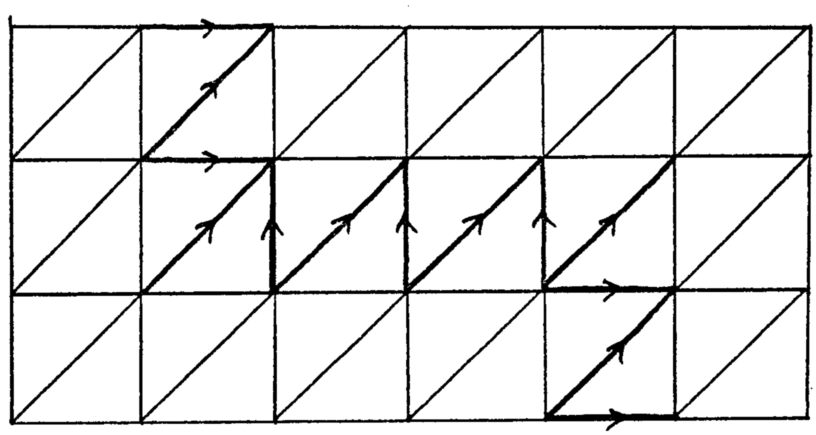
The simplicial homology $H_*(K)$ is the homology of $C_*(K)$, which is independent of the choices of orientations for the simplexes of K (cf. [HW], chapter 2).

Since K is finite, the group of simplicial cochains $C^i(K) = \mathrm{Hom}(C_i(K), \mathbb{Z})$ can be identified with $C_i(K)$, $\delta = \mathrm{Hom}(\partial, \mathbb{Z}) : C^{i-1}(K) \to C^i(K)$ is induced by

$$\delta\sigma = \sum_{\tau} [\sigma, \tau]\tau ,$$

and $\delta \circ \delta = 0$ is equivalent to (1.1). The simplicial cohomology $H^*(K)$ is the homology of $C^*(K)$.

Remark. On a finite complex, a simplicial chain and a simplicial cochain are the same thing - a sum of oriented simplexes. Much of the following discussion will explicitly use the geometry of simplicial cocycles:



If we replace $C_i(K)$ in the preceding definition by the group of formal sums of oriented simplexes with coefficients in any abelian group G, we obtain the simplicial homology $H_*(K; G)$ and the simplicial cohomology $H^*(K; G)$.

The description of Zeeman's spectral sequence involves the simplicial homology or cohomology of K with a "system of coefficients", the combinatorial analog of a sheaf. A simplicial complex K can be regarded as a category with objects the simplexes of K and morphisms the inclusions of faces. A <u>stack</u> on K is a functor from K to the category of abelian groups. If $\mathcal{L}$ is a covariant stack, the cohomology of K with coefficients in $\mathcal{L}$, $H^*(K; \mathcal{L})$, is the homology of the chain complex $C^*(K; \mathcal{L})$, where a cochain in $C^i(K; \mathcal{L})$ is a sum

$$\Sigma\, g_\sigma \sigma\,, \quad \sigma \text{ an } i\text{-simplex of } K, \quad g_\sigma \in \mathcal{L}(\sigma),$$

and

$$(1.2) \qquad \delta(\Sigma\, g_\sigma \sigma) = \sum_\tau\, [\sigma, \tau]\mathcal{L}(\sigma < \tau)(g_\sigma)\tau\,,$$

where $\mathcal{L}(\sigma < \tau) : \mathcal{L}(\sigma) \to \mathcal{L}(\tau)$ is the morphism corresponding to the inclusion $\sigma < \tau$. If $\mathcal{L}$ is contravariant, $H_*(K; \mathcal{L})$ is defined similarly.

For example, if G is an abelian group and $\mathcal{L}$ is the constant stack with $\mathcal{L}(\sigma) = G$ for all σ and $\mathcal{L}(\sigma<\tau) = \mathrm{id}_G$, $H^q(K;\mathcal{L}) = H^q(K; G)$ and $H_q(K;\mathcal{L}) = H_q(K; G)$.

The <u>local (integral) homology stack</u> $\mathcal{H}_p$ is defined as follows. $\mathcal{H}_p(\sigma) = H_p(\overline{\text{star }\sigma}, \partial\,\overline{\text{star }\sigma})$, and $\mathcal{H}_p(\sigma<\tau) : H_p(\overline{\text{star }\sigma}, \partial\,\overline{\text{star }\sigma}) \to H_p(\overline{\text{star }\tau}, \partial\,\overline{\text{star }\tau})$ is "restriction", so $\mathcal{H}_p$ is covariant. Here star σ consists of all simplexes of K which have σ as a face, $\overline{\text{star }\sigma}$ is its closure, and $\partial\,\overline{\text{star }\sigma}$ is $\overline{\text{star }\sigma}$ - star σ. If $\sigma < \tau$, star $\sigma \supset$ star τ, and $H_p(\sigma<\tau)$ is the composition

$$\begin{tikzcd}
H_p(\overline{\text{star }\sigma}, \partial\,\overline{\text{star }\sigma}) \arrow[d, "\text{excision}"] & H_p(\overline{\text{star }\tau}, \partial\,\overline{\text{star }\tau}) \arrow[d, "\text{excision}"] \\
H_p(K, K - \text{star }\sigma) \arrow[r, "\text{restriction}"] & H_p(K, K - \text{star }\tau)
\end{tikzcd}$$

Zeeman simplifies the definition of $\mathcal{H}_p$ by use of the classical convention that if S is any collection of simplexes of K, $H_*(S)$ is the (simplicial) homology of that part of $C_*(K)$ generated by S. If $K-S$ is a subcomplex of K, then $H_*(S)$ is the relative homology group $H_*(K, K-S)$. (For example, if S covers an open subset U of $|K|$, $H_*(S)$ is isomorphic to the singular homology of U based on infinite chains, and $H^*(S)$ is isomorphic to the singular cohomology of U with compact supports.) If S and T cover open sets $U \supset V$ of $|K|$, there is a simplicial restriction homomorphism

$$H_p(S) \to H_p(T)$$

and a simplicial inclusion homomorphism

$$H^p(T) \to H^p(S).$$

Thus $\mathscr{H}_p(\sigma) = H_p(\text{star } \sigma)$, and $\mathscr{H}_p(\sigma < \tau)$ is the restriction homomorphism

$$H_p(\text{star } \sigma) \to H_p(\text{star } \tau).$$

Similarly, the <u>local cohomology stack</u> $\mathscr{H}^p$ on K is defined by $\mathscr{H}^p(\tau) = H^p(\text{star } \tau)$, and if $\sigma < \tau$,

$$H^p(\text{star } \tau) \to H^p(\text{star } \sigma)$$

is induced by the inclusion $\text{star } \tau \subset \text{star } \sigma$.

The relation between stacks and sheaves can be summarized as follows. If $\mathcal{S}$ is any presheaf of abelian groups on $X = |K|$ (i.e. a functor from the category of open sets and inclusions of open sets of X to the category of abelian groups), define a stack $\mathcal{L}$ on K by $\mathcal{L}(\sigma) = \mathcal{S}(|\text{star } \sigma|)$. If $\mathcal{S}$ is contravariant, then $\mathcal{L}$ is covariant. Then if $\mathcal{S}$ is "invariant under subdivision", $\mathcal{S}$ is a sheaf and $H^*(X; \mathcal{S}) \cong H^*(K; \mathcal{L})$. $\mathcal{S}$ is invariant under subdivision means that

$$\mathcal{S}(|\text{star } \sigma| \supset |\text{star } \omega|) : \mathcal{S}(|\text{star } \sigma|) \to \mathcal{S}(|\text{star } \omega|)$$

is an isomorphism whenever $\sigma \in K$ and ω is a simplex of a subdivision of K with interior $\omega \subset$ interior σ. For example, if $\mathcal{S}$ is the presheaf $\mathcal{S}(U) = H_p(X, X-U)$ (singular homology), then $\mathcal{S}$ is invariant under subdivision, and the associated stack is $\mathcal{H}_p$. If $\mathcal{S}$ is covariant and invariant under subdivision, $H_*(X; \mathcal{S}) \cong H_*(K; \mathcal{L})$. (For related remarks, see the proof of Lemma 10 of [Ze 1], and §4 below, where a slightly different definition of stacks is given.)

Now Zeeman's homology spectral sequence is defined as follows. If $x = \Sigma\, n_\sigma \sigma$ is an integral simplicial chain, the <u>support</u> of x is the smallest subcomplex of K containing all the σ for which $n_\sigma \neq 0$. Let $D_{p,q}$ be the abelian group $C_p(K) \otimes C^q(K)$ modulo the subgroup $\{x \otimes y,\ y \mid \text{support } x = 0\}$.

Let $d : D_{p,q} \to D_{p-1,q} + D_{p,q+1}$ be induced by

$$d(x \otimes y) = \partial x \otimes y - (-1)^p x \otimes \delta y. \tag{1.3}$$

(The signs are justified in [Ze 1].) Let $D = \Sigma D_{p,q}$, and let $D_s = \sum_{p-q=s} D_{p,q}$. Then D is a double complex, bigraded by p and q, with a total grading $s = p-q$. $d : D_s \to D_{s-1}$, and $d \circ d = 0$. There are two spectral sequences associated to (D, d), got by filtering D with respect to p or q. (For a pleasant discussion of these spectral sequences, see [Go], page 86.) They both converge to $H_*(D)$, the homology of D with respect to d.

These spectral sequences are interesting because of the subtle "facing relation" defining D. If x and y are elementary chains σ and τ (i.e. x is the chain 1σ and y is the cochain $y(\tau) = 1$, $y(\rho) = 0$ for $\rho \neq \tau$), then $y \mid \text{support } x = 0$ just says that τ is not a face of σ. Thus $D_{p,q}$ can be identified with the free abelian group generated by all pairs of simplexes (σ_p, τ^q) such that $\sigma \geq \tau$. (Note however that the differential d on $C_*(K) \otimes C^*(K)$ doesn't preserve the facing relation. That is why D was defined as a <u>quotient</u> of $C_*(K) \otimes C^*(K)$.) Thus the "domain" of D is $\{(p, q),\ 0 \leq q \leq p \leq n\}$.

An easy argument (Lemma 1 of [Ze 1]) shows that the p-filtration spectral sequence of (D, d) collapses to an isomorphism $H_*(K) \xrightarrow{\approx} H_*(D)$. Zeeman's homology spectral sequence E for K is the q-filtration spectral sequence associated to D. He defines a "dual" double complex $\hat{D}$, for which the p-filtration spectral sequence collapses to an isomorphism $H^*(\hat{D}) \xrightarrow{\approx} H^*(K)$. Zeeman's cohomology spectral sequence $\hat{E}$ for K is the q-filtration spectral sequence associated to $\hat{D}$.

<u>Theorem</u> ([Ze 1], p. 159). 1) The spectral sequences of the finite simplicial complex K run

$$E^2_{p,q} \cong H^q(K; \mathcal{X}_p) \Longrightarrow H_{p-q}(K)$$

$$\hat{E}^2_{p,q} \cong H_q(K; \mathcal{X}_p) \Longrightarrow H^{p-q}(K).$$

2) The sequences are topological (although not homotopy type) invariants of $|K|$.

3) If K is a closed orientable combinatorial n-manifold, then both sequences collapse to the Poincaré duality isomorphism.

The proof of 1) for E goes as follows. Let $F^qD = \sum_{j \geq q} D_{i,j}$. F^qD is generated by all pairs of simplexes (σ_i, τ^j) such that $\sigma \geq \tau$ and $j \geq q$. Now

$$E^1_{p,q} = H_{p-q}(F^qD, F^{q+1}D),$$

where the homology is with respect to the differential induced by $d(\sigma\otimes\tau) = \partial\sigma\otimes\tau - (-1)^\rho\sigma\otimes\delta\tau$. Clearly $F^qD/F^{q+1}D$ can be identified with $\sum_{\tau^q} C_*(\text{star } \tau^q)$, where $C_*(\text{star } \tau^q)$ is the free abelian group generated by the simplexes σ of star τ (i.e. $\sigma \geq \tau$). Furthermore, the second summand of $d(\sigma_p\otimes\tau^q)$ (1.3) lies in $F^{q+1}D$, so d on $F^qD/F^{q+1}D$ corresponds to the simplicial boundary operator in each $C_*(\text{star } \tau^q)$. Thus

$$(1.4) \qquad E^1_{p,q} \cong \sum_{\tau\in C^q} H_p(\text{star } \tau) = \sum_{\tau\in C^q} H_p(\overline{\text{star } \tau}, \partial\,\overline{\text{star } \tau})$$

$$= C^q(K;\mathscr{H}_p)$$

Now $d^1 : E^1_{p,q} \to E^1_{p,q+1}$ is the boundary map $H_{p-q}(F^qD, F^{q+1}D) \to H_{p-q-1}(F^{q+1}D, F^{q+2}D)$ induced by d. Since the first summand of $d(\sigma_p\otimes\tau^q)$ (1.3) doesn't lie in $F^{q+1}D$, d^1 corresponds to $\pm\,\delta : C^q(K;\mathscr{H}_p) \to C^{q+1}(K;\mathscr{H}_p)$. (Compare the general discussion on pp. 86-87 of [Go].)

Zeeman proves the topological invariance of E (and $\hat{E}$) in §3 of [Ze 1]. Intuitively, his proof generalizes the proof that the simplicial cohomology of a complex K is isomorphic to the

Čech cohomology of the underlying space $|K|$. E is clearly not a homotopy invariant of $|K|$, because the sheaves $\mathcal{H}_p$ of local homology occurring in the E^2 term are not invariants of the global homotopy type of $|K|$. For example, Zeeman calculates E for K = the cone on a closed manifold M ([Ze 1], p. 160). $|K|$ is contractible, but the local homology at the cone point (the only singularity of $|K|$) is the homology of M.

If K is a closed combinatorial n-manifold, then $\overline{\text{star } \sigma}$ is a combinatorial n-ball for all $\sigma \in K$, so

$\mathcal{H}_p(\sigma) = \mathcal{H}_p(\text{star } \sigma, \partial \text{ star } \sigma) \cong \begin{cases} \mathbb{Z} & p = n \\ 0 & p \neq n \end{cases}$ i.e. K is a homology n-manifold (has the local homology of euclidean n-space). K is orientable means $\mathcal{H}_n$ is the constant sheaf $\mathbb{Z}$, so E collapses to the isomorphism

$$H^q(K) = H^q(K;\mathcal{H}_n) \xrightarrow{\approx} H_{n-q}(K).$$

Remark: Actually, Zeeman doesn't verify that this isomorphism is "the" Poincaré duality isomorphism. It's possible to prove this using the analysis in [Go] mentioned above - in fact this led me to the alternate description of E to come in §3.

Formally, E has the same domain as D, namely $0 \leq q \leq p \leq n = \dim K$. The differentials run

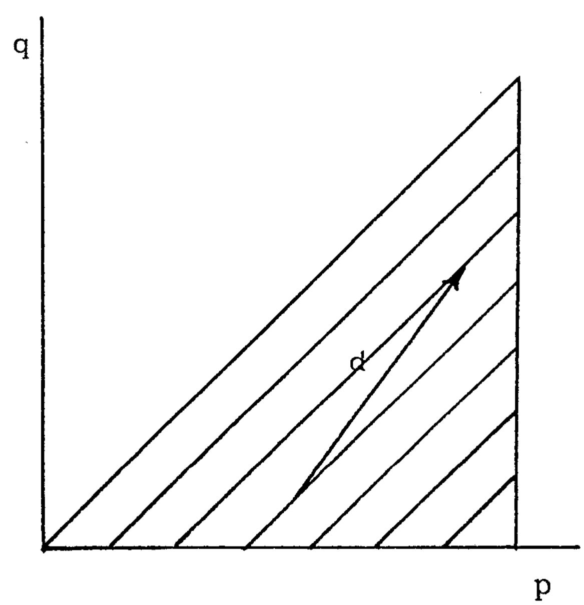
$$d^r : E^r_{p,q} \to E^r_{p+r-1,\, q+r} ,$$

and the $E^\infty_{s+q,\, q}$, $0 \leq q \leq n-s$, are the successive quotients of the filtration

$$(1.5) \qquad H_s(K) = F^0_s \supset F^1_s \supset \dots \supset F^{n-s}_s \supset 0$$

induced on $H_s(K)$ by the q-filtration of D. Precisely, F^q_s corresponds to $\mathrm{Im}(H_s(F^qD) \to H_s(D))$ under the isomorphism $H_s(K) \xrightarrow{\approx} H_s(D)$, and

$$E^\infty_{s+q,\, q} \cong F^q_s / F^{q+1}_s .$$



$\hat{E}$ has the same domain, and the differentials run in the opposite direction,

$$\hat{d}^r : \hat{E}^r_{p,q} \to \hat{E}^r_{p-r+1,\, q-r} .$$

The $E^\infty_{s+q,\, q}$, $0 \leq q \leq n-s$, are the successive quotients of the filtration

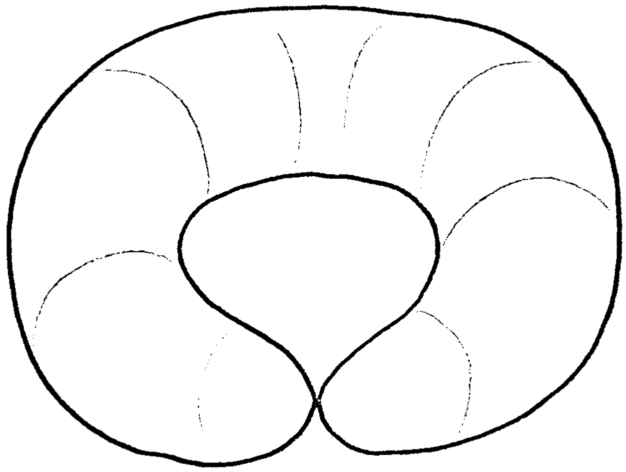
$$(1.6) \qquad 0 \subset \hat{F}^s_0 \subset \hat{F}^s_1 \subset \ldots \subset \hat{F}^s_{n-s} = H^s(K)$$

induced on $H^s(K)$ by the q-filtration of $\hat{D}$.

The filtrations (1.5) and (1.6) are in fact defined as topological invariants of any space X (since E and $\hat{E}$ can be defined topologically - see [Ze 1], §3).

<u>Example.</u> Let X be the "pinched torus"

$$X = S^1 \times S^1/\{p\} \times S^1, \quad p \in S^1.$$



X is homeomorphic to the curve $x^3 + y^3 = xyz$ in the complex projective plane, which has one singular point $(x = y = 0)$. Clearly $\mathcal{H}_0 \cong 0$; $\mathcal{H}_1 \cong 0$ except at the singular point x, where $\mathcal{H}_1 \cong \mathbb{Z}$; and $\mathcal{H}_2 \cong \mathbb{Z}$ except at x, where $\mathcal{H}_2 \cong \mathbb{Z} \oplus \mathbb{Z}$. The nonzero entires in the E^2 term are $E^2_{1,0} = H^0(X;\mathcal{H}_1) \cong \mathbb{Z}$, $E^2_{2,0} = H^0(X;\mathcal{H}_2) \cong \mathbb{Z}$, and $E^2_{2,2} = H^2(X,\mathcal{H}_2) \cong \mathbb{Z}$; and $d^2 = 0$. Therefore $E^2 = E^\infty$, and so $H_0(X) = F^2_0$, $H_2(X) = F^0_2$, and $H_1(X) = F^0_1$.

H^2			$\mathbb{Z}$
H^1		0	0
H^0	0	$\mathbb{Z}$	$\mathbb{Z}$
	$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{H}_2$

If $\alpha \in H_s(X)$, filtration $\alpha = \max\{q, \alpha \in F_s^q\}$. Thus the generator of $H_1(X)$ has filtration 0 instead of filtration 1 as it would in a 2-manifold. (If X is a manifold, $\mathcal{H}_p = 0$ for $p < n$, so the E^2 term is concentrated along the line $p = n$, and all cycles have maximum filtration, i.e. $H_s(X) = F_s^{n-s}$ for all s.)

Now $H_0(X) \cong H_1(X) \cong H_2(X) \cong \mathbb{Z}$, and so $H^0(X) \cong H^1(X) \cong H^2(X) \cong \mathbb{Z}$. Let $[X]$ be a generator of $H_2(X)$. The Poincaré duality map $H^i(X) \to H_{2-i}(X)$ is given by $\alpha \to \alpha \cap [X]$, $\cap$ = cap product.

Lemma. $\cdot \cap [X] : H^1(X) \to H_1(X)$ is the zero map.

Proof: X can also be considered as the two-sphere with the north and south poles identified. Let $f : S^2 \to X$ be the identification map. Let $[S^2]$ be a generator of $H_2(S^2)$ with $f_*[S^2] = [X]$. If $\beta \in H^1(X)$,

$$\beta \cap [X] = \beta \cap f_* [S^2] = f_*(f^*(\beta) \cap [S^2]) = f_*(0 \cap [S^2]) = 0.$$

The second equality holds by the "naturality" of cap product - cf. §3. $f^*(\beta) = 0$ because $H^1(S^2) = 0$.

Thus the generator of $H_1(X)$ is not in the image of the "duality map" $\cdot \cap [X]$, i.e. there is no cohomology class dual to it. This is the algebraic manifestation of its not having maximum filtration. We will see that, geometrically, this failure of duality stems from the fact that the generator of $H_1(X)$ does not have as many "degrees of freedom" as it would in a 2-manifold - any cycle representing it must pass through the singular point x.

It turns out that $\hat{E}$ of the pinched torus is "dual" to E, so the elements of $H^1(X)$ have filtration 0 (Filtration $\beta = \min\{q, \beta \in \hat{F}_q\}$). Algebraically, this corresponds to the fact that $\cdot \cap [X]$ is the zero map on $H^1(X)$; geometrically, to the fact that the generator of $H^1(X)$ can be represented by a cocycle which is supported by the singular point x. If X were a 2-manifold, any element of $H^1(X)$ would be given by the cocycle b = "intersection with a" for some 1-cycle a, and so the support of b would be 1-dimensional. (If b is a cocycle, support b is the smallest closed subset of X such that $b \mid \text{support } b \neq 0$.)

The ideas suggested by this example will be developed in the following sections. I will return to the pinched torus periodically, since it is the simplest example of a "geometric cycle with singularity."

A more sophisticated example is given by Zeeman on p. 181 of [Ze 1]. Let X be the quadric cone in $P_3(\mathbb{C})$,

$x^2 + y^2 + z^2 = 0$ in homogeneous coordinates (x, y, z, w). X is homeomorphic to the Thom space of the tangent bundle of the 2-sphere. X is a 4-manifold except for one singular point x, and a neighborhood of x is homeomorphic to the cone on real projective 3-space $P_3(\mathbb{R}) \cong$ the tangent circle bundle of the 2-sphere. The homology of X is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. The spectral sequence E shows that the generator β of $H_2(X)$ has filtration 0, but 2β has filtration 2. This is because any cycle representing β must pass through x, but 2β can be represented by a cycle in the complement of x. $\hat{E}$ shows that all elements of $H^2(X)$ have filtration 2. This reflects the fact that the support of any 2-cocycle on X is 2-dimensional. Algebraically, we have that

$$\cdot \cap [X] : H^2(X) \to H_2(X)$$

is "multiplication by 2", which is injective (so 2-cohomology classes have maximum filtration) but not surjective (so not all 2-homology classes have maximum filtration). I will give a complete analysis of E and $\hat{E}$ when X has isolated singularities in §§4 and 5.

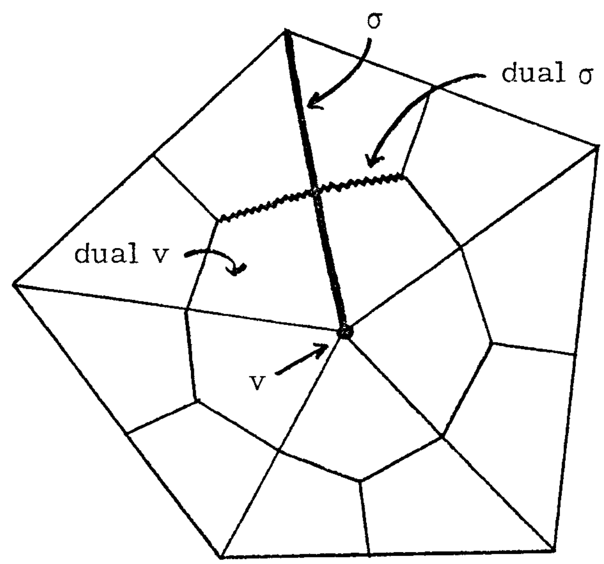
The spectral sequences of the quadric cone in $P_3(\mathbb{C})$ have particular interest because they show that $\hat{E}$ may not be dual to E - since $\hat{d}^2 \neq 0$ but $d^2 = 0$. Also, E and $\hat{E}$ may be significantly different with other coefficients than $\mathbb{Z}$. The definition of D and $\hat{D}$, and hence E and $\hat{E}$, can be made with coefficients in any abelian group G. Then the coefficients of the E^2 term are the stacks of local homology with coefficients in G, and E converges to $H_*(K; G)$ ($\hat{E} \Rightarrow H^*(K, G)$). For the spectral sequences of the quadric cone X with $\mathbb{Z}/2$ coefficients, the non-zero class in $H^2(X) \cong \mathbb{Z}/2$ has filtration zero. This reflects the fact that although the generator $\alpha \in H^2(X)$ can't be supported by the singular point x, its image under the coefficient homomorphism $H^2(X) \to H^2(X; \mathbb{Z}/2)$ can. (Moreover, notice that for rational or $\mathbb{Z}/p$ coefficients with p odd, X is a homology manifold, and so E and $\hat{E}$ collapse.)

In the process of generalizing E and $\hat{E}$ to arbitrary topological spaces, Zeeman defines a spectral sequence $E(f)$ associated to any continuous map $f : X \to Y$. The general sequences are described in Theorem 2 of [Ze 1]. $E^2_{p,q} \cong H^q(Y; f_*\mathcal{H}_p)$, where $\mathcal{H}_p$ is the sheaf of local homology on X, and $f_*\mathcal{H}_p$ is the sheaf induced by f_* on Y. $E(f) \Rightarrow H_*X$. (Zeeman's spectral sequence E of X is just E of the identity map on X.) I will give a geometrical discussion of E of a simplicial map in §§3 and 5.

2. The cap product

The first step in a geometrical analysis of the spectral sequence E is an interpretation of the isomorphism $H_*(K) \xrightarrow{\approx} H_*(D)$ ([Ze 1] Lemma 1). Zeeman shows that the inverse isomorphism is induced by cap product, and as a corollary he shows that there is a relation between cap product and the filtration of a homology class ([Ze 1] Theorem 3).

To understand the geometrical relation between cap product and duality, let us return to the days of yesteryear, and the classical proof of Poincaré duality for a combinatorial n-manifold K. K is a finite simplicial complex such that the closed star of each simplex σ is a combinatorial n-cell. (That is, $\overline{\text{star } \sigma}$ has a simplicial subdivision isomorphic to some subdivision of the standard n-simplex.) Let K' be the first barycentric subdivision of K. To each simplex in K is associated a "dual," which is a subcomplex of K'. If σ^i is an i-simplex of K, dual σ^i is a combinatorial $(n-i)$-cell.



Assume that K is an orientable n-manifold, i.e. its n-simplexes can be oriented coherently (in other words, so their sum is a cycle). Choose such an orientation for the n-simplexes of K, and orient the other simplexes arbitrarily. (A vertex has a canonical orientation.) This determines orientations of the dual cells so that if σ is an elementary cochain,

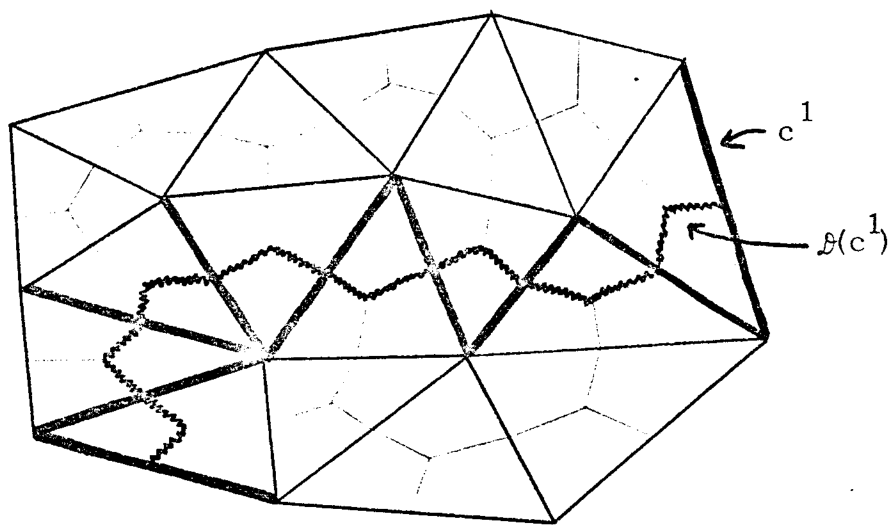
$$\text{dual}(\delta\sigma) = \partial(\text{dual}\ \sigma).$$

Thus taking duals induces a chain isomorphism

$$\mathcal{D} : C^*(K) \xrightarrow{\approx} D_{n-*}(K),$$

where $D_*(K)$ is the complex of cellular chains on K consisting of formal sums of oriented dual cells. Therefore $\mathscr{D}$ induces an isomorphism of homology. $H(C^*(K)) = H^*(K)$, and $H(D_*(K)) \cong H_*(K)$ (cellular homology $\cong$ simplicial homology, cf. [HW] §3.8). This is the classical description of the duality isomorphism

$$H^*(K) \xrightarrow{\approx} H_{n-*}(K).$$



(Of course, duality wasn't expressed in terms of cohomology originally - my aim is to bring out the geometrical content of the classical proof.)

The geometric duality $\sigma \to$ dual σ can be described in an arbitrary simplicial complex K. The vertexes of the derived complex K' are the barycenters of simplexes of K. Each simplex of K' can be written uniquely as $\langle \underline{\sigma}_0, \ldots, \underline{\sigma}_i \rangle$, where each

$\sigma_j \in K$, $\underline{\sigma}_j$ is its barycenter, and $\sigma_0 < \sigma_1 < \ldots < \sigma_i$. ($\sigma < \tau$ means σ is a proper face of τ.) Now let dual σ $= \{<\underline{\sigma}_0, \ldots, \underline{\sigma}_i> \in K', \ \sigma \leq \sigma_0\}$. Dual σ is a subcomplex of K' "transverse" to σ; if K is a combinatorial manifold, dual σ is the classical dual cell of σ (cf. [Ze 1], proof of Lemma 11).

Recall the basic subcomplexes of K and K' which describe the geometry of K at a simplex σ:

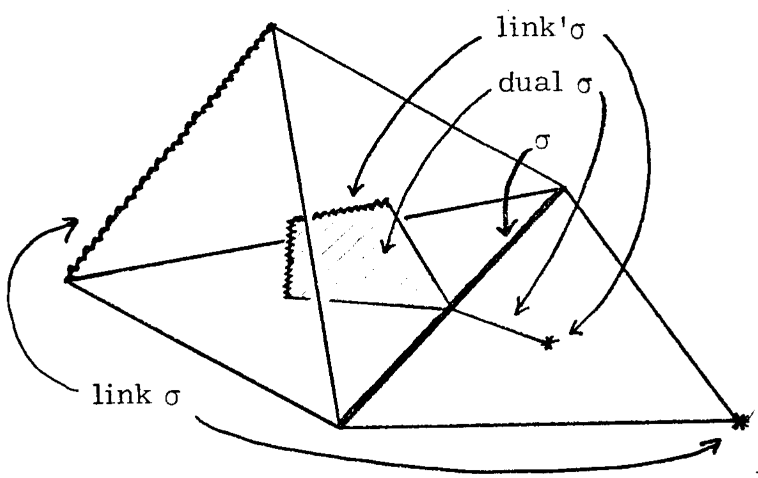
star $\sigma = \{\tau \in K, \ \sigma \leq \tau\}$

star$'\sigma = \{<\underline{\sigma}_0, \ldots, \underline{\sigma}_i> \in K', \ \sigma = \sigma_j$ for some $j\}$

link $\sigma = \{\tau \in K, \ \tau \in \overline{\text{star } \sigma}$ and τ has no vertexes in common with $\sigma\}$

link$'\sigma = \{<\underline{\sigma}_0, \ldots, \underline{\sigma}_i> \in K', \ \sigma < \sigma_0\}$

dual $\sigma = \{<\underline{\sigma}_0, \ldots, \underline{\sigma}_i> \in K', \ \sigma \leq \sigma_0\}$



If S is a collection of simplexes of K, $\overline{S}$ denotes its closure (the smallest subcomplex of K containing S), and S' denotes its first barycentric subdivision, so $S' \subset K'$. There is a canonical p.l. homeomorphism $\mathrm{star}'\sigma \xrightarrow{\approx} \mathrm{star}\ \sigma$ and a canonical simplicial isomorphism $\mathrm{link}'\sigma \xrightarrow{\approx} (\mathrm{link}\ \sigma)'$ (so it is customary to identify these complexes).

