A CHARACTERIZATION OF HOMOLOGY MANIFOLDS

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A homology n-manifold is a space which has the same local homology at each point as Euclidean n-space. The principal result of this paper is the characterization of triangulable homology manifolds by a *global* property: The n-circuit X is a homology manifold if and only if the diagonal cycle Δ in $X \times X$ is Poincaré dual to a cocycle with support Δ (Theorem 1). If X is a smooth manifold, this cocycle represents the Thom class of the tangent bundle of X. The homological properties of Thom classes have been studied by Milnor [**13**; §11] and Spanier [**14**; Chapter 6]. The proof is based on their techniques.

A corollary of this proof is that an n-circuit X satisfies Poincaré duality if and only if there is a class U dual to the diagonal which has a certain symmetry with respect to the canonical involution T on $X \times X$; namely $U \smile V = U \smile T^*V$ for all V (Proposition 1). Furthermore, for any n-circuit X, the diagonal cycle is dual to *some* cocycle U, if coefficients are in a field (Proposition 2). Thus $U|(X \times X - \Delta)$ is the "obstruction" to X being a homology manifold. Propositions 1 and 2 have been obtained independently by P. Holm [**8**].

The ideas of Lefschetz about intersection theory and the topology of algebraic varieties have been my constant guide (*cf.* [**15**]). Theorem 1 can be interpreted in terms of the intersection pairing (Theorem 3).

This paper is a revised version of part of my doctoral thesis at Brandeis University [**11**], written under the supervision of Professor Jerome Levine. I have also been helped by the questions and suggestions of P. Lynch, D. Stone, A. Landman, and especially D. Sullivan. My viewpoint has recently been influenced by the work of I. Fáry [3].

Homology will be singular homology throughout, with integer coefficients in §§1 and 2, and field coefficients in §§3 and 4. Sign conventions for products are those of [**14**].

1. *Thom classes*

In this section, all spaces X will be assumed to be *triangulable.* That is, X is homeomorphic with the geometric realisation of a locally finite simplicial complex.

An *n-circuit* is a pair $(X, [X])$, where X is a compact space and $[X] \in H_n(X)$, such that X is purely n-dimensional, with singularities of codimension at least two, and $[X]$ restricts to an orientation for the non-singular part of X. These conditions can be made precise using the *local homology groups* of X at a point $x \in X$, defined by $H_i(X)_x = H_i(X, X-\{x\})$. The space X is *purely n-dimensional* if and only if $H_i(X)_x = 0$ for $i > n$ and all $x \in X$, and $\{x \in X, H_n(X)_x \neq 0\}$ is dense in X. The *homological singularity set SX* is the closure of $\{x \in X, H_i(X)_x \not\cong H_i(R^n)_0$ for some $i\}$. The set SX is a sub-complex of every triangulation of X, since $H_i(X)_x \cong H_i(X)_y$ if x and y are points in the same simplex of a triangulation [3]. For X to be an n-circuit, the dimension of SX must be less than or equal to $n-2$, and the *orientation class* $[X] \in H_n(X)$ must restrict to a generator of $H_n(X)_x$ for all $x \in X - SX$.

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For example, let X be a complex projective variety of pure complex dimension k. Since the complex structure of X determines a canonical orientation, X is a $2k$-circuit. Circuits were introduced by Lefschetz around 1930, principally for this reason [**10**; Chapter VII]. (For a simple proof that a complex variety is triangulable, see [**6**] or [7].)

The n-circuit X is *irreducible* if X is not the union of two n-circuits unequal to X. Any n-circuit can be uniquely expressed as a union of irreducible n-circuits. An irreducible n-circuit is the same thing as Lefschetz's "oriented simple n-circuit" [**10**; p. 46], or Brouwer's "oriented pseudomanifold" [**14**; p. 150].

The space X is a *homology n-manifold* if $H_i(X)_x \cong H_i(R^n)_0$ for all i and all $x \in X$. Thus, the n-circuit X is a homology manifold if and only if SX is empty. My purpose here is to discuss global properties of (compact, oriented) homology manifolds which characterize them among all circuits.

For an n-circuit X, the *duality morphism*

$$\mathscr{D}_X : H^*(X) \to H_*(X)$$

is defined by $\mathscr{D}_X(a) = a \frown [X]$, where $\frown$ is the cap product [**14**; p. 254]. Thus $\mathscr{D}_X(H^i(X)) \subset H_{n-i}(X)$. If X is a homology manifold, $\mathscr{D}_X$ is the Poincaré duality isomorphism.

If X is an n-circuit, $X \times X$ is a $2n$-circuit with orientation class $[X \times X] = [X] \times [X]$. Let $d : X \to X \times X$ be the diagonal embedding $d(x) = (x, x)$, and let $\Delta = d(X)$, the *diagonal* of $X \times X$. The class $d_*[X] \in H_n(X \times X)$ is the *diagonal homology class* of $X \times X$.

