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Theorem.

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Homology Fibrations and the "Group-Completion" Theorem

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A topological monoid M has a classifying-space BM, which is a space with a base-point. There is a canonical map of H-spaces $M \to \Omega BM$ from M to the space of loops on BM, and it is a homotopy-equivalence if the monoid of connected components $\pi_0 M$ is a group. The "group-completion" theorem ([2-4, 6, 9]) describes the relationship between M and ΩBM in general. Let us regard $\pi = \pi_0 M$ as a multiplicative subset of the Pontrjagin ring $H_*(M)$, using singular integral homology. The map $M \to \Omega BM$ induces a homomorphism of Pontrjagin rings, and (because $\pi_0(\Omega BM)$ is a group) the image of π in $H_*(\Omega BM)$ consists of units.

Proposition 1. *If π is in the centre of $H_*(M)$ then*

$$H_*(M)[\pi^{-1}] \xrightarrow{\cong} H_*(\Omega BM).$$

Although several proofs of this theorem have appeared its importance for the process of "Quillenization"[1] perhaps justifies our publishing the present one, which is simple and conceptual. We shall prove, moreover, a stronger statement than Proposition 1 in the two respects described in Remarks 1 and 2 below. Our method was suggested by Quillen's second unpublished proof, and by conversations with him for which we are very grateful. The use of homology fibrations arose from [5]. We have listed some examples and applications of the theorem at the end.

Remark 1. In Proposition 1 one need not assume that π is in the centre of $H_*(M)$, but only that $H_*(M)[\pi^{-1}]$ can be constructed by right fractions. Recall that if π is a multiplicative subset of a ring A one says that $A[\pi^{-1}]$ can be constructed by right fractions if every element of it can be written $a p^{-1}$ with $a \in A$, $p \in \pi$, and if $a_1 p_1^{-1} = a_2 p_2^{-1}$ if and only if $a_1 p_1' = a_2 p_2'$ and $p_1 p_1' = p_2 p_2'$ for some $p_1', p_2' \in \pi$. A typical example is when π consists of the powers of an element $x \in A$ such that $ax = x\alpha(a)$ for all $a \in A$, where α is an endomorphism of A. This arises as the Pontrjagin ring of the monoid of all maps $S^n \to S^n$ whose degrees are powers of a prime p, as we shall see below.

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[1]   This word is due to I. M. Gel'fand.

We shall prove Proposition 1 by constructing a space M_∞ whose homology is obviously $H_*(M)[\pi^{-1}]$, and a homology equivalence $M_\infty \to \Omega BM$. The basic example is the case when $M = \coprod_{n \geq 0} B\Sigma_n$, where Σ_n is the nth symmetric group, and the monoid structure of M comes from juxtaposition $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. Then M_∞ will be $\mathbb{Z} \times B\Sigma_\infty$.

Remark 2. To say that a map $f: X \to Y$ is a homology equivalence may have at least two meanings. The weaker one is that f induces an isomorphism of integral homology. The stronger is that $f_*: H_*(X; f^*A) \xrightarrow{\cong} H_*(Y; A)$ for every coefficient system A of abelian groups on Y. The map $M_\infty \to \Omega BM$ we shall construct will be a homology equivalence in the stronger sense. Thus ΩBM, whose components have of course abelian fundamental groups, is a "Quillenization" of M_∞. The advantage of allowing twisted coefficient systems is that one can conclude that $\tilde{M}_\infty \to \widetilde{\Omega BM}$ is a homology equivalence as well as $M_\infty \to \Omega BM$, where $\widetilde{\Omega BM}$ is the universal covering space of ΩBM, and $\tilde{M}_\infty$ is its pull-back to M_∞. This means that the fundamental group of $\tilde{M}_\infty$ must be perfect, and so our method incorporates a general proof that the commutator subgroup of $\pi_1(M_\infty)$ is perfect. If isolated this would reduce to Wagoner's argument in [11].

Everything we say below is true if homology equivalence is given either of the above meanings. Nevertheless it will be convenient to adopt a middle definition, allowing only *abelian* coefficient systems A on Y, i.e. those such that for each $y \in Y$ the group of automorphisms of the coefficient group A_y at y induced by the action of $\pi_1(Y, y)$ is abelian. Of course any system coming from ΩBM is abelian.