If J and L are subcomplexes of K, $J*L$ denotes their simplicial join (when it's defined; cf. [Ze 2]). (Link $\sigma = \{\tau \in K,$ τ is joinable to $\sigma\}$.) Now we can describe the relations between star σ, link σ, and dual σ. By definition,

$$\mathrm{dual}\ \sigma = \underline{\sigma} * \mathrm{link}'\sigma,$$

i.e. dual σ is the simplicial cone on $\mathrm{link}'\sigma$ with cone point $\underline{\sigma}$, the barycenter of σ. Thus dual σ is called the <u>dual cone</u> of σ. Now

$$\overline{\mathrm{star}\ \sigma} = \sigma * \mathrm{link}\ \sigma,$$

and $\sigma*\mathrm{link}\ \sigma \overset{\mathrm{p.l.}}{\cong} \partial\sigma * \underline{\sigma} * \mathrm{link}'\sigma = \partial\sigma * \mathrm{dual}\ \sigma$, so we have an isomorphism of pairs

$$(2.1) \qquad (\overline{\mathrm{star}\ \sigma}, \partial\, \overline{\mathrm{star}\ \sigma}) \overset{\mathrm{p.l.}}{\cong} \partial\sigma * (\mathrm{dual}\ \sigma, \mathrm{link}'\sigma).$$

If σ is an i-simplex, $\partial\sigma$ is a combinatorial (i-1)-sphere, so $(\overline{\mathrm{star}\ \sigma}, \partial\, \overline{\mathrm{star}\ \sigma})$ is p.l. homeomorphic to the $i^{\underline{\mathrm{th}}}$ suspension of (dual σ, link'σ).

This structure is used in the classical proof of duality to show that if σ is an i-simplex of the combinatorial n-manifold K, dual σ is an (n-i)-cell. Since K is an n-manifold, $\overline{\text{star } \sigma}$ is an n-cell, so ∂ star σ = $\partial\sigma$ * link σ is an (n-1)-sphere. Since $\partial\sigma$ is an (i-1)-sphere, it follows that link σ is an (n-i-1)-sphere (cf. [Mo]). Thus dual σ = $\underline{\sigma}$ * link'σ is an (n-i)-cell.

Now according to Whitney ([Wh] 1938), if M is an oriented (closed) n-manifold, the Poincaré duality map $H^q(M) \to H_{n-q}(M)$ is given by cap product with the fundamental class [M]. That is, [M] is the generator of $H_n(M)$ corresponding to the given orientation; and if $\beta \in H^q(M)$, the dual of β is $\beta \cap [M] \in H_{n-q}(M)$. The essential step in understanding the relation of cap product to duality is to see how $\bullet \cap [M]$ fits into the classical proof when $M = |K|$. This will provide a bridge from the algebra of cap product to the rigid combinatorial geometry of K discussed above.

Let K be an oriented combinatorial n-manifold (with its n-simplexes coherently oriented so their sum is a fundamental cycle $[K] \in C_n(K)$, and with the lower dimensional simplexes arbitrarily oriented). If σ^i is an i-simplex of K, define a chain $D\sigma^i \in C_{n-i}(K')$ by

$$(2.2)\qquad D(\sigma^i) = \sum_{\sigma^i<\tau^{i+1}<\dots<\tau^n} \eta(\sigma^i,\dots,\tau^n)\langle \underline{\sigma}^i,\dots,\underline{\tau}^n\rangle,$$

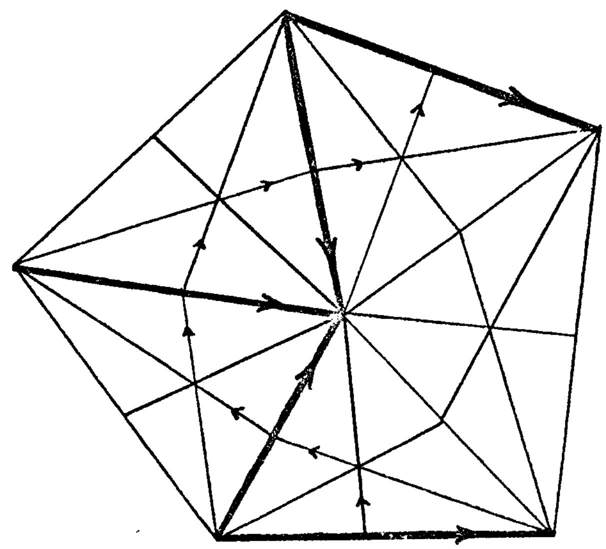
where the sum is taken over all increasing sequences $\sigma^i < \tau^{i+1} < \dots < \tau^n$ of simplexes of K, and $\eta(\sigma^i,\tau^{i+1},\dots,\tau^n) = [\sigma^i,\tau^{i+1}][\tau^{i+1},\tau^{i+2}]\dots[\tau^{n-1},\tau^n]$ ($[\ ,\ ]$ = incidence number, cf. §1). Note that $\eta(\sigma^i,\tau^{i+1},\dots,\tau^n) = 0$ unless τ^j is a top dimensional proper face of τ^{j+1} for each j, i.e. dimension $\tau^j = j$. Note also that we are using the *canonical* orientations of the simplexes of K' - the vertexes of a simplex K' are ordered by $\underline{\sigma} < \underline{\tau} \iff \sigma < \tau$. Clearly $D(\sigma)$ is the sum of all the $(n-i)$-simplexes in dual σ. I will show below that

$$D(\delta\sigma) = \partial(D\sigma),$$

from which it follows that D is a chain map $C^*(K) \to C_{n-*}(K')$, equal to the composition

$$C^*(K) \xrightarrow{\pm\partial} D_{n-*}(K) \xrightarrow{s} C_{n-*}(K'),$$

where s is the subdivision map.



Therefore D induces the Poincaré duality isomorphism

$$H^*(K) \xrightarrow{\approx} H_{n-*}(K') \cong H_{n-*}(K)$$

In fact, I will show that (2. 2) can be generalized to a formula for the cap product of a simplicial cochain and a simplicial chain, so that $D(\sigma) = \sigma \cap [K]$ (cf. [Fr]).

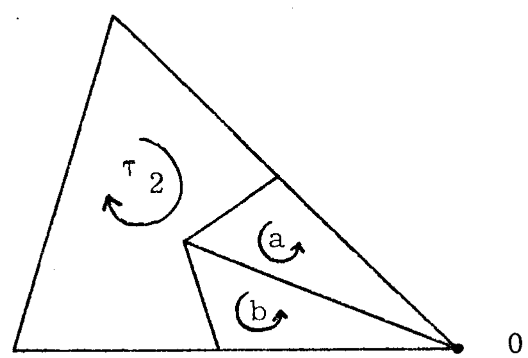
<u>Theorem 1A</u>. Let K be an (oriented) simplicial complex. The cap product pairing between the cohomology and the homology of K is induced by the pairing

$$C^q(K) \otimes C_p(K) \to C_{p-q}(K')$$

given by

$$(2.3)\quad \sigma^q \cap \tau_p = \Sigma\ \eta(\sigma, \omega_1, \ldots, \omega_{p-q-1}, \tau) <\underline{\sigma}, \underline{\omega}_1, \ldots, \underline{\omega}_{p-q-1}, \underline{\tau}>$$

Here σ^q is an elementary q-cochain, τ_p is an elementary p-chain, and the sum is taken over all sequences of simplexes $\sigma < \omega_1 < \ldots < \omega_{p-q-1} < \tau$ in K with $\dim \omega_i = i-q$.
($\sigma^q \cap \tau_p = 0$ if $p < q$, and $\sigma^p \cap \tau_p = \sigma(\tau)<\underline{\sigma}>$.)



$\sigma^0 \cap \tau_2 = a + b$

Remark. The immediate geometrical significance of this definition of cap product is that if y is a q-cochain and x is a p-chain, support $y \cap x \subset \bigcup_{\dim \sigma = q} \text{dual}\ \sigma$, and so it has a certain degree of "transversality" to the simplexes of K.

Proof: (Cf. [Wh].) I will show that (2.3) agrees up to sign with the definition of cap product in [Sp]. There is no standard definition of cap product - the various definitions differ by signs depending on the dimensions of the factors (cf. [G. Wh]). (I will come back to this point in the next section. Zeeman's sign conventions for the double complex D force a particular choice of signs for cap product which differs from mine.)

Let $x \in C_p(K)$ be a cycle, and let A_* be the chain complex

$$A_i = \begin{cases} 0 & i < 0 \\ C^{p-i}(K) & i \geq 0 \end{cases}$$

Let $\varepsilon_x : A_0 \to \mathbb{Z}$ be given by $\varepsilon_x(y) = y(x)$, where $y \in C^p(K) = \mathrm{Hom}(C_p(K), \mathbb{Z})$. ε_x is an augmentation for the chain complex A_*. Let $\varepsilon : C_0(K') \to \mathbb{Z}$ be the standard augmentation for $C_*(K')$ which assigns 1 to each vertex of K'.

Define an acyclic carrier Γ from A_* to $C_*(K')$ as follows. If σ is an elementary cochain of K, let $\Gamma(\sigma) = (\overline{\text{star } \sigma})' \subset K'$. If $\sigma < \tau$, $\overline{\text{star } \sigma} \supset \overline{\text{star } \tau}$, and $\overline{\text{star } \sigma}$ is acyclic, since it collapses to σ.

Now it follows from the classical acyclic carrier technique (cf. [HW]) that there is a chain map

$$\varphi_x : A_* \to C_*(K')$$

such that φ_x preserves augmentation and is carried by Γ, i.e. $\epsilon \varphi_x = \epsilon_x$ and $\operatorname{supp} \varphi(\sigma) \subset \Gamma(\sigma)$ for each elementary cochain σ of K. Furthermore, any two augmentation preserving chain maps carried by Γ are chain homotopic.

I claim that $\varphi_x(y) = y \cap x$, as defined in (2.3) is such a chain map. The following properties must be checked:

(2.4)

1) If $y \in C^q(K)$, and $q > p$, $y \cap x = 0$.

2) If $y \in C^p(K)$, $\epsilon(y \cap x) = y(x)$.

3) $\partial(y \cap x) = \delta y \cap x$

4) If σ is an elementary q-cochain, $\operatorname{supp}(\sigma \cap x) \subset (\overline{\text{star } \sigma})'$.

1), 2), and 4) are clear by (2.3).

<u>Lemma 1A.</u> If $x \in C_p(K)$ and $y \in C^q(K)$,

$$\partial(y \cap x) = \delta y \cap x + (-1)^{p-q} y \cap \partial x.$$

<u>Proof:</u> It suffices to prove the formula when x and y are elementary, say $x = \tau_p$ and $y = \sigma^q$. Let γ denote a sequence

$\sigma < \omega_1 < \ldots < \omega_{p-q-1} < \tau$ of simplexes of K with $\dim \omega_i = i-q$, and let $\eta(\gamma) = \eta(\sigma, \omega_1, \ldots, \omega_{p-q-1}, \tau)$. Thus (2.3) says

$$\sigma \cap \tau = \sum_{\gamma} \eta(\gamma)\langle \underline{\sigma}, \underline{\omega}_1, \ldots, \underline{\omega}_{p-q-1}, \underline{\tau}\rangle .$$

Then

$$\begin{aligned} \partial(\sigma \cap \tau) = {} & \sum_{\gamma} \eta(\gamma)\langle \underline{\sigma}, \underline{\omega}_1, \ldots, \underline{\omega}_{p-q-1}, \underline{\tau}\rangle . \\ & + \sum_{\gamma} \eta(\gamma) \sum_{i=1}^{p-q-1} (-1)^i \langle \underline{\sigma}, \underline{\omega}_1, \ldots, \hat{\underline{\omega}}_i, \ldots, \underline{\omega}_{p-q-1}, \underline{\tau}\rangle \\ & + \sum_{\gamma} \eta(\gamma)(-1)^{p-q} \langle \underline{\sigma}, \underline{\omega}_1, \ldots, \underline{\omega}_{p-q-1}\rangle \end{aligned}$$

Now $\delta\sigma = \sum_{\sigma<\omega_1} [\sigma, \omega_1]\omega_1$, so

$$\begin{aligned} \delta\sigma \cap \tau & = \sum_{\sigma<\omega_1} [\sigma, \omega_1]\omega_1 \cap \tau \\ & = \sum_{\sigma<\omega_1} [\sigma, \omega_1] \sum_{\omega_1 < \ldots \omega_{p-q-1} < \tau} \eta(\omega_1, \ldots, \tau)\langle \underline{\omega}_1, \ldots, \underline{\tau}\rangle \\ & = \sum_{\gamma} \eta(\gamma)\langle \underline{\omega}_1, \ldots, \underline{\tau}\rangle , \end{aligned}$$

since $\eta(\sigma, \omega_1, \ldots, \tau) = [\sigma, \omega_1]\eta(\omega_1, \ldots, \tau)$. Similarly,

$$\partial\tau = \sum_{\omega_{p-q-1}<\tau} [\omega_{p-q-1}, \tau]\omega_{p-q-1}, \quad \text{so}$$

$$\sigma \cap \partial\tau = \sum_{\gamma} \eta(\gamma)\langle\underline{\sigma}, \underline{\omega}_1, \ldots, \underline{\omega}_{p-q-1}\rangle.$$

Thus we have that

$$\partial(\sigma \cap \tau) = \delta\sigma \cap \tau + (-1)^{p-q}\sigma \cap \partial\tau$$

$$+ \sum_{i} (-1)^i \sum_{\gamma} \eta(\gamma)\langle\underline{\sigma}, \ldots, \hat{\underline{\omega}}_i, \ldots, \underline{\tau}\rangle,$$

so we must show this last term is zero. But the coefficient of $\langle\underline{\sigma}, \ldots, \hat{\underline{\omega}}_i, \ldots, \underline{\tau}\rangle$ is

$$(-1)^i \sum_{\omega} \eta(\sigma, \ldots, \omega_{i-1}, \omega, \omega_{i+1}, \ldots, \tau),$$

where the sum is over all ω such that $\omega_{i-1} < \omega < \omega_{i+1}$. This equals

$$(-1)^i \sum_{\omega} [\sigma, \omega_1] \ldots [\omega_{i-1}, \omega][\omega, \omega_{i+1}], \ldots, [\omega_{p-q-1}, \tau]$$

But $\sum_{\omega} [\omega_{i-1}, \omega][\omega, \omega_{i+1}] = 0$ (1.1), so the coefficient of $\langle\underline{\sigma}, \ldots, \hat{\underline{\omega}}_i, \ldots, \underline{\tau}\rangle$ is indeed zero. This completes the proof of the lemma, and therefore of property 3) of (2.4).

Now I will show that (the subdivision of) Spanier's cap product ([Sp]) satisfies (2.4) (up to sign), and so agrees (up to sign) with my cap product. If τ_p is an (oriented) simplex of K, say $\tau = \langle v_0, \ldots, v_p \rangle$, and $y \in C^q(K)$, Spanier (following Whitney [Wh]) defines cap product as the pairing ψ induced by

$$(2.5) \qquad \psi(y, \tau) = y(\langle v_{p-q}, \ldots, v_p \rangle) \langle v_0, \ldots, v_{p-q} \rangle,$$

i.e. $\psi(y, \tau)$ is y of the back q-face of τ times the front (p-q)-face of τ. (In other words, he uses the classical Alexander-Whitney "diagonal approximation" $\tau_p \to \sum_{i+j=p}$ (front i-face of $\tau \otimes$ back j-face of τ) to define $\cap$. See [Sp], p. 254.)

Let $s : C_*(K) \to C_*(K')$ be the subdivision chain map. We must show that $s\psi(y \cap x)$ satisfies (2.4). 1) and 2) are clear from (2.5). According to [Sp], p. 253,

$$\partial\psi(y^q, x_p) = (-1)^{p-q}\psi(\delta y, x) + \psi(y, \partial x).$$

So if $\partial x = 0$, $\partial s\psi(y^q, x_p) = (-1)^{p-q} s\psi(\delta y, x)$. Thus 3) holds up to sign. Finally, if $y = \sigma$, (2.5) implies that $\psi(y, \tau) = 0$ unless $\sigma < \tau$, in which case $\psi(y, \tau) = \pm\, \omega$, $\omega < \tau$. Thus $\psi(\sigma, \tau)$ lies in $\overline{\text{star } \sigma}$, as desired. This completes the proof of Theorem 1A.

The pairing of theorem 1 will be used in §3 to analyse the spectral sequence E. A certain amount of information about $\hat{E}$ will then follow from the formal duality between E and $\hat{E}$, but a complementary theorem on cap product is needed for a complete geometrical analysis of $\hat{E}$.

<u>Theorem 1B</u>. Let K be an (oriented) simplicial complex. The cap product pairing between the cohomology and the homology of K is induced by the pairing

$$C^q(K') \otimes C_p(K) \to C_{p-q}(K)$$

given by

$$(2.6) \qquad \langle \underline{\sigma}_0, \ldots, \underline{\sigma}_q \rangle \cap \tau = \begin{cases} \eta(\sigma_0, \ldots, \sigma_q)\sigma_0 & \text{if } \tau = \sigma_q \\ 0 & \text{if } \tau \neq \sigma_q \end{cases}$$

Here $\langle \underline{\sigma}_0, \ldots, \underline{\sigma}_q \rangle$ is an arbitrary q-simplex of K' (so $\sigma_0 < \ldots < \sigma_q$), and τ is an elementary p-chain of K. Note that $\eta(\sigma_0, \ldots, \sigma_q) = 0$ unless σ_i is a top dimensional proper face of σ_{i+1} for each i, so the only case in which $\langle \underline{\sigma}_0, \ldots, \underline{\sigma}_q \rangle \cap \tau \neq 0$ is when $\dim \sigma_i = i+p-q$ for each i

($p = \dim \tau$).

Proof: I will show that (2.6) agrees up to sign with the definition of cap product in [Sp].

Lemma 1B. If $x \in C_p(K)$ and $y \in C^q(K')$,

$$\partial(y \cap x) = \delta y \cap x + (-1)^q y \cap \partial x \, .$$

Proof. Let $y = <\underline{\sigma}_0, \dots, \underline{\sigma}_q>$ and $x = \tau$. Then by (2.5)

(a) $$\partial(<\underline{\sigma}_0, \dots, \underline{\sigma}_q> \cap \tau) = \begin{cases} \eta(\sigma_0, \dots, \sigma_q)\partial\sigma_0 & \text{if } \tau = \sigma_q \\ 0 \quad \text{if} \quad \tau \neq \sigma_q \, . \end{cases}$$

Now $\delta<\underline{\sigma}_0, \dots, \underline{\sigma}_q> = \sum_{\rho<\sigma_0} <\rho, \underline{\sigma}_0, \dots, \underline{\sigma}_q> + \sum_{\sigma_q<\omega} (-1)^{q+1}<\underline{\sigma}_0, \dots, \underline{\sigma}_q, \omega>$,

where ρ is a top dimensional proper face of σ_0 and σ_q is a top dimensional proper face of ω. Thus

$$\delta<\underline{\sigma}_0, \dots, \underline{\sigma}_q> \cap \tau = \begin{cases} \sum_{\rho<\sigma_0} \eta(\rho, \sigma_0, \dots, \sigma_q)\rho & \text{if} \quad \tau = \sigma_q \\ (-1)^{q+1}\eta(\sigma_0, \dots, \sigma_q, \omega)\sigma_0 & \text{if} \quad \tau = \omega \end{cases}$$

Now $\sum_{\rho<\sigma_0} \eta(\rho, \sigma_0, \ldots, \sigma_q)\rho = \eta(\sigma_0, \ldots, \sigma_q)\partial\sigma_0$ by (1.1) and the definition of η, so

$$\text{(b)} \qquad \delta<\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \tau = \begin{cases} \eta(\sigma_0, \ldots, \sigma_q)\partial\sigma_0 & \text{if} \quad \tau = \sigma_q \\ (-1)^{q+1}\eta(\sigma_0, \ldots, \sigma_q, \omega)\sigma_0 & \text{if} \quad \tau = \omega \end{cases}$$

Finally, $\partial\tau = \sum_{\sigma<\tau} [\sigma, \tau]\sigma$, so

$$<\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \partial\tau = \begin{cases} \eta(\sigma_0, \ldots, \sigma_q)[\sigma_q, \tau]\sigma_0 & \text{if} \quad [\sigma_q, \tau] \neq 0 \\ 0 \text{ otherwise,} \end{cases}$$

so

$$\text{(c)} \qquad <\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \partial\tau = \begin{cases} \eta(\sigma_0, \ldots, \sigma_q, \omega)\sigma_0 & \text{if} \quad \tau = \omega \\ 0 \text{ otherwise} \end{cases}$$

Comparing (a), (b), and (c), we have

$$\partial(<\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \tau) = \delta<\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \tau - (-1)^{q+1}<\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \partial\tau$$

$$= \delta<\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \tau + (-1)^{q}<\underline{\sigma}_0, \ldots, \underline{\sigma}_q> \cap \partial\tau ,$$

as desired.

Now the proof of theorem 1B follows from the acyclic carrier argument of theorem 1. Let $t : C^*(K') \to C^*(K)$ be the dual of subdivision (i.e. $(ty)(x) = y(sx)$), and let x be a fixed p-cycle of K. The cap product of (2.6) satisfies

1) If $y \in C^q(K')$, and $q > p$, $y \cap x = 0$

2) If $y \in C^p(K')$, $\varepsilon(y \cap x) = (ty)(x)$

3) $\partial(y \cap x) = \delta y \cap x$ (by the lemma)

4) $\mathrm{supp}(<\sigma_0, \ldots, \sigma_q> \cap x) \subset \overline{\mathrm{star}\ \sigma_q}$.

I claim that $y \to \psi(ty, x)$ also satisfies 1) - 4), where ψ is Spanier's cap product (2.5). 1) and 2) are clear, and 3) follows from his boundary formula. Finally, $\psi(t<\sigma_0, \ldots, \sigma_q>, \tau) = 0$ unless $<\sigma_0, \ldots, \sigma_q>$ lies in a face of τ, i.e. $\sigma_q < \tau$, in which case $\psi(t<\sigma_0, \ldots, \sigma_q>, \tau) = \pm\omega$, $\omega < \tau$, so $\psi(t<\sigma, \ldots, \sigma_q>, \tau)$ lies in $\overline{\mathrm{star}\ \sigma_q}$.

Therefore his cap product is chain homotopic to mine.

<u>Remark</u>. Steenrod (unpublished, cf. [G. Wh]) showed that cap product [and cup product] can be characterized as a pairing of homology theories satisfying certain axioms. These axioms are listed in [St 1] as follows. If X is a topological space and A_1, A_2 are subspaces, $\cap$ is a bilinear pairing

$$\cap : H^q(X, A_1) \times H_p(X, A_1 \cup A_2) \to H_{p-q}(X, A_2)$$

such that

0) $\epsilon(y \cap x) = \langle y, x \rangle$, ϵ = augmentation, $\langle\ ,\ \rangle$ = Kronecker pairing

1) $\partial(y \cap x) = (-1)^q i^* y \cap \partial x$, $i : A \to X$, $y \in H^q(X)$, $x \in H_p(X, A)$

2) $\delta y \cap x = (-1)^q i_*(y \cap \partial x)$, $i : A \to X$, $y \in H^q(A)$, $x \in H_p(X, A)$

3) $f_*(f^* y \cap x) = y \cap f_* x$, where $f : (X, A_1, A_2) \to (Y, B_1, B_2)$ is a continuous map, and $y \in H^q(Y, B_1)$, $x \in H_p(X, A_1 \cup A_2)$.

We have seen that the cap products (2.3) and (2.6) satisfy 0), 1), and 2). (The sign for (2.3) in 1) and 2) is $(-1)^{p-q}$ by Lemma 1A, and the sign for (2.6) is $(-1)^q$ by lemma 1B. This sign is in fact arbitrary.) It is an easy matter to check the naturality axiom (3) for a simplicial map. (Note that it's enough to check the axioms for triangulable spaces and maps.)

3. G. Whitehead's spectral sequence

If K is a simplicial complex let K_i denote the i-skeleton of K, the union of all the simplexes of dimension $\leq i$. Let $K^{(i)}$, the i-<u>coskeleton</u> of K, be the union of all the duals of simplexes σ with $\dim \sigma \geq i$. If K is n-dimensional, $K^{(i)}$ is an $(n-i)$-dimensional subcomplex of the first derived complex K' (Zeeman calls $K^{(i)}$ the $(n-i)^{\underline{th}}$ coskeleton of K [Ze 1].) In terms of K', we have

$$(K_i)' = \{<\underline{\sigma}_0, \ldots, \underline{\sigma}_j>, \ \dim \sigma_j \leq i\}$$

(3.1)

$$K^{(i)} = \{<\underline{\sigma}_0, \ldots, \underline{\sigma}_j>, \ \dim \sigma_0 \geq i\}.$$

If K is a combinatorial n-manifold, $K^{(i)}$ is (the first derived complex of) the $(n-i)$-skeleton of the dual cell complex to K. The following properties of $K^{(i)}$ for an arbitrary complex are immediate from the definition:

$$K' = K^{(0)} \supset K^{(1)} \supset \ldots \supset K^{(n)} \supset \emptyset\,, \qquad n = \dim K$$

$$\dim(K^{(i)} \cap (K_j)') = j - i$$

(3.2)

$$|K^{(i)}| \text{ is a deformation retract of } |K| - |K_{i-1}|, \text{ and}$$
$$|K_i| \text{ is a deformation retract of } |K| - |K^{(i+1)}|$$

Using the definition (2.3) of cap product, if x_p is a p-chain and y^q is a q-cochain on K, $y \cap x$ is a (p-q)-chain on $K^{(q)} \subset K'$.

Proposition 1. Let $z \in C_s(K')$. If supp $z \subset K^{(q)}$, then z is homologous to a chain

$$\widetilde{z} = \sum_{j \geq q} y^j \cap x_{s+j},$$

where each $y^j \in C^j(K)$ and $x_{s+j} \in C_{s+j}(K)$.

Proof: It is clear that any elementary chain $<\underline{\sigma}_0, \ldots, \underline{\sigma}_s>$ in $K^{(q)}$ (i.e. $\dim \sigma_0 \geq q$) is homologous to a chain $\sum_k \pm <\underline{\tau}_{k0}, \ldots, \underline{\tau}_{ks}>$, where $\dim \tau_{k0} \geq q$, and $\dim \tau_{k(i+1)} = (\dim \tau_{ki}) + 1$ for each k, i. But $<\underline{\tau}_{k0}, \ldots, \underline{\tau}_{ks}> = \eta(\tau_{k0}, \ldots, \tau_{ks})\tau_{k0} \cap \tau_{ks}$ by (2.3), so the proposition follows.

This proposition shows that the filtration of the complex K' by its coskeletons bears an interesting relation to the cap product. Let $\tilde{E}$ denote the homology spectral sequence associated to this filtration (3.2). Then $\tilde{E}$ converges to the simplicial homology $H_*(K')$, and $\tilde{E}^\infty$ is associated to the filtration

$$H_s(K') = \tilde{F}_s^0 \supset \tilde{F}_s^1 \supset \ldots \supset \tilde{F}_s^n \supset 0 ,$$

where $\tilde{F}_s^q = \mathrm{Im}[H_s(K^{(q)}) \to H_s(K')]$. This spectral sequence has been studied in a more general setting by G. Whitehead [G. Wh]. (See the remarks following theorems 2A and B.)

<u>Theorem 2A</u>. Zeeman's spectral sequence E of a simplicial complex K is isomorphic to Whitehead's spectral sequence $\tilde{E}$ of K' (up to sign). More precisely, there is a map of spectral sequences

$$E^r_{p,q} \xrightarrow{\approx} \tilde{E}^r_{p-2q,q}, \qquad r \geq 1$$

commuting up to sign with the differentials.

<u>Corollary 1</u>. $\tilde{E}$ is a topological invariant of $|K|$, and $\tilde{E}^2_{i,j} \cong H^j(|K|; \mathscr{H}_{i+2j})$.

Corollary 2. If $\alpha \in H_s(X)$, where $X = |K|$, the filtration of α associated to E is $\geq q$ if and only if α is represented by a simplicial cycle in $K^{(q)}$.

Remark. §5 is devoted to the geometrical application of Corollary 2.

Lemma. Let $f : K \to L$ be a simplicial map of finite complexes. The following conditions are equivalent:

1) $\dim f(\sigma) = \dim \sigma$ for all $\sigma \in K$
2) $|f|^{-1}(x)$ is finite for all $x \in |L|$.
3) $f'(K^{(q)}) \subset L^{(q)}$, where f' is a first derived of f.

Proof: See [Sta], pp. 83, 85.

Such a map is called nondegenerate. By 3), f' induces a map of Whitehead spectral sequences.

Corollary 3. A continuous map $g : X \to Y$ induces a map of Zeeman's spectral sequences if g is the realization of a nondegenerate simplicial map.

Proof of theorem: Consider the homomorphism

$$C_p(K) \otimes C^q(K) \to C_{p-q}(K')$$

given by $x \otimes y \to y \cap x$ (2.3). If $y \mid \operatorname{supp} x = 0$, $y \cap x = 0$, so this map induces a homomorphism

$$c : D_{p,q} \to C_{p-q}(K'),$$

where D is the double complex used to define E (§1). Unfortunately, c does not commute with the differentials, i.e. $cd \neq \partial c$, because Zeeman's sign convention (1.3) is different from mine (lemma 1A). However, it is easy to define isomorphisms $\varepsilon(p,q) : D_{p,q} \to D_{p,q}$, with $\varepsilon(p,q)$ equal to multiplication by ± 1, so that

$$d(\varepsilon(p,q)x_p \otimes y^q) = (-1)^{p-q}\varepsilon(p-1,q)\partial x \otimes y + \varepsilon(p,q+1)x \otimes \delta y.$$

Then if $\varepsilon = \sum_{p,q} \varepsilon(p,q) : D \xrightarrow{\approx} D$, we have that

$$c(\varepsilon d \varepsilon)(x_p \otimes y^q) = (-1)^{p-q}\partial x \quad y + x \quad \partial y = \partial c(x \otimes y).$$

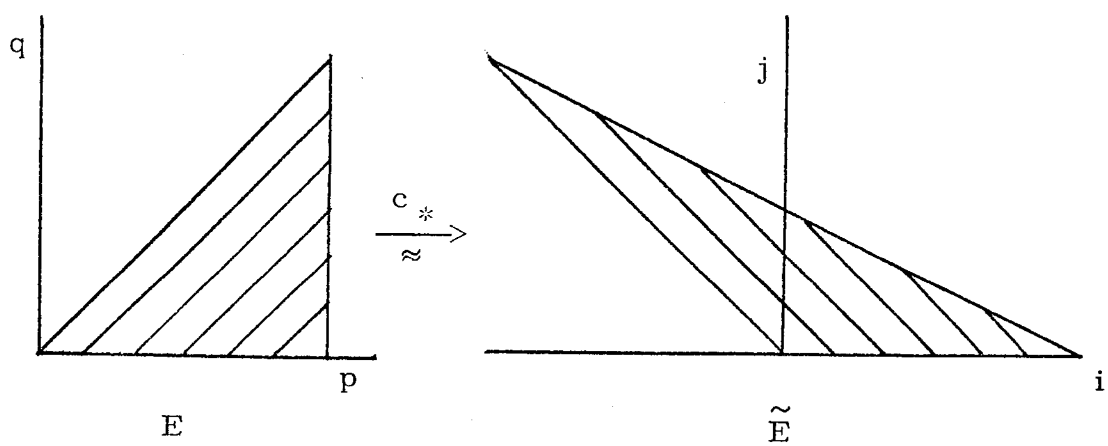
Thus c is a homomorphism of differential abelian groups

$$c : (D, \varepsilon d \varepsilon) \to (C_*(K), \partial).$$

Now by the definition (2.3) of $\cap$, $c(F^qD) \subset C_*(K^{(q)})$ (cf. the proposition above), i.e. c takes the filtration defining E to the filtration defining $\widetilde{E}$, so c induces a map of spectral sequences c_*. The total grading of D is given by $D_s = \sum_{p-q=s} D_{p-q}$, so $c(D_s) \subset C_s(K')$, i.e. c preserves the total grading as well. Thus

$$c_*^r : E^r_{p,q} \to \widetilde{E}^r_{i,j},$$

where $j = q$ (filtration degree) and $i + j = p - q$ (total degree), so $i = p - 2q$.



