Simplicial structures and transverse cellularity*

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1. Introduction

To every simplicial mapping $f: K \to L$, we associate two structures, the simplicial mapping cylinder C_f, and the dual structure induced on K by f. We investigate these structures and the relationship between them. One outcome of this investigation is that we can give an answer to the question which first prompted it: *Which simplicial mappings of combinatorial manifolds preserve piecewise linear structure*? (i.e., *when is* $f(M) \simeq M$?). The answer is that the *transversely cellular* mappings do this. In a sense to be described, this is the best possible result.

The simplicial mapping cylinder C_f is defined (§ 4) as the top half of Whitehead's cylinder. It is a subcomplex of the join of K' and L, where K' denotes a first barycentric subdivision of K.

The dual structure induced on K by f is introduced and the basic properties of the duals are derived in § 5. If α is a simplex of L, then $D(\alpha, f)$, *the dual to* α *with respect to* f, is a subcomplex of K' which is most easily defined as the inverse image under f of the subcomplex of L' dual to α. The duals are closely related to the point inverses $f^{-1}(x)$. If $b(\alpha)$ denotes the barycenter of α, and if $D(\alpha, f)$ is finite, then $D(\alpha, f)$ collapses simplicially to $f^{-1}b(\alpha)$. But the duals often behave better than the point inverses in that $D(\alpha, f)$ may be collapsible when $f^{-1}b(\alpha)$ is not, and in that $D(\alpha, f)$ is a combinatorial $(n - i)$-manifold whenever K is a combinatorial n-manifold and α is an i-simplex in the range of f.

If M is a combinatorial n-manifold, the simplicial mapping $f: M \to L$ is called *p.l. cellular* if $f^{-1}(x)$ is compact, and the regular neighborhood of $f^{-1}(x)$ in M is a combinatorial n-ball, for each x in $|L|$. For example, the collapsible simplicial mappings (each $f^{-1}(x)$ is collapsible) are p.l. cellular.

It is natural to conjecture that p.l. cellular mappings of manifolds without boundary preserve combinatorial structure. A topological theorem of R. Finney [5], combined with the *Hauptvermutung* in low dimensions [10], [11]

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does indeed show this to be true for closed manifolds of dimension $n \leqq 3$. Also, T. Homma has elegantly demonstrated [7] (in work done independently of and concurrently with the present work) that, for any n, collapsible simplicial mappings of closed manifolds preserve combinatorial structure. However, we prove (Proposition 6.1) that, if $n \geqq 5$, there are p.l. cellular mappings of the n-sphere onto complexes which are not combinatorial manifolds. The examples given essentially depend on the fact that, although f is p.l. cellular, the regular neighborhood of $f^{-1}(x)$ in a certain submanifold of M may fail to be a ball. This reveals the inadequacy of p.l. cellularity, and motivates the definition of transverse cellularity given below. On the other hand, Proposition 6.3 asserts that, if $n \leqq 4$ and M is a combinatorial n-manifold without boundary, then the p.l. cellular maps $f\colon M \to L$ coincide with the transversely cellular maps. Also, collapsible mappings of manifolds without boundary are clearly transversely cellular.

Definition. Let M be a combinatorial n-manifold, and let $f\colon M \to L$ be a simplicial mapping. Then f is *dual-collapsible* if and only if $D(\alpha, f)$ is a combinatorial $(n - i)$-ball, for each i-simplex α of L. f is *transversely cellular* if both f and $f \mid \partial M\colon \partial M \to f(\partial M)$ are dual-collapsible.

The main theorems of this paper are Theorems (7.1) and (8.1). Let M be a combinatorial n-manifold (finite or infinite, with or without boundary), and let $f\colon M \to L$ be a simplicial mapping. Suppose that K is a subcomplex of M such that $f \mid K$ is an isomorphism and $K = f^{-1}f(K)$.

THEOREM (7.1). *If f is dual-collapsible then there exists a* p.l. *homeomorphism $h\colon (M \times I, M \times 0) \to (C_f, M')$ such that $h \mid M \times 0 = 1$. If f is transversely cellular, there exists a* p.l. *homeomorphism*

$$h\colon (M \times I, M \times 1, M \times 0) \to (C_f, L, M')$$

such that $h \mid K \times 1 = f$ and $h \mid M \times 0 = 1$.

This gives a sharp answer to our original question. It is the best possible answer in the sense that the converse of each part of (7.1) is true. In fact we have the following.

THEOREM (8.1). *If C_f is a combinatorial $(n + 1)$-manifold and $L < \partial C_f$ then f is dual-collapsible. If, further, L is itself a combinatorial n-manifold, then f is transversely cellular.*

These results are used (§ 9) to derive the basic properties of simplicial mapping cylinders. Three applications are given in § 10. We show that the transverse cellularity of a mapping does not depend on the triangulation chosen. We prove a theorem about the topological quotient space of a complex K induced by a simplicial mapping of a subcomplex of K. (This applies in an interesting fashion to a "figure-eight decomposition" of S^4.) And we prove the following *join-cobordism* theorem.

THEOREM (10.4). *If V is an $(n+1)$-dimensional combinatorial manifold, M_1 and M_2 are disjoint n-dimensional combinatorial submanifolds of ∂V, and every simplex of V is the join of a simplex of M_1 and a simplex of M_2, then there is a* p.l. *homeomorphism* $h: (M_1 \times I, M_1 \times 1, M_1 \times 0) \rightarrow (V, M_1, M_2)$.

(We stress that M_1 and M_2 may be infinite and may have non-empty boundary.)

We turn finally to simplicial mappings for which the inverse image of each point is contractible. Theorem 11.1 is a *Vietoris theorem* which states that such a simplicial mapping of one finite complex onto another is a simple homotopy equivalence. Theorem 11.2 implies that such mappings of one closed combinatorial j-manifold onto another necessarily preserve piecewise linear structure for all $j \leqq n$ if and only if the Poincaré conjecture is true for all $j \leqq n$.

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2. Background and notation

Simplexes and complexes. By a *complex* we mean a countable rectilinear simplicial complex in some euclidean space R^q. The assertion $L < K$ signifies that L is a subcomplex of the complex K. A *partition* of K is a complex K_1 covering the same point set such that each simplex of K is the union of finitely many simplexes of K_1.

If K is a complex, then $|K|$, the *polyhedron* determined by K, is the topological space obtained by giving the weak topology to the point set covered by K. A *homeomorphism* (=piecewise linear homeomorphism) $h: K_1 \rightarrow K_2$ is a topological homeomorphism $h: |K_1| \rightarrow |K_2|$ such that, for each finite complex $L < K_1$, there is a partition L_1 of L and a complex L_2 such that $h: L_1 \rightarrow L_2$ is a simplicial isomorphism. If such a homeomorphism exists, we write $K_1 \simeq K_2$, and say that K_1 and K_2 are *homeomorphic* or *combinatorially equivalent*. For example, if $|K_1| = |K_2|$, (same point set, same topology) then the identity map is a homeomorphism of K_1 onto K_2.