Our main idea is that of a *homology fibration*. In [5] a homology fibration was defined as a map $p: E \to B$ such that for each $b \in B$ the natural map $p^{-1}(b) \to F(p, b)$ from the fibre at b to the homotopical fibre at b is a homology equivalence. ($F(p, b)$ is defined as the fibre-product $P_b \times_B E$, where P_b is the space of paths in B beginning at b.) In this language to obtain a homology equivalence $M_\infty \to \Omega BM$ it is enough to produce a homology fibration $E \to BM$ with E contractible and with fibre M_∞ at the base-point.

If M is a topological group which acts on a space X one often considers the space X_M fibred over BM with fibre X, associated to the universal bundle $EM \to BM$. But the construction of X_M makes sense even if M is only a topological monoid, for X_M can be described as the realization of the topological category whose space of objects is X and whose space of morphisms is $M \times X$, a pair (m, x) being thought of as a morphism from x to mx. (Here, and in constructing BM also, we use the "thick" realization of simplicial spaces, denoted by $\| \ \|$ in the appendix to [9].)

Our main result is

Proposition 2. *If M is a topological monoid which acts on a space X, and for each $m \in M$ the map $x \mapsto mx$ from X to itself is a homology equivalence, then $X_M \to BM$ is a homology fibration with fibre X.*

This should be compared with the fact that if $x \mapsto xm$ is a homotopy equivalence for each m then $X_M \to BM$ is a quasifibration. (When M is discrete this is a particular case of [7] (Lemma p. 98); in general it is a particular case of [9] (1.5).)

Notice that in the basic example the left action of $M = \coprod_{n \geq 0} B\Sigma_n$ on $M_\infty = \mathbb{Z} \times B\Sigma_\infty$ is essentially the "shift" maps $B\Sigma_\infty \to B\Sigma_\infty$ induced by embedding Σ_∞ in Σ_∞ as the permutations of $\{n, n+1, \ldots\}$. These are homology equivalences but not homotopy equivalences, even though they induce the identity on $[K; B\Sigma_\infty]$ for any compact space K. They would not be homology equivalences if we had allowed non-abelian coefficient systems.

To see how the group completion theorem follows from Proposition 2 let us begin with the case when $\pi_0 M$ is the natural numbers $\mathbb{N}$. Choose $m \in M$ in the component $1 \in \mathbb{N}$, and let X be the telescope M_∞ formed from the sequence $M \to M \to M \to \cdots$, where each map is right multiplication by m. The homology of M_∞ is the direct limit of

$$H_*(M) \to H_*(M) \to H_*(M) \to \cdots,$$

which is precisely $H_*(M)[\pi^{-1}]$ because we have assumed the latter can be formed by right fractions. For the same reason the action of M on M_∞ on the left is by homology equivalences. The space $(M_\infty)_M$ is the telescope of a sequence of copies of M_M, which is canonically contractible. (It is the standard EM of [8].) So $(M_\infty)_M$ is contractible, and the homotopical fibre of $(M_\infty)_M \to BM$ is ΩBM, and Proposition 2 yields Proposition 1.

The general case of Proposition 1 reduces at once to that where $\pi_0 M$ is finitely generated, for both $H_*(M)[\pi^{-1}]$ and $H_*(\Omega BM)$ are continuous under direct limits. But if $\{s_1, \ldots, s_k\}$ generate π then $H_*(M)[\pi^{-1}] = H_*(M)[s^{-1}]$, where $s = s_1 s_2 \ldots s_k$, and the preceding argument applies, defining M_∞ as the telescope generated by multiplication by any element m in the component s.

We come to the proof of Proposition 2. For technical convenience we shall adopt a stronger definition of homology-fibration than that of [5]. It is appropriate only for base-spaces B which are locally contractible in the sense that each point has arbitrarily small contractible neighbourhoods. But if M has this property then BM has; and restricting to such M is immaterial for our purposes, as both $H_*(M)$ and $H_*(\Omega BM)$ are unchanged if M is replaced by the realization of its singular complex.

Definition. A map $p: E \to B$ is a *homology-fibration* if each $b \in B$ has arbitrarily small contractible neighbourhoods U such that the inclusion $p^{-1}(b') \to p^{-1}(U)$ is a homology-equivalence for each b' in U.

To justify this definition we must show that such a map is a homology-fibration in the earlier sense. This will be done in Proposition 5 below.

The advantage of the new definition is that it makes the following proposition obvious. (Cf. [5] (5.2).)