I claim that $c_*^1 : E^1_{p,q} \to \widetilde{E}^r_{p-2q,q}$ is an isomorphism, and so c_* is an isomorphism of spectral sequences. Now

$$E^1_{p,q} = H_{p-q}(F^qD, F^{q+1}D) \cong \sum_{\dim \tau = q} H_p(\text{star } \tau) = \sum_{\tau^q} H_p(\overline{\text{star } \tau}, \partial\, \overline{\text{star } \tau});$$

$$\tilde{E}^1_{p-2q,q} = H_{p-q}(K^{(q)}, K^{(q+1)}) \cong \sum_{\tau^q} H_{p-q}(\mathring{\text{dual}}\ \tau)$$

$$= \sum_{\tau^q} H_{p-q}(\text{dual}\ \tau, \text{link}'\tau).$$

The top isomorphism was proved above (1.4). The bottom isomorphism holds because $K^{(q)} = \bigcup_{\tau^q} \text{dual}\ \tau$, $\text{dual}\ \tau = \underline{\tau} * \text{link}'\tau$, and $\text{link}'\tau^q = \bigcup_{\tau<\sigma} \text{dual}\ \sigma \subset K^{(q+1)}$. Now $c^1_* : H_{p-q}(F^qD, F^{q+1}D) \to H_{p-q}(K^{(q)}, K^{(q+1)})$ is induced by $x_p \otimes y^q \to y \cap x$, so on each direct summand $H_p(\overline{\text{star}\ \tau}, \partial\, \overline{\text{star}\ \tau})$, c^1_* is induced by $x_p \to \tau \cap x_p$, where τ is an elementary cochain. But

$$\tau \cap \cdot : H_p(\overline{\text{star}\ \tau}, \partial\, \overline{\text{star}\ \tau}) \to H_{p-q}(\text{dual}\ \tau, \text{link}'\tau)$$

is an isomorphism for each τ, the inverse of the suspension isomorphism (cf. (2.1)) - in fact $\tau \cap \cdot : C_*(\overline{\text{star}\ \tau}, \partial\, \overline{\text{star}\ \tau}) \to C_{*-q}(\text{dual}\ \tau, \text{link}'\tau)$ is a <u>chain</u> isomorphism, since $\sigma \to \tau \cap \sigma$ is a bijection between the oriented simplexes of $\text{star}\ \tau \subset K$ and the oriented simplexes of $\mathring{\text{dual}}\ \tau \subset K'$. This completes the proof of the theorem. (For a discussion of the isomorphism c^2_* on the E^2 terms, see §4.)

Let $\hat{\tilde{E}}$ be the cohomology spectral sequence associated to the filtration (3.2) of K'. $\hat{\tilde{E}}$ converges to the simplicial cohomology $H^*(K')$, and $\hat{\tilde{E}}^\infty$ is associated to the filtration

$$0 \subset \hat{\tilde{F}}^s_0 \subset \hat{\tilde{F}}^s_1 \subset \ldots \subset \hat{\tilde{F}}^s_n = H^s(K'),$$

where $\hat{\tilde{F}}^s_q = \operatorname{Ker}[H^s(K') \to H^s(K^{(q+1)})]$.

<u>Theorem 2B</u>. Zeeman's spectral sequence $\hat{E}$ of a simplicial complex K is isomorphic to the spectral sequence $\hat{\tilde{E}}$ of K' (up to sign). More precisely, there is a map of spectral sequences

$$\hat{\tilde{E}}^r_{p-2q,\, q} \longrightarrow \hat{E}^r_{p,\, q}, \qquad r \geq 1$$

commuting up to sign with the differentials.

<u>Proof</u>. This theorem follows by a completely algebraic argument from the proof of theorem 2A. $\hat{D} = \operatorname{Hom}(D, \mathbb{Z})$, $\hat{d} = \operatorname{Hom}(d, \mathbb{Z})$; $C^*(K') = \operatorname{Hom}(C_*(K'), \mathbb{Z})$, and $\delta = \operatorname{Hom}(\partial, \mathbb{Z})$. Moreover, the filtrations on $\hat{D}$ and $C^*(K')$ used to define $\hat{E}$ and $\hat{\tilde{E}}$ are induced from the filtrations on D and $C_*(K')$ - $\hat{F}^q = \{f : F \mid F^{q+1} = 0\}$. Thus $\operatorname{Hom}(c, \mathbb{Z}) : \operatorname{Hom}(C_*(K'), \mathbb{Z}) \to \operatorname{Hom}(D, \mathbb{Z})$ induces the desired map of spectral sequences, which is an isomorphism on the E^1 terms by the dual argument to that given for theorem 1A.

However, this proof obscures the geometry somewhat. It is more enlightening to use the definition (2.6) of cap product

$$C^q(K')\otimes C_p(K) \to C_{p-q}(K)$$

to compare $\hat{\tilde{E}}$ and E. Recall that $\hat{D}_{p,q}$ is the free abelian group generated by pairs of simplexes (σ^p, τ_q) with $\sigma \geq \tau$.

Define a map

$$\bar{c} : C^s(K') \to \sum_p \hat{D}_{p,p-s} \quad .$$

by $\bar{c}(\omega^s) = \sum_{\sigma\in K} (\sigma, \omega \cap \sigma)$ (using the definition (2.6) of $\cap$). As in the proof of theorem 2A, a little calculation shows there is an isomorphism $\varepsilon : \hat{D} \xrightarrow{\approx} \hat{D}$ which is multiplication by ± 1 on each $\hat{D}_{p,q}$, such that $\bar{c}(\delta\omega) = \varepsilon\hat{d}\varepsilon\bar{c}(\omega)$, so $\bar{c}$ is a homomorphism of differential groups

$$\bar{c} : (C^*(K), \delta) \to (\hat{D}, \varepsilon\hat{d}\varepsilon).$$

Now an elementary cochain $\omega \in C^s(K')$ has filtration $\leq q$ means that $\omega(\rho) = 0$ for $\rho \in K^{(q+1)}$, i.e. the simplex ω does not lie in $K^{(q+1)}$. If $\omega = <\underline{\sigma}_0, \ldots, \underline{\sigma}_s>$, this means that dim $\sigma_0 = \pm\, \sigma_0$ or 0, so $\bar{c}(\omega)$ is a sum of terms $(\sigma, \pm\, \sigma_0)$ with dim $\sigma_0 \leq q$, which lies in $F^q\hat{D} = \sum_{i\leq q} \hat{D}_{p,i}$. Therefore $\bar{c}$ is filtration preserving, so $\bar{c}$

induces a map $\bar{c}_*$ of spectral sequences,

$$\bar{c}^r_* : \hat{\tilde{E}}^r_{i,j} \to \hat{E}_{p,q}$$

with $j = q$ and $i+j = p-q$, so $i = p-2q$. The argument that $\bar{c}^1_*$ is an isomorphism is essentially the same as for c^1_*, with homology replaced by cohomology. This completes the proof of Theorem 2B.

The general setting of Whitehead [G. Wh] gives some perspective on the spectral sequences $\tilde{E}$ and $\hat{\tilde{E}}$. Since they are defined by a filtration on K, they can be defined for any generalized homology theory (G_*, G^*), with $\tilde{E}(K') \Longrightarrow G_*(K')$ and $\hat{\tilde{E}}(K') \Longrightarrow G^*(K')$. Since the E^2 terms can be calculated without using the dimension axiom for G (cf. §4), we have that if $X = |K|$,

$$\tilde{E}^2_{i,j} \cong H^j(X; \mathcal{S}_{i+2j}) \Longrightarrow G_{i+j}(X)$$

where $\mathcal{S}_p$ is the sheaf of local $p^{\underline{th}}$ G-homology on X (and the dual result holds for $\hat{\tilde{E}}$). This observation simplifies Whitehead's discussion of $\tilde{E}$ somewhat.

Furthermore, Whitehead's definition of $\tilde{E}$ involves two subspaces of X, so that for ordinary homology theory, the spectral sequence of a homology manifold collapses to the general duality isomorphism

$$H^q(A, B) \xrightarrow{\approx} H_{n-q}(X-B, X-A),$$

where A and B are subpolyhedra of $X = |K|$. For arbitrary K the spectral sequence $\tilde{E}$ has the following description. Let $P \supset Q$ be full subcomplexes of K. ($L \subset K$ is full means $\sigma \in L$ if and only if all the faces of σ are in L.) Let $N(P)$ and $N(Q)$ be their open derived neighborhoods in K'. ($N(L)$ is all simplexes of K' which have a face in L'). Then $\tilde{E}$ runs

$$\tilde{E}^2_{i,j} \cong H^j(P, Q; \mathscr{H}_{i+2j}) \Longrightarrow H_{i+j}(K'-N(Q), K'-N(P)),$$

where $\mathscr{H}_*$ is the local homology stack on K (restricted to P). If $(X, A, B) = (|K|, |P|, |Q|)$, we have

$$\tilde{E}^2_{i,j} \cong H^j(A, B; \mathscr{H}_{i+2j}) \Longrightarrow H_{i+j}(X-B, X-A),$$

from which all the classical duality theorems follow. (For example, if $A = X$ and $X-B$ is an open homology manifold, one obtains the Lefschetz duality $H^s(X, B) \xrightarrow{\approx} H_{n-s}(X-B)$.)

In this setting, the definition of $\tilde{E}$ still uses a filtration by coskeletons, and is only slightly different from the "absolute" case $X = A$, $B = \emptyset$.

Finally, $\tilde{E}$ can be generalized to a spectral sequence defined for any simplicial map $s : K \to L$, and $\tilde{E}$ is isomorphic to Zeeman's spectral sequence E of $|s|$ (see the end of §1). Let $s' : K' \to L'$ be a first derived of f (i.e. barycentrically subdivide L, and then choose "barycenters" in K so that s subdivides to a simplicial map, cf. [Co]). Now since $L^{(q)}$ is a subcomplex of L', $(s')^{-1}L^{(q)}$ is a subcomplex of K' for each q. Let $\tilde{E}(s')$ be the homology spectral sequence associated to this filtration of K', so $\tilde{E} \Longrightarrow H_*(K')$.

<u>Theorem 2C</u>. If $s : K \to L$ is a simplicial map, and $f = |s|$, Zeeman's spectral sequence $E(f)$ is isomorphic (up to sign) to $\tilde{E}(s')$.

<u>Corollary 1.</u> $\tilde{E}(s')$ is a topological invariant of f, and $\tilde{E}^2_{i,j} \cong H^j(|L|, f_*\mathscr{H}_{i+2j})$, where $\mathscr{H}_*$ is the sheaf of local homology on $|K|$, and $f_*\mathscr{H}_*$ is the sheaf induced by f on $|L|$.

<u>Corollary 2.</u> If s is a triangulation of $f : X \to Y$, the filtration of $\alpha \in H_s(X)$ associated to $E(f)$ is $\leq q$ if and only if α is

represented by a simplicial cycle a in K' such that $s'(a)$ lies in $L^{(q)}$.

Remarks 1. There is of course a definition of $\hat{\tilde{E}}(f')$, and $\hat{\tilde{E}}(f') \cong \hat{E}(f)$.

2. Corollary 2 will be interpreted geometrically in §5.

4. Elementary properties

In this section I will begin the geometrical discussion of the spectral sequence $\widetilde{E}$. The $\widetilde{E}^2$ term can be calculated directly, without reference to Zeeman's spectral sequence E, and the progression from $\widetilde{E}^1$ to $\widetilde{E}^\infty$ relates the local "obstructions to duality" to the global ones. The edge morphism $e_X : H^q(X;\mathscr{V}_n) \to H_{n-q}(X)$, which is defined for any n-dimensional space X and which reduces to the duality map when X is a manifold, can be interpreted for arbitrary (triangulable) X as cap product with a canonical class $\langle X\rangle \in H_n(X;\mathscr{V}^n)$. If X is an n-circuit with fundamental class $[X]$, $\cdot \cap [X] : H^q(X) \to H_{n-q}(X)$ factors through e_X; furthermore, e_X can be described in terms of a topological "normalization" of X.

4A. The $\widetilde{E}^2$ term

Recall that the definition of the homology of a complex K with coefficients in a stack $\mathscr{L}$ (§1) required that the simplexes of K be oriented arbitrarily, so that incidence numbers could be used in the boundary formula (1.2). Suppose the covariant functor $\mathfrak{M} : K \to$ (abelian groups) has the property that $\mathfrak{M}(\sigma < \tau) = 0$ unless σ is a top dimensional proper face of τ, and if $\rho, \tau \in K$

$$(4.1) \qquad \sum_{\sigma \in K} \mathcal{M}(\rho < \sigma) \circ \mathcal{M}(\sigma < \tau) = 0$$

the zero homomorphism from $\mathcal{M}(\rho)$ to $\mathcal{M}(\tau)$. Then the cohomology of K with coefficients in $\mathcal{M}$, $H^*(K;\mathcal{M})$, can be defined to be the homology of the chain complex $C^*(K;\mathcal{M})$, where a cochain in $C^i(K;\mathcal{M})$ is a sum

$$\Sigma\, g_\sigma \sigma, \qquad \sigma \in K_i, \qquad g_\sigma \in \mathcal{M}(\sigma),$$

and

$$\delta(\Sigma\, g_\sigma \sigma) = \sum_\tau \mathcal{M}(\sigma < \tau)(g_\sigma)\tau$$

(Compare (1.2).) I will call such a functor $\mathcal{M}$ an "oriented stack". Note that if $\mathcal{L}$ is a stack, an orientation of K determines an oriented stack $\widetilde{\mathcal{L}}$ such that $C^*(K;\mathcal{L}) = C^*(K;\widetilde{\mathcal{L}})$ by letting $\widetilde{\mathcal{L}}(\sigma) = \mathcal{L}(\sigma)$ and $\widetilde{\mathcal{L}}(\sigma < \tau) = [\sigma, \tau]\mathcal{L}(\sigma < \tau)$.

If K is a simplicial complex, define a covariant oriented stack h_p on K as follows. If $\sigma_q \in C_q(K)$, $h_p(\sigma) = H_{p-q}(\mathring{\text{dual}}\,\sigma) = H_{p-q}(\text{dual}\,\sigma, \text{link}'\sigma)$. If $\sigma_q < \tau_{q+1}$,

$h_p(\sigma < \tau) : h_p(\sigma) \to h_p(\tau)$ is the map

$$\begin{tikzcd}
H_{p-q}(\text{dual } \sigma, \text{link}'\sigma) \arrow[d, "\partial_*"] & H_{p-q-1}(\text{dual } \tau, \text{link}'\tau) \\
H_{p-q-1}(\text{link}'\sigma) \arrow[r] & H_{p-q-1}(\text{link}'\sigma, \bigcup\limits_{\substack{\sigma<\rho \\ \rho\neq\tau}} \text{dual } \rho) \arrow[u, "\text{excision}"']
\end{tikzcd}$$

(Recall that $\text{link}'\sigma = \bigcup\limits_{\sigma<\rho} \text{dual } \rho$.) h_p satisfies (4.1) because of the ∂_* in $h_p(\sigma < \tau)$.

Now h_p is isomorphic to the oriented stack associated to the stack $\mathscr{H}_p$, for the isomorphism is given by

$$\sigma \cap \cdot : \mathscr{H}_p(\sigma) = H_p(\overline{\text{star } \sigma}, \partial\, \overline{\text{star } \sigma}) \to H_{p-q}(\text{dual } \sigma, \text{link}'\sigma) = h_p(\sigma).$$

<u>Proposition 1</u>. $\tilde{E}^2_{i,j}$ is canonically isomorphic to $H^j(K, h_{i+2j})$, and the isomorphism

$$\begin{tikzcd}
E^2_{p,q} \arrow[r, "c^2_*"] \arrow[d, equal] & \tilde{E}^2_{p-2q,q} \arrow[d, equal] \\
H^q(K;\mathscr{H}_p) & H^q(K, h_p)
\end{tikzcd}$$

of theorem 2A is induced by the isomorphism of coefficients $\mathscr{H}_p \cong h_p$.

Proof: $\tilde{E}^1_{i,j} = H_{i+j}(K^{(j)}, K^{(j+1)}) \cong \sum_{\tau^j} H_{i+j}(\text{dual } \tau, \text{link}'\tau)$,

$\cong H^j(K; h_{i+2j})$. $d^1 : \tilde{E}^1_{i,j} \to \tilde{E}^1_{i-2,j+1}$ is induced by the boundary map of $C_*(K')$, and thus induces $\delta : C^j(K; h_{i+2j}) \to C^{j+1}(K; h_{i+2j})$ by the definition of h_*. The description of c^2_* is clear from the proof of theorem 2A.

Remark. This discussion brings out the technical point that the definition of E depends on a choice of orientations for the simplexes of K, whereas the definition of $\tilde{E}$ doesn't (the simplexes of K' being canonically oriented.)

4B. The $\tilde{E}^r$ term.

To gain a feeling for the workings of $\tilde{E}$, consider the groups $\tilde{Z}^r$. First, recall that $\tilde{E}$ is defined to be the spectral sequence associated to the filtration

$$K^1 = K^{(0)} \supset K^{(1)} \supset \dots \supset K^{(q)} \supset K^{(q+1)} \supset \dots \supset K^{(n)} \supset 0.$$

Now, following the discussion in [Mc L] (Ch. XI, §3),

$$\tilde{Z}^r_{i,j} = \{c; c \in C_{i+j}(K^{(j)}),\ \partial c \in C_{i+j-1}(K^{(j+r)})\}.$$

In other words, $\tilde{Z}^r_{i,j}$ is the "approximate $i+j$ cycles of level r" - an element of $\tilde{Z}^r$ is a chain c whose boundary lies r stages lower down in the filtration than c does. One can think of $|K|$ as being broken up into a lot of little disjoint pieces - the open duals cones to the simplexes of K, each of which is "transverse" to its dual simplex. Now a simplicial chain c wending its way through K' will have a certain "degree of transversality" to K, equal to the minimum integer j such that c intersects the dual to some j-simplex. The integer j can also be thought of as the "geometric codimension" of c, since c must intersect each simplex of K in codimension at least j (3.2). The chain c will also have a "depth of coherence" r, which measures how much of a cycle c is, i.e. how well the parts of c fit together. c has depth of coherence $\geq r$ if (geom. cod. ∂c) $\geq$ (geom. cod. c) - r. In other words, c can be written uniquely as $c = \Sigma c_\sigma$, where c_σ is a chain in $\mathring{\text{dual}}\ \sigma$. The geometric codimension $j = \min\{\dim \sigma, c_\sigma \neq 0\}$. c has depth of coherence $\geq r$ means that if $\dim \tau < q+r$, the boundaries of the chains c_σ for $\sigma < \tau$ cancell each other out when restricted to $\mathring{\text{dual}}\ \tau$. (If $\sigma < \tau$, $\text{dual}\ \tau \subset \text{link}'\sigma = \partial\ \text{dual}\ \sigma$, so there are "boundary maps" $C_*(\mathring{\text{dual}}\ \sigma) \to C_*(\mathring{\text{dual}}\ \tau)$ for $\sigma < \tau$, discussed above in the definition of the stack h_*.) For example, let $c \in \tilde{Z}^2_{i,j}$, i.e. let c be an $(i+j)$-chain of K' with geometric codimension j and depth of coherence 2. Consider the sum $x = \sum_{\sigma^j} c_\sigma \sigma$. c has depth of coherence 2 says that for all $\tau \in C_{j+1}(K)$, $\sum_{\sigma<\tau} \partial c_\sigma \mid \mathring{\text{dual}}\ \tau = 0$, which is precisely the condition that x is a <u>cocycle</u> of $C^j(K; h_{i+2j})$. (Compare Proposition 1.)

Now as $r \to \infty$, the chains of $\tilde{Z}^r$ have more and more coherence; that is, they come closer to being cycles. At ∞, we are just dealing with cycles, and the structure of $\tilde{E}^\infty$ is determined by the filtrations (geometric codimensions) of these cycles. Each term $\tilde{E}^r$ can be thought of as a group of "obstructions to duality." $\tilde{E}^1$ is completely local, being the direct sum of the local homology of K at each simplex. (Cycles in E^1 have no coherence.) E^∞ is completely global, measuring the "degrees of freedom" of cycles on K up to homology (see §5). $\tilde{E}^2$ (and $\tilde{E}^r$ for $2 < r < \infty$) is a mixture of local and global information about K. In practice, $\tilde{E}^2$ can be calculated by grouping the places with the same local homology together into strata, and analyzing the way the global homology of these pieces fits together (cf. the examples at the end of §5.)

4C. The edge morphisms

For any topological space X, there are two edge morphisms associated to E:

$$e_X : H^q(X;\mathcal{N}_n) \cong E^2_{n,q} \longrightarrow\!\!\!\!\to E^\infty_{n,q} \cong F^q(H_{n-q}(X)) \subset H_{n-q}(X)$$

$$f_X : H_p(X) = F^0(H_p(X)) \longrightarrow\!\!\!\!\to E^\infty_{p,0} \subset E^2_{p,0} \cong H^0(X;\mathcal{N}_p)$$

(Associated to $\hat{E}$, we have $\hat{e}_X : H^q(X) \to H_{n-q}(X;\mathcal{N}^n)$ and $\hat{f}_X : H_0(X;\mathcal{N}^p) \to H^p(X)$.) The map f_X is well known in sheaf theory - it assigns to a homology class the corresponding section of the local homology sheaf. Since f_X is geometrically uninteresting, I won't discuss it here. (See the example of isolated singularities below.)

On the other hand, the edge morphism e_X is the "Poincaré duality" map for the space X. We will see that if X is a manifold, e_X is the classical duality isomorphism. In general, the image of e_X is the group of homology classes of maximum filtration in E, so e_X is surjective if and only if $E^\infty \cong H_*(X)$. Thus E gives an algebraic criterion for e_X to be a surjection. In the following discussion, I will analyze the geometry of e_X by comparing it with the map $\cdot \cap [X]$ when X is a geometric cycle.

<u>Proposition 2</u>. Let X be a triangulable space. Then

$e_X = \cdot \cap \langle X\rangle : H^q(X;\mathcal{X}_n) \to H_{n-q}(X)$, and

$\hat{e}_X = \cdot \cap \langle X\rangle : H^q(X) \to H_{n-q}(X;\mathcal{X}^n)$, where

$\langle X\rangle = \hat{e}_X(1) \in H_n(X;\mathcal{X}^n)$.

<u>Remark</u>. This is proved in a more general setting in [Br] (corollary 10.2).

Before the proof, some background on cap product with stack coefficients is necessary. (Presumably the following definitions agree with [Br] for stacks which come from sheaves.) Let $\mathcal{A}$ be a covariant stack and $\mathcal{B}$ a contravariant stack on K, and suppose there is a pairing $\varphi : \mathcal{A} \otimes \mathcal{B} \to \mathbb{Z}$, i.e. a homomorphism $\varphi_\sigma : \mathcal{A}(\sigma) \otimes \mathcal{B}(\sigma) \to \mathbb{Z}$ for each $\sigma \in K$ such that the following diagram "commutes" for $\sigma < \tau$:

(4.2)
$$\begin{tikzcd}
\mathcal{A}(\sigma) \otimes & \mathcal{B}(\sigma) \arrow[d] \arrow[dr, "\varphi_\sigma"] & \\
& & \mathbb{Z} \\
\mathcal{A}(\tau) \otimes \arrow[uu] & \mathcal{B}(\tau) \arrow[ur, "\varphi_\tau"'] &
\end{tikzcd}$$

That is, if $a \in \mathcal{A}(\sigma)$ and $b \in \mathcal{B}(\tau)$, $\varphi_\sigma(a \otimes \mathcal{B}(\sigma < \tau)b)$ $= \varphi_\tau(\mathcal{A}(\sigma < \tau)a \otimes b)$, so $\varphi(a \otimes b)$ is well-defined. Then φ determines a cap-product pairing

$$(4.3) \qquad H^q(K;\mathcal{A}) \otimes H_p(K;\mathcal{B}) \to H_{p-q}(K')$$

defined on the chain level by

$$(\sum_\sigma a_\sigma \sigma) \otimes (\sum_\tau b_\tau \tau) \to \sum_{\sigma,\tau} \varphi(a_\sigma \otimes b_\tau)\sigma \cap \tau,$$

where $\cap : C^q(K) \otimes C_p(K) \to C_{p-q}(K')$ is cap product (2.3). (Any chain map representing cap product may be used here.) If $\mathcal{B}$ is any contravariant stack, there is also a cap product pairing

$$(4.4) \qquad H^q(K') \otimes H_p(K;\mathcal{B}) \to H_{p-q}(K;\mathcal{B})$$

defined on the chain level by

$$(\sum n_\omega \omega) \otimes (\sum b_\tau \tau) \to \sum n_\omega b_\tau \, \omega \cap \tau,$$

where $\cap : C^q(K') \otimes C_p(K) \to C_{p-q}(K)$ is the cap product of (2.6).

Proof of Proposition 2: The first step is to describe e_X in terms of the spectral sequence $\tilde{E}(K')$, $|K| = X$.

$$e_K : H^q(K;\mathcal{N}_n) \cong \tilde{E}^2_{n-2q-q} \longrightarrow\!\!> \tilde{E}^\infty_{n-2q,\,q} \cong F^q(H_{n-q}(K')) \subset H_{n-q}(K').$$

By the proof of proposition 1, the isomorphism $H^q(K;\mathscr{N}_n) \cong \tilde{E}^2_{n-2q,q}$ is induced by the map

$$\sum_{\tau^q} [c_\tau]\tau \longrightarrow \sum_{\tau^q} \tau \cap c_\tau ,$$

where c_τ is an n-cycle of $\overline{\text{star } \tau}$ mod $\partial \overline{\text{star } \tau}$. Furthermore $\delta(\Sigma\, c_\tau \tau) = 0 \Longrightarrow \partial(\Sigma\, \tau \cap c_\tau) = 0$, and $\Sigma\, \tau \cap c_\tau$ represents $e_K(\Sigma\, c_\tau \tau)$.

Similarly we have

$$\hat{e}_K : H^q(K') = \hat{F}^{n-q}(H^q(K')) \longrightarrow\!\!> \hat{\tilde{E}}_{n-2q,q} \subset \dot{\hat{\tilde{E}}}{}^2_{n-2q,q} = H_{n-q}(K, \mathscr{N}^n)$$

induced by $\Sigma\, n_\sigma \sigma \to \sum_{\omega_n} n_\sigma [\omega](\sigma \cap \omega)$ (ω is an n-cocycle on star$(\sigma \cap \omega)$).

Thus $\langle K\rangle = \hat{e}_x(1)$ is represented by $\sum_{\omega_n} [\omega]\omega \in C_n(K;\mathscr{N}^n)$.

Let $\varphi : \mathscr{N}_n \otimes \mathscr{N}^n \to \mathbb{Z}$ be the Kronecker pairing $\varphi(\alpha_\sigma \otimes \beta_\sigma) = \langle\beta_\sigma, \alpha_\sigma\rangle = \beta_\sigma(\alpha_\sigma)$. (4.2) clearly commutes, so $\cdot \cap \langle X\rangle$ indeed gives homomorphisms

$$\cdot \cap \langle K\rangle : H^q(K;\mathscr{N}_n) \to H_{n-q}(K')$$

$$\cdot \cap \langle K\rangle : H^q(K') \to H_{n-q}(K;\mathscr{N}^n)$$

by (4.3) and (4.4) respectively.

Now let $\sum_{\tau^q} [c_\tau]\tau \in C^q(K; \mathscr{N}_n)$. By definition (4.3)

$$(\Sigma\, [c_\tau]\tau) \cap \langle K \rangle = (\Sigma\, [c_\tau]\tau) \cap \Sigma\, [\omega]\omega$$

$$= \Sigma \langle \omega, c_\tau \rangle \tau \cap \omega$$

$$= \Sigma\, \tau \cap c_\tau = e_K(\Sigma\, [c_\tau]\tau)$$

Similarly, let $\Sigma\, n_\sigma \sigma \in C^q(K')$. By definition (4.4)

$$(\Sigma\, n_\sigma \sigma) \cap \langle K \rangle = (\Sigma\, n_\sigma \sigma) \cap \Sigma [\omega]\omega$$

$$= \Sigma\, n_\sigma [\omega]\sigma \cap \omega = \hat{e}_K(\Sigma\, n_\sigma \sigma).$$

This completes the proof of proposition 2.

<u>Corollary.</u> If M is a closed oriented combinatorial n-manifold, e_M is the classical Poincaré duality map.

<u>Proof</u>. $\mathscr{N}_n$ is just the "orientation sheaf" of M - M is orientable if and only if $\mathscr{N}_n$ is isomorphic to the constant sheaf $\mathbb{Z}$. Under the isomorphism $\mathscr{N}_n \cong \mathbb{Z}$ determined by the fundamental

class $[M] \in H_n(M) \cong \mathbb{Z}$, $<M>$ corresponds to $[M]$, and $e_M = \cdot \cap [M] : H^q(M) \to H_{n-q}(M)$, which can be identified with (the subdivision of) the classical duality map using the definition (2.3) of cap product.

If M is not orientable, i.e. $\mathscr{K}_n$ is not constant, $e_M : H^q(M; \mathscr{K}_n) \xrightarrow{\approx} H_{n-q}(M)$ is the usual "twisted Poincaré duality."

4D. Circuits and normalization

Many spaces with singularities carry "fundamental homology classes," so the natural "duality map" for such spaces is cap product with the fundamental class. This duality map bears a simple relation to the edge morphism e.

An <u>n-circuit</u>, or <u>geometric n-cycle</u>, is a compact n-dimensional space X such that $H_n(X) \cong \mathbb{Z}$, together with a generator $[X]$ of $H_n(X)$. Furthermore, X can be triangulated as a finite simplicial complex K such that

(i) every simplex of K is a face of some n-simplex (X is "purely n-dimensional"), and

(ii) every $(n-1)$-simplex of K is a face of exactly two n-simplexes (the singular set of X has codimension ≥ 2).

[X] is called the <u>fundamental class</u> of X. A map $f: X \to Y$ of n-circuits has <u>degree k</u> if $f_*[X] = k[Y]$.

If X is an n-circuit, then any triangulation K of X satisfies (i) and (ii) (see [ST], where n-circuits are called "oriented n-pseudomanifolds"). The term "n-circuit" was introduced by Lefschetz; "geometric cycle" is due to Sullivan.

<u>Examples.</u> 1. Any singular cycle in a space Y can be represented by a map $f: X \to Y$, where X is a curcuit. Furthermore, a singular homology can be represented by a map of a "circuit with boundary" into Y. One thus obtains an elementary geometric picture of ordinary homology theory. The topological and combinatorial properties of geometric cycles in Y can provide subtle invariants of Y ([Su 1]).

2. A complex projective algebraic variety V of complex dimension k is a 2k-circuit ([Le], [Za]). V is triangulable ([Loj]), and the singular set SV is a subvariety of complex codimension at least one. Furthermore, V - SV is a connected open complex manifold, so the complex structure gives V - SV a canonical orientation. (In fact, any complex analytic space is a circuit.) A simple example is the singular curve $x^3 + y^3 = xyz$ in $P_2(\mathbb{C})$, the "pinched torus" discussed in §1.

3. A real algebraic variety (or, more generally, a real analytic

set) is usually <u>not</u> a circuit, but a "mod 2 circuit." That is, $H_n(V) \cong \mathbb{Z}/2$, and $V = |K|$, where every $(n-1)$-simplex of K is a face of an <u>even</u> number of n-simplexes. Any purely n-dimensional connected compact real analytic set is a mod 2 circuit ([BH]).