A *diagonal cohomology class* for the n-circuit X is a class $U \in H^n(X \times X)$ such that $\mathscr{D}_{X \times X}(U) = d_*[X]$.

THEOREM 1. *The n-circuit X is a homology manifold if and only if X has a diagonal cohomology class U which is supported by the diagonal*; *i.e., the restriction of U to $X \times X - \Delta$ is zero.*

Proof. If the n-circuit X is a homology manifold, then the $2n$-circuit $X \times X$ is also a homology manifold, so $\mathscr{D}_{X \times X}$ is an isomorphism. Therefore X has a unique diagonal cohomology class $U \in H^n(X \times X)$. The restriction of U to $X \times X - \Delta$ is zero if and only if there is a class $U' \in H^n(X \times X, X \times X - \Delta)$ such that $j^*U' = U$, where $j : (X \times X, \phi) \to (X \times X, X \times X - \Delta)$ is the inclusion. Consider the commutative diagram

$$\begin{tikzcd} H^n(X \times X) \arrow[r, "\mathscr{D}_{X \times X}"] & H_n(X \times X) \\ H^n(X \times X, X \times X - \Delta) \arrow[u, "j^*"] \arrow[r, "D"] & H_n(\Delta) \arrow[u, "i_*"'] \end{tikzcd}$$

where $i : \Delta \to X \times X$ is the inclusion and D is the duality isomorphism (*cf.* [**14**; 6.2.17]). Let $U' = D^{-1}[\Delta]$. Then $\mathscr{D}_{X \times X} j^*U' = i_*DU' = i_*[\Delta] = d_*[X]$, so $j^*U' = U$.

To prove the converse of this theorem, I will first state an equivalent theorem, using the class U'.

Let X be an arbitrary n-circuit. A *Thom class* for X is a class

$$U' \in H^n(X \times X, X \times X - \Delta)$$

such that, if U_x' is the restriction of U' to

$$H^n(\{x\} \times X, \{x\} \times X - \{(x, x)\}) = H^n(X, X - \{x\}),$$

and $[X]_x$ is the restriction of $[X]$ to $H_n(X, X-\{x\})$, then $\langle U_x', [X]_x \rangle = 1$ for all $x \in X - SX$. A Thom class is a generalization of a "cohomology orientation" in the sense of Spanier [**14**; p. 294].

LEMMA 1. *Let X be an n-circuit, and let*

$$U \in H^n(X \times X) \text{ and } U' \in H^n(X \times X, X \times X - \Delta)$$

be arbitrary cohomology classes with $j^*U' = U$, where

$$j : (X \times X, \phi) \to (X \times X, X \times X - \Delta)$$

is the inclusion. The following conditions are equivalent:

(1) *U is a diagonal cohomology class for X*;

(2) *U' is a Thom class for X*;

(3) $U/[X] = 1$, *where / is the slant product* [**14**; p. 287].

Proof. (1)$\Leftrightarrow$(3). By [**14**; 6.1.6], $(U/[X]) \frown [X] = (p_1)_*(U \frown [X \times X])$, where $p_1 : X \times X \to X$ is projection onto the first factor. Thus $U \frown [X \times X] = d_*[X]$ implies $U/[X] = 1$. The converse follows from the fact that $U \frown [X \times X]$ is in the image of d_*.

To see this, let N be a neighbourhood of Δ in $X \times X$ such that Δ is a deformation retract of N. (For example, let N be the stellar neighbourhood of Δ in a sufficiently fine triangulation of $(X \times X, \Delta)$.) Let $i : (N, N - \Delta) \to (X \times X, X \times X - \Delta)$, and $i' : N \to X \times X$ be the inclusion maps. Now

$$i_* : H_*(N, N - \Delta) \to H_*(X \times X, X \times X - \Delta)$$

is an isomorphism, by excision. Let $[X \times X]' = (i_*)^{-1} j_*[X \times X]$. Then, by [**14**; 5.6.16],

$$\begin{aligned} U \frown [X \times X] = j^*U' \frown [X \times X] &= U' \frown j_*[X \times X] \\ &= U' \frown i_*[X \times X]' = i_*'(i^*U' \frown [X \times X]'). \end{aligned}$$

So $U \frown [X \times X] \in \text{Image}\,(i_*') = \text{Image}\,(d_*)$.

(2)$\Leftrightarrow$(3). If $x \in X$, let $f_x : X \to X \times X$ be the map $f_x(y) = (x, y)$. Let $\iota \in H_0(X)$ be represented by the 0-cycle $\{x\}$. Then

$$\langle U/[X], \iota \rangle = \langle U, \iota \times [X] \rangle = \langle U, (f_x)_*[X] \rangle = \langle (f_x)^*U, [X] \rangle,$$

so $U/[X] = 1$ if and only if $\langle (f_x)^*U, [X] \rangle = 1$ for all $x \in X$. Since $(f_x)^*U$ is the restriction of U to $\{x\} \times X$, the conclusion follows.