Lower case Greek letters represent (closed) simplexes unless otherwise stipulated. Thus $\sigma < K$ means that σ is a simplex of the complex K; $\dot{\sigma}$ is the boundary of σ. The symbol $\emptyset$ denotes the empty simplex, and it is understood that $\emptyset < K$ for every complex K.

Joins. Two simplexes $\sigma = v_0 v_1 \cdots v_m$ and $\tau = x_0 x_1 \cdots x_n$ in R^q are said to be joinable if $\{v_0, v_1, \cdots, v_m, x_0, \cdots, x_n\}$ is a set of $m + n + 2$ linearly independent points. The join of σ and τ, written $\sigma\tau$, is the $(n + m + 1)$-simplex spanned by this set of vertices. We define $\emptyset\sigma = \sigma\emptyset = \sigma$.

More generally, two complexes K and L are *joinable* if

(1) for every $\sigma < K$ and $\tau < L$, σ and τ are joinable, and

(2) $(\sigma\tau) \cap (\sigma_1\tau_1) = (\sigma \cap \sigma_1)(\tau \cap \tau_1)$, for all $\sigma, \sigma_1 < K$ and $\tau, \tau_1 < L$. The *join of* K *and* L, written KL, is defined as $\{\sigma\tau \mid \sigma < K, \tau < L\}$. Given complexes K and L, one can always find isomorphic complexes K_1 and L_1, in some R^n, which are joinable. Whenever we speak of a subcomplex of KL, we mean a complex isomorphic to a subcomplex of K_1L_1.

Notational schemes. If $L < K$ and $\emptyset \neq \alpha < K$, we define:

$N(L, K) = \{\sigma < K \mid \exists\, \tau < K \cdot \ni : \sigma < \tau \text{ and } \tau \cap L \neq \emptyset\}$,

$\dot{N}(L, K) = \{\sigma < N(L, K) \mid \sigma \cap L = \emptyset\}$,

$\mathrm{Lk}(\alpha, K) = \{\tau < K \mid \alpha \text{ and } \tau \text{ are joinable, } \alpha\tau < K\}$,

$\mathrm{St}(\alpha, K) = \{\tau < K \mid \alpha < \tau\}$.

In order to avoid awkward descriptions of complexes, such as those given above, we shall use two different schemes. At times, we use the Alexander notation [2], $K = \sigma P + Q$, where K is written as the sum of its principal simplexes, $\sigma < K$, $P = \mathrm{Lk}(\sigma, K)$ and $\sigma \not< Q$. At other times, we use the fact that a complex is given once any set containing all principal simplexes is given. For example we may define $N(L, K)$ by stipulating that it is a complex and writing

$$N(L, K) = \{\sigma < K \mid \sigma \cap L \neq \emptyset\} .$$

Barycentric subdivision. If σ is a simplex, then $b(\sigma)$ denotes the barycenter of σ, a distinguished interior point. If $\sigma < \lambda < \cdots < \tau$ (all distinct), then $b(\sigma)b(\lambda) \cdots b(\tau)$ is the simplex spanned by the barycenters of these simplexes. Whenever a simplex $b(\alpha)b(\beta) \cdots b(\gamma)$ is written, it is understood that $\alpha < \beta < \cdots < \gamma$.

If $L < K$, then $B(K/L)$, *the barycentric subdivision of* K *modulo* L, is defined by

$$B(K/L) = L + \{\alpha b(\sigma) \cdots b(\tau) \mid \alpha < L, \sigma \not< L, \alpha < \sigma < \cdots < \tau < K\} .$$

$B(K/L)$ is a partition of K, and hence $B(K/L) \simeq K$. If $L = \emptyset$, we write $B(K/L) = K'$.

If $\sigma < K$ then *the dual to* σ *in* K, written $D(\sigma, K)$, and its subcomplex $\dot{D}(\sigma, K)$ are defined by:

$$D(\sigma, K) = \{b(\psi) \cdots b(\tau) \mid \sigma < \psi < \cdots < \tau < K\} < K' ,$$

$$\dot{D}(\sigma, K) = \{b(\psi) \cdots b(\tau) \mid \sigma \lneqq \psi < \cdots < \tau < K\} < K' .$$

Clearly $D(\sigma, K) = b(\sigma)\dot{D}(\sigma, K)$. It is a classical fact that

$$\dot{D}(\sigma, K) \cong \mathrm{Lk}(\sigma, K)' ,$$

under the isomorphism taking $b(\sigma\alpha)b(\sigma\beta) \cdots b(\sigma\gamma)$ onto $b(\alpha)b(\beta) \cdots b(\gamma)$.

Manifolds and collapsibility. By an n-ball (n-sphere) we mean a complex K which is homeomorphic to an n-simplex (the boundary of an $(n+1)$-simplex). We write $K \simeq B^n (K \simeq S^n)$. The complex M is an *n-manifold* (=combinatorial n-manifold) if $\mathrm{Lk}(\alpha, M)$ is an $(n - i - 1)$-ball or sphere for each i-simplex $\alpha < M$. We let ∂M denote $\{\sigma < M \mid \mathrm{Lk}(\sigma, M)$ is a non-empty ball$\}$. By a *polyhedral n-manifold*, we mean a polyhedron $X = |M|$ where M is an n-manifold. If $X = |M| = |K|$ is a polyhedral n-manifold, then it follows that K is an n-manifold. (See [16, Lem. 9], and beware of the different terminology.)

We shall assume the theory of collapsibility and regular neighborhoods as developed by Whitehead [14] and expounded by Zeeman [16]. In particular, $K \searrow L$ and $K \overset{s}{\searrow} L$ denote the facts that K collapses to L, and K collapses simplicially to L.

Functions. We call $f: X \to Y$ *compact*, *contractible*, or *collapsible* if $f^{-1}(x)$ is compact, contractible, or collapsible for each x in Y.

3. Stellar neighborhoods and standard extensions of simplicial mappings

This section contains some useful preliminary lemmas.

Collapsing reduced joins. Recall that $L < K$ is *full* in K if every simplex of K having all its vertices in L lies completely in L. In this situation every simplex of $N(L, K)$ can be uniquely written in the form $\sigma\tau$ where $\sigma < L$ and $\tau < \dot{N}(L, K)$.

If K and L are complexes, then a *reduced join from K to L* is a complex $N < KL$ such that (1) $K < N$ and $L < N$ and (2) Every principal simplex of N meets L. Notice that K and L are both necessarily full in N.

Example 1. If L is a full subcomplex of P, then $N(L, P)$ is a reduced join from $\dot{N}(L, P)$ to L. (Actually $N(L, P)$ is isomorphic to a reduced join, but we ignore this distinction.)

Example 2. If L is a full subcomplex of P, and if $\sigma < \dot{N} = \dot{N}(L, P)$, then $\mathrm{Lk}(\sigma, N)$ is a reduced join from $\mathrm{Lk}(\sigma, \dot{N})$ to $\mathrm{Lk}(\sigma, N) \cap L$.

Example 3. If f is a simplicial mapping of K onto L then we shall see (§ 4) that C_f is a reduced join from K' to L and from L to K'.