<u>Proposition 3</u>. Let X be an n-circuit. There is a canonical homomorphism $\theta : H^q(X) \to H^q(X;\mathscr{N}_n)$ such that the following diagram commutes:

$$\begin{tikzcd}
H^q(X) \arrow[rr, "\bullet \cap {[X]}"] \arrow[dr, "\theta"'] & & H_{n-q}(X) \\
& H^q(X;\mathscr{N}_n) \arrow[ur, "e_X = \bullet \cap \langle X \rangle"'] &
\end{tikzcd}$$

(The analogous result holds for $\hat{e}_X$.)

<u>Proof</u>: Let $X = |K|$. I will always assume the n-simplexes of K are coherently oriented to represent $[X]$.

$\theta : C^q(K) \to C^q(K,\mathscr{N}_n)$ is defined by

$$\theta(\sum_i n_i \sigma_i) = \sum_i [\sum_j n_i \omega_{ij}]\sigma_i$$

where $\sigma \in C_q(K)$, and the sum is taken over all $\omega_{ij} \in C_n(K)$ such that $\sigma_i \leq \omega_{ij}$. Since K is an n-circuit, $\sum_j n_i\omega_{ij} = n_i \sum_j \omega_{ij}$ is a cycle in $(\overline{\text{star } \sigma_i}, \partial\, \overline{\text{star } \sigma_i})$ for each i, so $[\sum_j n_i\omega_{ij}] \in H_n(\overline{\text{star } \sigma_i}, \partial\, \overline{\text{star } \sigma_i}) = \mathcal{K}_n(\sigma_i)$. I claim that θ is a chain map, i.e. $\delta\theta = \theta\delta$.

$$\delta\theta(\sum_i n_i\sigma_i) = \delta \sum_i [\sum_j n_i\omega_{ij}]\sigma_i$$

$$= \sum_k [c_k]\tau_k ,$$

summed over all τ_k which have some σ_i as a face. Clearly

$$c_k = \sum_{ij} [\sigma_i, \tau_k] n_i\omega_{ij} ,$$

summed over all i, j such that $\sigma_i \leq \tau_k \leq \omega_{ij}$. On the other hand,

$$\theta\delta(\sum_i n_i\sigma_i) = \theta \sum_{ik} [\sigma_i, \tau_k] n_i\tau_k$$

$$= \sum_{ik} [\sigma_i, \tau_k] n_i \sum_j [\omega_{kj}]\tau_k,$$

summed over all j such that $\tau_k \leq \omega_{kj}$. Thus $\delta\theta = \theta\delta$, so θ induced a map on homology. It remains to show that

$\theta(u) \cap \langle K \rangle = u \cap [K]$ for $u \in H^q(K)$. Let u be represented by the cocycle $\sum_i n_i \sigma_i$

$$\theta(\sum_i u_i \sigma_i) \cap \langle K \rangle = (\sum_i [\sum_j n_i \omega_{ij}] \sigma_j) \cap \langle K \rangle$$

$$= \sum_{ij} n_i (\sigma_j \cap \omega_{ij}),$$

where the sum is over all ω_{ij} such that $\sigma_i \leq \omega_{ij}$ (cf. the proof of proposition 2). Now

$$(\sum_i n_i \sigma_i) \cap [K] = (\sum_i n_i \sigma_i) \cap [\sum_k \omega_k]$$

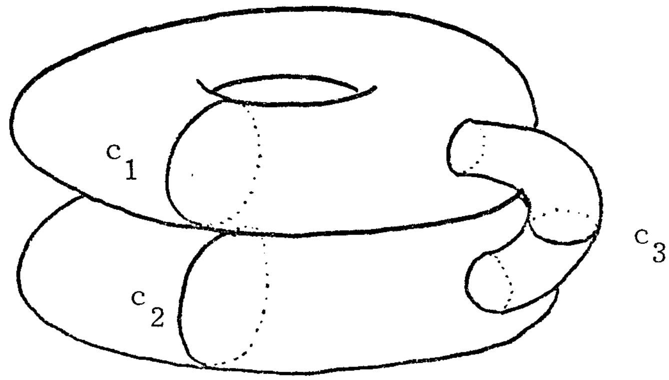
$$= \sum_{ik} n_i (\sigma_i \cap \omega_k),$$

where the sum is over all $\omega_k \in K_n$. Since $\sigma_i \cap \omega_k = 0$ if $\sigma_i \not\leq \omega_k$, $\theta(\sum_i n_i \sigma_i) \cap \langle K \rangle = (\sum_i n_i \sigma_i) \cap [K]$, q. e. d.

θ is clearly not injective in general, since θ injective $\Longrightarrow$ $\cdot \cap [K]$ injective (and $\cdot \cap [K]$ isn't injective for the pinched torus, for example). Neither is θ surjective in general, as is shown by the following example.

Let $X = (S^1_\alpha \times S^2) \# (S^1_\beta \times S^2) \big/ S^1_\alpha \times \{x\} = S^1_\beta \times \{x\}$, where $\#$ is connected sum (by a tube not touching $S^1_\alpha \times \{x\}$ or

$S^1_\beta \times \{x\}$).



X is a 3-circuit with singularity $S(X) \cong S^1$.

$H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, generated by cycles c_1, c_2, c_3 representing the two 2-spheres in Y and the transverse 2-sphere of the tube. Now the sheaf $\mathcal{N}_3$ has stalk $\mathbb{Z}$ except along the singular circle, where the stalk is $\mathbb{Z} \oplus \mathbb{Z}$. Thus $H^q(X;\mathcal{N}_3) \cong H^q(S^1 \times S^2 \# S^1 \times S^2)$.

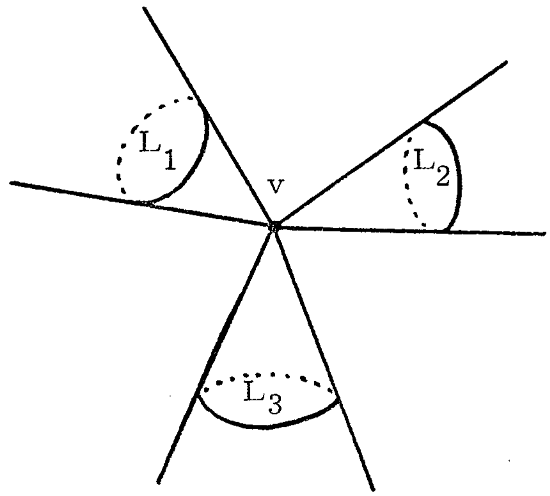
It is clear that we have

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[dr, phantom, "\cong" sloped] & & & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \\
& H^1(X) \arrow[rr, "\cdot \cap [X]"] \arrow[dr, "\theta"'] & & H_2(X) \arrow[ur, phantom, "\cong" sloped] \\
& & H^1(X;\mathcal{N}_3) \arrow[ur, "\cong", "e_X"'] \arrow[d, phantom, "\cong" sloped] & \\
& & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} &
\end{tikzcd}$$

e_X is an isomorphism, but $\cdot \cap [X]$ and θ aren't surjective. In fact, if $u \in C^1(X)$ is the Kronecker dual of the 1-cycle represented by the singular circle SX, and $v \in C^1(X)$ is the dual of the 1-cycle running around the tube, u and v represent the generators of $H^1(X)$, and $[u] \cap [X] = [c_1] + [c_2]$, $[v] \cap [X] = [c_3]$.

This example suggests that it is possible to analyze e_X by partially resolving the singularities of X to make the sheaf $\mathcal{N}_n$ constant. The following discussion of this idea combines ideas of Kaup [Ka] with a simple resolution technique of Sullivan (unpublished, cf. [Su 2]).

If X is an algebraic variety in $P_n(\mathbb{C})$ there is an algebraic normalization $\bar{X} \xrightarrow{\pi} X$, such that $\pi^{-1}(X)$ is finite, $\bar{X}$ has only one "branch" at each point, and the points of $\pi^{-1}(X)$ correspond to the branches of X at x (see [Mu], §§8 and 9 of chapter 3). If X is triangulated as a complex K, and x is a vertex of K, the k branches of X at x are the cones $x * L_i$ $1 \leq i \leq k$, where $L_1, \ldots, L_k$ are the components of link v - S(link v), S = singular set.



The branches of X at a point $x \in \operatorname{int} \sigma$, where σ is a simplex of codimension ≥ 2 in K, are the sets $\sigma * L_i$, where the L_i are the closures of the components of link σ - S(link σ). Algebraic normalization corresponds to "pulling apart" the branches of X at each point x in a canonical way.

Lemma. If X is an n-circuit, the following conditions are equivalent:

(i) link σ - S(link σ) is connected for $\sigma \in K_{n-2}$, K any triangulation of X

(ii) $\mathcal{Y}_n \cong \mathbb{Z}$, the constant sheaf with stalk $\mathbb{Z}$.

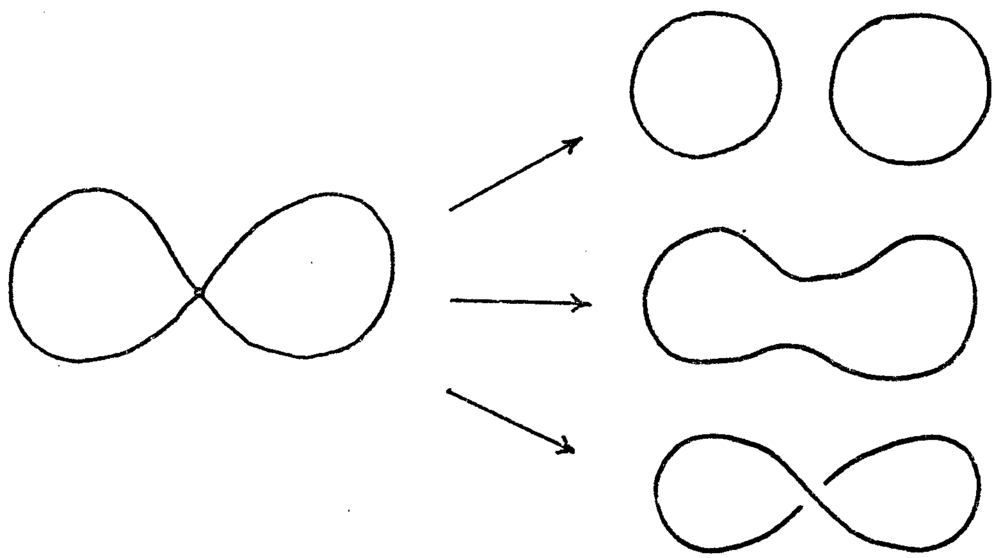
Proof: First note that if $X = |K|$ is an n-circuit, and σ is an i-simplex of K, link σ is purely $(n-i-1)$-dimensional, the singularity S(link σ) is of codimension ≥ 2 in link σ, and each component of link σ - S(link σ) is oriented. This follows from the fact that $\overline{\text{star }\sigma} = \sigma * \text{link }\sigma$. Now if $x \in \text{int }\sigma$, the stalk $\mathscr{H}_n(X) = H_n(X, X - \{x\}) \cong H_n(\overline{\text{star }\sigma}, \partial\, \overline{\text{star }\sigma}) \cong$ $H_{n-i}(\text{dual }\sigma, \text{link}'\sigma) \cong \tilde{H}_{n-i-1}(\text{link}'\sigma)$ (since dual $\sigma = \underline{\sigma} * \text{link}'\sigma$). If $i \leq n-2$, $n-i-1 > 0$, so $\tilde{H}_{n-i-1}$ is a free abelian group on the components of link σ - S(link σ).

As a corollary of the proof, the stalk of $\mathscr{H}_n$ at $x \in X$ is a free abelian group on the branches of X at x. If $\mathscr{H}_n \cong \mathbb{Z}$, i.e. X has but one branch at each point, we will say X is <u>locally irreducible</u>, or <u>normal</u>. "Zariski's Main Theorem" says that a normal complex algebraic variety is indeed normal in this sense (cf. [Mu]).

<u>Proposition 4</u> (Sullivan). Any n-circuit X has a unique "normalization", i.e. there is a locally irreducible n-circuit $\overline{X}$ and a degree one map $\pi : \overline{X} \to X$ such that $\pi^{-1}(X)$ is finite for each $x \in X$, and the canonical sheaf map $\pi_* \mathscr{H}_n(\overline{X}) \to \mathscr{H}_n(X)$ is an isomorphism. Moreover, if $p : Y \to X$ is another normalization of X, there is a unique homeomorphism $h : \overline{X} \to Y$ such that $\pi = p \circ h$, and h is degree 1.

<u>Remarks</u>. By the proof of the lemma, $\pi_* \mathscr{H}_n(\overline{X}) \xrightarrow{\approx} \mathscr{H}_n(X)$ says that the points of $\pi^{-1}(x)$ are in one-to-one correspondence with

the branches of X at x (in a continuous fashion as x varies). As a corollary of the proof, we will obtain an explicit geometrical construction for the algebraic normalization of a variety. It should be noted that real varieties have algebraic normalizations, but the pulling apart of the codimension one singularities is not unique topologically (Sullivan):



Proof: First I will sketch Sullivan's step-by-step construction, and then construct an abstract "lattice" describing it.

Let $X = |K|$, and suppose $\dim SX = s$. If σ is an s-simplex of SX, link σ will be an $(n-s-1)$-manifold - possibly with several components, say link $\sigma = L_1 \cup \ldots \cup L_k$. Now replace the neighborhood star $\sigma \cong \sigma * \text{link } \sigma$ of σ by $\bigcup_{i=1}^{k} \sigma_i * L_i$, identifying each $\partial\sigma_i$ to $\partial\sigma$. The resulting space $\tilde{X}$ contains k copies of σ, $\tilde{X}$ has just one branch at each of these simplexes, and there is a canonical map $p : \tilde{X} \to X$ got by identifying all the

σ_i's to σ.

Now one continues this "pulling apart" process down through the skeletons of K. If link σ is connected for all σ of dimension $> t$, then the link of a t-simplex τ will be a disjoint union of (n-t-1)-circuits, and so the normalization process can be repeated for τ. Thus $\pi : \overline{X} \to X$ is constructed in a finite number of steps.

Now consider the sheaf $\mathcal{N}_n$, which (by the proof of the last lemma) reflects the "branching pattern" of X. If $X = |K|$, $\mathcal{N}_n$ is a stack on K with a particularly simple description. If $\sigma \in K$, $\mathcal{N}_n(\sigma)$ is a free abelian group. Let $\alpha_j(\sigma)$, $1 \leq j \leq k_\sigma$, be generators for $\mathcal{N}_n(\sigma)$. If $\sigma < \tau$, then $\mathcal{N}_n(\sigma) \to \mathcal{N}_n(\tau)$ is given by a matrix

$$\alpha_j(\sigma) \to \sum_i \epsilon_{ij}^{\sigma\tau} \; \alpha_i(\tau).$$

$(\epsilon_{ij}^{\sigma\tau})$ satisfies

(i) each $\epsilon_{ij}^{\sigma\tau} = \pm 1$ or 0

(ii) for each i, there is a unique j such that $\epsilon_{ij}^{\sigma\tau} \neq 0$.

Let $\epsilon_{ij}^{\sigma\tau} = 0$ if σ is not a face of τ. Properties (i) and (ii) hold because the generators $\alpha_i(\sigma)$ correspond to the components $L_i(\sigma)$ of link'σ - S(link'σ), and $\epsilon_{ij}^{\sigma\tau} \neq 0 \iff L_i(\tau) \subset L_j(\sigma)$ (by the proof of the lemma and the definition of the stack $h_* \cong \mathcal{N}_*$).

Let $S = \{\alpha_j(\sigma),\ 1 \leq j \leq k_\sigma\ ,\ \sigma \in K\}$. S is partially ordered by the relation $\alpha_j(\sigma) \leq \alpha_i(\tau) \iff \epsilon_{ij}^{\sigma\tau} \neq 0$ (which is transitive by (ii). Let η be the nerve of $(S, \leq)$, i.e. η is the finite simplicial complex whose i-simplexes are ordered strings $s_0 < \ldots < s_i$ of elements of S. Now the first derived complex K' is the nerve of $(K, \leq)$, and there is a canonical simplicial map $\eta \to K'$ which sends the vertex $\alpha_i(\sigma)$ of η to the barycenter of σ for all i, σ . I claim that the geometric realization $\pi : |\eta| \to |K'| = X$ of this map is a normalization of X. In fact, it's easy to interpret Sullivan's construction so that it produces $\eta \to K'$ from K.

More generally, suppose $p : Y \to X = |K|$ is any normalization of X. I will construct a homeomorphism $h : |\eta| \to Y$ such that $\pi = p \circ h$ by induction up the skeletons of K. The sheaf isomorphism $\Phi : p_* \mathscr{H}_n(Y) \to \mathscr{H}_n(X)$ determines a function $\xi : S \to Y$ as follows. $\alpha_i(\sigma) \in S$ is an element of $H_n(\overline{\text{star}\ \sigma}, \partial\ \overline{\text{star}\ \sigma})$, which is isomorphic (by restriction) to the stalk $\mathscr{H}_n(\underline{\sigma})$ of $\mathscr{H}_n$ at the barycenter $\underline{\sigma} \in X$. Now $\Phi_{\underline{\sigma}} : \sum_{p(y)=\underline{\sigma}} \mathscr{H}_n(y) \xrightarrow{\approx} \mathscr{H}_n(\underline{\sigma})$, and $\mathscr{H}_n(y)$ is a free abelian group, since Y is locally irreducible. Now let $\xi(\alpha_i(\sigma))$ be the point of Y over $\underline{\sigma}$ corresponding to the summand of $\mathscr{H}_n(\underline{\sigma})$ generated by $\alpha_i(\sigma)$. I claim that ξ extends (uniquely) to a homeomorphism

$h : |\eta| \to Y$. Suppose $h_{t-1} : \pi^{-1}(|K_{t-1}|) \xrightarrow{\approx} p^{-1}(|K_{t-1}|)$ has been defined extending ξ (i.e. $h_{t-1}(\alpha_i(\sigma)) = \xi(\alpha_i(\sigma))$ for $\dim \sigma \leq t-1$), and let τ be a t-simplex of K. Now $p^{-1}(\mathring{\tau}) \to \mathring{\tau}$ is a trivial covering space of $\mathring{\tau}$ (since $\mathcal{K}_n(X)$ is constant on $\mathring{\tau}$), i.e. $p^{-1}(\mathring{\tau}) \cong \mathring{\tau} \times p^{-1}(\underline{\tau})$. The same is true of $\pi^{-1}(\mathring{\tau}) \to \mathring{\tau}$, for property (ii) of $(\epsilon_{ij}^{\sigma\tau})$ says that for each face σ of τ and each vertex $\alpha_i(\tau)$ of η over $\underline{\tau}$, there is a unique vertex $\alpha_j(\sigma)$ of η over $\underline{\sigma}$. Thus $\pi^{-1}(\mathring{\tau})$ is simplicially isomorphic to $\underline{\tau} \times (\mathring{\tau})'$. Thus h_{t-1} extends over $\pi^{-1}(\mathring{\tau})$ for each t-simplex τ of K. (As a corollary, any normalization Y of X can be triangulated so that the projection map p is simplicial.)

Examples. 1. If X is the pinched torus $S^2/\{n, s\}$, where n and s are the north and south poles of the 2-sphere, the normalization of X is the identification map $S^2 \to X$.

2. If $X = (S^1_\alpha \times S^2) \# (S^1_\beta \times S^2) / S^1_\alpha \times \{x\} = S^1_\beta \times \{x\}$ (described above), the normalization of X is the identification map $(S^1_\alpha \times S^2) \# (S^1_\beta \times S^2) \to X$.

Proposition 5 (cf. [Ka]). Let $\pi : \overline{X} \to X$ be the normalization of the n-circuit X. There is a canonical isomorphism $\psi : H^q(\overline{X}) \to H^q(X; \mathcal{K}_n)$ for each q such that the following diagram commutes:

$$\begin{tikzcd}
 & H^q(\bar{X}) \arrow[r, "\cdot \cap [\bar{X}]"] \arrow[dd, "\psi"] & H_{n-q}(\bar{X}) \arrow[dd, "\pi_*"] \\
H^q(X) \arrow[ur, "\pi^*"] \arrow[dr, "\theta"'] & & \\
 & H^q(X; \mathscr{H}_n) \arrow[r, "e_X"] & H_{n-q}(X)
\end{tikzcd}$$

(Recall that $e_X \circ \theta = \cdot \cap [X]$ by proposition 3.)

Proof: Let $f : L \to K$ be a triangulation of π. Notice that f is nondegenerate; that is, $f(\sigma)$ has the same dimension as σ for each $\sigma \in K$, since $\pi^{-1}(x)$ is finite for each $x \in X$. Now the fundamental class $[\bar{X}]$ determines an isomorphism $\mathscr{H}_n(\bar{X}) \cong \mathbb{Z}$, i.e. for each simplex $\sigma \in L$, $[\bar{X}]$ restricted to star σ gives an n-cycle $c_\sigma = \sum_{\omega_n} \omega$, $\sigma < \omega$, representing a generator of $\mathscr{H}_n(\sigma) \cong \mathbb{Z}$. ($\bar{X}$ is locally irreducible.) Define ψ by

$$\psi\Big(\sum_{\sigma^q} n_\sigma \sigma\Big) = \sum_{\sigma^q} n_\sigma f_*[c_\sigma](f_*\sigma).$$

$\tau = f_*\sigma$ is a q-simplex of K, and $f_*[c_\sigma] \in \mathscr{H}_n(\tau)$. It's easy to check that $\delta\psi = \psi\delta$, so ψ induces a map on cohomology. ψ is in fact a chain isomorphism, because $f_*\mathscr{H}_n(L) \cong \mathscr{H}_n(K)$ since f is a normalization.

It remains to show that the diagram commutes, and it's enough to check this for elementary cochains. If $\tau \in C^q(K)$,

$f^*\tau = \sum_i \epsilon_i \sigma_i$, where the σ_i are all the (oriented) q-simplexes of L such that $f_*\sigma_i = \epsilon_i \tau$, $\epsilon_i = \pm 1$. Thus $\psi(f^*\tau)$

$= \sum_i \epsilon_i f_*[c_{\sigma_i}](f_*\sigma) = \sum_i f_*[c_{\sigma_i}]\tau$. But $c_{\sigma_i} = \sum_{\omega_n < \sigma} \omega_n$, so

$\sum_i f_*(c_{\sigma_i}) = \sum_{\substack{\sigma \\ f_*\sigma = \pm\tau}} \sum_{\omega_n > \sigma} f_*\omega_n = \sum_{\rho_n > \tau} \rho_n$, since

$\mathscr{N}_n(\tau) \cong \sum_{f_*\sigma = \pm\tau} \mathscr{N}_n(\sigma)$. Thus $\psi(f^*\tau) = [\sum_{\rho_n > \tau} \rho_n]\tau$, which equals

$\theta(\tau)$ by definition (see the proof of proposition 3).

Now let $\sigma \in C^q(L)$. I claim $e_X(\psi(\sigma)) = f_*(\sigma \cap [L])$.

$e_X(\psi(\sigma)) = e_X([\sum_{\omega_n > \sigma} f_*\omega_n] f_*\sigma) = \sum_\gamma \eta(\gamma) \langle \underline{\tau}, \underline{\rho}_{q+1}, \ldots, \underline{\rho}_n \rangle$, a cycle in $C_q(K')$, where $\tau = f_*\sigma$, $\rho_n = f_*\omega_n$, $\omega_n > \sigma$, and γ runs over all sequences $\tau < \rho_{q+1} \ldots < \rho_n$, ρ_i an i-simplex of K. On the other hand,

$$f_*(\sigma \cap [L]) = f_*(\sum_\beta \eta(\beta) \langle \sigma, \underline{\omega}_{q+1}, \ldots, \underline{\omega}_n \rangle)$$

$$= \sum_\beta \eta(\beta) \langle f_*\underline{\sigma}, f_*\underline{\omega}_{q+1}, \ldots, f_*\underline{\omega}_n \rangle,$$

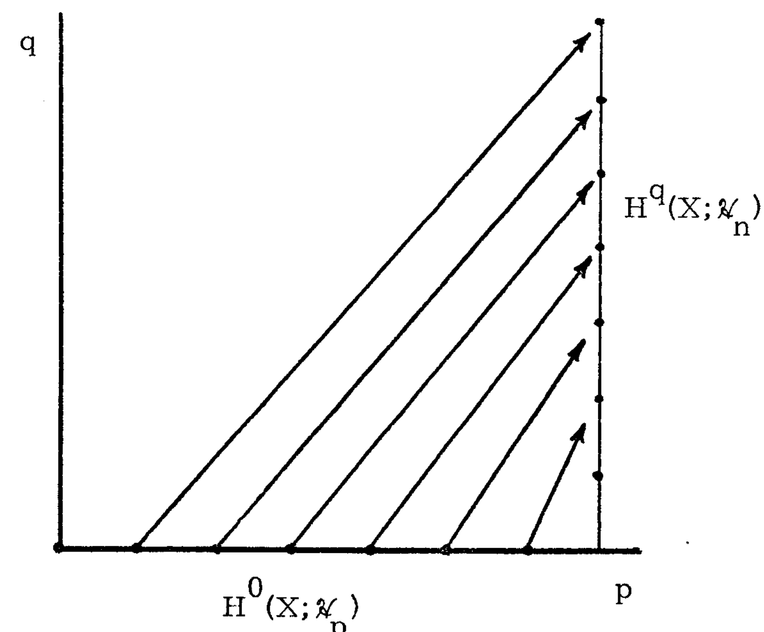
where β runs over all sequences $\sigma < \omega_{q+1} < \ldots < \omega_n$ in L. Now $[f_*\omega_i, f_*\omega_{i+1}] = [\omega_i, \omega_{i+1}]$, so

$$\eta(\sigma, \omega_{q+1}, \ldots, \omega_n) = \eta(f_*\sigma, f_*\omega_{q+1}, \ldots, f_*\omega_n).$$

Furthermore, each sequence $\sigma < \rho_{q+1} < \ldots < \rho_n$ lifts to a sequence in L, so the proof is complete.

As an application of some of the ideas in this section, I will analyze the spectral sequence E of an n-circuit X with isolated homological singularities (e.g. the pinched torus, or the quadric cone in $P_3(\mathbb{C})$, cf §1).

Let X be an n-circuit such that, except for a finite number of singular points, the stalk of $\mathcal{H}_i$ is 0 for $i < n$ and $\mathbb{Z}$ for $i = n$ (i.e. X is a closed homology manifold with isolated singularities). Then $H^q(X;\mathcal{H}_p) = 0$ for $q > 0$ and $p < n$, so E^2 has one nonzero row $(q = 0)$ and one nonzero column $(p = n)$.

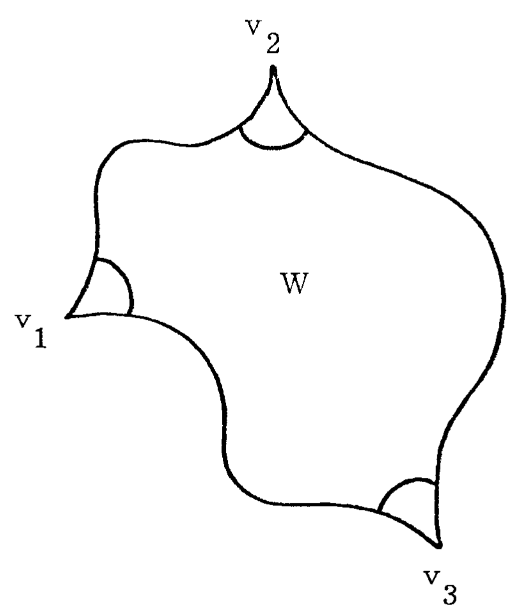


There are "transgression" homomorphisms $t = d^{n-p+1} : H^0(X;\mathcal{H}_p) \to H^{n-p+1}(X;\mathcal{H}_n)$ which fit into the long exact sequence

$$(4.5) \quad \ldots \to H^0(X; \mathcal{X}_{n-q-1}) \xrightarrow{t} H^q(X; \mathcal{X}_n) \xrightarrow{e_X} H_{n-q}(X) \xrightarrow{f_X} H^0(X; \mathcal{X}_{n-q}) \to \ldots$$

where e_X and f_X are the edge morphisms of E (cf. [Ka]).

Let K be a triangulation of X. The singular points of X must be vertexes of K because $\mathcal{X}_*$ is constant on the interior of each simplex of K. Let K'' be the second derived complex of K, and let $C = \bigcup_{i=1}^{s} (\overline{\text{star } v_i}, K'')$, where $v_1, \ldots, v_s$ are the singular points of X. Then $W = X - |\mathring{C}|$ is an oriented homology manifold with boundary $\partial W = \bigcup_{i=1}^{s} (\text{link } v_i, K'')$.



<u>Proposition 6</u>. The long exact sequence (4.5) is isomorphic to the long exact homology sequence of the pair $(W, \partial W)$.

<u>Proof</u>: I will define the isomorphisms in the following ladder, and leave the reader to check commutativity (using $\tilde{E}$).

$$\begin{array}{ccccccccc}
\cdots \to H^0(X; H_{n-q+1}) & \xrightarrow{t} & H^q(X; \mathcal{H}_n) & \xrightarrow{e_X} & H_{n-q}(X) & \xrightarrow{f_X} & H^0(X; \mathcal{H}_{n-q}) \to \cdots \\
\downarrow \wr\wr & & \alpha_q \downarrow \wr\wr & & \beta_q \downarrow \wr\wr & & \gamma_q \downarrow \wr\wr \\
\cdots \to H_{n-q}(\partial W) & \longrightarrow & H_{n-q}(W) & \longrightarrow & H_{n-q}(W, \partial W) & \longrightarrow & H_{n-q-1}(\partial W) \to \cdots
\end{array}$$

α_q is the composition $H^q(X; \mathcal{H}_n) \cong H^q(X, C) \cong H^q(W, \partial W) \cong H_{n-q}(W)$. $\beta_q : H_{n-q}(X) \cong H_{n-q}(X, C) \cong H_{n-q}(W, \partial W)$.

$\gamma_q : H^0(X; \mathcal{H}_{n-q}) = \sum_i \mathcal{H}_{n-q}(v_i) = \sum H_{n-q}(\overline{\text{star } v_i}, \partial\, \overline{\text{star } v_i}) \cong \sum_i H_{n-q-1}(\partial\, \overline{\text{star } v_i}) = H_{n-q-1}(\partial W)$.

The ladder begins with β_0 and ends with α_n.

<u>Corollary.</u> If the n-circuit X has at most finitely many homological singular points, and e_X is an isomorphism (e.g. if X is normal and $\cdot \cap [X] = e_X$ is an isomorphism), X is a homology manifold.

5. A geometrical interpretation of the filtration

By Theorem 2 (§ 3), the filtration (1.5) induced on the homology of a space X by the spectral sequence E comes from the filtration of a triangulation K of X by its coskeletons (3.2). In this section, I will use the fact that the coskeletons of K are "transverse" to the skeletons of K, to characterize the filtration of a homology (or cohomology) class topologically.

Let X be a triangulable space of dimension n. A "transverse p-cycle" in X is a singular p-cycle c such that for some triangulation K of X,

$$\dim((\text{support } c) \cap \sigma) \leq p + \dim \sigma - n$$

for all $\sigma \in K$. (One might also say that c is in "general position" in K, or that c is "transimplicial" to K.) A basic lemma in classical intersection theory is that, if X is a (homology) manifold, any cycle in X can be approximated by a transverse cycle (cf. [Le], [ST]). However, if X has singularities, there may be some cycles in X which aren't homologous to transverse cycles. (For example, any 1-cycle running around the pinched torus must intersect the singular point, which is a vertex of any triangulation.)

In fact, a homology class in a singular space is represented by a transverse cycle if and only if it has maximum filtration in Zeeman's spectral sequence. Recall that if $\alpha \in H_p(X)$, and $\dim X = n$, filtration $\alpha \leq n-p$. By Corollary 2 to Theorem 2A, α has filtration $n-p$ if and only if α is represented by a simplicial cycle in $K^{(n-p)}$, $|K| = X$. But $\dim(K^{(n-p)} \cap \sigma) = p + \dim \sigma - n$ for all $\sigma \in K$ (3.2), so such a cycle is transimplicial to K. Conversely, suppose α is represented by a transverse cycle c, say c is transimplicial to K. Then $\operatorname{supp} c \cap K_{n-p-1} = \emptyset$, so $\operatorname{supp} c \subset X - |K_{n-p-1}|$. But $|K^{(n-p)}|$ is a deformation retract of $X - |K_{n-p-1}|$ (3.2), so c is homologous to a cycle in $K^{(n-p)}$, so filtration $\alpha = n-p$. (This argument will be generalized in section A (below.)