Proposition 3. *If*

$$
\begin{array}{ccccc}
E_1 & \longleftarrow & E_0 & \longrightarrow & E_2 \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_2} \\
B_1 & \xleftarrow{\ f_1\ } & B_0 & \xrightarrow{\ f_2\ } & B_2
\end{array}
$$

is a commutative diagram in which p_0, p_1, p_2 *are homology-fibrations, and* $p_0^{-1}(b) \to$ $p_i^{-1}(f_i(b))$ *is a homology-equivalence for each* $b \in B_0$, *then the induced map of double-mapping-cylinders*

$$p: \ \mathrm{cyl}(E_1 \leftarrow E_0 \to E_2) \to \mathrm{cyl}(B_1 \leftarrow B_0 \to B_2)$$

is a homology-fibration.

Proof. Each point of the lower cylinder has arbitrarily small neighbourhoods U in the form of mapping-cylinders of maps $V_0 \to V_i$ ($i = 0, 1$ or 2), and $p^{-1}(U)$ is the mapping-cylinder of $p_0^{-1}(V_0) \to p_i^{-1}(V_i)$.

Exactly as in [9] (1.6) one deduces

Proposition 4. *If* $p: E \to B$ *is a map of simplicial spaces such that* $E_k \to B_k$ *is a homology-fibration for each* $k \geq 0$, *and for each simplicial operation* $\theta: [k] \to [l]$ *and each* $b \in B_l$ *the map* $p^{-1}(b) \to p^{-1}(\theta^* b)$ *is a homology-equivalence, then the map of realizations* $\|E\| \to \|B\|$ *is a homology-fibration.*

Proof. This follows from Proposition 3 because the realizations $\|E\|$ and $\|B\|$ can be made up skeleton by skeleton, and $\|B\|_{(k)}$ is the double-mapping-cylinder of ($\|B\|_{(k-1)} \leftarrow \Delta^k \times B_k \to \Delta^k \times B_k$), and so on.

Proposition 2 is a particular case of Proposition 4, for X_M and BM are the realizations of simplicial spaces E and B such that $E_k = X \times B_k$ and $B_k = M^k$.

To conclude we need the following justifying proposition.

Proposition 5. *If* B *is a paracompact locally contractible space, and* $p: E \to B$ *is a homology-fibration, then* $p^{-1}(b) \to F(p, b)$ *is a homology-equivalence for each* $b \in B$.

Proof. Let P be the space of paths in B beginning at b, and let $f: P \to B$ be the end-point map, a Hurewicz fibration. Then $f^* E$ is $F(p, b)$. Choose a basis $\mathscr{B}$ for the topology of B consisting of contractible sets. Then there is a basis $\mathscr{B}^*$ for the topology of P consisting of contractible sets U such that $f(U) \in \mathscr{B}$ and $f: U \to f(U)$ is a Hurewicz fibration. $\mathscr{B}^*$ consists of sets $P(t_1, \ldots, t_k; U_1, \ldots, U_k; V_1, \ldots, V_k)$, where $0 = t_0 < t_1 < \cdots < t_k = 1$, and $U_1 \supset V_1 \subset U_2 \supset V_2 \subset \cdots \subset U_k \supset V_k$ belong to $\mathscr{B}$; a path α belongs to this set if $\alpha(t_i) \in V_i$ and $\alpha([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \ldots, k$. Because $f: U \to f(U)$ is both a homotopy-equivalence and a Hurewicz fibration when $U \in \mathscr{B}^*$, the pull-back $f^* E|U$ is homotopy-equivalent to $E|f(U)$. Thus $f^* E \to P$ is a homology-fibration in our sense, and Proposition 5 follows from the particular case:

Proposition 6. *If* $p: E \to B$ *is a homology-fibration (with* B *paracompact and locally contractible), and* B *is contractible, then* $p^{-1}(b) \to E$ *is a homology-equivalence for each* $b \in B$.

Proof. Let $\mathscr{B}$ be a basis for B consisting of contractible sets U such that $p^{-1}(b) \to p^{-1}(U)$ is a homology equivalence for each $b \in U$. There is a Leray spectral sequence for the covering of E by the $p^{-1}(U)$. One obtains it as in [8] by forming a space $E_{\mathscr{B}}$ homotopy-equivalent to E which maps to the nerve $|\mathscr{B}|$ so that above a point of the open simplex $[U_0 \subset U_1 \subset \cdots \subset U_p]$ of the nerve one has $p^{-1}(U_0)$.