LEMMA (3.1). *If N is a reduced join from the finite complex K to the finite complex L, and if* $\mathrm{Lk}(\sigma, N) \cap L \searrow 0$ *for each $\sigma < K$, then $N \searrow L$.*

PROOF. This lemma is identical with [14, Th. 2], except that the latter theorem assumes that $\mathrm{Lk}(\sigma, N) \cap L \overset{s}{\searrow} 0$, and concludes that $N \overset{s}{\searrow} L$. The main lemma used by Whitehead in proving Theorem 2 is the following:

($*$) *If* $K = \sigma P + Q$ *is a finite complex,* $P \neq \varnothing$, *and* $P \overset{s}{\searrow} 0$, *then* $K \overset{s}{\searrow} \dot{\sigma} P + Q$.

Our assertion can be proved exactly as Whitehead proved his theorem, once we show that ($*$) is true when we replace *simplicial collapse* by *collapse*.

Suppose $K = \sigma P + Q$ is a finite complex, $P \neq \varnothing$, and $P \searrow 0$. By [16, Th. 4], there is a partition $\hat{\pi}(P) \overset{s}{\searrow} 0$. Then $\sigma\hat{\pi}(P)$ is a partition of σP and, by [16, Lem. 3], this can be extended to a partition π of K. Thus

$$\pi(K) = \pi(\sigma P + Q) = \sigma\pi(P) + \pi(Q) ,$$

where $\pi(P) = \mathrm{Lk}(\sigma, \pi(K)) \overset{s}{\searrow} 0$. Hence ($*$) implies that

$$\pi(K) \overset{s}{\searrow} \dot{\sigma}\pi(P) + \pi(Q) = \pi(\dot{\sigma} P + Q) .$$

Therefore $K \searrow \dot{\sigma} P + Q$, q.e.d.

COROLLARY (3.2). *If N is a reduced join from K to L such that* $\mathrm{Lk}(\sigma, N)$ *is finite and* $\mathrm{Lk}(\sigma, N) \cap L \searrow 0$ *for each* $\sigma < K$, *then*

$$\mathrm{Lk}(\sigma, N) \searrow \mathrm{Lk}(\sigma, N) \cap L \searrow 0$$

for each $\sigma < K$.

PROOF. As pointed out in Example 2, $\mathrm{Lk}(\sigma, N)$ is a reduced join from $\mathrm{Lk}(\sigma, K)$ to $\mathrm{Lk}(\sigma, N) \cap L$. But if $\tau < \mathrm{Lk}(\sigma, K)$, then

$$\begin{aligned} \mathrm{Lk}(\tau, \mathrm{Lk}(\sigma, N)) \cap [\mathrm{Lk}(\sigma, N) \cap L] &= \mathrm{Lk}(\sigma\tau, N) \cap \mathrm{Lk}(\sigma, N) \cap L \\ &= \mathrm{Lk}(\sigma\tau, N) \cap L . \end{aligned}$$

The latter collapses to 0 because $\sigma\tau < K$. Hence by (3.1),

$$\mathrm{Lk}(\sigma, N) \searrow \mathrm{Lk}(\sigma, N) \cap L \searrow 0 , \qquad \text{q.e.d.}$$

Standard extensions. We say that the subcomplex L of K is *well situated in* K if L is full in K, and if, further, $\mathrm{Lk}(\sigma, K) \cap L$ is a simplex for each simplex $\sigma < \dot{N}(L, K)$.

Example. If L is full in K, then L is well situated in $B(K/L)$. For L is clearly full in $B(K/L)$, and if $A < \dot{N}(L, B(K/L))$, say $A = b(\sigma) \cdots b(\tau)$, then $\mathrm{Lk}(A, B(K/L)) \cap L = \sigma \cap L$. The latter is a simplex because L is full in K.

Suppose that L is well situated in $K = N(L, K) + P$, and that $f: L \to L_1$ is a simplicial mapping. (We assume that L_1 and K are joinable.) Then the *standard extension* F of f is the mapping $F: K \to \dot{N}L_1 + P$ defined by the conditions

(1) $F \mid L = f$,

(2) $F \mid P + \dot{N}(L, K) = 1$,

(3) F is simplicial.

Thus $F(\sigma) = \sigma$ if $\sigma \cap L = \emptyset$ and $F(\sigma\tau) = \sigma f(\tau)$ if $\sigma < \dot{N}$, $\tau < L$ and $\sigma\tau < K$. The usefulness of standard extensions is due to the following lemma.

LEMMA (3.3). *Assume that L is a well-situated subcomplex of*

$$K = N(L, K) + P\ ,$$

that $F\colon K \to K_1$ is the standard extension of $f\colon L \to L_1$ and that $y \in |F(K)|$. Then $F^{-1}(y) = f^{-1}(y)$ if $y \in |L_1|$, $F^{-1}(y) = y$ if $y \in |P + \dot{N}(L, K)|$, and $F^{-1}(y)$ is a convex cell otherwise.

PROOF. The first two assertions are immediate. Thus we must prove that $F^{-1}(y)$ is a convex cell when $y \in F(|N(L, K)| - |\dot{N}(L, K)| - |L|)$.

Let α be the carrier of y in $F(K)$; that is, the unique simplex containing y in its interior. Then $\alpha = \sigma\tau$ where $\sigma < \dot{N}(L, K)$, $\tau < L_1$, $\sigma \neq \emptyset \neq \tau$. Consider any point $x \in F^{-1}(y)$. Carrier $x = \hat{\sigma}\hat{\tau}$ where $\hat{\sigma} < \dot{N}(L, K)$, $\hat{\tau} < L$. But $\hat{\sigma} f(\hat{\tau}) = F(\hat{\sigma}\hat{\tau}) = F(\text{carrier } x) = \text{carrier } F(x) = \text{carrier } y = \sigma\tau$. Therefore $\sigma = \hat{\sigma}$ and $x \in \sigma\hat{\tau}$, where $\hat{\tau} < \mathrm{Lk}(\sigma, K) \cap L = \eta$, a simplex. Thus $F(x) = y$ implies $x \in \sigma\eta$, so that $F^{-1}(y) = (F \mid \sigma\eta)^{-1}(y)$. Since $\sigma\eta$ is a simplex, this is a convex cell, q.e.d.

Neighborhoods in manifolds. We now state

LEMMA (3.4). *If L is a full non-empty subcomplex of the n-manifold M^n, then*

(1) *$N = N(L', K')$ is an n-manifold;*

(2) *$\partial N = \dot{N}(L', K') + N(L' \cap \partial K', \partial K')$;*

(3) *$\dot{N}(L', K')$ and $N(L' \cap \partial K', \partial K')$ are $(n - 1)$-manifolds, if they are not empty;*

(4) *$\partial\dot{N}(L', K') = \dot{N}(L' \cap \partial K', \partial K') = \partial N(L' \cap \partial K', \partial K')$.*

This is a consequence of the more general Proposition 5.6, and the remark which follows the proof of that proposition.

4. Simplicial mapping cylinders

Suppose that f is a simplicial mapping of K into L. Whitehead [14] defined the simplicial mapping cylinder (at least for finite complexes) as follows:

(1) Triangulate the cell complex $K \times I$ by starring, in order of increasing dimension, the convex cells $\sigma \times I$ at their centroids $b(\sigma) \times 1/2$.