Now if X is a normal n-circuit, we have seen that α has maximum filtration if and only if $\alpha = \beta \cap [X]$, i.e. α is dual to a cohomology class β. (Hence it is a direct consequence of the definition (2.3) of cap product that α is represented by a transverse cycle.)

Thus the homology classes of maximum filtration in a space are, geometrically, those represented by transverse cycles; and, algebraically, those dual to cohomology classes.

In this section I will show that the filtration of a homology class measures how its topological "freedom of movement" is restricted in X. Dually, the filtration of a cohomology class shows how "elusive" it is; $\beta \in H^p(X)$ has maximum filtration if and only if $\beta \cap [X] \neq 0$ (X normal), so β is quite "spread out", being supported by its dual cycle.

If X is a polyhedron, and consequently is equipped with a rigid geometrical decomposition into equisingular strata, I will show that a homology class has filtration $\geq q$ if and only if it is represented by a cycle which intersects the strata in codimension $\geq q$.

5A. The degrees of freedom of a homology class

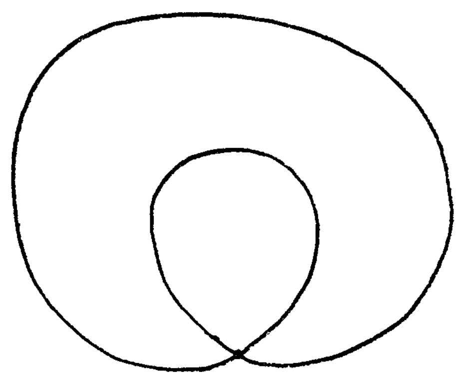
Recall that a topological space A has Čech dimension $< q$ if any open cover of A has a refinement U such that $U_0 \cap \ldots \cap U_q = \emptyset$ for any $q + 1$ open sets $U_0, \ldots, U_q$ in U (i.e. the nerve of U has no q simplexes).

Definition. If $\alpha \in H_s(X)$ is a singular homology class, α has $\geq q$ degrees of freedom in X, $df(\alpha) \geq q$, if

$$\alpha \in \text{Image}[H_s(X-A) \to H_s(X)]$$

for all closed subspaces $A \subset X$ of Čech dimension $< q$. (In other words, any cycle representing α can be moved off A by a homology.)

<u>Examples</u>. 1. If X is the annulus $S^1 \times I$, the generator of $H_1(X)$ has 1 degree of freedom. If $\{s\} \times I$ is collapsed to a point for some $s \in S^1$, to produce a homotopy equivalent space Y, the generator of $H_1(Y)$ has 0 degrees of freedom, since any cycle representing it must pass through the singular point.



2. Similarly, if X is the pinched torus $S^1 \times S^1/\{w\} \times S^1$ (§ 1), the generator of $H_1(X)$ has 0 degrees of freedom, since it can't be moved off the singular point.

3. If K is a combinatorial n-manifold, any $\alpha \in H_s(|K|)$ has $n-s$ degrees of freedom, since α is represented by a simplicial s-cycle a, which can be moved off any closed $A \subset |K|$ of dimension $< n-s$ by "general position" (cf. the lemma below).

However, (assuming a is of infinite order for simplicity) a can't be moved off the support of its dual cycle b, which is $(n-s)$-dimensional.

Note that if $\alpha, \beta \in H_s(X)$, $df(n\alpha) \geq df(\alpha)$, and $df(\alpha+\beta) \geq \min(df(\alpha), df(\beta))$. Thus the homology classes of X with $\geq q$ degrees of freedom form a subgroup of $H_s(X)$ for each q. This yields a filtration of $H_s(X)$ by subgroups

$$0 \subset F_s^n \subset \ldots \subset F_s^{q+1} \subset F_s^q \subset \ldots \subset F_s^0 \subset H_s(X)$$

$(n = \dim X)$, which is a topological invariant of X by definition.

Theorem 3A. Let $\alpha \in H_s(X)$ be a singular homology class. If X is triangulable, then the degrees of freedom of α equals the filtration of α in Zeeman's spectral sequence.

Proof: Let K be a triangulation of X. I will show the equivalence of the following conditions:

1) $df(\alpha) \geq q$
2) $\alpha \in \operatorname{Im}[H_s(X-|K_{q-1}|) \to H_s(X)]$ (α is represented by a cycle in the complement of the $(q-1)$-skeleton of K)
3) $\alpha \in \operatorname{Im}[H_s(|K^{(q)}|) \to H_s(X)]$ (α is represented by a cycle in the q-coskeleton on K)

4) filtration $\alpha \geq q$ in E

5)* α is represented by a (simplicial) cycle which can be moved off any closed $A \subset X$ of $\dim < q$ by an arbitrarily small ambient isotopy of X.

Clearly 1) $\Longrightarrow$ 2) (let $A = |K_{q-1}|$). 2) $\Longrightarrow$ 3) because $|K^{(q)}|$ is a deformation retract of $X - |K_{q-1}|$ (3.2). Now let $\tilde{\alpha} \in H_s(|K^{(q)}|)$ be such that $\tilde{\alpha} \to \alpha$ under the inclusion map. Then a simplicial cycle in $K^{(q)} \subset K'$ representing $\tilde{\alpha}$ will also represent α. Therefore, by corollary 2 to theorem 2A, 3) $\Longleftrightarrow$ 4) (in other words, by the isomorphism $E \cong \tilde{E}$, where $\tilde{E}$ is got by filtering X by the coskeletons of K). Clearly 5) $\Longrightarrow$ 1). I claim that 3) $\Longrightarrow$ 5),* which will complete the proof. What I will actually show is that any closed A of dimension $< q$ can be ambient ϵ-isotoped off $|K^{(q)}|$. (This is a variation on the usual "general position" theorem - compare the p.l. proposition below.) Recall that $\dim(K^{(q)} \cap K_i) = i-q$ for all i (3.2), so $K^{(q)} \cap K_{q-1} = \emptyset$. Suppose that A has been ambient ϵ-isotoped so that $|K^{(q)} \cap K_i| \cap A = \emptyset$ for $i < j$. Let σ be a j-simplex of K. I will define an isotopy h_t $\overline{\text{star } \sigma}$ keeping $\partial\, \overline{\text{star } \sigma}$ fixed which moves A off $|K^{(q)} \cap \sigma|$ (and which can be made arbitrarily small). Repeating this construction for each $\sigma \in K$ in order of increasing dimension will produce the desired isotopy.

*See Erratum.

Lemma 1.* Let B, C be closed subspaces of the j-disc D^j. Suppose that there is a triangulation of D with B as an r-dimensional subcomplex, and suppose C has Čech dimension s. If $r + s < j$, and $B \cap C \cap \partial D = \emptyset$, there is an arbitrarily small isotopy f_t of D such that $f_t \mid \partial D$ is the identity and $B \cap f_1(C) = \emptyset$.

Before proving the lemma, let's use it to construct h_t. Let $D^j = \sigma^j$, $B = |K^{(q)}| \cap \sigma$, and $C = A \cap \sigma$. $\dim B + \dim C \leq (j-q) + (q-1) = j - 1$, and $B \cap C \cap \partial D = \emptyset$ (by inductive hypothesis), so there is an ϵ-isotopy f_t of σ rel $\partial\sigma$ such that $|K^{(q)} \cap \sigma| \cap f_1(A \cap \sigma) = \emptyset$. Now $(\overline{\text{star}\ \sigma}, \partial\, \overline{\text{star}\ \sigma}) \cong (\sigma * \text{link}\ \sigma, \partial\sigma * \text{link}\ \sigma)$, so $h_t = f_t * 1$ is an extension of f_t to an isotopy of $\overline{\text{star}\ \sigma}$ rel $\partial\, \overline{\text{star}\ \sigma}$ which moves A off $|K^{(q)} \cap \sigma|$, and if f_t is small, h_t will be small.

Proof of lemma: The proof reduces to the case $B = D^r \subset D^j$ (standard inclusion) as follows. Let L be a triangulation of D^j with B as an r-subcomplex (e.g. the first barycentric subdivision of σ^j in the proof of the theorem). Now apply the lemma to τ, $C \cap \overline{\text{star}\ \tau} \subset \overline{\text{star}\ \tau}$ for the simplexes τ of L_r, in order of increasing dimension, to move C off L_r.

So suppose $B = D^r \subset D^j = D^r \times D^{j-r}$, and $C \subset D^j$ is of dimension $< j-r$, with $B \cap C \cap \partial D^j = \emptyset$. If $x \in \mathring{D}^{j-r}$,

*See Erratum.

consider the isotopy of D^{j-r} rel ∂ defined by pushing the center of D^{j-r} linearly to x, and extending this motion conically over D^{j-r} to ∂D^{j-r}. Multiplying this isotopy by the identity on D^r, we obtain an isotopy h_t^x of D^j rel $D^j \times \partial D^{r-j}$. I claim that, given $\varepsilon > 0$, there is an $x_0 \in \mathring{D}^{r-j}$ such that $|x_0| < \varepsilon$ and $h_1^{x_0}(C) \cap D^r = \emptyset$. Suppose this isn't true, i.e. there is an $\varepsilon > 0$ such that for all $x \in \mathring{D}^{r-j}$ with $|x| < \varepsilon$, there exists a $c \in C$ such that $h_1^x(c) \in D^r$. Let $p : D^j \times D^{r-j} \to D^{r-j}$ be the projection. Then there is an $\varepsilon' > 0$ such that $p(c) \supset \{x \in D^{r-j}, \ |x| \leq \varepsilon'\}$, so the projection of C has dimension $r-j$. This implies that $\dim C \geq r-j$ (see [Na], theorem 21-2), which is a contradiction. Now if $|x_0| < \varepsilon$ and $h_1^{x_0}(C) \cap D^r = \emptyset$, it's easy to modify h_t so that it keeps ∂D^j fixed and still moves C off D^r (by changing h_t in a small collar of ∂D^j). This completes the proof of theorem 3A.

<u>Corollary 1.</u> Let $\alpha \in H_s(X)$, X triangulable. If there exist $\beta \in H^q(X)$ and $\gamma \in H_{s+q}(X)$ such that $\beta \cap \gamma = \alpha$, then $df(\alpha) \geq q$.

<u>Proof:</u> By the definition (2.3) of cap product, α is represented by a cycle in $K^{(q)}$ for any triangulation K of X. Thus the

corollary follows by the proof that 3) $\Longrightarrow$ 5) above.

Remark. Since degrees of freedom = filtration in E, corollary 1 is equivalent to Theorem 3 of [Ze 1]. As Zeeman points out, filtration $\alpha = q$ does not imply that $\alpha = \sum_i \beta_i \cap \gamma_i$ with $\dim \beta_i \geq q$ for each i. However, this is true on the chain level by Proposition 1 of §3.

Corollary 2. If X is triangulable, and $\dim X = n$, then $df(\alpha) \leq n-s$ for all $\alpha \in H_s(X)$.

Corollary 3. If X is triangulable, and $\alpha \in H_s(X)$, then $\alpha \in \mathrm{Im}[e_X\colon H^{n-s}(X; \mathcal{K}_n) \to H_s(X)]$ if and only if $df(\alpha) = n-s$.

Corollary 4. Let X be a normal n-circuit. Then X is a duality space (i.e. $\cdot \cap [X]$ is an isomorphism) if and only if each homology class in X has the maximum degree of freedom (i.e. equal to its codimension).

Proof: By corollary 2, we can only conclude that $\cdot \cap [X]$ is surjective. But in fact $\cdot \cap [X]$ surjective implies $\cdot \cap [X]$ is injective (though not conversely - e.g. the quadric cone in $P_3(\mathbb{C})$, §1). This is a general fact which follows from the "universal coefficient theorem":

Lemma 2. Let (C_*, ∂) be a chain complex of free abelian groups, and let (C^*, δ) be the dual complex $C^q = \mathrm{Hom}(C_q, \mathbb{Z})$, $\delta = \mathrm{Hom}(\partial, \mathbb{Z})$. Let $D : C^* \to C_{n-*}$ be a chain map, i.e. $D_q : C^q \to C_{n-q}$ for all q, and $D_q \circ \delta = \partial \circ D_q$. Then if the induced map $D_* : H(C^*) \to H(C_{n-*})$ on homology is surjective, it is an isomorphism.

Proof: Write $H_q(C^*) = H^q(C) \cong F^q \oplus T^q$, and $H_q(C_*) = H_q(C)$ $= F_q \oplus T_q$, where F = free part and T = torsion part. By the universal coefficient theorem ([Sp] (5.5.3)),

$$F^q \cong F_q \quad \text{and} \quad T^q \cong T_{q-1} \qquad \text{for all} \quad q.$$

Now $D_*(F^q) \subset F_{n-q}$, so since D_* is surjective, $D_*(F^q) = F_{n-q}$, which implies $\mathrm{rank}(F_q) \geq \mathrm{rank}(F_{n-q})$. Replacing q by $n-q$, we have $\mathrm{rank}(F_{n-q}) \geq \mathrm{rank}(F_q)$, so $\mathrm{rank}(F_q) = \mathrm{rank}(F_{n-q})$, and

$$D_* \mid F^q : F^q \to F_{n-q}$$

is an isomorphism for all q. (A surjection of free abelian groups of equal rank is an isomorphism.)

Now we also have $D_*(T^q) = T_{n-q}$, which implies order $(T^q) \geq$ order(T_{n-q}), so order$(T_{q-1}) \geq$ order(T_{n-q}). Replacing q by $n-q+1$, order$(T_{n-q}) \geq$ order(T_{q-1}), so order(T^q) = order(T_{n-q}), and

$$D_* \mid T^q : T^q \to T_{n-q}$$

is an isomorphism for all q. (A surjection of finite abelian groups of the same order is an isomorphism.) Thus $D_* : H^q(C) \cong F^q \oplus T^q \to F_{n-q} \oplus T_{n-q} \cong H_q(C)$ is an isomorphism for all q.

<u>Remarks on Theorem 3A</u>. 1. If X is not triangulable, I can show filtration $\alpha \leq$ degrees of freedom α, but I don't know

whether equality always holds.

2. The definition of degrees of freedom makes sense for coefficients in any abelian group G, as does the definition of E. The same proof shows theorem 3A is true for G coefficients. If $\varphi : G \to H$ is a homomorphism of groups and $\alpha \in H_s(X; G)$, clearly $df(\varphi_* \alpha) \geq df(\alpha)$. Sometimes the equality is strict (see [Ze 1], p. 181). (For example, X may be a G-homology manifold but not a $\mathbb{Z}$-duality space.)

Now suppose that X is a polyhedron; that is, X is equipped with a piecewise linear structure, consisting of a maximal family of p.l. related triangulations. (For a general disucssion of p.l. structures, see [Ze 2], chapter 1.) This rigid geometry provides a good framework in which to develop the theory of general position and transversality.

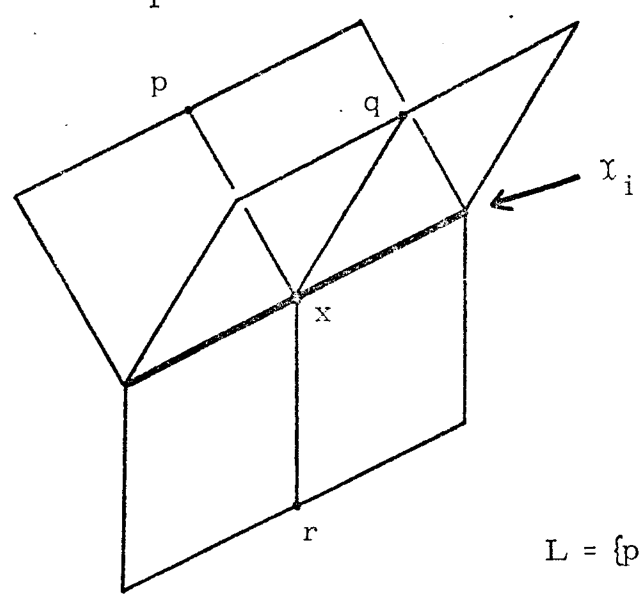
Many polyhedra, such as algebraic varieties, have natural geometrical "stratifications."

Definition. A p.l. stratification of an n-dimensional polyhedron X is a filtration

$$X = X_n \supset X_{n-1} \supset \dots \supset X_0 \supset \emptyset$$

by subpolyhedra such that $X_i - X_{i-1}$ is an (open) p.l. i-manifold for each i, and each point $x \in X_i - X_{i-1}$ has a neighborhood N in (X, X_i) p.l. homeomorphic to $((\text{cone } L) \times D^i, D^i)$ for some polyhedron L. Furthermore, for each $j > i$, $X_j \cap N$ corresponds to $(\text{cone } L_j) \times D^i$ for some subpolyhedron L_j of L.

The manifolds $\mathfrak{X}_i = X_i - X_{i-1}$ are called the strata, and L is the link of $\mathfrak{X}_i$ at x. The description of N says that X has "locally constant normal geometry" along $\mathfrak{X}_i$; in other words, the stratum $\mathfrak{X}_i$ is "equisingular".



$L = \{p, q, r\}$

D. Stone calls such an object a "p.l. variety," reserving the name "p.l. stratification" for a p.l. variety with block-bundles along the strata (see [Sto]).

Examples. 1. Let $f : |K| \to X$ be a p.l. triangulation of X, and let $X_n = f(|K_n|)$, the n-skeleton. $\mathfrak{X}_i$ is homeomorphic to a union of open i-simplexes, so it's an i-manifold. The equisingularity condition holds because $\overline{\text{star}\ \sigma} \cong c(\text{link}\ \sigma) \times \sigma$ for each $\sigma \in K$.

2. Let X_i be the intersection of the i-skeletons of all p.l. triangulations of X. The resulting filtration is called the intrinsic stratification of X (Zeeman and Armstrong). $\mathfrak{X}_i$ is clearly an i-manifold. For a proof of equisingularity, see [Ak].

3. It is probably true that any smooth stratified set in the sense of Thom can be triangulated as a p.l. stratification, but no one has written down a proof.

The following result puts the proof of Theorem 3A in a geometric context.

<u>Proposition.</u> Let A, B be subpolyhedra of X, and let $\{X_i\}$ be a p.l. stratification of X. There is an arbitrarily small p.l. ambient isotopy h_t of X such that, for each stratum $\mathfrak{X}_i$, $h_t(\mathfrak{X}_i) = \mathfrak{X}_i$ for all t, and

$$\dim(h_1(A) \cap B \cap \mathfrak{X}_i) \leq \dim(A \cap \mathfrak{X}_i) + \dim(B \cap \mathfrak{X}_i) - i.$$

In other words, h_t moves $A \cap \mathfrak{X}_i$ into "general position" with respect to $B \cap \mathfrak{X}_i$ in each stratum $\mathfrak{X}_i$. This proposition generalizes Zeeman's general position theorem for subpolyhedra of a p.l. manifold ([Ze 2], chapter 6, theorem 15).

Proof: The isotopy is constructed by induction up the strata. Suppose that $\dim(A \cap B \cap \mathfrak{X}_j) \leq \dim(A \cap \mathfrak{X}_j) + \dim(B \cap \mathfrak{X}_j) - j$ for $j < i$. (This is always true for $j = 0$.) Zeeman describes a procedure for moving $A \cap \mathfrak{X}_i$ into general position with respect to $B \cap \mathfrak{X}_i$ in $\mathfrak{X}_i$ by a sequence of "local shifts" ([Ze 2], chapter 6, p. 11). I claim that such a local shift extends to an ambient stratum-preserving isotopy of X which leaves X_{i-1} fixed, from which the proposition follows by induction. Let K be a triangulation of X with A, B, and all the X_j as subcomplexes, say $X_j = |K_{(j)}|$. Given $\sigma \in K_{(i)} - K_{(i-1)}$, a local shift of $\mathfrak{X}_i$ with respect to this triangulation is a carefully chosen isotopy f_t of the i-disk $\overline{\text{star}}(\underline{\sigma}, K'_{(i)})$, keeping its boundary fixed. Now $\text{star}(\underline{\sigma}, K') \cong D^i \times c(L) \cong D^i * L$, where L is the link of $\mathfrak{X}_i$ at $\underline{\sigma}$. Thus $f_t * 1_L$ is a topological isotopy of $\overline{\text{star}}(\underline{\sigma}, K')$ keeping its boundary fixed, and equal to f_t on $\overline{\text{star}}(\underline{\sigma}, K'_{(i)})$. However, $f_t * 1_L$ is not piecewise linear (a variation of "the standard mistake", cf. [Ze 2], chapters 1 and 2).

Lemma. Let P, Q be polyhedra. If $F : P \times I \to P \times I$ is a p.l. isotopy with $F \mid P \times \{0\}$ the identity, there is a p.l. isotopy $G : (P * Q) \times I \to (P * Q) \times I$ with $G \mid (P * Q) \times \{0\} \cup Q \times I$ the identity, and $G \mid P \times I = F$. (P and Q sit inside $P * Q$ as the two "ends" of the join.)

Proof: It is clear from the definition of simplicial join (and of the product of a complex with a 1-simplex) that

$$(P * Q) \times I = \left[(P \times I) * (Q \times \{1\})\right] \cup \left[(P \times \{0\}) * (Q \times I)\right],$$

where the union is along the common subpolyhedron $(P \times \{0\}) \cup (Q \times \{1\})$. Thus we can let $G \mid (P \times I) * (Q \times \{1\}) = F * 1$, and $G \mid (P \times \{0\}) * (Q \times I)$ be the identity. (G is clearly level-preserving since F is.)

Applying this lemma to the proof of the proposition, we obtain a p.l. isotopy g_t of $\overline{\text{star}}(\underline{\sigma}, K')$ keeping its boundary fixed, and equal to f_t on $\overline{\text{star}}(\underline{\sigma}, K'_{(i)})$. Extend g_t by the identity outside $\overline{\text{star}}(\underline{\sigma}, K')$. The resulting ambient isotopy keeps X_{i-1} fixed, and preserves the strata by the construction of g_t, since each X_j intersects $\text{star}(\underline{\sigma}, K')$ in a subpolyhedron corresponding to $D^i * L_j \subset D^i * L$.

Corollary 6. Let X be a polyhedron with a p.l. stratification $\{X_i\}$, and let $\alpha \in H_s(X)$. Then $df(\alpha) \geq q$ if and only if α is represented by a p.l. cycle a such that $\dim(\text{support } a \cap \mathfrak{X}_i) \leq i-q$ for each i. (A p.l. cycle is a simplicial cycle in some p.l. triangulation.)

Proof: Let K be a p.l. triangulation of X with all the X_i as subcomplexes. If $df(\alpha) \geq q$, α is represented by a simplicial cycle a in $K^{(q)}$ (condition 3 in the proof of theorem 3A.) But $K^{(q)}$ intersects every simplex of K in codimension q, so the support of a intersects $\mathfrak{X}_i$ in codimension at least q for all i. Conversely, suppose α is represented by a p.l. cycle whose support A intersects each $\mathfrak{X}_i$ in codimension $\geq q$. By the proposition, there is a (p.l.) isotopy h_t of X such that $h_1(A) \cap |K_{q-1}| \cap \mathfrak{X}_i = \emptyset$ for all i, i.e. $h_1(A) \cap |K_{q-1}| = \emptyset$. Thus α is represented by a cycle in the complement of the $(q-1)$-skeleton $|K_{q-1}|$, so $df(\alpha) \geq q$ by condition 2 in the proof of theorem 3A.

Remark.* The proof of theorem 3A contains a topological version of the above "general position" proposition (with $\{X_i\}$ the skeletons of a topological triangulation of the space X). It follows from that proof that "p.l. cycle" can be replaced by

*See Erratum.

"singular cycle" in the corollary, since a singular cycle whose support intersects each stratum of a p.l. stratification in Čech codimension q can be moved off the $(q-1)$-skeleton of any triangulation by a topological isotopy.

This corollary gives a precise geometrical condition for a homology class to have q degrees of freedom. This condition can be further sharpened by introducing blockbundles into the stratification (as in [Sto]). For example, the homology classes of maximum filtration in a polyhedron are those represented by cycles "blocktransverse" to the strata (see [Mc C]).

It would be interesting to relate the filtration of a class to a topologically invariant "stratification" of a space. The most appealing tack would be to use the local homology sheaves $\mathcal{H}_p$ to define this stratification, since the information carried by these sheaves is all that is necessary to compute Zeeman's spectral sequence. Such a "homological stratification" might also help to answer the question of whether the filtration of a class equals its degree of freedom in a nontriangulable space.

Examples.

1. Let X be an n-circuit with one singular point x. (The same analysis will hold for a finite number of singularities.) Examples are the 2-circuit $x^3 + y^3 = xyz$ in $P_2(\mathbb{C})$ (the pinched torus) or

the 4-circuit $x^2 + y^2 + z^2 = 0$ in $P_3(\mathbb{C})$ (the quadric cone). By corollary 6, a class $\alpha \in H_s(X)$ will have n-s degrees of freedom unless the support of every cycle representing α contains x, in which case $df(\alpha) = 0$. The spectral sequence E was analyzed completely in §4. From the E^2 term, it is apparent that α must have filtration n-s or 0, and filtration α = n-s if and only if $\alpha \in \mathrm{Im}(e_X)$, where e_X is the edge morphism of E (4.5). But by proposition 6 of §4, $\mathrm{Im}(e_X) = \mathrm{Im}[H_s(X - \{x\}) \to H_s(X)]$. Thus filt α = n-s if and only if α is represented by a cycle in the complement of the singular point x.

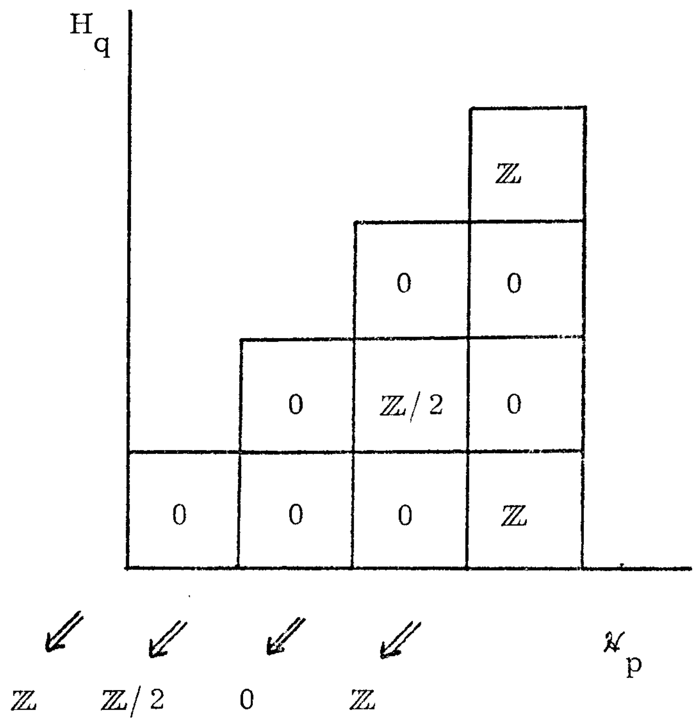
2. Let X be the 3-circuit got from the 3-sphere by identifying antipocal points of a circle:

$$X = S^3/x = -x, \quad x \in S^1.$$

The homology of X is $\mathbb{Z} \quad \mathbb{Z}/2 \; 0 \; \mathbb{Z}$, the generator α of $H_1(X) \cong \mathbb{Z}/2$ being represented by the singular circle S(X). I claim that α has one degree of freedom - α is represented by a cycle in the complement of any given point, but any cycle representing α must intersect S(X).

A neighborhood of S(X) in X is the total space of the nontrivial bundle over S^1 with fiber the cone on two circles.

Thus the link of a point $x \in S(X)$ is $S^2 \cup S^2$, i.e. the neighborhood $\overline{\text{star}}(x) \cong \text{cone}(S^2 \cup S^2) = D^3 \vee D^3$, with x as the cone point. Thus $\mathcal{H}_3$ has stalk $\mathbb{Z} \oplus \mathbb{Z}$ along $S(X)$, and the two summands are interchanged in passing around $S(X)$. Since $\mathcal{H}_3 \mid X - S(X) \cong \mathbb{Z}$, $H^*(X;\mathcal{H}_3) \cong \mathbb{Z}\ 0\ 0\ \mathbb{Z}$. (This also follows from Proposition 5 of §4, since the normalization of X is the identification map $S^3 \to X$. Thus $H^*(X;\mathcal{H}_3) \cong H^*(S^3)$.) $\mathcal{H}_2$ is supported by $S(X)$, where its stalk is $\mathbb{Z}$, and the generators of $\mathbb{Z}$ are interchanged in passing around $S(X)$. Thus $H^*(X;\mathcal{H}_2) \cong 0\ \mathbb{Z}/2$. Clearly $\mathcal{H}_1$ and $\mathcal{H}_0$ are zero, so $E^2(X)$ is



It follows that $d^2 = 0$, so $E^\infty = E^2$, and α indeed has filtration 1.

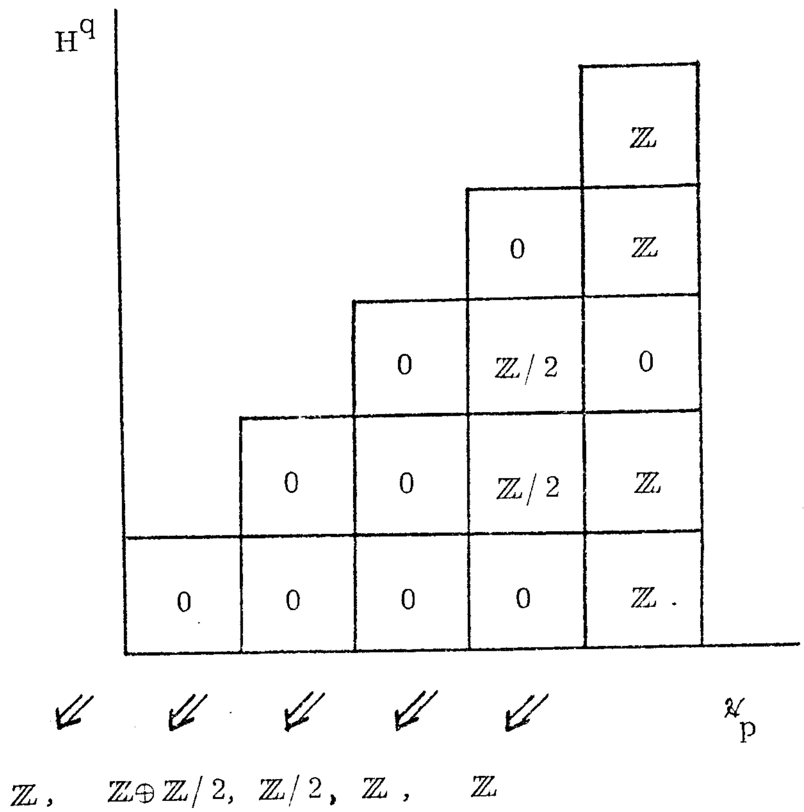
3. By jacking example 2 up a dimension, we can better see how the "twisting" of X along $S(X)$ disturbs duality when $S(X)$ is a manifold. Let A be the total space of the bundle over the torus $S^1 \times S^1$ with fiber $c(S^1 \cup S^1)$, twisted along one factor of the torus:

$$A = D^2 \times S^1 \times S^1/(0, x, y) = (0, -x, y)$$

Let X be the union of A with $B = S^1 \times S^1 \times D^2$, identified along their common boundary. X is a 4-circuit with $S(X)$ a torus. The Mayer-Vietoris sequence of $(X; A, B)$ shows that $H_*(X) \cong \mathbb{Z},\ \mathbb{Z} \oplus \mathbb{Z}/2,\ \mathbb{Z}/2,\ \mathbb{Z},\ \mathbb{Z}$. Let α, β be generators of the two summands $\mathbb{Z}$, $\mathbb{Z}/2$ of $H_1(X)$. β is represented by the circle $C \subset S(X)$ along which the twisting occurs, and α is represented by the other factor of $S(X)$. The generator γ of $H_2(X)$ is represented by $S(X)$. I claim that $df(\beta) = 2$, since any cycle representing β must intersect $S(X)$; and $df(\gamma) = 1$, since any cycle representing γ must intersect C.