The spectral sequence comes from the filtration of $E_\mathscr{B}$ by the inverse-images of the skeletons of $|\mathscr{B}|$. It is $H_p(|\mathscr{B}|; \mathscr{H}_q) \Rightarrow H_*(E)$, where $\mathscr{H}_q$ is the local coefficient system $U \mapsto H_q(p^{-1}(U))$ on $\mathscr{B}$. But $|\mathscr{B}|$ is homotopy-equivalent to B, which is contractible, so $H_0(|\mathscr{B}|; \mathscr{H}_q) \cong H_q(E)$, as we want.

Examples. (i) If M is a discrete monoid whose enveloping group is G, and G can be constructed from M as the set of formal fractions $m_1 m_2^{-1}$ with m_1 and m_2 in M, then Proposition 2 implies that $BM \simeq BG$.

(ii) The case $M = \coprod_{n \geq 0} B\Sigma_n$, where Σ_n is the nth symmetric group, has already been mentioned. It is closely related to the basic example of algebraic K-theory, where $M = \coprod_P B \operatorname{Aut}(P)$, and P runs through the finitely generated projective modules over a fixed discrete ring A, and the composition law in M comes from the direct sum of modules. Then M_∞ can be taken to be $K_0(A) \times BGL_\infty(A)$, as one can form the telescope $M \to M \to M \to \cdots$ by successively adding the free A-module on one generator. As with Σ_∞ the shifts $GL_\infty(A) \to GL_\infty(A)$ induce homology isomorphisms because they are conjugate to the identity on each $GL_n(A)$.

(iii) If $M = \coprod_{k \geq 0} G_n(p^k)$, where $G_n(p^k)$ is the space of maps $S^{n-1} \to S^{n-1}$ of degree p^k (for some prime p), and the composition is composition of maps, then one has an example where π is not in the centre of $H_*(M)$. Each component of M is the telescope of

$$G_n(1) \to G_n(p) \to G_n(p^2) \to \cdots,$$

where the maps are composition on the left with a standard map of degree p. This telescope is the same up to homotopy as one component of the space of maps from S^{n-1} to the telescope $S^{n-1} \to S^{n-1} \to S^{n-1} \to \cdots$ whose maps have degree p, i.e. as one component of $\operatorname{Map}(S^{n-1}; S^{n-1}[p^{-1}])$, where $S^{n-1}[p^{-1}]$ is S^{n-1} localized away from p. Comparing homotopy groups one finds that M_∞ can be identified with $\mathbb{Z} \times G_n(1)[p^{-1}]$. The right-hand action of M on M_∞ is by homotopy equivalences, so the homology fibration of Proposition 2 is actually a quasifibration, and $M_\infty \simeq \Omega BM$. Thus enlarging the monoid of homotopy equivalences of S^{n-1} to the monoid of maps of degree p^k has the effect of localizing the classifying space, a result essentially equivalent to the "mod p Dold theorem" of Adams [1].

In this example because the right-hand action of M on M_∞ is by homotopy equivalences $H_*(M)[\pi^{-1}]$ can be formed by left fractions. But it cannot be formed by right fractions. For example $G_2(p^k)$ is homotopically a circle, and composition on the right with a map of degree p is a homotopy equivalence $G_2(p^k) \to G_2(p^{k+1})$, and the telescope formed from it is not local for the left action.

(iv) A closely related example is $M = \coprod_{k \geq 0} B\Sigma_{p^k}$, where composition comes from the cartesian product of permutations. Then $M_\infty \simeq \mathbb{Z} \times B\Pi$, where $\Pi = \varinjlim \Sigma_{p^k}$ is the group of periodic permutations of $\mathbb{Z}$ whose period is a power of p. But ΩBM is $\mathbb{Z} \times Q[p^{-1}]$, where Q is one component of $\Omega^\infty S^\infty$. This follows from the Barratt-Priddy-Quillen homology isomorphism $B\Sigma_\infty \to Q$; for $B\Sigma_{p^k}$ has the homology of Q up to a dimension tending to infinity with k, and in the telescope

defining M_∞ the map $B\Sigma_{p^k} \to B\Sigma_{p^{k+1}}$ corresponds to multiplying by p in the H-space structure of Q.

Examples (iii) and (iv) have been studied by Tornehave and Snaith in works to appear.

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