(2) Thinking of $K \times I$ and L as disjoint formal complexes, identify each simplex $\sigma \times 1$ with $f(\sigma)$.

An argument by induction on the dimension of K shows that the result of this process is the complex

$$W_f = \{(\sigma \times 0)(b(\tau) \times 1/2) \cdots (b(\mu) \times 1/2) \mid \sigma < \tau < \cdots < \mu < K\}$$
$$+ \{f(\sigma)(b(\tau) \times 1/2) \cdots (b(\mu) \times 1/2) \mid \sigma < \tau < \cdots < \mu < K\} + L.$$

Essentially we define our simplicial mapping cylinder, C_f, as the upper half of W_f. This change simplifies the notation, but it is not very significant since we shall prove (9.4) that $C_f \simeq W_f$.

Definition. Suppose that f is a simplicial mapping of K into L where K and L are joinable in R^q. Let L' be the standard barycentric subdivision of L and let K' be a barycentric subdivision of K chosen so that f is also simplicial with respect to K' and L'. Then $C_f < K'L$ is defined by

$$C_f = \{\alpha b(\sigma) \cdots b(\tau) \mid \alpha < f(\sigma), \sigma < \cdots < \tau < K\} + L$$
$$= \{f(\psi)b(\sigma) \cdots b(\tau) \mid \psi < \sigma < \cdots < \tau < K\} + L .$$

Remarks. (1) f is not usually simplicial with respect to K' and L' if barycenters in K' are chosen as centroids (e.g., take $K = \sigma^2$, $L = \sigma^1$). However if, for each $\sigma < K$, $b(\sigma)$ is chosen as an interior point of the convex cell $(f \mid \sigma)^{-1}b(f(\sigma))$, then $f: K' \to L'$ is simplicial. Notice then, that for $\alpha < L$ we have

$$f^{-1}(b(\alpha)) = \{b(\sigma) \cdots b(\tau) \mid f(\sigma) = \cdots = f(\tau) = \alpha, \tau < K\} .$$

(2) Since K and L are joinable in R^q, K' and L are joinable, and $|KL| = |K'L|$. In fact, if $\pi(K)$ is any partition of K, and if C is any subcomplex of KL, then $\pi(K)$ and L are joinable and

$$\pi_*(C) = \{\sigma\beta \mid \beta < L, \exists\, \alpha < K \cdot \ni : \sigma < \pi(\alpha) \text{ and } \alpha\beta < C\} < \pi(K)L$$

is a partition of C. (The proof is tedious but straight-forward.) We shall use this fact again later, considering $\pi(L) = L'$, $C_f < K'L$ and $\pi_*(C_f) < K'L'$.

LEMMA (4.1). *If $f: K \to L$ is a simplicial mapping, then C_f is a reduced join from K' to L. If f is onto then C_f is also a reduced join from L to K'.*

LEMMA (4.2). *If $f: K \to L$ is simplicial, then L is well situated in C_f.*

PROOF. If $A \cap L = \varnothing$, then $A < K'$. Let $A = b(\sigma) \cdots b(\tau)$. Then

$$\mathrm{Lk}(A, C_f) \cap L = f(\sigma) , \qquad \text{q.e.d.}$$

LEMMA (4.3). *If $K \times I$ is triangulated as C_{1_K}, then the simplicial projection $\pi_f: (K \times I) \to C_f$ given by $\pi_f(\psi b(\sigma) \cdots b(\tau)) = f(\psi)b(\sigma) \cdots b(\tau)$ is the standard extension to $K \times I$ of $f: K \to L$.*

For, $K = K \times 1$ is well situated by (4.2). This is just the definition of standard extension.

5. The dual structure determined by a simplicial mapping

In this section, we define the duals determined by a simplicial mapping and give their basic properties. We assume throughout that $f\colon K \to L$ is a simplicial map, that $f\colon K' \to L'$ is also simplicial, and that $\alpha = \alpha^i$ is an i-dimensional simplex of L, ($i \geqq 0$).

Definition. $D(\alpha, f)$, *the dual to* α *with respect to* f, *and* $\dot{D}(\alpha, f)$ *are defined by*

$$D(\alpha, f) = \{b(\sigma) \cdots b(\tau) \mid \alpha < f(\sigma), \sigma < \cdots < \tau < K\} ,$$
$$\dot{D}(\alpha, f) = \{b(\sigma) \cdots b(\tau) \mid \alpha \lneqq f(\sigma), \sigma < \cdots < \tau < K\} .$$

PROPOSITION (5.1). $D(\alpha, f) = \mathrm{Lk}(\alpha, C_f) \cap K'$.

This follows directly from the definitions of the sets.

PROPOSITION (5.2). $D(\alpha, f) = f^{-1}D(\alpha, L)$ and $\dot{D}(\alpha, f) = f^{-1}\dot{D}(\alpha, L)$.

PROOF. Suppose that $A = b(\sigma) \cdots b(\tau) < D(\alpha, f)$. $f\colon K' \to L'$ is simplicial. Hence

$$f(A) = f(b(\sigma)) \cdots f(b(\tau)) = b(f(\sigma)) \cdots b(f(\tau)) = b(\beta) \cdots b(\gamma)$$

where $\alpha < \beta < \cdots < \gamma < L$. Therefore $f(A) < D(\alpha, L)$ and $A < f^{-1}D(\alpha, L)$.

On the other hand, suppose that

$$A = b(\beta) \cdots b(\gamma) < D(\alpha, L) .$$

If $b(\sigma) \cdots b(\tau) < f^{-1}(A)$, then $b(\beta)$ precedes (or equals) $b(f(\sigma))$ in A. Therefore $\beta < f(\sigma)$ and $\alpha < \beta$. Thus $\alpha < f(\sigma)$, so $b(\sigma) \cdots b(\tau) < D(\alpha, f)$.

The second assertion is proved similarly, q.e.d.

COROLLARY (5.3). $D(\alpha, gf) = f^{-1}D(\alpha, g)$ *and* $\dot{D}(\alpha, gf) = f^{-1}\dot{D}(\alpha, g)$.

PROPOSITION (5.4). *If* $\alpha^{i-1} < \alpha^i$, *then*

(i) $f^{-1}(b(\alpha^i))$ *is a full subcomplex of* $\dot{D}(\alpha^{i-1}, f)$

(ii) $D(\alpha^i, f) = N(f^{-1}(b(\alpha^i)), \dot{D}(\alpha^{i-1}, f))$

(iii) $\dot{D}(\alpha^i, f) = \dot{N}(f^{-1}(b(\alpha^i)), \dot{D}(\alpha^{i-1}, f))$.

(We set $\alpha^{-1} = \varnothing$ *and* $\dot{D}(\alpha^{-1}, f) = K'$).

PROOF. Recalling that

$$f^{-1}(b(\alpha^i)) = \{b(\sigma) \cdots b(\tau) \mid f(\sigma) = \cdots = f(\tau) = \alpha^i\} ,$$

we see that $f^{-1}(b(\alpha^i)) < D(\alpha^i, f) < \dot{D}(\alpha^{i-1}, f)$.