Note that C is not determined by the geometry of X. By Corollary 6 above, $df(\gamma) = 1 \iff$ any cycle representing γ intersects $S(X)$ in codimension ≤ 1. The characterization of degrees of freedom as codimension with respect to a stratification is intrinsic to the geometry of X.

The computation of E^2 is similar to example 2. The normalization of X is just $\overline{X} = D^2 \times S^1 \times S^1 \underset{\partial}{\cup} S^1 \times S^1 \times D^2$, so $H^*(X; \mathcal{H}_4) \cong H^*(X) = \mathbb{Z}\ \mathbb{Z}\ 0\ \mathbb{Z}\ \mathbb{Z}$. $\mathcal{H}_3$ is supported by $S(X)$, where its stalk is $\mathbb{Z}$, twisted along C. Thus E^2 is



Again $d^2 = 0$, so $E^2 = E^\infty$, and $df(\beta) = 2$, $df(\gamma) = 1$.

<u>Remark</u>. If X is an n-circuit with $S(X)$ a manifold, let $\xi/S(X)$ be the "normal bundle" along the singularity. $|\xi|$ is a regular neighborhood of $S(X)$, and the "fiber" of ξ is cL, where L is the link of $S(X)$. (In example 3, $S(X) \cong S^1 \times S^1$, $|\xi| = A$, and $L \cong S^1 \cup S^1$.) The effect of the "twisting" (i.e. non-triviality) of ξ on the homology of $|\xi|$ is measured by the Serre spectral sequence of ξ. Sullivan has suggested that Zeeman's spectral sequence E of a stratified space contains the information of the Serre spectral sequences of the

fibrations along the singularities and the Mayer-Vietorrs sequences for glueing all the strata together. It seems clear to me that this information determines E, but that much of it is lost in passing to E. However, any attempt to use E in a more subtle way than as a computational tool should involve such a "picking apart" of E (cf. §6).

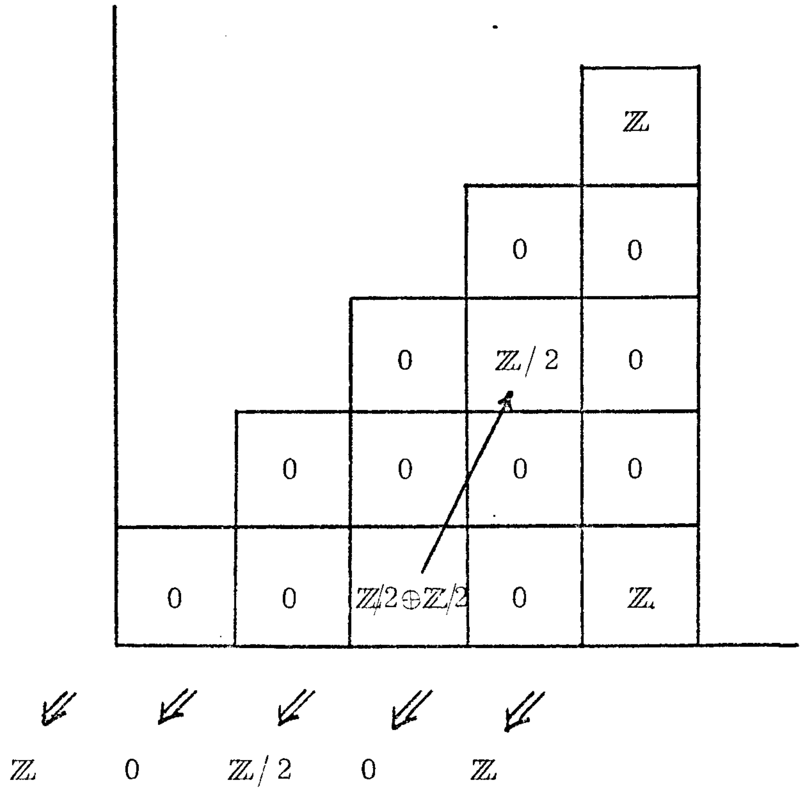
It is difficult to compute Zeeman's spectral sequence for spaces one finds "in nature" (e.g. algebraic varieties), for one really has to know a good bit about the homology of X, as well as the geometry of its decomposition into equisingular strata, in order to fathom E(X). However, E provides an excellent framework in which to organize and test one's knowledge about X, and it can often be useful in computing $H_*(X)$.

4. Let V be the variety $\{xy^2 = wz^2\}$ in complex projective 3-space with homogeneous coordinates (x, y, z, w). The intersection of V with affine 3-space is the variety $W = \{xy^2 = z^2\}$, known as the "pinch point" (which describes its geometry in a neighborhood of 0). V is a 4-circuit with $S(V) = \{y = z = 0\} \cong P_1(\mathbb{C}) \cong S^2$. Now the stratum $S(V) \subset V$ fails to be equisingular at the two points $p = \{x = y = z = 0\}$ (The origin of $\mathbb{C}^3$) and $q = \{y = z = w = 0\}$ (the point at infinity). (So $V \supset S(V) \supset \{p, q\}$ is a p.l. stratification of V). To see this, it suffices to consider $W = V - q$, since

$V - q \cong V - p$. $S(W)$ is the x-plane $\{y = z = 0\}$, and the "normal fiber" to a point $(a, 0, 0) \in S(W) - 0$ can be identified with a neighborhood of the origin in $\{ay^2 = z^2\}$, a cone in $\mathbb{C}^2$. Thus the normal fiber is the cone on two circles $c(S^1 \cup S^1)$. (So the link of $S(V)$ away from p and q is $S^1 \cup S^1$). As $(a, 0, 0)$ moves once around the unit circle in the x-plane, this cone flips over. Therefore $S(W) \subset W$ is not equisingular at the origin, and the link of the origin in W is just the 3-circuit X of example 2. In fact, it is clear that V is just the suspension of X, i.e. the join of X with the two points $\{p, q\}$. Thus $H_*(V) = \mathbb{Z}\ 0\ \mathbb{Z}/2\ 0\ \mathbb{Z}$, and $H_2(V) \cong \mathbb{Z}/2$ is generated by the homology class α of the algebraic cycle $S(V)$. The sheaves $\mathcal{H}_p$ on V are:

	$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$
$V - S(V)$	0	0	0	0	$\mathbb{Z}$
$S(V) - \{p, q\}$	0	0	0	$\tilde{\mathbb{Z}}$	$\mathbb{Z} \oplus \mathbb{Z}$
p or q	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$

Here $\tilde{\mathbb{Z}}$ denotes that $\mathcal{H}_3$ is not constant - the generators of the stalk $\mathbb{Z}$ are flipped as one runs around the equator of $S(V) - \{p, q\}$. It follows that E^2 is



(The normalization of V is homeomorphic to S^4.) Now the generators of $H^0(X;\mathcal{X}_1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ correspond to the generators of $H_1(\text{link } p)$ and $H_1(\text{link } q)$. It follows from the description of $\tilde{E}^2$ (§4A and B) that d^2 of each of these generators is the generator of $H^2(X;\mathcal{X}_3) \cong \mathbb{Z}/2$. Since this is the only possible non-zero differential of E, E^∞ has the same 4th column as E, and the only other non-zero term is $E^\infty_{2,0} \cong \mathbb{Z}/2$. Hence $\alpha \in H_2(X)$ has filtration 0. Geometrically, the cycle $S(V)$ can be moved off p by a homology if and only if its intersection with link p is null-homologous in link p. But $S(V) \cap \text{link } p$ represents the generator of

$H_1(\text{link } p) \cong H_1(X) \cong \mathbb{Z}/2$ (cf. example 2).

5B. The extent of a cohomology class

This section is "dual" to the last section, but there are some interesting differences which point out the geometrical disparity between homology and cohomology on singular spaces.

Definition. If $\beta \in H^s(X)$ is a singular cohomology class, β has extent $\leq q$, $\mathrm{ex}(\beta) \leq q$, if

$$\beta \in \mathrm{Kernel}[H^s(X) \to H^s(X-A)]$$

$(\beta \mid X - A = 0)$ for some closed subspace $A \subset X$ of Čech dimension $\leq q$.

Examples. 1. If X is the annulus $S^1 \times I$, the generator of $H^1(X)$ has extent 1 (it vanishes on $X - \{s\} \times I$, but not on $X - \{x\}$ for any $x \in X$). If $\{s\} \times I$ is collapsed to a point to produce a homotopy equivalent space Y, the generator of $H^1(Y)$ has extent 0, since it vanishes on the complement of the singular point. (This is Zeeman's example.)

2. Similarly, if X is the pinched torus $S^1 \times S^1/\{w\} \times S^1$,

the generator of $H^1(X)$ has extent 0, since it vanishes on the complement of the singular point.

3. If K is a combinatorial n-manifold, any $\beta \in H^s(|K|)$ has extent n-s. Assume K is closed and oriented, and let α be the Poincaré dual homology class of β (i.e. $\alpha = \beta \cap [K]$). Let a_{n-s} be a simplicial (n-s)-cycle representing α. Then a (singular) cocycle b^s representing β is given by $b^s(c_s) = a_{n-s} \cdot c_s$, where $\cdot$ is intersection. (The singular cycle c_s is first approximated by a simplicial cycle "transverse" to a_{n-s}). Now $b^s \mid |K| - \sup(a_{n-s}) = 0$, so extent $\beta \leq n-s$. However, since any s-cycle can be moved off a set of dimension $< n-s$, $\beta \mid X - A \neq 0$ for all A of dim $< n-s$.

Zeeman's terminology for the extent of a cohomology class is "codimension" ([Ze 1], p. 177). If b^s is a singular cochain on X, let $\sup b^s = \{x \in X;$ for every neighborhood U of x, $b^s \mid U \neq 0\}$. Then codimension b^s is the Čech dimension of $\sup b^s$ (a closed set by definition). The <u>codimension</u> of a cohomology class β is the minimum codimension of any cocycle representing it. The equivalence of codimension and extent is a corollary of the following observation.

<u>Lemma.</u> If b^s is a singular cochain on X with $\delta(b^s) = 0$ (b^s is a cocycle), then $\sup b^s$ is the smallest closed set $A \subset X$ such that $b^s \mid X - A = 0$.

<u>Remark.</u> The lemma is false if b^s is an arbitrary cochain, since b^s may have empty support without being zero.

<u>Proof:</u> First we must show that there is such a smallest closed set, i.e. if $b \mid X - A_i = 0$, $i \in I$, then $b \mid X - \bigcap_I A_i = 0$. Suppose $f : \Delta^s \to X - \bigcap_I A_i = \bigcup_I (X - A_i)$ is a singular simplex. Choose a subdivision (e.g. barycentric) of Δ^s fine enough so that $f(\sigma^s)$ is contained in some $X - A_i$ for each little s-simplex σ^s in the subdivision. If each σ^s has its orientation induced from Δ^s, then f is homologous to $\Sigma f \mid \sigma^s$. Since b is a cocycle, $b(f) = b(\Sigma f \mid \sigma^s) = \Sigma\, b(f \mid \sigma^s) = 0$.

Now it is clear that $\sup b$ is the intersection of all closed $A \subset X$ such that $b \mid X - A = 0$.

<u>Remark.</u> There isn't a good definition of the support of a simplicial cocycle, which accounts for our use of singular cohomology. This reflects a difference in personality between cohomology and homology, which is more at home in the discrete simplicial world than cohomology.

Note that if $\beta, \gamma \in H^s(X)$, $ex(n\beta) \leq ex(\beta)$, and $ex(\beta+\gamma) \leq \max\{ex(\beta), ex(\gamma)\}$. Thus the cohomology classes of X with extent $\leq q$ form a subgroup of $H^s(X)$ for each q. This yields a filtration of $H^s(X)$ by subgroups

$$0 \subset \hat{F}_0^s \subset \ldots \subset \hat{F}_q^s \subset \hat{F}_{q+1}^s \subset \ldots \subset \hat{F}_n^s = H^s(X)$$

($n = \dim X$), which is a topological invariant of X by definition. Note also that if $\beta, \gamma \in H^*(X)$, $ex(\beta \cup \gamma) \leq \max\{ex(\beta), ex(\gamma)\}$, so $F_q = \sum_s F_q^s$ is a subring of $H^*(X)$. In other words, the above filtration respects the product structure of $H^*(X)$. (To see that $ex(\beta \cup \gamma) \leq \max\{ex(\beta), ex(\gamma)\}$, recall that if $\beta \mid U = 0$ and $\gamma \mid V = 0$, where U, V are open subsets of X, then $\beta \cup \gamma \mid U \cup V = 0$.)

<u>Theorem 3B</u>. Let $\beta \in H^s(X)$ be a singular cohomology class. If X is triangulable, then the extent of β equals the filtration of β in Zeeman's spectral sequence. (This was conjectured by Zeeman, [Ze 1] p. 178.)

<u>Proof</u>: Let K be a triangulation of X. I will show the equivalence of the following conditions:

1) $ex(\beta) \leq q$

2) $\beta \in \mathrm{Ker}[H^s(X) \to H^s(X - |K_q|)$ (β vanishes off the q-skeleton of K)

3) $\beta \in \mathrm{Ker}[H^s(X) \to H^s(|K^{(q+1)}|)]$ (β vanishes on the (q+1)-coskeleton of K)

4) filtration $\beta \leq q$ in $\hat{E}$.

Clearly 2) $\Longrightarrow$ 1) (let $A = |K_q|$). 2) $\Longleftrightarrow$ 3) since $|K^{(q+1)}|$ is a deformation retract of $X - |K_q|$ (3.2). By theorem 2B and the definition of Whitehead's spectral sequence $\hat{\tilde{E}}$, 4) $\Longleftrightarrow$ $\beta \in \mathrm{Ker}[H^s(K') \to H^s(\ K^{(q+1)}\ )]$ as a simplicial cohomology class, so 4) $\Longleftrightarrow$ 3). Finally, 1) $\Longrightarrow$ 3).* Let $A \subset X$ be a closed subspace of Čech dimension $\leq q$ such that $\beta \mid X - A = 0$. By the proof of theorem 3A, there is an ε-ambient isotopy h_t of X such that $|K^{(q+1)}| \cap h_1(A) = \emptyset$. Let g be the inverse of h_1. $\beta = g^*\beta$, but $g^*\beta \mid X - h_1(A) = 0$, and $X - h_1(A) \supset |K^{(q+1)}|$, so $\beta \mid |K^{(q+1)}| = 0$. This completes the proof of theorem 3B.

<u>Corollary 1.</u> Let $\beta \in H^s(X)$, X triangulable. If $ex(\beta) \leq q$, then $\beta \cap \mu = 0$ for all $\mu \in H_t(X)$ with $t > s + q$.

<u>Proof:</u> By condition 3) of the above proof, β is represented by a simplicial cocycle b on K' such that $b \mid K^{(q+1)} = 0$, for any triangulation K of X. Now by the definition (2.6) of cap

*See Erratum.

product, $b \cap \tau = 0$ for any elementary chain τ of K of dimension $> s + q$.

Corollary 2. If X is triangulable, and $\beta, \gamma \in H^*(X)$,

$$\mathrm{filt}(\beta \cup \gamma) \leq \max\{\mathrm{filt}(\beta), \mathrm{filt}(\gamma)\} .$$

In other words, the filtration induced on $H^*(X)$ by the spectral sequence $\hat{E}$ respects the ring structure.

Corollary 3. If X is triangulable, and $\dim X = n$, then $\mathrm{ex}(\beta) \leq n-s$ for all $\beta \in H^s(X)$.

Corollary 4. If X is triangulable, and $\beta \in H^s(X)$, then $\beta \in \mathrm{Ker}[\hat{e}_X : H^s(X) \to H_{n-s}(X;\mathscr{H}^n)]$ if and only if $\mathrm{ex}(\beta) < n-s$.

For example, if X is a locally irreducible n-circuit, the kernel of the duality map $\cdot \cap [X]$ is the cohomology classes with extent $< n-s$. Therefore $\cdot \cap [X]$ is injective $\iff$ every cohomology class in X has maximum extent. Recall, however, that $\cdot \cap [X]$ may be injective without being surjective.

Remarks. 1. If X is not triangulable, Zeeman has shown that filtration $\beta \leq$ extent β. I don't know whether they are always equal.

2. Extent can be defined with any coefficient group, as can $\hat{E}$, and theorem 3B is true for arbitrary coefficients by the same proof.

G. L. Gordon ([Gor 1], [Gor 2]) has made a study of the cohomology of stratified spaces with applications to the theory of residues on complex analytic varieties, using the classical "tubular cycles" or "pseudocycles" of Lefschetz - forerunners of the more abstract cocycles of today. The following results are interpretations of Gordon's work.

Proposition 1. Let $X_0 \subset X_1 \subset \ldots \subset X_n = X$ be a filtration of X such that

i) X is triangulable as a complex K with each X_i covered by a subcomplex

ii) for each i, $\mathcal{H}^p \mid X - X_i = 0$ for $p \leq i$

Then if $\beta \in H^s(X)$ has extent $\leq q$, $\beta \mid X - X_{s+q} = 0$. (For example, let $\{X_i\}$ be a p.l. stratification of X (with respect to some p.l. structure).

Proof: Let $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \ldots \subset \mathcal{K}_n = K$ be full subcomplexes of K with $|\mathcal{K}_i| = X_i$. Let $N(\mathcal{K}_{s+q})$ be the stellar neighborhood of $\mathcal{K}_{s+q}$ in K',

$$N(\mathcal{K}_{s+q}) = \{\rho \in K', \ \rho > \omega \ \text{ for some } \ \omega \in \mathcal{K}'_{s+q}\}.$$

Since $\mathcal{K}_{s+q}$ is full in K', $|\mathcal{K}_{s+q}|$ is a deformation retract of the open neighborhood $|N(\mathcal{K}_{s+q})|$, so it suffices to represent β by a simplicial cocycle b on K' such that $b \mid K' - N(\mathcal{K}_{s+q}) \mid = 0$. To this end, we use the decomposition of K'

$$K' = \bigcup_{\sigma \in K} \mathring{\text{dual}}\, \sigma \qquad\qquad \partial \text{ dual } \sigma = \text{link } \sigma = \bigcup_{\sigma < \tau} \mathring{\text{dual}}\, \tau .$$

By theorem 3A, extent $\beta^s \leq q$ implies β is represented by a simplicial s-cocycle b on K' such that $b \mid K^{(q+1)} = 0$. Now $K^{(q+1)} = \bigcup_{\dim \sigma > q} \text{dual}\, \sigma$, and $N(\mathcal{K}_{s+q}) = \bigcup_{\sigma \in K_{s+q}} \mathring{\text{dual}}\, \sigma$.

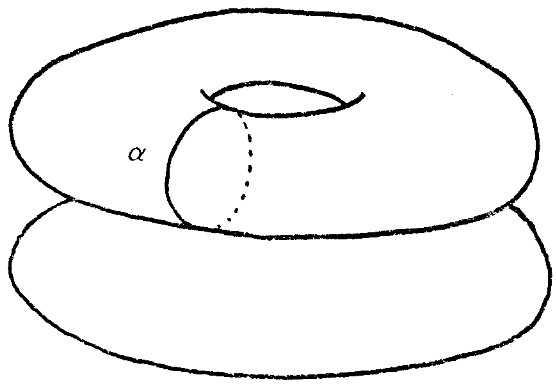
I will construct an $(s-1)$-cochain c on K' such that $\delta c - b$ vanishes off $N(\mathcal{K}_{s+q})$, by defining c on the open complexes $\mathring{\text{dual}}\, \sigma$ one at a time, in order of decreasing dimension of σ. Suppose $c \in C^{s-1}(K')$ has been defined so that

$\delta c - b \mid \mathring{\text{dual}}\, \tau = 0$ for all $\tau > \sigma$, and $\sigma \notin \mathcal{K}_{s+q}$, $\dim \sigma < q$.

Since $\delta c - b \mid \partial$ dual $\sigma = 0$, $\delta c - b$ is a chain in $C^s(\text{dual } \sigma, \text{link } \sigma)$. But condition (ii) on the filtration $\{X_i\}$ implies that $H^j(\text{dual } \sigma, \text{link } \sigma) = 0$ for $j \leq (s+q) - \dim \sigma$ (cf. (2.1)). $(s+q) - \dim \sigma \geq s$, so $H^s(\text{dual } \sigma, \text{link } \sigma) = 0$. Thus there exists $d \in C^{s-1}(\text{dual } \sigma, \text{link } \sigma)$ such that $\delta d = (\delta c - b) \mid \text{dual } \sigma$. Viewing d as a cochain in K', we have $(\delta(c-d) - b) \mid \overset{\circ}{\text{dual}}\, \sigma = 0$. Thus we have extended the definition of c over $\overset{\circ}{\text{dual}}\, \sigma$, which completes the proof.

Even if $\{X_i\}$ is a p.l. stratification of X, the converse of the proposition is false. For example, let

$$X = S^1 \times S^1 \bigcup_{\{s\} \times S^1} S^1 \times S^1.$$



Let X_0 be empty, and let X_1 be the singular circle of X. Then $X_0 \subset X_1 \subset X_2 = X$ is a (smooth) stratification of X. Let $\beta \in H^1(X)$ be the algebraic dual of the homology class α represented by one of the circles "transverse" to the singularity X_1. Clearly $\beta \mid X - X_1 = 0$, but extent $\beta = 1$, not 0.

Gordon studies the cohomology of a space X by embedding it in an m-manifold M, and using the Lefschetz duality

$$H^s(X) \xrightarrow{\approx} H_{m-s}(M, M-X) \cong H_{m-s}(N, \partial N),$$

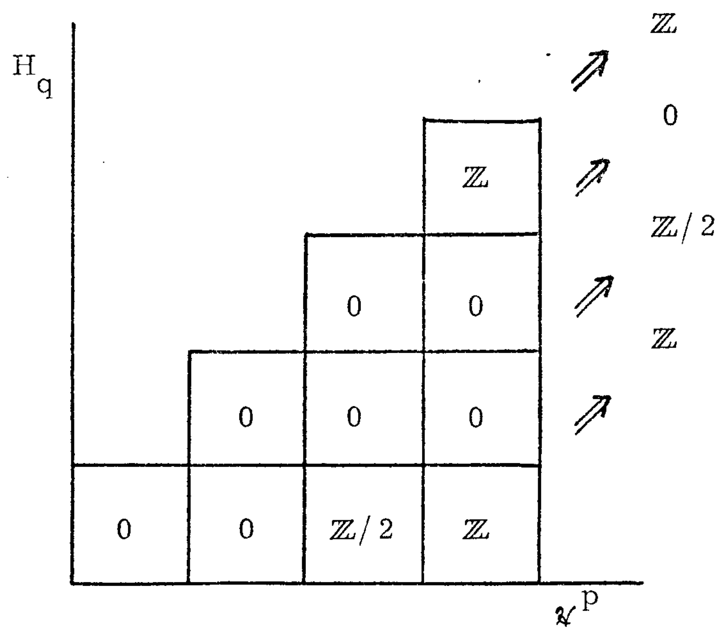
where N is a regular neighborhood of X in M. Cocycles on X can then be thought of as transversal intersections of cycles in $C_*(N, \partial N)$ with X, so-called "tubular cycles."

Proposition 2. Let K be a full subcomplex of the combinatorial m-manifold Q. Let N be the stellar neighborhood of K in Q. Then if $\beta \in H^s(K)$ has extent $\leq q$, its Lefschetz dual $\alpha \in H_{m-s}(N, \partial N)$ is represented by a (simplicial) cycle which intersects K in a subcomplex of dimension $\leq q$.

Proof: extent $\beta \leq q \iff \beta$ is represented by a simplicial cocycle b on K' such that $b \mid K^{(q+1)} = 0$. Now α is represented by the simplicial cycle $\pi^*(b) \cap [N] \in C_{m-s}(N, \partial N)$, where $[N]$ is the fundamental class of N, $\pi : N \to K$ is a simplicial retraction, and $\pi^*(b) \cap [N]$ is given by (2.6). Now the simplexes of K' occurring in $\pi^*(b)$ are just the simplexes of b, since π is the identity on K. But $b \mid K^{(q+1)} = 0$, so all simplexes $<\underline{\sigma}_0, \ldots, \underline{\sigma}_s>$ of K' occurring in $\pi^*(b)$ have $\dim \sigma_0 \leq q$. Therefore, by formula (2.6), the simplexes of K occurring in $\pi^*(b) \cap [N]$ must all have dimension $\leq q$.

Examples. 1. Let X be an n-circuit with one singular point x (or a finite number of singular points). Then the transgression homomorphisms of $\hat{E}$ fit into a long exact sequence dual to (4.5), which is isomorphic to the cohomology long exact sequence of the pair $(W, \partial W)$, $W = X$ - star x. Thus it is clear that a cohomology class β^s has filtration n-s or 0, and filtration $\beta^s = 0 \iff \beta^s \mid X - \{x\} = 0$. In other words, β^s can have extent n-s or 0, and will have extent $0 \iff$ it is represented by a cocycle supported by the singular point $\{x\}$ (cf. [Ze 1], p. 183).

2. Let $X = S^3/x = -x, \ x \in S^1$, example 2 of §5A. $H^*(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2\,\mathbb{Z}$. The analysis of X in §5A yields that $\hat{E}^2$ is



Hence the generator β of $H^2(X)$ has filtration 0. This corresponds geometrically to the fact that β can be supported by any point in the singular circle S(X).

It is also easy (and informative) to compute $\hat{E}$ for examples 3 and 4 of §5A.

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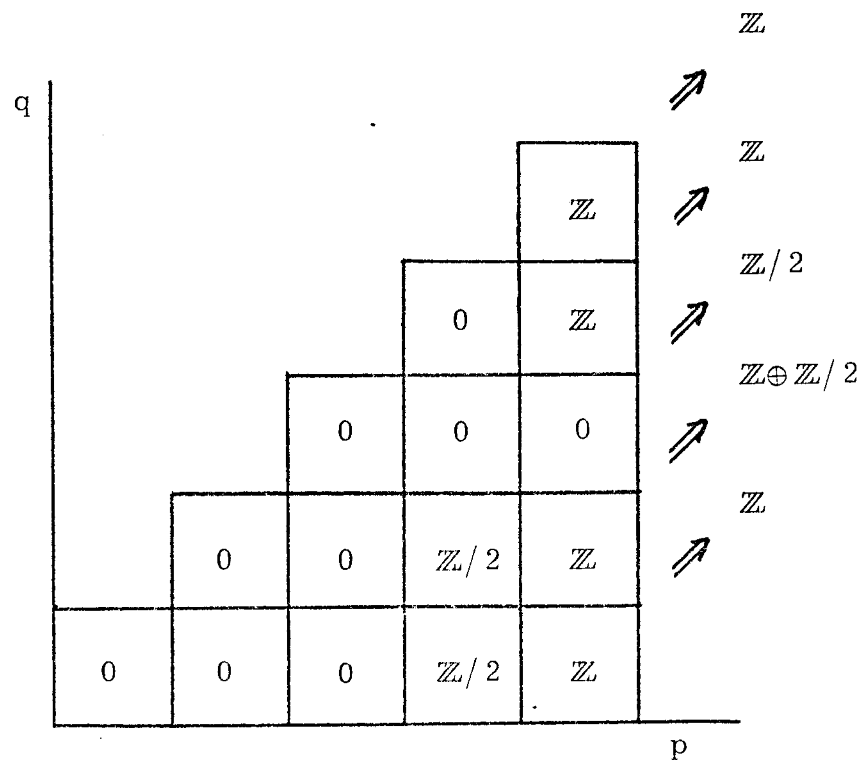
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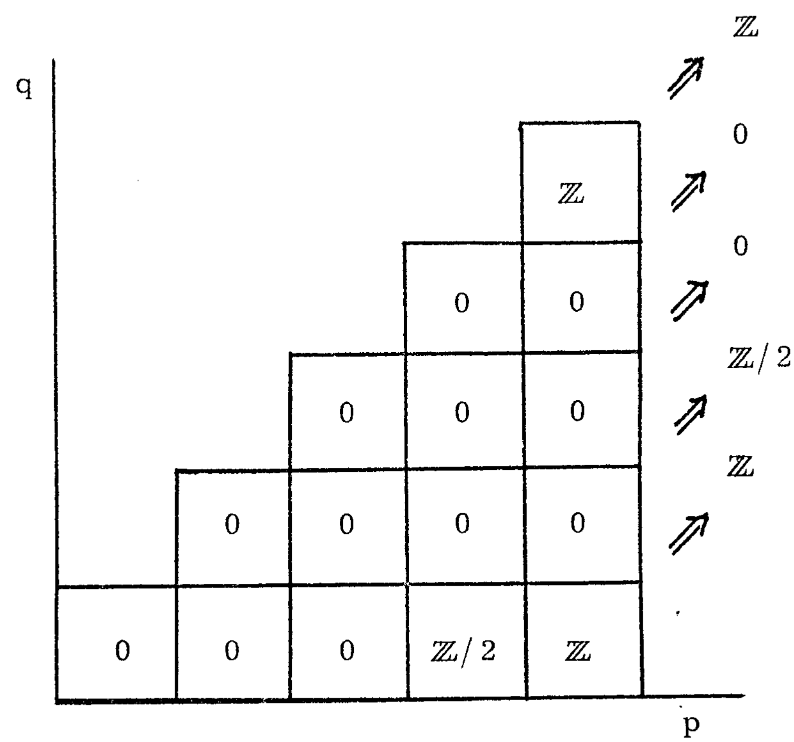
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5C. Degrees of freedom over a map

This section generalizes § 5A to an analysis of (triangulable) maps. (There is a dual generalization of § 5B.)

Definition. If $\alpha \in H_s(X)$ is a singular homology class, and $f : X \to Y$ is a continuous map, α has $\geq q$ degrees of freedom over f if

$$\alpha \in \text{Image}[H_s(X - f^{-1}(A)) \to H_s(X)]$$

for all closed subspaces $A \subset Y$ of Čech dimension $< q$.

Note that the degree of freedom of α over f is less than or equal to the degree of freedom of $f_*(\alpha)$ in Y. If f is closed (e.g. X compact), and $\dim f(B) \leq \dim B$ for all closed $B \subset X$ (e.g. if f is triangulable), then the degree of freedom of α over f is also less than or equal to the degree of freedom of α in X.

Theorem 3C. Let $\alpha \in H_s(X)$ be a singular homology class. If $f : X \to Y$ is a triangulable map, then the degrees of freedom of α over f equals the filtration of α in Zeeman's spectral sequence for f.

The proof of this theorem is a generalization of the proof of theorem 3A. By theorem 2C, α has filtration $\geq q$ in $E(f)$ if and only if α is represented by a simplicial cycle in $(s')^{-1}(L^{(q)})$, where $s : K \to L$ is a triangulation of f. Thus we need only show

(i) $f^{-1}(|L^{(q)}|)$ is a deformation retract of $f^{-1}(|L_q|)$ (compare (3. 2))

(ii) If $A \subset Y$, and $\dim A < q$, $f^{-1}(A)$ can be ε-isotoped off $f^{-1}(|L^{(q)}|)$.

Every simplex in K' is the join of a simplex in $(s')^{-1}(L^{(q)})$ with a simplex in $(s')^{-1}(L'_q)$. Thus the deformation retraction (i) is given by "sliding along the join lines" in each simplex of K'. By the proof of theorem 3A, if $A \cap |L^{(q)}| \cap \partial\tau = \emptyset$, $\tau \in L$, $A \cap \tau$ can be ε-isotoped off $|L^{(q)}| \cap \tau$ in τ, keeping $\partial\tau$ fixed. Since f is triangulable, $f^{-1}(\mathring{\tau}) \cong f^{-1}(\underline{\tau}) \times \mathring{\tau}$, and f corresponds to projection onto $\mathring{\tau}$ (see [Mi], p. 125, lemma 3). Thus the isotopy of $A \cap \tau$ in τ can be lifted to an isotopy moving $f^{-1}(A \cap \tau)$ off $f^{-1}(|L^{(q)}| \cap \tau)$ in $f^{-1}(\tau)$. Condition (ii) follows by induction on $\dim \tau$.

<u>Remarks</u>. This theorem needs to be proved for a wider class of maps. (Most algebraic maps aren't triangulable, for example.)

The spectral sequence should be interesting when X and Y are (smooth) manifolds and f is a (differentiable) mapping with singularities.