THEOREM 2. *The n-circuit X is a homology manifold if and only if X has a Thom class.*

Lemma 1 implies that Theorem 2 is equivalent to Theorem 1, since the restriction of U to $X \times X - \Delta$ is zero if and only if there exists a class U' with $j^*U' = U$.

I have already shown that if the n-circuit X is a homology manifold, then X has a Thom class. There are three steps in the proof of the converse. (a) If X has a Thom class, then $\mathscr{D}_X$ is an isomorphism. For field coefficients, this is due to Milnor [**13**; §11]. (b) If $\mathscr{D}_X$ is an isomorphism, then X cannot have isolated homological singularities. This was apparently first observed (for "locally irreducible" spaces) by L. Kaup [**9**; Kor. 4.3]. (c) If X has a Thom class, then a "generic hyperplane section" of X also has a Thom class. This was suggested to me by Sullivan. Therefore, if X could have a Thom class without being a homology manifold, using (c) several times would produce a subspace Y of X with isolated singularities and a Thom class. Then $\mathscr{D}_Y$ would be an isomorphism by (a), which would contradict (b).

The following lemmas justify the principles (a), (b) and (c).

LEMMA 2. *If the n-circuit X has a Thom class, then $\mathscr{D}_X$ is an isomorphism.*

Proof. I will show that $U/\cdot : H_*(X) \to H^*(X)$ is the inverse of $\mathscr{D}_X = \cdot \frown [X]$ (up to sign).

Let $T : X \times X \to X \times X$ be the involution $T(x, y) = (y, x)$. Then

$$T^*U \frown [X \times X] = T_*(U \frown T_*[X \times X]) = T_*(U \frown (-1)^n [X \times X]) = (-1)^n d_*[X];$$

so $(-1)^n T^*U$ is a diagonal class.

Let N be a neighbourhood of Δ in $X \times X$ as in the proof of Lemma 1. Let $\alpha \in H_k(X \times X)$, $k \geqslant n$. By the proof of Lemma 1, $U \frown \alpha = i_*'(i^*U' \frown \alpha')$, where $\alpha' = (i_*)^{-1} j_*\alpha$. Now let p_2 be the projection of $X \times X$ to the second factor. As $p_1|\Delta = p_2|\Delta$, and Δ is a deformation retract of N, it follows that $p_1 \circ i'$ is homotopic with $p_2 \circ i'$. Therefore,

$$(p_1)_*(U \frown \alpha) = (p_1)_* i_*'(i^*U' \frown \alpha') = (p_2)_* i_*'(i^*U' \frown \alpha') = (p_2)_*(U \frown \alpha).$$

So if $a \in H_i(X)$, then

$$\begin{aligned} (U/a) \frown [X] &= (p_1)_*(U \frown ([X] \times a)) && [\mathbf{14}; 6.1.6] \\ &= (p_2)_*(U \frown ([X] \times a)) \\ &= (p_1)_* T_*(U \frown ([X] \times a)) \\ &= (-1)^{ni}(p_1)_*(T^*U \frown (a \times [X])) \\ &= (-1)^{ni}(T^*U/[X]) \frown a && [\mathbf{14}; 6.1.6] \\ &= (-1)^{n+ni} a. \end{aligned}$$

Similarly, an argument of Milnor [**13**; 11.8] shows that if $b \in H^i(X)$, then $U \smile (1 \times b) = U \smile (b \times 1)$; so

$$\begin{aligned} U/(b \frown [X]) &= (U \smile (1 \times b))/[X] && [\mathbf{14}; 6.1.5] \\ &= (U \smile (b \times 1))/[X] \\ &= (-1)^{ni}((b \times 1) \smile U)/[X] \\ &= (-1)^{ni} b \smile (U/[X]) && [\mathbf{14}; 6.1.4] \\ &= (-1)^{ni} b. \end{aligned}$$

Remark. The second half of the proof can be omitted by appealing to Lemma 8 below.

LEMMA 3. *Let X be an n-circuit with* $\dim(SX) \leqslant 0$. *If $\mathscr{D}_X$ is an isomorphism, then SX is empty.*

Proof. The set SX must be finite, since it is a subcomplex of any triangulation of X. Suppose that $SX \subset \{x_1, \ldots, x_v\}$. For each j, let N_j be the open star of x_j, in a triangulation fine enough so that $\bar{N}_i \cap \bar{N}_j = \varnothing$ for $i \neq j$. Let $N = N_1 \cup \ldots \cup N_v$, $M = X - N$, $L = M \cap \bar{N}$, and $L_j = M \cap \bar{N}_j$, the link of x_j. Now M is a homology n-manifold with boundary L (that is, $M - L$ is a homology n-manifold and L is collared in M). Thus L is a homology $(n-1)$-manifold. I will show that if $\mathscr{D}_X$ is an isomorphism, L_j has the homology of an $(n-1)$-sphere for each j, so SX is in fact empty.