Because $f\colon K' \to L'$ is simplicial, $f^{-1}(b(\alpha^i))$ is full in K', and *a fortiori* it is full in $\dot{D}(\alpha^{i-1}, f)$. This proves (i).

If $A = b(\sigma) \cdots b(\tau)$ is a simplex of $D(\alpha^i, f)$, then $\alpha^i < f(\sigma)$. Therefore $\alpha^i = f(\psi)$ for some face ψ of σ. Then $b(\psi)b(\sigma) \cdots b(\tau)$ is a simplex of $\dot{D}(\alpha^{i-1}, f)$ meeting $f^{-1}(b(\alpha^i))$. Hence $D(\alpha^i, f) < N(f^{-1}b(\alpha^i), \dot{D}(\alpha^{i-1}, f))$.

On the other hand, suppose that $A = b(\sigma)\cdots b(\tau)$ is a simplex of $\dot{D}(\alpha^{i-1}, f)$ meeting $f^{-1}b(\alpha^i)$. Then some vertex of A, say $b(\mu)$, lies in $f^{-1}b(\alpha^i)$. It follows that $\alpha^{i-1} \lneqq f(\sigma) < f(\mu) = \alpha^i$, so $f(\sigma) = \alpha^i$. Hence $A < D(\alpha^i, f)$. This proves (ii). The proof of (iii) is similar, q.e.d.

PROPOSITION (5.5). *If $D(\alpha^i, f)$ is finite, then $D(\alpha^i, f) \searrow f^{-1}b(\alpha^i)$.*

PROOF. $D(\alpha^i, f)$ is a reduced join from $\dot{D}(\alpha^i, f)$ to $f^{-1}b(\alpha^i)$ by the previous proposition. This assertion will follow from (3.1) if we show that

$$\mathrm{Lk}(A, D(\alpha^i, f)) \cap f^{-1}b(\alpha^i) \searrow 0$$

for each $A < \dot{D}(\alpha^i, f)$. Let $A = b(\sigma)\cdots b(\tau)$. $f(\sigma) \gneqq \alpha^i$. Then

$$\begin{aligned}\mathrm{Lk}(A, D(\alpha^i, f)) &\cap f^{-1}b(\alpha^i)\\ &= \{b(\mu)\cdots b(\nu) \mid f(\mu) = \cdots = f(\nu) = \alpha^i, \nu < \sigma\}\\ &= (f \mid \sigma)^{-1}b(\alpha^i).\end{aligned}$$

The latter is a convex cell and so it is collapsible, q.e.d.

PROPOSITION (5.6). *If $K = K^n$ is a combinatorial n-manifold and $\alpha^i < f(K)$, then*

(1) *$D(\alpha^i, f)$ is a combinatorial $(n-i)$-manifold,*

(2) *$\partial D(\alpha^i, f) = \dot{D}(\alpha^i, f) + D(\alpha^i, f \mid \partial K)$,*

(3) *$\dot{D}(\alpha^i, f)$ is empty or is an $(n-i-1)$-manifold. $D(\alpha^i, f \mid \partial K)$ is an $(n-i-1)$-manifold if $\alpha^i < f(\partial K)$, and is empty otherwise,*

(4) *$\partial\dot{D}(\alpha^i, f) = \dot{D}(\alpha^i, f \mid \partial K) = \dot{D}(\alpha^i, f) \cap \partial K' = \partial D(\alpha^i, f \mid \partial K)$.*

PROOF. We proceed by induction on n, the result being trivial if $n = 0$. Suppose $n > 0$ and the result is true in dimensions less than n.

We prove (1) and (2) by considering a simplex $A = b(\sigma_0)\cdots b(\sigma_q)$ of $D(\alpha^i, f)$, and showing that $\mathrm{Lk}(A, D(\alpha^i, f))$ is always an $(n-q-i-1)$-ball or sphere and that it is a ball if and only if $A < \dot{D}(\alpha^i, f) + D(\alpha^i, f \mid \partial K)$. Let $n_i =$ dimension σ_i.

Case I. Assume $A \cap f^{-1}b(\alpha^i) \neq \emptyset$, i.e., $\alpha^i = f(\sigma_0)$. A typical simplex in $\mathrm{Lk}(A, D(\alpha^i, f))$ is composed of a sequence of vertices which precede $b(\sigma_0)$, followed by a sequence of vertices which may be interspersed between the vertices of A, followed by a sequence of vertices which follows $b(\sigma_q)$. Thus

$$\begin{aligned}\mathrm{Lk}(A, D(\alpha^i, f)) &= \{b(\sigma)\cdots b(\tau) \mid \alpha^i = f(\sigma) = \cdots = f(\tau), \tau < \dot{\sigma}_0\}\dot{D}(\sigma_0, \dot{\sigma}_1)\\ &\qquad\cdots \dot{D}(\sigma_{q-1}, \dot{\sigma}_q)\dot{D}(\sigma_q, K)\\ &= [(f \mid \dot{\sigma}_0)^{-1}b(\alpha^i)]\dot{D}(\sigma_0, \dot{\sigma}_1)\cdots \dot{D}(\sigma_{q-1}, \dot{\sigma}_q)\dot{D}(\sigma_q, K).\end{aligned}$$

Since $f(\sigma_0) = \alpha^i$, $(f \mid \dot{\sigma}_0)^{-1}b(\alpha^i)$ is the boundary of the convex $(n_0 - i)$-cell $(f \mid \sigma_0)^{-1}b(\alpha^i)$. Thus the first term in the above join is an $(n_0 - i - 1)$-sphere. $\dot{D}(\sigma_{i-1}, \dot{\sigma}_i) \cong \mathrm{Lk}(\sigma_{i-1}, \dot{\sigma}_i)'$ is an $(n_i - n_{i-1} - 2)$-sphere. Finally $\dot{D}(\sigma_q, K) \cong$

$\mathrm{Lk}(\sigma_q, K)'$ is an $(n - n_q - 1)$ ball or sphere according to whether or not σ_q (and hence A) is in the boundary of K.

Thus if $A \cap f^{-1}b(\alpha^i) \neq \emptyset$, we see that $\mathrm{Lk}(A, D(\alpha^i, f))$ is a ball or sphere depending on whether $A < \partial K'$ or not; i.e., depending on whether

$$\begin{aligned} A < D(\alpha^i, f) \cap \partial K' &= f^{-1}D(\alpha^i, L) \cap \partial K' \\ &= (f \mid \partial K)^{-1} D(\alpha^i, L) = D(\alpha^i, f \mid \partial K) . \end{aligned}$$

As the join of $q + 2$ complexes whose dimensions add up to $n - i - 2q - 2$, this link has dimension $n - i - q - 1$.