6. Applications

In §5A we have seen that Zeeman's spectral sequence relates the local homology of a space X to the filtration on its global homology which measures the restriction of movement of cycles caused by the singularities of X. For example, a normal n-circuit X is a Poincaré duality space if and only if each homology class α_s has degree of freedom equal to its codimension $n-s$. (If X is a polyhedron, this is equivalent to α being represented by a cycle which intersects each stratum of the intrinsic stratification of X in codimension $n-s$ (Corollary 6 of §5A).)

With this geometry under our belt, there are several directions to explore (which were described to me by D. Sullivan when he first introduced me to the spectral sequence). One is to find generalizations, for spaces with singularities, of facts about manifolds which only depend on Poincaré duality. For example, the euler characteristic of an odd-dimensional manifold is zero. Sullivan has shown recently that any space which can be stratified using only odd-dimensional strata has zero euler characteristic. My analysis of Zeeman's spectral sequence led to a particularly simple proof of this fact (§A).

A similar (but much more difficult) problem is to generalize the fact that the boundary of an oriented manifold has signature zero (cf. [Su 2], p. 202). In other words, what singularities can one introduce into manifold theory and retain the cobordism invariance of the signature? (Note that it isn't clear what the definition of the signature of an n-circuit X should be. One choice is the signature of the total cup product pairing on $H^*(X)$, $(\alpha, \beta) \to \langle \alpha \cup \beta, [X] \rangle$.) At present, this problem seems too hard to solve without adding significant further machinery to the spectral sequence.

Another direction is to study "characteristic homology classes" of spaces with singularities. In chapter II, I will analyze the diagonal cycle Δ in $X \times X$ for an n-circuit X. §§3 and 4 in particular will shed some light on the concept of degrees of freedom of a cycle. The spectral sequence, however, is only used in a mundane way (Lemma 4 of § II.3).

A quite provocative problem is to analyze the degrees of freedom of the Stiefel homology classes of a variety. (For their interesting history, see [Su 3] and [H].) In §B I will make some elementary observations about this problem. A harder problem is to analyze the Chern homology classes of an algebraic variety, recently discovered by R. MacPherson (unpublished).

A. The euler characteristic χ

If M is a closed, oriented $(2k+1)$-dimensional manifold, $\chi(M) = 0$. For if β_i is the ith Betti number of M, $\beta_i = \beta_{2k+1-i}$ by Poincaré duality, so

$$\chi(M) = \sum_i (-1)^i \beta_i = \sum_i (-1)^i \beta_{2k+1-i} = -\sum_j (-1)^j \beta_j = -\chi(M).$$

If X is any space and $x \in X$, let χ_x, the local euler characteristic of X at x, be the euler characteristic of $H_*(X, X - \{x\})$, i.e.

$$\chi_x = \sum_i (-1)^i \operatorname{rank} H_i(X, X - \{x\}).$$

For example, if X is an n-manifold, $\chi_x = 1$ for all $x \in X$ if n is even, and $\chi_x = -1$ for all $x \in X$ if n is odd.

Lemma. If X is triangulable and $\chi_x = -1$ for all $x \in X$, then $\chi(X) = 0$.

Proof: Let K be a triangulation of X. Recall the decomposition

$$K' = \bigcup_{\sigma \in K} \mathring{\text{dual}}\, \sigma,$$

where $\text{dual}\, \sigma = \{<\underline{\tau}_0, \ldots, \underline{\tau}_i> \in K', \;\; \tau_0 = \sigma\}$. Now

$$\begin{aligned}
\chi(X) = \chi(K') &= \sum_\sigma \chi(\mathring{\text{dual}}\, \sigma) \\
&= \sum_\sigma (-1)^{\dim \sigma} \chi(\mathring{\text{star}}\, \sigma) \qquad \text{by (2.1)} \\
&= \sum_\sigma (-1)^{\dim \sigma} \chi_{\underline{\sigma}} \, . \\
&= - \sum_\sigma (-1)^{\dim \sigma} \\
&= - \chi(K) = - \chi(X).
\end{aligned}$$

(A similar proof has been given by Banchoff [Ba].)

Remark. This lemma can be proved using the spectral sequence $\widetilde{E}$ in the same way that one proves the multiplicativity of χ for fibrations with the Serre spectral sequence (cf. [Sp] 9.3.1). One defines $\chi(\widetilde{E}^r) = \sum_s (-1)^s \operatorname{rank} \widetilde{E}^r_s$, where $\widetilde{E}^r_s = \sum_{p-q=s} \widetilde{E}^r_{p,q}$. Then $\chi(\widetilde{E}^r) = \chi(\widetilde{E}^{r+1})$ for $1 \leq r$, so $\chi(\widetilde{E}^1) = \chi(\widetilde{E}^\infty)$, and $\chi(\widetilde{E}^\infty) = \chi(X)$. The proof of the lemma amounts to the calculation that $\chi(\tilde{E}^1) = 0$.

Proposition (Sullivan). If X has a (p.l.) stratification with only odd-dimensional strata, then $\chi(X) = 0$.

Outline of proof: If x lies in an open stratum S consider the transverse link L to S at x. (The fiber of the normal bundle along S is the cone on L.) The stratification on X induces one on L. Furthermore, the strata of L are odd dimensional. Since L has fewer strata than X, we can assume by induction that $\chi(L) = 0$. It follows that $\chi_x = -1$, so $\chi(X) = 0$ by the lemma.

Corollary (Sullivan). If V is an algebraic variety in complex projective n-space, and W is a subvariety, then $\chi(V) = \chi(W) + \chi(V-W)$.

Outline of proof: Let N be a regular neighborhood of W in V.

$$\chi(V) = \chi(N) + \chi(V-N) - \chi(T),$$

where T is the boundary of N. Since W is a deformation retract of N, $\chi(W) = \chi(N)$ and $\chi(V-W) = \chi(V-N)$. Since T is a bicollared subspace of V, and V has a stratification with only even-dimensional strata (Thom), the induced stratification on T has only odd-dimensional strata, so $\chi(T) = 0$ by the proposition.

B. The Stiefel homology classes s_i

Let K be a finite simplicial complex. Let c_i be the sum of all the i-simplexes in K', an unoriented simplicial chain in K'. A combinatorial calculation shows that $\partial c_i = 0$ for all i if and only if χ_x is odd for all $x \in X = |K|$. Such a space X is called a (mod 2) <u>Euler space</u>. The classes $s_i(X) = [c_i] \in H_i(X; \mathbb{Z}/2)$ are called the <u>Stiefel homology classes</u> of X. They are piecewise linear invariants of X, and if X is a smooth n-manifold, s_i is the Poincaré dual of w^{n-i}, the $(n-i)$th Stiefel-Whitney cohomology class of X. (An n-manifold X is an Euler space since $\chi_x = (-1)^n$ for all $x \in X$.) In fact, any real analytic space is an Euler space. For discussion and proofs of these results, see [Su 3] and [HT].

Note that if X is an Euler space, so is $|\text{link } v|$ for all vertexes v of a triangulation of X.

<u>Lemma.</u> If $X = |K|$ is an Euler space, then $s_i(X)$ has > 0

degrees of freedom if and only if $s_{i-1}(|\text{link } v|) = 0$ for all $v \in K$.

Proof: By the proof of theorem 3A, $df(s_i(X)) > 0 \iff c_i(K')$ can be moved off the 0-skeleton of K by a simplicial homology in K'. If $v \in K_0$, let $\text{link}'v = \text{link}(v, K')$ (cf. §2). Link'v is simplicially isomorphic to $(\text{link } v)'$, so I will identify them. Now $c_i(K')$ restricted to $\text{star}'v = \text{int}(v * \text{link}'v)$ is equal to $v * c_{i-1}(\text{link}'v)$. Furthermore, if b_i is a chain in link'v such that $\partial b_i = c_{i-1}(\text{link}'v)$, $v * b_i$ is a homology of $c_i(K')$ off v. Conversely, a homology of $c_i(K')$ off v, restricted to star'v, is the join of v with a homology of $c_{i-1}(\text{link}'v)$ to 0. Thus $c_i(K')$ can be moved off $v \iff c_{i-1}(\text{link}'v) \sim 0$, q.e.d.

Example. Let X be the suspension of the Klein bottle B. $\chi(B) = 0$, so X is an Euler space. If K is a triangulation of B, the join of K with two points p, q is a triangulation of X, and $|\text{link } p| = |\text{link } q| = B$. Thus $df(s_2(X)) = 0$, since $s_1(B) \neq 0$, being dual to $w^1(B)$, the obstruction to orienting B.

Thus $df(s_i(X)) < n-i$ for some Euler spaces X - in particular there may be no class $w^{n-i} \in H^{n-i}(X, \mathbb{Z}/2)$ such that $w^{n-i} \cap s_n(X) = s_i(X)$ ($s_n(X)$ is the "mod 2 fundamental class" of X).

One should be able to say something more about the filtration of $s_i(X)$ in Zeeman's spectral sequence ($\mathbb{Z}/2$ coefficients), since s_i and $\tilde{E}$ are both defined in a simple combinatorial fashion using the first barycentric subdivision of a triangulation. Nevertheless, such an analysis has eluded me, and I now believe that one must have some geometrical understanding of what the s_i mean before their degrees of freedom can be computed.

II. Thom classes

If X is an n-circuit, the freedom of movement of the diagonal Δ in $X \times X$ is severely restricted by any singularities in X. Consequently, the homological properties of Δ reflect the global geometry of X in a subtle way.

Suppose X is a p.l. n-manifold, and let $U \in H^n(X \times X)$ be the Poincaré dual of $[\Delta] \in H_n(X \times X)$. Then $U \mid X \times X - \Delta = 0$, so U is the image of a class $\tilde{U} \in H^n(X \times X, X \times X - \Delta)$. $\tilde{U}$ corresponds to the Thom class of the tangent bundle of X via the isomorphism

$$H^n(X \times X, X - \Delta) \cong H^n(TX, TX - X) \cong H^n(D(TX), S(TX)).$$

(Here the tangent bundle TX is identified with a regular neighborhood of Δ in $X \times X$, and D, S denote the associated disc and sphere bundles.)

If X is an arbitrary n-circuit, a <u>Thom class</u> is a class $U \in H^n(X \times X)$ such that $U \cap [X \times X] = [\Delta]$. I will show in § 1 that with field coefficients, any n-circuit X has a Thom class, so $[\Delta]$ has the maximum degree of freedom in $X \times X$ (equal to its codimension, n). This is geometrically surprizing, since it follows that the diagonal is homologous to a cycle transverse to the singularities of $X \times X$ (Corollary 6 of §5A).

Thus we are led to study the algebraic properties of Thom classes. In §2, I will show that the n-circuit X is a duality space (i.e. $\cdot \cap [X]$ is an isomorphism) if and only if X has a Thom class U such that

$$U \cup T^* v = U \cup v$$

for all $V \in H^*(X \times X)$, where T is the involution on $X \times X$, $T(x, y) = (y, x)$. In §3, I will show that if X is a normal n-circuit (i.e. the "orientation sheaf" $\mathcal{X}_n$ is constant), then X is a homology manifold if and only if X has a Thom class U such that $U \mid X \times X - \Delta = 0$. In other words, <u>$X$ is a homology manifold if and only if the diagonal Δ is homologous to a transverse cycle which lies in a regular neighborhood of Δ</u>.

§4 contains an application to homology intersection theory, where the use of the diagonal was introduced classically by Lefschetz (cf. [St 2]). If X is a (homology) manifold, and α, β are homology classes,

$$\alpha \cdot \beta = U \cap (\alpha \times \beta) \in H_*(\Delta) \cong H_*(X).$$

In general, the existence of an "intersection pairing" on $H_*(X)$ is equivalent to the existence of a Thom class for X. In particular, I will give simple axioms for an intersection pairing φ on $H_*(X; F)$, F a field, such that φ exists if and only if X is an F-homology manifold.

The notation in this chapter will be Spanier's [Sp]. In particular, I will use his sign convention for cap product, in order to refer to his formulas relating cup, cap, and slant products. My techniques are based on a proof of Poincaré duality for smooth manifolds given by Milnor [Mi], and Spanier's discussion of duality for topological manifolds [Sp].

1. Degrees of freedom of the diagonal

Let X be an n-circuit with fundamental class $[X]$, and let $d : X \to X \times X$ be the diagonal map $d(x) = (x, x)$. The diagonal $\Delta = d(X) \subset X \times X$ represents a homology class $[\Delta] = d_*[X] \in H_n(X \times X)$. A <u>Thom class</u> for X is a cohomology class $U \in H^n(X \times X)$ such that $U \cap [X \times X] = [\Delta]$. ($X \times X$ is a $2n$-circuit with fundamental class $[X \times X] = [X] \times [X]$.)

<u>Proposition 1.</u> Let X be an n-circuit, and let $h : \mathbb{Z} \to F$ be a homomorphism from the integers to a field F. There is a class $U \in H^n(X \times X; F)$ such that $U \cap h_*[X \times X] = h_*[\Delta]$.

<u>Corollary 1.</u> $h_*[\Delta]$ has n degrees of freedom in $X \times X$. (This follows from I, Theorem 3A, Corollary 1.)

For example, if F is the rational numbers or the integers modulo a prime p, there is a canonical coefficient homomorphism h, so we can simply say that Δ has n degrees of freedom as a cycle with coefficients in $\mathbb{Q}$ or $\mathbb{Z}/p$.

<u>Proof of the proposition:</u> For simplicity of notation, I will omit h_*, and write $[\Delta]$, $[X]$, etc., for their images under h_*.

Let $\{\alpha_i\}$ be a basis for $H^*(X; F)$ over F, and let $\{\beta_i\}$ be the algebraically dual basis of $H_*(X, F)$, i.e. $<\alpha_i, \beta_j> = \delta_{ij}$. Then by the Künneth theorem, $\{\alpha_i \times \alpha_j\}$ is a basis for $H^*(X \times X; F)$, and $\{\beta_i \times \beta_j\}$ is the dual basis for $H_*(X \times X; F)$, i.e. $<\alpha_i \times \alpha_j, \beta_k \times \beta_\ell> = \delta_{ik}\delta_{j\ell}$.

Now let $[\Delta] = \Sigma\ a_{ij}(\beta_i \times \beta_j)$. We have

$$\begin{aligned} a_{ij} &= <\alpha_i \times \alpha_j, [\Delta]> \\ &= <\alpha_i \times \alpha_j, d_*[X]> \\ &= <d^*(\alpha_i \times \alpha_j), [X]> \\ &= <\alpha_i \cup \alpha_j, [X]> . \end{aligned}$$

Thus (a_{ij}) is the cup product pairing matrix for X with respect to $\{\alpha_i\}$.

We're looking for a class $U \in H^n(X \times X; F)$ such that $U \cap [X \times X] = [\Delta]$. Let $U = \sum_{k\ell} b_{k\ell}(\alpha_k \times \alpha_\ell)$. Now

$$U \cap [X \times X] = [\Delta]$$

$$\iff \langle \alpha_i \times \alpha_j, U \cap [X \times X]\rangle = \langle \alpha_i \times \alpha_j, [\Delta]\rangle \quad \text{for every } i, j$$

$$\iff \langle (\alpha_i \times \alpha_j) \cup U, [X \times X]\rangle = a_{ij}.$$

Now

$$(\alpha_i \times \alpha_j) \cup U = (\alpha_i \times \alpha_j) \cup (\Sigma\, b_{k\ell}\alpha_k \alpha_\ell)$$

$$= \Sigma\, b_{k\ell}(\alpha_i \times \alpha_j) \cup (\alpha_k \times \alpha_\ell)$$

$$= \Sigma\, b_{k\ell}(-1)^{\underline{j}\underline{k}}(\alpha_i \cup \alpha_k) \times (\alpha_j \cup \alpha_\ell)$$

by [Sp] (5.6.13), where $\alpha_i \in H^{\underline{i}}(X; F)$.

We have $\langle(\alpha_i \times \alpha_j) \cup U, [X \times X]\rangle$
$= \epsilon[((\alpha_i \times \alpha_j) \cup U) \cap [X \times X]]$, ϵ = augmentation, and

$$((\alpha_i \times \alpha_j) \cup U) \cap [X \times X] = (\Sigma\, b_{k\ell}(-1)^{\underline{j}\underline{k}}(\alpha_i \cup \alpha_k) \times (\alpha_j \cup \alpha_\ell)) \cap ([X] \times [X])$$

$$= \Sigma\, b_{k\ell}(-1)^{\underline{j}\underline{k}}(-1)^{(\underline{i}+\underline{k})(n-\underline{j}+\underline{\ell})}((\alpha_i \cup \alpha_k) \cap [X]) \times ((\alpha_j \cup \alpha_\ell) \cap [X]),$$

by [Sp] (5.6.21). Thus, letting $s = \underline{j}\underline{k} + (\underline{i}+\underline{k})(n-\underline{j}+\underline{\ell})$,

$$<(\alpha_i \times \alpha_j) \cup U, [X \times X]> = \Sigma\ (-1)^s b_{k\ell} <\alpha_i \cup \alpha_k, [X]> \cdot <\alpha_j \cup \alpha_\ell, [X]>$$

$$= \Sigma\ (-1)^s b_{k\ell} a_{ik} a_{j\ell}$$

$$= \Sigma\ (-1)^{s+\underline{j\ell}} a_{ik} b_{k\ell} a_{\ell j} \,.$$

Since $\underline{i} + \underline{k} = \underline{k} + \underline{\ell} = \underline{\ell} + \underline{j} = n$, a short calculation shows $s + \underline{j\ell} = n\underline{k}$. Therefore

$$(1.1) \qquad U \cap [X \times X] = [\Delta] <\!\!\Longrightarrow \sum_{k\ell} (-1)^{n\underline{k}} a_{ik} b_{k\ell} a_{\ell j} = a_{ij} \,.$$

Let A, B be the matrices (a_{ij}), (b_{ij}). If $C = (c_{ij})$ satisfies $ACA = A$, then $b_{ij} = (-1)^{n\underline{i}} c_{ij}$ satisfies (1.1), so X has a Thom class U (F coefficients) $<\!\!\Longrightarrow$ A has a "quasi-inverse" C.

But over a field, A always has a quasi-inverse, for let P, Q be nonsingular matrices such that PAQ is diagonal with 1's and 0's on the diagonal. Let $C = QP$. Then

$$(PAQ)(PAQ) = PAQ,$$

so $$AQPA = A,$$

or $$ACA = A.$$

This completes the proof.

Remark. Note that if C is a quasi-inverse for A, C may have nonzero entries c_{ij} with $\underline{i} + \underline{j} \neq n$. Thus $\Sigma\,(-1)^{n\underline{i}} c_{ij}\alpha_i \times \alpha_j$ is an inhomogeneous element of $H^*(X \times X)$. Therefore we let

$$U = \sum_{\underline{i}+\underline{j}=n} (-1)^{n\underline{i}} c_{ij}\alpha_i \times \alpha_j .$$

If $v = \sum_{\underline{i}+\underline{j}\neq n} (-1)^{n\underline{i}} c_{ij}\alpha_i \times \alpha_j$, the proof of the proposition shows that $(U+v) \cap [X \times X] = [\Delta] \in H_n(X \times X)$. But $(U+v) \cap [X \times X]$ $= U \cap [X \times X] + v \cap [X \times X]$, so $U \cap [X \times X] = [\Delta]$ and $v \cap [X \times X] = 0$.

The following is an amusing corollary of this analysis. Let $M(d)$ be the space of $d \times d$ matrices with entries in the field F. Given a matrix $A \in M(d)$, let $N_A = \{D \in M(d), ADA = 0\}$. N_A is a linear subspace of $M(d)$. It is easy to show that N_A has index r^2 in $M(d)$, where $r = \operatorname{rank} A$, i.e. if $k = \dim_F N_A$,

$$k^2 + r^2 = d^2$$

(Note that $\{C \in M(d), ACA = A\}$ is a coset of N_A, so this equation implies for example that A has a unique quasi-inverse $\Longleftrightarrow$ A is nonsingular.) Now if A is the cup product matrix of a space X, $d^2 = \dim_F H^*(X \times X)$, $k^2 = \dim_F \mathrm{Ker}(\cdot \cap [X \times X])$, and A is the matrix of $\cdot \cap [X]$. Thus $\cdot \cap [X]$ is an isomorphism if and only if $\cdot \cap [X \times X]$ is (F coefficients). Therefore X is a duality space if and only if $X \times X$ is.

<u>Corollary 2.</u> Suppose X is an n-circuit such that $H_*(X; \mathbb{Z})$ is torsion free. Then the following are equivalent.

1) X has a Thom class

2) the cup product pairing matrix of X has a quasi-inverse

3) $\mathrm{Coker}(\cdot \cap [X])$ is torsion free.

<u>Proof:</u> Since $H_*(X)$ is torsion free, the proof of the proposition <u>verbatim</u> shows that 1) $\Longleftrightarrow$ 2). Now the matrix A of $\cdot \cap [X]$ is just the cup product pairing matrix, for a_{ij} is defined by

(1.2) $$\alpha_j \cap [X] = \Sigma\, a_{ij}\beta_i,$$

$$<=> \quad \langle \alpha_k, \alpha_j \cap [X]\rangle = \langle \alpha_k, \Sigma\, a_{ij}\beta_i\rangle,$$

$$<=> \quad \langle \alpha_k \cup \alpha_j, [X]\rangle = a_{kj}$$

Now any integral matrix A can be diagonalized by elementary row and column operations, say PAQ is diagonal, where P and Q are compositions of elementary matrices. Clearly A has a quasi-inverse if and only if PAQ does. Furthermore, P and Q correspond to changes of basis in the target and source of $\cdot \cap [X]$, so PAQ will also represent $\cdot \cap [X]$. Thus it suffices to prove 2) <=> 3) when A is diagonal. But then A has a quasi-inverse <=> all its entries are ± 1 <=> $\mathrm{Coker}(\cdot \cap [X])$ is torsion free.

<u>Remark</u>. An integral matrix A has a quasi-inverse if and only if the greatest common divisor $g(A)$ of the determinants of the $r \times r$ minors of A is 1, where $r = \mathrm{rank}\, A$. This is because elementary row and column operations don't change $g(A)$, and if A is diagonal, $g(A) = 1$ <=> all the entries of A are ± 1 or 0.

<u>Examples</u>. 1. If X^2 is the pinched torus $S^1 \times S^1/\{x\} \times X^1$,

the homology of X is $\mathbb{Z}\ \mathbb{Z}\ \mathbb{Z}$, and the cup product matrix A is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Thus A has a quasi-inverse C, for example $C = A$. Therefore, if ι is the generator of $H_0(X)$ such that $\epsilon(\iota) = 1$,

$$[\Delta] = \iota \times [X] + [X] \times \iota ,$$

so we can take $U = \mu \times 1 + 1 \times \mu$, where $\langle \mu, [X] \rangle = 1$. Thus the diagonal in $X \times X$ has the maximum degree of freedom, in spite of the singularity.

2. Let X^4 be the quadric cone in $P_3(C)$, which is homeomorphis to the Thom space of the tangent bundle of S^2. The homology of X is $\mathbb{Z}\ 0\ \mathbb{Z}\ 0\ \mathbb{Z}$, and the cup product matrix A is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$g(A) = \det A = 2$, so A has no quasi-inverse. Thus X has no Thom class. Since X is locally irreducible ($\mathcal{X}_n = \mathbb{Z}$), so is $X \times X$, so $[\Delta]$ has 0 degrees of freedom in $X \times X$ (I, Cor. 3 to Thm. 3A). In other words, Δ isn't homologous to a cycle lying in the complement of the singular point of $X \times X$. It is interesting to note that X is a homology manifold with $\mathbb{Q}$ or $\mathbb{Z}/p$ coefficients for p an odd prime, since the local homology at the singular point is $0\ 0\ 0\ \mathbb{Z}/2\,\mathbb{Z}$. (The link of the singular point is $P_3(\mathbb{R}) \cong$ the tangent circle bundle of S^2.)

2. Symmetry of the Thom class

Suppose the n-circuit X is a duality space; that is, $\cdot \cap [X]$ is an isomorphism, where $[X]$ is the fundamental class of X. Then $X \times X$ is also a duality space ($\cdot \cap [X \times X]$ is an isomorphism for all field coefficients by the Künneth theorem, and hence $\cdot \cap [X \times X]$ is an isomorphism for integral coefficients), so X has a unique Thom class. However, it is easy to construct an n-circuit with a unique Thom class, which is not a duality space (cf. the proof of prop. 1, §1, and the remark following it).

When $\cdot \cap [X]$ is an isomorphism, U has a strong "symmetry" property, which in fact characterizes duality spaces.

<u>Proposition.</u> Let X be a space, and let $[X] \in H_n(X)$, $U \in H^n(X \times X)$ for some n, with $U \cap ([X] \times [X]) = d_*[X]$, where $d : X \to X \times X$ is the diagonal map $d(x) = (x, x)$. Let $T : X \times X \to X \times X$ be the map $T(x, y) = (y, x)$. Then the following are equivalent:

1) $\cdot \cap [X] : H^q(X) \to H_{n-q}(X)$ is an isomorphism for all q
2) $U \cup T^*v = U \cup v$ for all $v \in H^*(X \times X)$
3) $U \cap T_*z = (-1)^n U \cap z$ for all $z \in H_*(X \times X)$

4) $U/(\alpha \cap [X]) = (-1)^{nq}\alpha$ for $\alpha \in H^q(X)$

5) $(U/\beta) \cap [X] = (-1)^{n+np}\beta$ for $\beta \in H_p(X)$

Here $/$ is the slant product - cf. [Sp].

Corollary. An n-circuit X is a duality space if and only if X has a Thom class U such that $U \cup T^*v = U \cup v$ for all $v \in H^*(X \times X)$, or equivalently, $U \cap T_*z = (-1)^n U \cap z$ for all $z \in H_*(X \times X)$.

Note that the proposition is purely "algebraic", i.e. all the conditions are homotopy invariants of X. Condition 1) just says that X is a Poincaré duality space of formal dimension n, with fundamental class $[X]$ (cf. [Bro]). If X satisfies Poincaré duality with local coefficients, and X has the homotopy type of a CW complex, then there is a homotopy equivalence $f : X \to Y$, where Y is an n-circuit, and $f_*[X] = [Y]$, the fundamental class of Y. ([Wa], proof of Cor. 2.3.2). (If X only satisfies ordinary Poincaré duality, $\pi_1(X)$ may not be finitely generated, in which case X can't be homotopy equivalent to an n-circuit.)

The geometric content of the corollary is that the n-circuit X is a duality space if and only if the diagonal cycle Δ in $X \times X$ is homologous to a transverse cycle $\tilde{\Delta}$ such that

$\tilde{\Delta} \cdot T_* z$ is homologous to $(-1)^n \tilde{\Delta} \cdot z$ for all cycles z in $X \times X$. (See I.5.) It should be emphasized that this condition is stronger than $T_* \tilde{\Delta} \sim (-1)^n \tilde{\Delta}$. In other words, <u>$U$ may satisfy $T^*U = (-1)^n U$ but $U \cap T_* z \neq (-1)^n U \cap z$ for some z.</u> Note that if X is a duality space, $T^*U = (-1)^n U$, for then it suffices to show

$$(T^*U) \cap [X \times X] = (-1)^n [\Delta]$$

$$\Longleftrightarrow \quad T_*((T^*U) \cap [X \times X]) = (-1)^n T_*[\Delta]$$

$$\Longleftrightarrow \quad U \cap T_*[X \times X] = (-1)^n [\Delta] .$$

But $T_*[X \times X] = T_*([X] \times [X]) = (-1)^n [X \times X]$, and $U \cap [X \times X] = [\Delta]$. However, if one only knows that $T^*U = (-1)^n U$, one can only conclude that $U \cap T_* z = T_*(T^*U \cap z) = (-1)^n T_*(U \cap z)$.

For example, consider the pinched torus X^2 (example 1 of §1). $[\Delta] = \iota \times [X] + [X] \times \iota$, where ι is the canonical generator of $H_0(X)$, so X has a Thom class $U = \mu \times 1 + 1 \times \mu$. Thus $T_*[\Delta] = [\Delta]$ and $T^*U = U$, but X is not a duality space.

Now the proof of the proposition follows the scheme

$$\begin{tikzcd}
4 \arrow[rr] & & 1 \arrow[dl] \arrow[dr, bend left] & & 5 \arrow[ll] \\
& 2 \arrow[ul] \arrow[rr, bend left] & & 3 \arrow[ur] &
\end{tikzcd}$$

1) $\Longrightarrow$ 2): Let $[X \times X] = [X] \times [X]$. It suffices to show

$$(T^*v \cup U) \cap [X \times X] = (v \cup U) \cap [X \times X],$$

since $\cdot \cap [X \times X]$ is an isomorphism (and $U \cup v = (-1)^{ni} v \cup U$ for $v \in H^i(X \times X)$). Now

$$\begin{aligned}
(T^*v \cup U) \cap [X \times X] &= T^*v \cap (U \cap [X \times X]) \\
&= T^*v \cap d_*[X] \\
&= d_*(d^*T^*v \cap [X]) \\
&= d_*(d^*v \cap [X]), \qquad \text{since } Td = d \\
&= v \cap d_*[X] \\
&= v \cap (U \cap [X \times X]) \\
&= (v \cup U) \cap [X \times X].
\end{aligned}$$

1) and 2) $\Longrightarrow$ 3): Given $z \in H_*(X \times X)$, let $v \in H^*(X \times X)$ be such that $v \cap [X \times X] = z$. By 2), $U \cup T^*v = U \cup v$. Thus

$$(U \cup T^*v) \cap [X \times X] = (U \cup v) \cap [X \times X],$$

so $\quad U \cap (T^*v \cap [X \times X]) = U \cap (v \cap [X \times X]) = U \cap z.$

$$\begin{aligned}
\text{But} \quad U \cap (T^*v \cap [X \times X]) &= U \cap T_*[T_*(T^*v \cap [X \times X])] \\
&= U \cap T_*(v \cap T_*[X \times X]) \\
&= U \cap T_*(v \cap (-1)^n[X \times X]) \\
&= (-1)^n U \cap T_* z.
\end{aligned}$$

2) $\Longrightarrow$ 4): First of all, $U \cap [X \times X] = d_*[X] \Longrightarrow U/[X] = 1$. It suffices to show $(U/[X]) \cap [X] = [X]$. Let $p : X \times X \to X$ be projection onto the first factor.

$$\begin{aligned}
(U/[X]) \cap [X] &= p_*(U \cap ([X] \times [X])) \qquad \text{[Sp] (6.1.6)} \\
&= p_* d_*[X] \\
&= [X].
\end{aligned}$$

Now let $\alpha \in H^q(X)$.

$$U/(\alpha \cap [X]) = (U \cup (1 \times \alpha))/[X] \qquad \text{[Sp] (6.1.5)}$$

$$= (U \cup (\alpha \times 1))/[X] \qquad \text{by 2)}$$

$$= (-1)^{nq}((\alpha \times 1) \cup U)/[X]$$

$$= (-1)^{nq}\alpha \cup U/[X] \qquad \text{[Sp] (6.1.4)}$$

$$= (-1)^{nq}\alpha \cup 1$$

$$= (-1)^{nq}\alpha .$$

3) ==> 5): Let $\beta \in H_p(X)$.

$$(U/\beta) \cap [X] = p_*(U \cap ([X] \times \beta)) \qquad \text{[Sp] (6.1.6)}$$

$$= (-1)^{n+np} p_*(U \cap (\beta \times [X])) \qquad \text{by 3)}$$

$$= (-1)^{n+np}(U/[X]) \cap \beta \qquad \text{[Sp] (6.1.6)}$$

$$= (-1)^{n+np} 1 \cap \beta$$

$$= (-1)^{n+np}\beta$$

5) $\Rightarrow$ 1): 5) implies that $\cdot \cap [X] : H^{n-p}(X) \to H_p(X)$ is surjective for all p, so $\cdot \cap [X]$ is an isomorphism by lemma 2 of I. 5 A.

4) $\Rightarrow$ 1): 4) implies that $U/\cdot : H_{n-q}(X) \to H^q(X)$ is surjective for all q, so $U/\cdot$ is an isomorphism by lemma 2 of I. 5A. But 4) says that $U/\cdot$ is a left inverse for $\cdot \cap [X]$ (up to sign), so $\cdot \cap [X]$ is also an isomorphism. This completes the proof of the proposition.