For each integer i, there is a commutative diagram

$$\begin{tikzcd}
 & H_i(M, L) & H_i(X) \arrow[l, "g_i"'] \\
H_i(M) \arrow[ur, "f_i"] & & \\
 & H^{n-i}(M, L) \arrow[ul, "\mathscr{D}_M", "\cong"'] \arrow[r, "h_i"] & H^{n-i}(X) \arrow[uu, "\mathscr{D}_X"', "\cong"]
\end{tikzcd}$$

where $\mathscr{D}_M$ is the Poincaré–Lefschetz duality isomorphism for M. The map f_i is induced by inclusion, g_i is the composition $H_i(X) \to H_i(X, SX) \cong H_i(M, L)$, and h_i is the composition $H^{n-i}(M, L) \cong H^{n-i}(X, SX) \to H^{n-i}(X)$. The map g_i is an isomorphism for $i > 1$, and a monomorphism for $i = 1$. The map h_i is an isomorphism for $i < n-1$ and an epimorphism for $i = n-1$. Thus, since $\mathscr{D}_X$ is an isomorphism, f_i is an isomorphism for $1 < i < n-1$, and if $n > 2$, f_i is a monomorphism for $i = 1$ and an epimorphism for $i = n-1$. Therefore, the long exact sequence of the pair (M, L) implies that $H_i(L) = 0$ for $0 < i < n-1$ (if $n > 2$).

Finally, I shall show that L_j is connected for $j = 1, \ldots, v$. Therefore L_j is a homology $(n-1)$-sphere, and $H_i(X)_{x_j} \cong H_i(R^n)_0$ for $j = 1, \ldots, v$ and all i.

An n-circuit Y is irreducible if and only if $H_n(Y)$ has rank one. As the map $g_n : H_n(X) \to H_n(M, L)$ is an isomorphism, it follows that the inclusion of M in X induces a bijection between the components of M and the irreducible n-circuits contained in X. But since $\mathscr{D}_X : H^0(X) \to H_n(X)$ is an isomorphism, each component of X is irreducible, so each component of X contains a unique component of M.

Now let x and y be the points in L_j. Since x and y are in the same component of X, they are in the same component of M. Combining an arc from x to y in M with the cone from x_j on $\{x, y\}$ yields a 1-cycle c in X. Since h_1 (defined above) is onto, $H_1(M) \to H_1(X)$ is onto, so c is homologous to a cycle in M. Consequently, x and y lie in the same component of L_j. (There is a similar geometric proof that $H_i(L) = 0$ for $0 < i < n-1$, using "geometric cocycles"; *cf.* §5 below.)

Y is a *stable* subspace of X if there is a neighbourhood U of Y in X such that the pair (U, Y) is homeomorphic with $(Y \times R^k, Y \times \{0\})$ for some k.

LEMMA 4. *If Y is a stable codimension k subspace of the n-circuit X, then Y is an $(n-k)$-circuit, with $SY = Y \cap SX$. If X has a Thom class, so does Y.*

Proof. Since Y is a stable subspace of X, we have

$$H_i(Y, Y-\{y\}) = H_{i+k}(X, X-\{y\})$$

for all $y \in Y$. Therefore, as X is purely n-dimensional, it follows that Y is purely $(n-k)$-dimensional. As $\dim(SX) \leqslant n-2$, it follows that $\dim(SY) \leqslant n-k-2$. Clearly $SY = Y \cap SX$. Let N be a closed neighbourhood of Y in X so that (N, Y) is homeomorphic with $(Y \times D^k, Y \times \{0\})$, and let B be the frontier of N in X. Let $[Y]$ be the image of $[X]$ under the composition

$$H_n(X) \to H_n(X, X-N) \cong H_n(N, B) \cong H_{n-k}(Y).$$

Then $[Y]$ is an orientation for Y.

Now consider the following diagram:

$$\begin{tikzcd}
U_X' \in H^n(X \times X, X \times X - \Delta_X) \arrow[r] \arrow[d] & H^n(\{y\} \times (X, X-\{y\})) \arrow[d] \\
H^n(Y \times N, Y \times N - \Delta_Y) \arrow[r] & H^n(\{y\} \times (N, N-\{y\})) \\
U_Y' \in H^{n-k}(Y \times Y, Y \times Y - \Delta_Y) \arrow[r] \arrow[u, "f", "\wr"'] & H^{n-k}(\{y\} \times (Y, Y-\{y\})) \arrow[u, "g", "\wr"']
\end{tikzcd}$$