Case II. Assume $A \cap f^{-1}b(\alpha^i) = \emptyset$. Thus $A < \dot{D}(\alpha^i, f)$. Reasoning as before,

$$\begin{aligned} \mathrm{Lk}(A, D(\alpha^i, f)) &= \{b(\sigma) \cdots b(\tau) \mid \alpha^i < f(\sigma), \tau < \dot{\sigma}_0\}M \\ &= D(\alpha^i, f \mid \dot{\sigma}_0)M , \end{aligned}$$

where M is a sphere or a ball of the appropriate dimension. But by induction hypothesis $D(\alpha^i, f \mid \dot{\sigma}_0)$ is a manifold, and by (5.5) this collapses to $(f \mid \dot{\sigma}_0)^{-1}b(\alpha^i)$. But $f(\sigma_0) \gneqq \alpha^i$, so

$$(f \mid \dot{\sigma}_0)^{-1}b(\alpha^i) = (f \mid \sigma_0)^{-1}b(\alpha^i)$$

is a convex cell. Thus $D(\alpha^i, f \mid \dot{\sigma}_0)$ is a ball, being a collapsible manifold.

Hence if $A < \dot{D}(\alpha^i, f)$, we see that $\mathrm{Lk}(A, D(\alpha^i, f))$ is a ball. This completes the proof of (1) and (2).

If $\alpha^i \nless f(\partial K)$, then $D(\alpha^i, f \mid \partial K) = \emptyset$. Thus $\dot{D}(\alpha^i, f)$ is either empty or is an $(n - i - 1)$-manifold without boundary, by (2). In this case, (3) and (4) are trivially satisfied.

If $\alpha^i < f(\partial K)$ then, by induction hypothesis, $D(\alpha^i, f \mid \partial K)$ is an $(n-i-1)$-manifold with $\partial D(\alpha^i, f \mid \partial K) = \dot{D}(\alpha^i, f \mid \partial K)$. But

$$\partial D(\alpha^i, f) = \dot{D}(\alpha^i, f) + D(\alpha^i, f \mid \partial K) ,$$

and

$$\begin{aligned} \dot{D}(\alpha^i, f) \cap D(\alpha^i, f \mid \partial K) &= \dot{D}(\alpha^i, f) \cap D(\alpha^i, f) \cap \partial K' \\ &= \dot{D}(\alpha^i, f) \cap \partial K' = \dot{D}(\alpha^i, f \mid \partial K) . \end{aligned}$$

Since the complement in a j-manifold without boundary of a j-manifold with boundary is again a j-manifold with the same boundary, we see that $\dot{D}(\alpha^i, f)$ is empty or is an $(n - i - 1)$-manifold with $\partial\dot{D}(\alpha^i, f) = \dot{D}(\alpha^i, f \mid \partial K)$, q.e.d.

REMARK. Suppose L is a full subcomplex of $K = N(L, K) + P$, and let $v = \alpha^0$ be a vertex which is joinable with K. Let $f\colon K \to v\dot{N}(L, K) + P$ be the simplicial map such that $f(L) = v$ and $f \mid P + \dot{N} = 1$. Notice that

$$D(\alpha^0, f) = N(L', K') ,$$
$$\dot{D}(\alpha^0, f) = \dot{N}(L', K') ,$$
$$D(\alpha^0, f \mid \partial K) = N(L' \cap \partial K', \partial K') ,$$

and

$$\dot{D}(\alpha^0, f \mid \partial K) = \dot{N}(L' \cap \partial K', \partial K') .$$

This shows that (3.4) is a special case of (5.6).

PROPOSITION (5.7). *Suppose that L is a well-situated subcomplex of the locally finite complex K, that $f: L \to L_1$ is a compact simplicial mapping, and that $F: K \to K_1$ is the standard extension of f, then*

(1) *If $\alpha < K_1$ and $\alpha \not< L_1$ then $D(\alpha, F) \searrow 0$;*

(2) *If $\alpha < L_1$, then $D(\alpha, F) \searrow D(\alpha, f)$.*

PROOF. F is a compact map by (3.3). K' is locally finite and $D(\alpha, F) = N(F^{-1}b(\alpha), D(\alpha, F)) < K'$. Therefore $D(\alpha, F)$ is finite.

If $\alpha \not< L_1$, then $D(\alpha, F) \searrow F^{-1}b(\alpha) \searrow 0$ by (5.5) and (3.3).

If $\alpha < L_1$, then $D(\alpha, f)$ is a full subcomplex of the finite complex

$$D(\alpha, F) = N\big(D(\alpha, f), D(\alpha, F)\big) = N .$$

We shall show that $\mathrm{Lk}(A, N) \cap D(\alpha, f) \searrow 0$ for each simplex $A < \dot{N}$.

Let $A = b(\sigma) \cdots b(\tau)$. Note that $\sigma \cap L \neq \emptyset$, since $\alpha < F(\sigma)$, and $\sigma \cap L = \sigma_0$ is a simplex because L is full in K. Thus,

$$\begin{aligned}\mathrm{Lk}(A, N) \cap D(\alpha, f) &= \{b(\mu) \cdots b(\nu) \mid \alpha < f(\mu), \nu < \sigma \cap L\} \\ &= D(\alpha, f \mid \sigma_0) \searrow 0 .\end{aligned}$$

Therefore $D(\alpha, F) \searrow D(\alpha, f)$, q.e.d.

6. P.l. cellularity and transverse cellularity

PROPOSITION (6.1). *If $n \geqq 5$ there is a* p.l. *cellular map of the n-sphere S^n onto a complex K^n which is not a combinatorial manifold.* (For the definition of p.l. cellularity see § 1.)

PROOF. Let Q^{n-1} be a compact contractible combinatorial manifold such that $Q^{n-1} \times I \simeq B^n$ and ∂Q^{n-1} is not simply connected. Such examples are known to exist for $n \geqq 5$. (See [4], [12].) Then $S^{n-1} \simeq 2Q^{n-1}$ (the double of Q^{n-1}) and we triangulate S^{n-1} as $S^{n-1} = Q_1 + Q_2$, where $Q_1 \simeq Q_2 \simeq Q^{n-1}$, Q_1 is well situated in S^{n-1} and $N = N(Q_1, S^{n-1}) \simeq Q_1$. Let $S^{n-1} = N + P$. Let F be the standard extension of the mapping $f: Q_1 \to v$ where v is a single point. $F(S^{n-1}) = v\dot{N} + P$ where $\pi_1 \dot{N} \neq 1$. $F^{-1}(v) = Q_1$ and by (3.3) $F^{-1}(x) \searrow 0$ if $x \neq v$. Notice that the regular neighborhood of $F^{-1}(v)$ in S^{n-1} is not a ball.

Now let $(v_0 + v_1)S^{n-1} = S^n$ be the suspension of S^{n-1}, and let

$$S(F): (v_0 + v_1)S^{n-1} \longrightarrow (v_0 + v_1)(v\dot{N} + P) = K$$

be the suspension of F. We claim that $S(F)$ is p.l. cellular, but that K is not a combinatorial manifold.