Remark. Milnor's proof of duality for smooth manifolds (with coefficients in a field) in [Mi] is essentially 2) $\Rightarrow$ 1). 3) $\Rightarrow$ 5) is Spanier's proof that $\cdot \cap [X]$ is the inverse of his duality isomorphism $U/\cdot$ for a topological manifold ([Sp] (6.3.12)). 2) and 3) are true for manifolds because U vanishes off the diagonal. (This is the tack Milnor and Spanier take.) In the next section I will show that if U vanishes off the diagonal, X is in fact a homology manifold.

3. A characterization of homology manifolds

A locally irreducible n-circuit X is a duality space if and only if every cycle in X is homologous to a transverse cycle (I. 5). If X is a p.l. manifold, these homologies can be made arbitrarily small - that is, any cycle can be closely approximated by a transverse cycle. Now (at least with field coefficients), the diagonal $\Delta \subset X \times X$ is homologous to a transverse cycle for any n-circuit X. When X is a manifold, this cycle can be found arbitrarily close to Δ - in other words, $U \mid X \times X - \Delta = 0$, where U is the Thom class. Conversely, $U \mid X \times X - \Delta = 0$ implies X is a duality space, since U will then satisfy the symmetry property $U \cup T^{*}v = U \cup v$ of § 2. This observation led Sullivan to conjecture that X will in fact be homology manifold, since Poincaré duality is given by a natural quasi-geometric condition.

<u>Theorem 4.</u> Let X be a normal n-circuit. X is a homology n-manifold if and only if X has a Thom class U such that $U \mid X \times X - \Delta = 0$.

A normal (locally irreducible) n-circuit is just a connected purely n-dimensional polyhedron with a given isomorphism of the "orientation sheaf" $\mathcal{H}_n$ with the constant sheaf $\mathbb{Z}$ (cf. I.4D). In other words, $H_n(X) \cong \mathbb{Z}$, and there is given a generator $[X]$ of $H_n(X)$ such that the image of $[X]$ under the restriction map $H_n(X) \to H_n(X, X-\{x\})$ is an isomorphism for each $x \in X$. Thus an oriented homology n-manifold is a normal n-circuit.

Proof: If X is an oriented homology n-manifold, $X \times X$ is an oriented homology 2n-manifold. Consider the diagram

$$\begin{array}{ccc} H^n(X \times X, X \times X - \Delta) & \xrightarrow{j^*} & H^n(X \times X) \\ \Big\downarrow \cdot \cap [X \times X] & & \Big\downarrow \cdot \cap [X \times X] \\ H_n(\Delta) & \xrightarrow{i_*} & H_n(X \times X) \end{array} \tag{3.1}$$

The vertical arrows are Lefschetz and Poincaré duality, respectively, and the horizontal arrows are induced by the inclusion of Δ in $X \times X$. If $\tilde{U} \in H^n(X \times X, X \times X - \Delta)$ is dual to $[\Delta] \in H_n(\Delta)$, $j^*\tilde{U}$ is dual to $[\Delta] \in H_n(X \times X)$, i.e. $U = j^*\tilde{U}$ is the Thom class of X. Thus $U \mid X \times X - \Delta = 0$.

The proof of the converse is broken into five lemmas.

Lemma 1. Let X be a normal n-circuit, and let $\tilde{U} \in H^n(X \times X, X \times X - \Delta)$, $U = j^*\tilde{U} \in H^n(X \times X)$. The following are equivalent:

1) $U \cap [X \times X] = [\Delta]$, i.e. U is a Thom class for X

2) $U/[X] = 1$

3) $U \mid \{x\} \times X$ is the generator of $H^n(\{x\} \times X) \cong H^n(X)$ corresponding to μ, where $\langle \mu, [X] \rangle = 1$, for all $x \in X$.

4) $\tilde{U} \mid \{x\} \times (X, X - \{x\})$ is the generator corresponding to $\mu_x \in H^n(X, X - \{x\})$, where $\langle \mu_x, [X]_x \rangle = 1$, $[X]_x$ = the restriction of $[X]$ to $(X, X - \{x\})$. (In other words, $\tilde{U}$ is an orientation for X in the sense of Spanier ([Sp] 6.2).)

Lemma 2. If the n-circuit X has a Thom class U such that $U \mid X \times X - \Delta = 0$, then X is a duality space.

Lemma 3. Suppose that the connected subpolyhedron Y of X has a neighborhood N in X such that $(N, Y) \cong (Y \times D^k, Y \times 0)$ for some k. (D^k is the standard k-disc with center 0.) Then

a) if X is a normal n-circuit, Y is a normal (n-k)-circuit

b) if X has a Thom class U_X such that $U_X \mid X \times X - \Delta_X = 0$, then Y has a Thom class U_Y such that $U_Y \mid Y \times Y - \Delta_Y = 0$.

I will say that Y is a "tubular" subpolyhedron of X. For example, if X is a p.l. manifold, its tubular subpolyhedra are all submanifolds with trivial normal bundles.

Lemma 4. Suppose that Y is a normal m-circuit with isolated homological singularities (points y such that $H_i(Y, Y - \{y\}) \neq 0$ for some $i < n$). If Y is a duality space, then Y is a homology manifold - i.e. Y has no homological singularities.

Lemma 5. Let X be a normal n-circuit which is not a homology manifold. Let $S(X)$ be the set of homological singularities of X. Then there is a tubular subpolyhedron $Y \subset X$ such that $S(Y) = Y \cap S(X)$ consists of isolated points.

Now the theorem follows easily from the lemmas, for suppose X has a Thom class vanishing off the diagonal. If X is not a homology manifold, we can choose Y as in lemma 5 - a tubular supbolyhedron of X with isolated singularities. But Y has a Thom class which vanishes off the diagonal (lemma 3), so Y is a duality space (lemma 2), so $S(Y) = \emptyset$ (lemma 4), contradicting the choice of Y.

Proof of lemma 1:

1) ==> 2) occurs in the proof of the proposition in §2 above (see "2) ==> 4)" of that proof).

2) ==> 1): $(U/[X]) \cap [X] = p_*(U \cap [X \times X])$ [Sp] (6.1.6),

so $[X] = p_*(U \cap [X \times X])$ by 2),

so $d_*[X] = d_*p_*(U \cap [X \times X])$.

Now $d_*p_*(U \cap [X \times X]) = U \cap [X \times X]$, since $U \cap [X \times X]$ is in the image of $i_* : H_n(\Delta) \to H_n(X \times X)$, namely $U \cap [X \times X] = i_*(\widetilde{U} \cap [X \times X])$, where $\widetilde{U} \in H^n(X \times X, X \times X - \Delta)$ and $j^*\widetilde{U} = U$ (see the diagram (3.1) above).

2) <==> 3): Let $x_0 \in X$, and let $f_0 : X \to X \times X$ be the map $f_0(x) = (x, x_0)$. Then $U \mid \{x_0\} \times X = (f_0)^*U$. Let $\iota \in H_0(X)$ be such that $\varepsilon(\iota) = 1$ (ε = standard augmentation). ι is represented by the 0-cycle $\{x_0\}$. Now

$$\begin{aligned} \varepsilon(U/[X]) &= \langle U/[X], \iota\rangle \\ &= \langle U, [X] \times \iota\rangle \\ &= \langle U, (f_0)_*[X]\rangle \\ &= \langle (f_0)^*U, [X]\rangle \end{aligned}$$

Thus $U/[X] = 1 \iff \langle (f_0)^*U, [X] \rangle = 1$. (Note that this argument shows that $U \mid \{x\} \times X$ is a generator for some x implies it is a generator for all x.)

$3 \iff 4$): This is clear from the diagram

$$\begin{array}{ccc} H^n(X \times X, X \times X - \Delta) & \xrightarrow{j^*} & H^n(X \times X) \\ \downarrow & & \downarrow \\ H^n(\{x\} \times (X, X - \{x\})) & \xrightarrow{\approx} & H^n(\{x\} \times X) \end{array}$$

The bottom arrow is an isomorphism since X is locally irreducible, so $H^n(X, X - \{x\}) \xrightarrow{\approx} H^n(X) \cong \mathbb{Z}$ for all $x \in X$. The vertical arrows are restriction maps, and $j^*\widetilde{U} = U$. This proves lemma 1.

<u>Proof of lemma 2:</u> I claim it suffices to show

$$(*) \qquad (p_1)_*(U \cap z) = (p_2)_*(U \cap z) \text{ if } z \in H_*(X \times X),$$

where $p_1, p_2 : X \times X \to X$ are the first and second projections, respectively. By the proposition of §2, it suffices to show that $(U/\beta) \cap [X] = (-1)^{n+np}\beta$ for $\beta \in H_p(X)$. But

$$\begin{aligned}
(U/\beta) \cap [X] &= (p_1)_*(U \cap ([X] \times \beta)) && \text{[Sp] (6.1.6)} \\
&= (p_2)_*(U \cap ([X] \times \beta)) && \text{by } (*) \\
&= (p_1)_* T_*(U \cap ([X] \times \beta)) \\
&= (p_1)_*(T^*U \cap T_*([X] \times \beta)) \\
&= (-1)^{np}(p_1)_*(T^*U \cap (\beta \times [X])) \\
&= (-1)^{np}(T^*U/[X]) \cap \beta \\
&= (-1)^{n+np}\beta
\end{aligned}$$

(This last step uses that $(-1)^n T^*U/[X] = 1$, which is clear since U is a Thom class for the n-circuit $X \iff (-1)^n T^*U$ is, and $U/[X] = 1$ for any Thom class U.) Thus it remains to show that $U \mid X \times X - \Delta = 0 \implies (p_1)_*(U \cap z)$.

Let N be a regular neighborhood of the diagonal in $X \times X$. Δ is a deformation retract of N, and $p_1 \mid \Delta = p_2 \mid \Delta$, so $p_1 \mid N$ is homotopic to $p_2 \mid N$. Thus it is enough to show that $U \cap z$ is represented by a cycle in N. This follows from the diagram

$$
\begin{tikzcd}
H^*(X \times X) \otimes H_*(X \times X) \arrow[rr, "\cap"] \arrow[d, shift left=2, "j_*"] & & H_*(X \times X) \\
H^*(X \times X, X \times X - \Delta) \otimes H_*(X \times X, X \times X - \Delta) \arrow[u, shift left=2, "j^*"] \arrow[urr, "\cap"] \arrow[d, shift right=2, "i^*"', "\cong"] & & \\
H^*(N, N - \Delta) \otimes H_*(N, N - \Delta) \arrow[u, shift right=2, "i_*", "\cong"'] \arrow[rr, "\cap"] & & H_*(N) \arrow[uu, "i_*"']
\end{tikzcd}
$$

and the fact that $U = j^* \widetilde{U}$. In other words,

$$
\begin{aligned}
U \cap z &= j^* \widetilde{U} \cap z \\
&= \widetilde{U} \cap j_* z \\
&= i_*(i^* \widetilde{U} \cap (i_*)^{-1} j_* z).
\end{aligned}
$$

(This argument is due to Milnor [Mi].)

<u>Proof of lemma 3:</u>

a): If $Y \subset X$ is tubular, and $y \in Y$, then y has a neighborhood A in X such that $(A, A \cap Y) \cong ((A \cap Y) \times D^k, A \cap Y)$. It follows that $H_i(Y, Y - \{y\}) \cong H_{i+k}(X, X - \{y\})$ for all i. Thus if X is purely n-dimensional and $H_n(X, X - \{x\}) \cong \mathbb{Z}$ for all $x \in X$, Y is purely $(n-k)$-dimensional and

$H_{n-k}(Y, Y - \{y\}) \cong \mathbb{Z}$ for all $y \in Y$. In fact, it is clear that the sheaf $\mathcal{H}_{n-k}(Y)$ is isomorphic to the restriction of the sheaf $\mathcal{H}_n(X)$ to Y, so if $\mathcal{H}_n(X)$ is constant, so is $\mathcal{H}_{n-k}(Y)$.

b): I will show that if $\widetilde{U}_X$ satisfies condition 4) of lemma 1, there is a class $\widetilde{U}_Y$ satisfying it for Y. It is helpful to think of the tube N as the total space of a trivial k-dimensional bundle over Y. Thus there is a Thom isomorphism

$$H^i(Y) \xrightarrow{\approx} H^{i+k}(N, N-Y)$$

for $i > 0$. Now consider the following diagram:

$$\begin{tikzcd}
\widetilde{U}_X \in \arrow[dd] & H^n(X \times X, X \times X - \Delta_X) \arrow[r] \arrow[d] & H^n(\{y\} \times (X, X - \{y\})) \arrow[d, "\wr"] \\
 & H^n(Y \times N, Y \times N - \Delta_Y) \arrow[r] & H^n(\{y\} \times (N, N - \{y\})) \\
\widetilde{U}_Y \in & H^{n-k}(Y \times Y, Y \times Y - \Delta_Y) \arrow[r] \arrow[u, "f"', "\wr"] & H^{n-k}(\{y\} \times (Y, Y - \{y\})) \arrow[u, "g"', "\wr"]
\end{tikzcd}$$

The map f is the relative Thom isomorphism for the trivial D^k bundle over $Y \times Y$:

$$H^{n-k}(Y \times Y, Y \times Y - \Delta) \overset{\approx}{\to} H^n(Y \times Y \times D^k, (Y \times Y \times (D^k - 0)) \cup ((Y \times Y - \Delta_Y) \times D^k))$$

$$\| $$

$$H^n(Y \times N, Y \times N - \Delta_Y)$$

g is the obvious isomorphism, and all the other arrows are restriction maps. The diagram clearly commutes, so if we define $\tilde{U}_Y$ to be the image of U_X, and $\tilde{U}_X \mid \{y\} \times (X, X - \{y\})$ is a generator, then $\tilde{U}_Y \mid \{y\} \times (Y, Y - \{y\})$ is a generator.

Proof of lemma 4: This is just the corollary to Proposition 6 of I.4.

Proof of lemma 5: Y is essentially a hyperplane section of X in some large Euclidean space (this was Sullivan's idea). Embed X piecewise linearly in some Euclidean space R^N. Let $\tilde{K}$ be a triangulation of X such that each simplex of $\tilde{K}$ is embedded linearly in R^N. Suppose $\dim S(X) = s$, and let σ be an s-dimensional simplex of $\tilde{K}$ such that $\mathring{\sigma}$ lies in $S(X)$. (If any point in $\mathring{\sigma}$ lies in $S(X)$, all of $\mathring{\sigma}$ does.) Let $P \subset R^N$ be the s-dimensional plane containing σ. Choose a triangulation $\tilde{L}$ of R^N so that $\tilde{L} \mid X$ is a subdivision of $\tilde{K}$ and P is covered by a subcomplex of $\tilde{L}$. Now let τ be an s-simplex of $\tilde{L}$ with $\mathring{\tau} \subset \mathring{\sigma}$. Assume that the barycenter $\underline{\tau}$ is the origin of

R^N, and let $p : R^N \to P$ be projection onto the linear subspace P. Let $\pi : L \to J$ be a simplicial approximation to p relative to τ, i.e. L is some subdivision of $\tilde{L}$, and $\pi \mid \tau = p \mid \tau$ = identity (cf. [Ze 3]).

Now let $\pi' : L' \to J'$ be a first derived of π (say J' is the barycentric subdivision of J, and the first derived L' is chosen so that $|\pi'| = |\pi|$). Let $Q = |(\pi')^{-1}(\underline{\tau})|$, so Q is a "p.l. approximation" to the $(N\text{-}s)$-hyperplane perpendicular to P. Let $Y = X \cap Q$ be the "hyperplane section" of X by Q. Thus $Y = |(f')^{-1}(\underline{\tau})|$, where f is the simplicial map got by restricting π to $L \mid X = K$.

For any simplicial map $f : K \to J$, $(f')^{-1}(J^{(s)}) \subset K^{(s)}$, where $J^{(s)}$, $K^{(s)}$ are the s-coskeletons of J, K. (This is an easy consequence of the definition (3.1).) Therefore, since $\underline{\tau} \in J^{(s)}$, $Y \subset |K^{(s)}|$. But $S(X) \subset |K_s|$, and $\dim(K_1^{(s)} \cap K_s) = 0$, so $\dim(Y \cap S(X)) = 0$; that is Y intersects $S(X)$ in isolated points.

Furthermore, for any simplicial map $f : K \to J$ and any s-simplex $\tau \in J$, $f^{-1}(\mathring{\tau})$ is p.l. homeomorphic to $(f')^{-1}(\underline{\tau}) \times D^s$. (The proof is elementary - see [Mi], p. 125, lemma 3.) Thus $Y = f^{-1}(\underline{\tau})$ is a tubular subpolyhedron of X. (By throwing away the components of Y not containing $\underline{\tau}$, we can make Y connected.) This completes the proof of the theorem.

4. Intersection pairings

To illuminate the geometry of Thom classes, I will discuss their relation to intersection pairings. If an intersection pairing on homology is any pairing compatible with the cup product pairing on cohomology, then the existence of an intersection pairing on $H_*(X)$ is equivalent to the existence of a Thom class for X. Furthermore, X is a homology manifold if and only if there is an intersection pairing on $H_*(X)$ satisfying a simple quasi-geometric axiom.

Throughout this section, X will be an n-circuit, $H_*(X)$ will denote the integral homology of X, and $H_*(X; F)$ will denote the homology of X with coefficients in the field F. I will assume there is given a homomorphism $h : \mathbb{Z} \to F$, and I will write $[X] = h_*[X] \in H_n(X; F)$, etc. (cf. prop. 1 of §1).

<u>Definition.</u> An <u>intersection pairing</u> on the n-circuit X is a homomorphism

$$\varphi : H_p(X) \otimes H_q(X) \to H_{p+q-n}(X)$$

defined for all p, q such that if $v \in H^r(X)$ and $w \in H^s(X)$,

$$(4.1) \qquad \varphi((v \cap [X]), (w \cap [X])) = (v \cup w) \cap [X]$$

If X is a duality space, (4.1) determines φ, which is the classical intersection pairing of Lefschetz when X is a p.l. n-manifold.

Proposition 1. If U is a Thom class for X, then X has an intersection pairing φ_U defined by

$$\begin{aligned}\varphi_U(\beta, \gamma) &= (-1)^{n+np}(U/\beta) \cap \gamma \\ &= (-1)^{n+np} p_*(U \cap (\gamma \times \beta)),\end{aligned}$$

where $\beta \in H_p(X)$, $\gamma \in H_q(X)$, and $p : X \times X \to X$ is projection onto the first factor.

Remark. If X is a manifold, $U \cap (\gamma \times \beta)$ is the intersection of $\gamma \times \beta$ with the diagonal $\Delta \subset X \times X$ (à la Lefschetz).

Proof: We must verify (4.1). Let $v \in H^r(X)$, $w \in H^s(X)$.

$\varphi_u((v \cap [X]), (w \cap [X]))$

$$= (-1)^{nr} p_*[U \cap ((w \cap [X]) \times (v \cap [X]))]$$

$$= (-1)^{nr+s(n-r)} p_*[U \cap ((w \times v) \cap [X \times X])] \qquad \text{[Sp] (5.7.21)}$$

$$= (-1)^{nr+s(n-r)} p_*[(U \cup (w \times v)) \cap [X \times X]]$$

$$= (-1)^{nr+s(n-r)+n(r+s)} p_*[((w \times v) \cup U) \cap [X \times X]]$$

$$= (-1)^{rs} p_*((w \times v) \cap (U \cap [X \times X]))$$

$$= (-1)^{rs} p_*((w \times v) \cap d_*[X])$$

$$= (-1)^{rs} p_* d_*(d^*(w \times v) \cap [X])$$

$$= (-1)^{rs}((w \cup v) \cap [X])$$

$$= (v \cup w) \cap [X], \qquad \text{q.e.d.}$$

Now if F is a field, $H_*(X; F)$ clearly has an intersection pairing, since (4.1) determines the pairing on the direct summand $\mathrm{Im}(\cdot \cap [X])$, and the pairing on the complementary summand can be arbitrary.

Proposition 2. Any intersection pairing φ on $H_*(X; F)$ yields a Thom class $U_\varphi \in H^n(X \times X; F)$ defined by

$$U_\varphi = \sum_{ij} (-1)^{\underline{i}} \epsilon\,\varphi(\beta_i, \beta_j)\alpha_i \times \alpha_j$$

where $\{\alpha_i\}$ is a basis for $H^*(X; F)$, $\dim \alpha_i = \underline{i}$, and $\{\beta_i\}$ is the algebraically dual basis for $H_*(X; F)$, i.e. $\langle\alpha_i, \beta_j\rangle = \delta_{ij}$.

Proof: By (1.1), we must show that if $c_{ij} = \epsilon\,\varphi(\beta_i, \beta_j)$,

$$\sum_{ij} (-1)^{\underline{i}\,\underline{j}} a_{pi} c_{ij} a_{jq} = a_{pq},$$

where $a_{pq} = \langle\alpha_p \cup \alpha_q, [X]\rangle$. Now $\langle\alpha_p \cup \alpha_q, [X]\rangle = \epsilon((\alpha_p \cup \alpha_q) \cap [X])$, so by (4.1),

$$a_{pq} = \epsilon\,\varphi((\alpha_p \cap [X]), (\alpha_q \cap [X])).$$

But $\alpha_j \cap [X] = \sum_i a_{ij}\beta_i$ (1.2). Thus

$$a_{pq} = \epsilon\, \varphi[(\sum_i a_{ip}\beta_i), (\sum_j a_{jq}\beta_j)]$$

$$= \epsilon \sum_{ij} a_{ip}a_{jq}\varphi(\beta_i, \beta_j)$$

$$= \sum_{ij} a_{ip}a_{jq}c_{ij}$$

$$= \sum_{ij} (-1)^{\underline{p}\underline{i}} a_{pi}c_{ij}a_{jq}$$

$$= \sum_{ij} (-1)^{\underline{i}\underline{j}} a_{pi}c_{ij}a_{jq}, \qquad \text{q. e. d.},$$

since $\underline{p} + \underline{i} = n = \underline{i} + \underline{j}$ for all nonzero a_{pi}, c_{ij}.

Lemma. U is a Thom class for $X^n \iff (-1)^n T^*U$ is a Thom class for X.

Proof: $T_*(T^*U \cap [X \times X]) = U \cap T^*[X \times X] = (-1)^n U \cap [X \times X]$. Since $T_*T_* = 1$ and $T_*[\Delta] = [\Delta]$, the lemma follows.

Proposition 3. $U_{(\varphi_u)} = (-1)^n T^*U$.

Proof: Let $U = \sum_{pq} b_{pq}\alpha_p \times \alpha_q$.

$$U_{(\varphi_u)} = \sum_{ij} (-1)^{\underline{i}}\, \varepsilon\, [(-1)^{n+n\underline{i}} p_*(U \cap (\beta_j \times \beta_i))](\alpha_i \times \alpha_j)$$

$$= \sum (-1)^{\underline{i}+n\underline{j}}\, \varepsilon\, p_*(U \cap (\beta_j \times \beta_i))(\alpha_i \times \alpha_j)$$

$$= \sum (-1)^{\underline{i}+n\underline{j}}\, \varepsilon\, p_*[\sum_{pq} b_{pq}(\alpha_p \times \alpha_q) \cap (\beta_j \times \beta_i)](\alpha_i \times \alpha_j)$$

$$= \sum (-1)^{\underline{i}+n\underline{j}}\, \varepsilon\, p_*[\sum_{pq} b_{pq}(-1)^{\underline{p}(\underline{i}-\underline{q})}(\alpha_p \cap \beta_j) \times (\alpha_q \cap \beta_i)](\alpha_i \times \alpha_j)$$

$$= \sum (-1)^{\underline{i}+n\underline{j}} \sum_{pq} b_{pq}\delta_{pj}\delta_{qi}(\alpha_i \times \alpha_j) .$$

$$= \sum (-1)^{\underline{i}+n\underline{j}} b_{ji}\alpha_i \times \alpha_j$$

$$= (-1)^n \sum (-1)^{\underline{ij}} b_{ji}\alpha_i \times \alpha_j, \quad \text{since } \underline{i} + \underline{j} = n \text{ for } b_{ji} \neq 0,$$

$$= (-1)^n \sum b_{ji} T^*(\alpha_j \times \alpha_i)$$

$$= (-1)^n T^*U, \qquad \text{q.e.d.}$$

One should not expect that $\varphi = \pm \varphi_{(U_\varphi)}$, since the definition of U_φ only uses the values of $\varphi(\beta, \gamma)$ for $\beta \in H_p(X)$, $\gamma \in H_q(X)$, with $p + q = n$. However, this much of the pairing can be recovered from U_φ.

Proposition 4. If $\beta \in H_p(X; F)$, $\gamma \in H_q(X; F)$, $p + q = n$, $\varphi_{(U_\varphi)}(\beta, \gamma) = (-1)^{pq}\varphi(\gamma, \beta)$.

Proof: It suffices to check that $\varepsilon\,\varphi_{(U_\varphi)}(\beta_p, \beta_q) = (-1)^{\underline{pq}}\,\varepsilon\;\varphi(\beta_q, \beta_p)$, where $\{\beta_i\}$ is the basis for $H_*(X; F)$ used to define U_φ.

$$\varepsilon\varphi_{(U_\varphi)}(\beta_p, \beta_q) = (-1)^{n+n\underline{p}}\,\varepsilon\;p_*(U_\varphi \cap (\beta_q \times \beta_p))$$

$$= (-1)^{n\underline{q}}\,\varepsilon\;p_*[\sum_{ij} (-1)^{\underline{i}}\,\varepsilon\;\varphi(\beta_i, \beta_j)(\alpha_i \times \alpha_j) \cap (\beta_q \times \beta_p)]$$

$$= (-1)^{n\underline{q}}\,\varepsilon\;p_*[\sum_{ij} (-1)^{\underline{i}}\,\varepsilon\;\varphi(\beta_i, \beta_j)(-1)^{\underline{i(p-j)}}(\alpha_i \cap \beta_q) \times (\alpha_j \cap \beta_p)]$$

$$= (-1)^{n\underline{q}} \sum_{ij} (-1)^{\underline{i}}\,\varepsilon\;\varphi(\beta_i, \beta_j)\delta_{iq}\delta_{jp}$$

$$= (-1)^{n\underline{q}+\underline{q}}\,\varepsilon\;\varphi(\beta_q, \beta_p)$$

$$= (-1)^{\underline{pq}}\,\varepsilon\;\varphi(\beta_q, \beta_p), \qquad \text{q. e. d.}$$

If the intersection pairing φ on $H_*(X; F)$ satisfies

$$(4.2) \qquad \varphi(\beta, \gamma) = (-1)^{pq}\varphi(\gamma, \beta), \quad \dim \beta = p, \quad \dim \gamma = q,$$

then it follows easily that $T^*U_\varphi = (-1)^n U_\varphi$. Conversely, if the Thom class U satisfies $T^*U = (-1)^n U$, it is clear that (4.2) holds for φ_u.

Thus, if φ is "symmetric" (satisfies (4.2)), U_φ is "weakly symmetric" ($T^*U_\varphi = (-1)^n U_\varphi$), and $\varphi_{(U_\varphi)} = \varphi$. If U is symmetric, then so is φ_u, and $U_{(\varphi_u)} = U$.

We have seen in § 2 that the Thom class of a duality space is symmetric in this sense. However, $T^*U = (-1)^n U$ does not imply X is a duality space, so the axioms (4.1) and (4.2) for the intersection pairing do not characterize duality spaces.

Proposition 5. If the intersection pairing φ on $H_*(X; F)$ satisfies

$$\varphi((\alpha \cap [X]), \beta) = \alpha \cap \beta, \tag{4.3}$$

for all $\alpha \in H^*(X; F)$ and $\beta \in H_*(X; F)$, then X is an F-duality space. (Note that (4.3) $\Longrightarrow$ (4.1).)

Proof: It suffices to show that

$$\cdot \cap [X] : H^*(X; F) \to H_*(X; F)$$

is injective. But $\alpha \cap [X] = 0$ implies $\alpha \cap \beta = 0$ for all β by (4.3), so $\alpha = 0$. (If $\alpha = \sum n_i \alpha_i$, $\epsilon(\alpha \cap \beta_i) = \langle\alpha, \beta_i\rangle = n_i$, where $\{\alpha_i\}$, $\{\beta_i\}$ are dual bases for $H^*(X; F)$, $H_*(X; F)$.)

Corollary. If the n-circuit X has an intersection pairing satisfying (4.3), then $\mathrm{Ker}(\cdot \cap [X])$ is all torsion.

The fundamental geometric property of the homology intersection pairing $\varphi(\beta, \gamma) = \beta \cdot \gamma$ on a manifold is

(4.4) If β and γ are represented by cycles with disjoint support, then $\varphi(\beta, \gamma) = 0$.

It may be true that any n-circuit X which has a pairing satisfying (4.1) and (4.4) is a homology manifold. I will prove this by replacing (4.4) by a stronger and less geometric condition. If X is a manifold,

$$\beta \cdot \gamma = \pm p_*((\beta \times \gamma) \cdot \Delta).$$

In fact, it's easy to show that

$$\varphi(\beta, \gamma) = (-1)^{(n-p)q} p_* \psi(\beta \times \gamma, \Delta)$$

for any n-circuit X, where $p = \dim \beta$, $q = \dim \gamma$, φ is any intersection pairing on $H_*(X)$, and ψ is any intersection pairing on $H_*(X \times X)$. Thus the intersection pairing φ on a manifold satisfies

(4.5) If $\Sigma\, n_{ij} \beta_i \times \gamma_j \in H_n(X \times X)$ is represented by a cycle not meeting the diagonal, then $\Sigma\, (-1)^{\underline{i}} \varphi(\beta_i, \gamma_j) = 0$.

<u>Proposition 6</u>. Let X be a normal n-circuit with an intersection pairing φ on $H_*(X; F)$, F a field. If φ satisfies (4.5), then X is an F-homology manifold.

<u>Corollary</u>. If X has intersection pairings φ_F on $H_*(X; F)$ for $F = \mathbb{Q}$, $\mathbb{Z}/p$ for all p, then X is an integral homology manifold.

<u>Proof:</u> By proposition 2,

$$U = \sum_{ij} (-1)^{\underline{i}}\, \varepsilon\, \varphi(\beta_i, \beta_j) \alpha_i \times \alpha_j$$

is a Thom class for X (F coefficients). Now

$$<U, \sum_{k\ell} n_{k\ell}\beta_k \times \beta_\ell> = \sum_{ijk\ell} (-1)^{\underline{i}} \varepsilon\, \varphi(\beta_i, \beta_j) n_{k\ell} <\alpha_i \times \alpha_j, \beta_k \times \beta_\ell>$$

$$= \sum (-1)^{\underline{i}} \varepsilon\, \varphi(\beta_i, \beta_j) n_{k\ell}\, \delta_{ik} \delta_{j\ell}$$

$$= \sum (-1)^{\underline{k}} n_{k\ell}\, \varphi(\beta_k, \beta_\ell)$$

Thus (4.5) says that if $\theta = \sum n_{k\ell}\beta_k \times \beta_\ell$ is represented by a cycle not meeting the diagonal, $<U, \theta> = 0$. In other words $U \mid X \times X - \Delta = 0$, since the coefficients are in a field. Therefore X is an F-homology manifold by Theorem 4 (§3).

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Erratum

The proof of lemma 1, p. 91, is incorrect. The mistake lies in concluding that $\dim p(C) = r-j$ implies $\dim(C) \geq r-j$ (p. 92). This is false in general (and the theorem of [Na] referred to is irrelevant). I don't know whether the lemma is true, though I doubt it.*
Thus theorems 3A and 3B are unproved as they stand.

However, the general position proposition of p. 98 implies that theorems 3A and 3B are true if the definitions of "degrees of freedom" and "extent" are changed to require the subspaces A of X to be subcomplexes of some triangulation.

In fact, both theorems are true as they stand. The proof of theorem 3B shows that extent $\beta \leq$ filtration β for any triangulable space, but theorem 4 of [Ze 1] asserts that filtration $\beta \leq$ extent β for any compact Hausdorff space. Thus extent β = filtration β for any compact triangulable space. Similarly, the proof of theorem 3A shows that degree of freedom $\alpha \leq$ filtration α, and the proof of theorem 4 of [Ze 1] can be dualized in a straightforward manner to show filtration $\alpha \leq$ degree of freedom α. Thus degree of freedom α = filtration α for any compact triangulable space.

* The lemma is false - c.f. "Antoine's necklace"