The map f is the composition

$$\begin{aligned}
&H^{n-k}(Y \times Y, Y \times Y - \Delta_Y) \\
&\qquad \cong H^n\Big(Y \times Y \times D^k, \big(Y \times Y \times (D^k - \{0\})\big) \cup \big((Y \times Y - \Delta_Y) \times D^k\big)\Big) \\
&\qquad \cong H^n(Y \times N, Y \times N - \Delta_Y).
\end{aligned}$$

The map g is the suspension isomorphism, and all the other arrows are restriction maps. Let U_Y' be the image of the Thom class U_X' of X. Since $[X]_y$ corresponds to $[Y]_y$ under the isomorphism $H_n(X, X-\{y\}) \cong H_{n-k}(Y, Y-\{y\})$, we have

$$\langle (U_Y')_y, [Y]_y \rangle = \langle (U_X')_y, [X]_y \rangle = 1$$

for all $y \in Y - SY$.

LEMMA 5. *Let X be an n-circuit such that SX is non-empty. There is a stable subspace Y of X such that $Y \cap SX$ is a non-empty set of isolated points.*

Proof. Let $k = \dim(SX)$. If $k = 0$, let $Y = X$. If $k > 0$, triangulate X, and let $f: X \to R^N$ be an embedding which is linear on each simplex of X. Let Q be a k-plane in R^N so that if $\pi: R^N \to Q$ is orthogonal projection, the map πf collapses no k-simplexes of X. Subdivide the triangulation of X so that πf is simplicial, and choose $q \in Q$ in the interior of $\pi f(\sigma)$ for some k-simplex σ of SX. (Recall that SX is a subcomplex of any triangulation of X.) Let $Y = (\pi f)^{-1}(q)$. Then $Y \cap SX$ is finite and non-empty by construction, and Y is a stable subspace of X by [**13**; 20.5]. Note that $f(Y)$ is the intersection of $f(X)$ with the $(n-k)$-plane through q orthogonal to Q.

Remarks. Theorems 1 and 2 can be generalized in several ways. The conditions of orientability (and compactness) can be dropped in the definition of an n-circuit, in which case Theorem 2 is true with twisted coefficients. If n-circuits and homology manifolds are defined with coefficients in an arbitrary commutative ring with unit, Theorems 1 and 2 still hold, with the same proofs.

2. *Symmetry of diagonal classes*

This section is an elaboration of the proof of Lemma 2. A *Poincaré duality space* of formal dimension n is a pair $(X, [X])$, where X is a space (not necessarily triangulable) and $[X] \in H_n(X)$, such that $\cdot \frown [X] : H^*(X) \to H_*(X)$ is an isomorphism [**1**]. For simplicity, I assume that X has the homotopy type of a finite CW complex.

I shall use the terminology of §1 (orientation class, diagonal cohomology class, etc.) in this more abstract setting.

LEMMA 6. *A Poincaré duality space has a unique diagonal cohomology class.*

Proof. This follows from the fact that if $(X, [X])$ is a Poincaré duality space, then $(X \times X, [X] \times [X])$ is too. For coefficients in a field, if $\cdot \frown [X]$ is an isomorphism then $\cdot \frown ([X] \times [X])$ is an isomorphism by the Künneth formula and the compatibility of cap and cross products [**14**; 5.7.21]. But if $\cdot \frown ([X] \times [X])$ is an isomorphism with coefficients in the rationals and the integers modulo any prime, then $\cdot \frown ([X] \times [X])$ is an isomorphism with integer coefficients.

It is easy to construct an n-circuit with a unique diagonal cohomology class which is not a Poincaré duality space.

LEMMA 7. *If U is the diagonal cohomology class of a Poincaré duality space, then $T^*U = (-1)^n U$.*

Proof. By the proof of Lemma 2, $(-1)^n T^*U$ is a diagonal cohomology class for X.

If X is an n-circuit with a diagonal cohomology class U such that $T^*U = (-1)^n U$, then X is not necessarily a Poincaré duality space. (See example 1.) However, the stronger symmetry property of U used in the proof of Lemma 2 does characterize duality spaces.

PROPOSITION 1. *Let X be a space, and let $[X] \in H_n(X)$, and $U \in H^n(X \times X)$ for some n, with $U \frown ([X] \times [X]) = d_*[X]$. The following conditions are equivalent:*

(1) *$(X, [X])$ is a Poincaré duality space;*

(2) *$U \frown T_*\alpha = (-1)^n U \frown \alpha$ for all $\alpha \in H_*(X \times X)$;*

(3) *$U \smile T^*\beta = U \smile \beta$ for all $\beta \in H^*(X \times X)$.*

LEMMA 8. *Let C be a chain complex of free abelian groups, and let $\hat{C} = \mathrm{Hom}\,(C, \mathbb{Z})$ be the dual cochain complex* [**14**; p. 234]. *Let $D : \hat{C} \to C$ be a chain map such that $D : \hat{C}^i \to C_{n-i}$. Suppose that C is of finite type. Then if the induced homology morphism $H(D) : H(\hat{C}) \to H(C)$ is an epimorphism, it is an isomorphism.*

Proof. This is an easy consequence of the universal coefficient theorem [**14**; 5.5.3]. The details are left to the reader.