$S(F)^{-1}(x, t) = F^{-1}(x) \times t$ $(-1 < t < 1)$. Thus if $x \neq v$ or if $t = \pm 1$, $S(F)^{-1}(x, t) \searrow 0$ and the regular neighborhood of $S(F)^{-1}(x, t)$ is a ball. On the other hand, if $-1 < t < 1$, then $S(F)^{-1}(v, t)$ has as regular neighborhood $N \times [t - \varepsilon, t + \varepsilon]$ for some $\varepsilon = \varepsilon(t) > 0$. Since $N \times I \simeq Q \times I \simeq B^n$, we conclude that $S(F)$ is p.l. cellular.

But K is not a combinatorial manifold since $\mathrm{Lk}(vv_0, K) = \dot{N}$ is not simply connected, q.e.d.

REMARK. It is not known whether K is a topological sphere.

In the above example the regular neighborhood of $S(F)^{-1}(v, t)$ in S^n is a ball, but its regular neighborhood in $S^{n-1} \times t$ is not. This indicates that we should worry about regular neighborhoods in certain submanifolds. As pointed out in the introduction this leads us to the duals, and we define $f: M \to L$ to be transversely cellular if $D(\alpha^i, f) \simeq B^{n-i}$ for each $\alpha^i < L$ and $D(\alpha^i, f \mid \partial M) \simeq B^{n-i-1}$ for each $\alpha^i < f(\partial M)$, $(i \geqq 0)$.

Example (6.2). Assume that L is a well-situated subcomplex contained in the interior of M, and $f: L \to L_1$ is a mapping of L onto L_1 such that $D(\alpha, f) \searrow 0$ for each $\alpha < L$. Then the standard extension F of f is transversely cellular. This is because $D(\beta, F) \simeq B^{n-i}$ for every i-simplex $\beta < F(M)$, by (5.6) and (5.7) and because $F \mid \partial M = 1_{\partial M}$.

PROPOSITION (6.3). *If M^n is a manifold without boundary, $n \leqq 4$, and f is a* p.l. *cellular simplicial mapping of M^n onto L, then f is transversely cellular.*

PROOF. We give the proof for $n = 4$. If $\alpha^0 < L$, then

$$D(\alpha^0, f) = N(f^{-1}(\alpha^0)', M')$$

is a regular neighborhood of $f^{-1}(\alpha^0)$ in M by (5.5) and (5.6). Since f is p.l. cellular, $D(\alpha^0, f) \simeq B^4$.

Suppose that $\alpha^1 < L$, and choose a vertex $\alpha^0 < \alpha^1$. Now $f^{-1}b(\alpha^1)$ is contractible since its regular neighborhood in M is a ball. But from (5.4)–(5.6) we see that $D(\alpha^1, f)$ is the regular neighborhood of $f^{-1}b(\alpha^1)$ in the 3-sphere $\dot{D}(\alpha^0, f)$. Thus $D(\alpha^1, f)$ is a compact contractible 3-manifold in S^3. Then $\partial D(\alpha^1, f) \simeq S^2$, so by the 3-dimensional Schoenflies theorem [1], $D(\alpha^1, f) \simeq B^3$.

Finally, if $i \geqq 2$, $D(\alpha^i, f)$ is a compact contractible $(4 - i)$-manifold, so $D(\alpha^i, f) \simeq B^{4-i}$, q.e.d.

7. Consequences of transverse cellularity

This section is devoted to the proof of

THEOREM (7.1). *Assume that M is an n-manifold, and that $f\colon M \to L$ is simplicial. Then*

(A_n). *If f is dual-collapsible, there exists a homeomorphism*

$$h\colon (M \times I, M \times 0) \longrightarrow (C_f, M')$$

such that $h \mid M \times 0 = 1$.

(B_n). *If f is transversely cellular, $K < M$, $f \mid K$ is an isomorphism, and $f^{-1}f(K) = K$, then there exists a homeomorphism*

$$h\colon (M \times I, M \times 0, M \times 1) \longrightarrow (C_f, M', L)$$

such that $h \mid M \times 0 = 1$, and $h \mid K \times 1 = f$.

PROOF OF (A_n) *assuming* (B_{n+1}). Consider the canonical simplicial mapping $\pi_f\colon M \times I \to C_f$ defined in (4.3) where $M \times I$ is triangulated as C_{1_M} and $|M \times 0|$ is triangulated as M'. Clearly $\pi_f \mid M' = 1$ and $\pi_f^{-1}\pi_f(M') = M'$. Thus (A_n) will follow from (B_{n+1}) once we show that π_f is transversely cellular.

Since $D(\alpha^i, f)$ is an $(n - i)$-ball for each $\alpha^i < L$, and π_f is the standard extension of f to $M \times I$, (5.6) and (5.7) imply that $D(\alpha^i, \pi_f)$ is an $(n - i + 1)$-ball for each $\alpha^i < C_f$. But $M = M \times 1$ is well-situated in $\partial(M \times I)$, and $\pi_f \mid \partial(M \times I)$ is the standard extension of f to $\partial(M \times I)$. Hence, by (6.2) $\pi_f \mid \partial(M \times I)$ is transversely cellular. Thus π_f is transversely cellular and the result (A_n) follows, q.e.d.

The proof that (B_n) is true for all n proceeds by induction. If $n = 0$, M is just a discrete set of points, and f is an isomorphism. So the proposition is trivial.

We assume that $n > 0$ and the proposition is known for integers less than n. The proof is rather long and will be given in a series of lemmas. Throughout this section, M denotes an n-manifold and f denotes a simplicial mapping.

LEMMA (7.2). *If $f\colon M \to L$ and if A is a j-simplex of C_f meeting both M' and L then $\mathrm{Lk}(A, C_f)$ is an $(n - j)$ ball or sphere according to whether $A < C_{f|\partial M}$ or not.*

PROOF. Let $A = v_0 v_1 \cdots v_r b(\sigma_0) \cdots b(\sigma_q) = v_0 b(\sigma_q) B$. Let $N = \dim \sigma_q$. Notice that $A < C_{f|\partial M}$ if and only if $\dot{D}(\sigma_q, M)$ is a ball. But

$$\mathrm{Lk}(A, C_f) = \mathrm{Lk}(A, C_{f|\sigma_q})\dot{D}(\sigma_q, M)\ .$$

Thus it will suffice to show that $\mathrm{Lk}(A, C_{f|\sigma_q})$ is an $(N - j)$-sphere. Because $\mathrm{Lk}(A, C_{f|\sigma_q}) = \mathrm{Lk}\big(B, \mathrm{Lk}[v_0 b(\sigma_q), C_{f|\sigma_q}]\big)$, our problem is reduced to showing that $\mathrm{Lk}\big(v_0 b(\sigma_q), C_{f|\sigma_q}\big)$ is an $(N - 1)$-sphere.