COROLLARY. *Let $[X] \in H_n(X)$. If $\cdot \frown [X]$ is an epimorphism, then $(X, [X])$ is a Poincaré duality space.*

However, $\cdot \frown [X]$ may be a monomorphism but not an isomorphism. (See Example 2.)

Proof of Proposition 1. (2) $\Rightarrow$ (1). (2) implies that $(U/a) \frown [X] = (-1)^{n+ni} a$ for all $a \in H_i(X)$, by the proof of Lemma 2. Thus $\cdot \frown [X]$ is an epimorphism, so $(X, [X])$ is a duality space by Lemma 8.

(3) $\Rightarrow$ (1). (3) implies that $U/(b \frown [X]) = (-1)^{ni} b$ for all $b \in H^i(X)$, by the proof of Lemma 2. Thus $U/\cdot$ is an epimorphism, so it is an isomorphism by Lemma 8. Therefore $(X, [X])$ is a duality space.

(1) $\Rightarrow$ (3). Let $\beta \in H^*(X \times X)$, and let $[X \times X] = [X] \times [X]$. A short calculation using [**14**; 5.6.18, 5.6.16] shows that $(T^*\beta \smile U) \frown [X \times X] = (\beta \smile U) \frown [X \times X]$, so $T^*\beta \smile U = \beta \smile U$, since $\cdot \frown [X \times X]$ is an isomorphism.

(1) and (3) $\Rightarrow$ (2). Given $\alpha \in H_*(X \times X)$, let $\beta \in H^*(X \times X)$ be dual to α; that is, $\beta \frown [X \times X] = \alpha$. A calculation using [**14**; 5.6.18, 5.6.16] shows that

$$U \frown \alpha = (U \smile T^*\beta) \frown [X \times X] = (-1)^n U \frown T_*\alpha.$$

3. *Existence of diagonal classes*

In this section, all coefficients will be in a fixed *field* F. I again assume that X has the homotopy type of a finite CW complex.

PROPOSITION 2. *Let X be a space, and $a \in H_n(X)$. There exists $u \in H^n(X \times X)$ such that $u \frown (a \times a) = d_*(a)$.*

COROLLARY. *Any n-circuit has a diagonal cohomology class (field coefficients).*

The corollary is not true with integer coefficients. (See Example 2.)

Let $a_1, \ldots, a_r$ be a basis for the F vector space $H_*(X)$, and let $b_1, \ldots, b_r$ be the dual basis for $H^*(X)$; that is, $\langle b_i, a_i \rangle = \delta_{ij}$.

LEMMA 9. *If $a \in H_n(X)$, then $d_*(a) = \sum_{i,j} \langle b_i \smile b_j, a \rangle\, a_i \times a_j = \sum_j (b_j \frown a) \times a_j$.*

Proof. If $d_*(a) = \sum b_{ij}\, a_i \times a_j$, then

$$b_{ij} = \langle b_i \times b_j, d_*(a) \rangle = \langle d^*(b_i \times b_j), a \rangle = \langle b_i \smile b_j, a \rangle.$$

Furthermore, $\langle b_i \smile b_j, a \rangle = \langle b_i, b_j \frown a \rangle$, so $\sum b_{ij}\, a_i \times a_j = \sum_j (b_j \frown a) \times a_j$.

If $a = [X]$, the orientation class of the compact homology manifold X, this lemma says that the coefficient matrix of the diagonal homology class in $X \times X$ is the intersection matrix of X (note that these two matrices are with respect to *dual* bases of $H_*(X)$), or equivalently that $d_*[X] = \sum \hat{a}_j \times a_j$, where $\{\hat{a}_j\}$ is the basis for $H_*(X)$ Poincaré dual to $\{a_j\}$; that is, $\hat{a}_i \cdot a_j = \delta_{ij}$, where $\cdot$ is the intersection product. This is a key step in the proof of the Lefschetz fixed point theorem [**15**].

Proof of Proposition 2. Choose a homogeneous basis $\{a_i\}$ for the graded vector space $H_*(X)$ so that $\{a_1, \ldots, a_s\}$ is a basis for the image of the morphism $\cdot \frown a$. Let b_i be the dual basis of $H^*(X)$. If $i > s$, then $b_i \frown a = 0$; for if $b \in H^k(X)$, then $\langle b, b_i \frown a \rangle = \langle b \smile b_i, a \rangle = \pm \langle b_i \smile b, a \rangle = \pm \langle b_i, b \frown a \rangle$. Now for each $i < s$, choose a homogeneous element $c_i \in H^*(X)$ with $c_i \frown a = a_i$. Let

$$u = \sum_{i \leqslant s} (-1)^{\deg a_i}\, b_i \times c_i.$$

Then, by [**14**; 5.6.21],

$$u \frown (a \times a) = \sum_{i \leq s} (b_i \frown a) \times (c_i \frown a) = \sum_{i \leq s} (b_i \frown a) \times a_i = \sum_i (b_i \frown a) \times a_i = d_*(a).$$

As a corollary of the proof, if $u \frown (a \times a) = d_*(a)$, then $\langle u \smile u, a \times a \rangle = \langle u, d_*(a) \rangle$ is the Euler characteristic of the image of $\cdot \frown a$. (This is true for the class u chosen above, and it is easy to check that $\langle u, d_*(a) \rangle$ does not depend on the choice of u.)

A different proof is based on the calculation that if $u = \sum_{i,j} c_{ij} b_i \times b_j$, then $u \frown (a \times a) = d_*(a)$ if and only if the matrix $C = \left((-1)^{n(\deg a_i)} c_{ij}\right)$ is a quasi-inverse for the matrix $B = (b_{ij})$; that is, $BCB = B$. It follows that if integer coefficients are used, and $H_*(X)$ has no torsion, then u exists if and only if the cokernel of $\cdot \frown a$ is torsion-free. (Compare Example 2.)

4. *Intersection pairings*

In this section, all coefficients will be in fixed *field F*. By an *intersection pairing*

$$\mathcal{I} : H_i(X) \times H_j(X) \to H_{i+j-n}(X),$$

I shall mean simply a bilinear pairing defined for all i and j.

PROPOSITION 3. *The space $(X, [X])$ is a Poincaré duality space over a field F if and only if there exists an intersection pairing $\mathcal{I}$ on the homology of X which is compatible with cap product; that is, if $a \in H_*(X)$ and $b \in H^*(X)$,*

$$\mathcal{I}(b \frown [X], a) = b \frown a.$$

Proof. If X is a Poincaré duality space, set $\mathcal{I}(a_1, a_2) = (b_1 \smile b_2) \frown [X]$, where $b_i \frown [X] = a_i$. Then $\mathcal{I}(b_1 \frown [X], a_2) = (b_1 \smile b_2) \frown [X] = b_1 \frown (b_2 \frown [X]) = b_1 \frown a_2$. Conversely, if such a pairing $\mathcal{I}$ exists, and $b \in H^i(X)$, then $b \frown [X] = 0$ implies that $\langle b, a \rangle = \varepsilon(b \frown a) = \varepsilon \mathcal{I}(b \frown [X], a) = 0$ for all $a \in H_i(X)$, so $b = 0$. (Here ε is the augmentation.) Thus $\cdot \frown [X] : H^i(X) \to H_{n-i}(X)$ is a monomorphism for all i, so it is an isomorphism, since $H^i(X) \cong H_i(X)$.

The pairing $\mathcal{I}$ on a Poincaré duality space X is related to the diagonal cohomology class U by the formula

$$\mathcal{I}(a_1, a_2) = p_*(U \frown (a_1 \times a_2)),$$

where p is projection to the first factor. For example, if X is a homology manifold, the intersection pairing $\cdot$ satisfies the classical formula of Lefschetz,

$$a_1 \cdot a_2 = p_*(d_*[X] \cdot (a_1 \times a_2)).$$

By the Künneth theorem, a bilinear pairing $\mathcal{I} : H_i(X) \times H_j(X) \to H_{i+j-n}(X)$ can be identified with a morphism $H_k(X \times X) \to H_{k-n}(X)$, $k = i+j$.

THEOREM 3. *The n-circuit X is an F-homology manifold if and only if there is an intersection pairing $\mathcal{I}$ on $H_*(X)$, compatible with cap product, such that $\mathcal{I}(\alpha) = 0$ for all classes $\alpha \in H_*(X \times X)$ which are represented by cycles in the complement of the diagonal.*

For example, if $a, b \in H_*(X)$ are represented by disjoint cycles in X, then $a \times b$ is represented by a cycle off the diagonal.

Proof. By Proposition 3, X is a Poincaré duality space. Let U be the diagonal cohomology class. If $\alpha \in H_n(X \times X)$, then $\langle U, \alpha\rangle = \varepsilon \mathscr{I}(\alpha)$. Thus $\mathscr{I}(\alpha) = 0$ for all $\alpha \in \operatorname{Im}[H_n(X \times X - \Delta) \to H_n(X \times X)]$ if and only if

$$U \in \operatorname{Ker}[H^n(X \times X) \to H^n(X \times X - \Delta)].$$

This is equivalent to X being a homology manifold, by Theorem 1.

Proposition 3 and Theorem 3 can be viewed as negative results about the existence of geometric intersection pairings on the *homology* of an n-circuit. But M. Goresky and R. MacPherson have recently defined an interesting "intersection homology theory", which contains ordinary homology and cohomology, and which admits a pairing extending the cap product pairing **[5]**.

5. *Examples*

Let X be a stratified n-circuit. (For example, a triangulation or a cellulation is a stratification.) The class $a \in H_i(X)$ is in the image of the duality map

$$\mathscr{D}_X : H^{n-i}(X) \to H_i(X)$$

if and only if a is represented by a cycle in X which is transverse to the strata **[12**; Theorem 5.2]. In fact, the cohomology of X can be defined as the group of transverse cycles (or *geometric cocyles*) in X, modulo transverse homologies. Cycles transverse to a piecewise-linear cellulation of a polyhedron are Buoncristiano, Rourke and Sanderson's *mockbundles* **[2]**. Cycles transverse to the strata of a Whitney stratified space are Goresky's *π-fibre subobjects* **[4]**.

In these terms, the corollary to Proposition 2 says that the diagonal Δ of an n-circuit is always homologous with a transverse cycle $\tilde{\Delta}$ (field coefficients). The proof shows the reason to be that the class of the diagonal can be expressed in terms of classes in the image of $\mathscr{D}_X$.

Theorem 1 says that $\tilde{\Delta}$ can be chosen to be arbitrarily close to Δ if and only if X is a homology manifold.

The following examples will illustrate these remarks. If X is an n-circuit, I shall write $[\Delta] = d_*[X] \in H_n(X \times X)$, suppressing the inclusion $\Delta \subset X \times X$.

Example 1. *The pinched torus.* Let X be the complex projective curve $x^3 + y^3 = xyz$ in homogeneous co-ordinates $[x, y, z]$. The space X is a 2-circuit with an isolated singular point $p = [0, 0, 1]$. It is homeomorphic with a torus with a meridian circle pinched to a point, or a sphere with two distinct points identified. The homology and cohomology groups of X are infinite cyclic in dimensions 0, 1 and 2. Let $\iota \in H_0(X)$, $a \in H_1(X)$, $[X] \in H_2(X)$, $1 \in H^0(X)$, $b \in H^1(X)$, $\mu \in H^2(X)$ be generators, with $\langle \iota, 1\rangle = 1$, $\langle b, a\rangle = 1$, and $\langle \mu, [X]\rangle = 1$. Then $\mathscr{D}_X(\mu) = \iota$, $\mathscr{D}_X(b) = 0$, and $\mathscr{D}_X(1) = [X]$. Thus $[\Delta] = (\iota \times [X]) + ([X] \times \iota)$, so X has a diagonal cohomology class $U = (\mu \times 1) + (1 \times \mu)$. (The class $(\mu \times 1) + (b \times b) + (1 \times \mu)$ is also diagonal.) Now $T^*U = U$, but $U \smile (1 \times b) \neq U \smile (b \times 1)$ (*cf.* Proposition 1).

The space $X \times X$ has four intrinsic strata, whose closures are $\{(p, p)\}$, $\{p\} \times X$, $X \times \{p\}$, and $X \times X$. The cycle Δ is homologous with the transverse cycle

$$\tilde{\Delta} = (\{x\} \times X) \cup (X \times \{x\}),$$

where $x \neq p$. Clearly, Δ is not a transverse cycle, since it contains (p, p).

Example 2. *The quadric cone.* Let X be the algebraic surface $x^2+y^2+z^2 = 0$ in complex projective 3-space with homogeneous co-ordinates $[x, y, z, w]$. The space X is a 4-circuit with an isolated singular point $p = [0, 0, 0, 1]$. It is homeomorphic with the Thom space of the tangent bundle of the 2-sphere. Both the homology and cohomology of X are infinite cyclic in dimensions 0, 2 and 4, and zero otherwise. If $a \in H_2(X)$ and $b \in H^2(X)$ are generators with $\langle b, a\rangle = 1$, then $\mathscr{D}_X(b) = 2a$. (The class b is represented by the zero section and a is represented by the fibre.) Thus

$$[\Delta] = (\imath \times [X]) + 2(a \times a) + ([X] \times \imath),$$

so X does not have a diagonal cohomology class U with integer coefficients (*cf.* the remark at the end of §3). But U does exist with coefficients in the rationals $\mathbb{Q}$ or in $\mathbb{Z}/k\mathbb{Z}$ for k odd.

The local homology of X at the singular point p is $H_0(X)_p = 0$, $H_1(X)_p = 0$, $H_2(X)_p = \mathbb{Z}/2\mathbb{Z}$, $H_3(X)_p = 0$, $H_4(X)_p = \mathbb{Z}$. Thus X is an F-homology manifold for $F = \mathbb{Q}$ or $F = \mathbb{Z}/k\mathbb{Z}$, k odd.

The quadric cone occurs as an example in Zeeman's thesis (1954, **[16]**). He displays the relation between the local homology groups of X and the duality map $\mathscr{D}_X$ using a wondrous spectral sequence, which is further investigated in **[11]**.

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