Let $\mu = (f \mid \sigma_q)^{-1}(v_0)$. Let $\nu = \mathrm{Lk}(\mu, \sigma_q)$. Notice that $\mu \neq \emptyset$ because $v_0 < f(\sigma_q)$. Therefore ν is a simplex of $\dot{\sigma}_q$. Now

$$\begin{aligned}\mathrm{Lk}(v_0 b(\sigma_q), C_{f|\sigma_q}) &= \{\beta b(\sigma)\cdots b(\tau) \mid v_0\beta < f(\sigma), \tau < \dot{\sigma}_q\} + f(\nu \\ &= \{\beta b(\sigma)\cdots b(\tau) \mid \beta < f(\sigma \cap \nu), \sigma \cap \mu \neq \varnothing, \tau < \dot{\sigma}_q\} + f(\nu) \\ &= \{f(\psi) b(\sigma)\cdots b(\tau) \mid \psi < \sigma \cap \nu, \sigma \cap \mu \neq \varnothing, \tau < \dot{\sigma}_q\} + f(\nu)\ .\end{aligned}$$

But notice that

$$B(\dot{\sigma}_q/\nu) = \{\psi b(\sigma)\cdots b(\tau) \mid \psi < \sigma \cap \nu, \sigma \cap \mu \neq \varnothing, \tau < \dot{\sigma}_q\} + \nu\ .$$

Therefore $\mathrm{Lk}(v_0 b(\sigma_q), C_{f|\sigma_q})$ is the image of the sphere $B(\dot{\sigma}_q/\nu)$ under the standard extension of $f \mid \nu$. This standard extension is transversely cellular by (6.2). Since the statement (B_{N-1}) is known by induction hypothesis, we see that the link in question is indeed an $(N-1)$-sphere, q.e.d.

LEMMA (7.3). *If $f: M \to L$, and if A is a j-simplex of M', then* $\mathrm{Lk}(A, C_f)$ *is an $(n-j)$-ball.*

If B is any non-empty simplex of M', then $\mathrm{Lk}(B, C_f) \cap L$ is non-empty and collapsible by (4.2). Using this fact, the proof of (7.3) is the same as the proof of (7.4).

LEMMA (7.4). *If $f: M \to L$ is dual-collapsible then* $\mathrm{Lk}(\alpha^i, C_f) \simeq B^{n-i}$ *for each $\alpha^i < L$.*

PROOF. We suppose the assertion is true for all simplexes of L of dimension greater than i (a justified supposition if there are no such simplexes), and prove the assertion for α^i by showing that $\mathrm{Lk}(\alpha^i, C_f)$ is a collapsible $(n-i)$-manifold.

By hypothesis $D(\beta, f)$ is a non-empty ball for each $\beta < L$. This implies that f is onto, so C_f is a reduced join from L to M'. It implies that $\mathrm{Lk}(\beta, C_f)$ is finite for each $\beta < L$, and that $\mathrm{Lk}(\beta, C_f) \cap M' = D(\beta, f) \searrow 0$. Thus by (3.2), $\mathrm{Lk}(\alpha^i, C_f) \searrow 0$.

Suppose $\beta^j < \mathrm{Lk}(\alpha^i, C_f)$. If $\beta^j < L$, then $\alpha^i\beta^j$ is a simplex of L of dimension greater than i, so $\mathrm{Lk}(\beta^j, \mathrm{Lk}(\alpha^i, C_f)) = \mathrm{Lk}(\alpha^i\beta^j, C_f)$ is an $(n-i-j-1)$-ball. On the other hand, if $\beta^j \cap M' \neq \varnothing$, then

$$\mathrm{Lk}(\beta^j, \mathrm{Lk}(\alpha^i, C_f)) = \mathrm{Lk}(\alpha^i\beta^j, C_f)$$

is an $(n-i-j-1)$-ball or sphere by (7.2). Thus $\mathrm{Lk}(\alpha^i, C_f)$ is an $(n-i)$-manifold, q.e.d.

LEMMA (7.5). *If $f: M \to L$ is dual-collapsible, then C_f is an $(n+1)$-manifold and $\partial C_f = M' + C_{f|\partial M} + L$.*

This is an immediate consequence of (7.2)–(7.4).

LEMMA (7.6). *If $f: M^n \to L$ is transversely cellular, then L is an n-dimensional submanifold of ∂C_f, and $\partial L = f(\partial M)$.*

PROOF. From the previous lemma, C_f is an $(n+1)$-manifold and $\partial C_f = M' + L + C_{f|\partial M}$. By definition $f|\partial M$ is transversely cellular; so (B_{n-1}) implies that $C_{f|\partial M} \simeq (\partial M) \times I$. Thus ∂C_f is an n-manifold without boundary, $M' + C_{f|\partial M}$ is an n-dimensional submanifold of ∂C_f, $\partial C_f = (M' + C_{f|\partial M}) + L$ and $L \cap (M' + C_{f|\partial M}) = f(\partial M) = \partial(M' + C_{f|\partial M})$. It follows that L is an n-manifold with $\partial L = f(\partial M)$, q.e.d.

The reason that transversely cellular maps preserve structure and the plan of attack can now be explained as follows. Since L is a combinatorial manifold, $\{D(\alpha, L) \mid \alpha < L\} \cup \{D(\alpha, \partial L) \mid \alpha < \partial L\}$ *yields a decomposition of* $|L|$ *as a cell complex, where the cells are combinatorial balls. By the hypothesis of transverse cellularity,* $\{D(\alpha, f) \mid \alpha < L\} \cup \{D(\alpha, f \mid \partial M) \mid \alpha < \partial L\}$ *gives a decomposition of M into combinatorial balls. These cell complexes are isomorphic under the correspondence* $D(\alpha, f) = f^{-1}D(\alpha, L)$, *and this isomorphism allows us to define a homeomorphism.* C_f *is homeomorphic to* $M \times I$ *in so nice a manner because, for each* $\alpha^i < L$, *we can find an* $(n - i + 1)$*-ball in* $|C_f|$ *stretching from* $D(\alpha^i, f)$ *to* $D(\alpha^i, L)$ *which corresponds precisely to the cell* $D(\alpha^i, f) \times I$ *in* $|M \times I|$.

We assume for the rest of §7 that $f\colon M \to L$ is a transversely cellular mapping. Let $C^* = C_f^*$ be the partition induced on C_f by barycentrically subdividing L (see Remark 2 of §4), and if $J < C_f$, let J^* denote the corresponding subcomplex of C^*. Thus

$$\begin{aligned} C^* &= \{b(\beta) \cdots b(\gamma)b(\sigma) \cdots b(\tau) \mid \exists\, \alpha < L \ni : \gamma < \alpha < f(\sigma)\} < M'L' \\ &= \{b(\beta) \cdots b(\gamma)b(\sigma) \cdots b(\tau) \mid \gamma < f(\sigma),\ \tau < M\} < M'L' \,. \end{aligned}$$

Define for each $\alpha < L$, $\alpha \neq \varnothing$, the following subcomplexes of C^*:

$$\begin{aligned} Q(\alpha, f) &= \{b(\beta) \cdots b(\gamma)b(\sigma) \cdots b(\tau) \mid \alpha < \beta\} \,, \\ \dot{Q}(\alpha, f) &= \{b(\beta) \cdots b(\gamma)b(\sigma) \cdots b(\tau) \mid \alpha < \dot{\beta}\} \,. \end{aligned}$$

LEMMA (7.7). *The Q's have these properties:*

(a) $Q(\alpha^i, f)$ *is a homogeneous* $(n - i + 1)$*-complex.*

(b) $Q(\alpha, f) \cap M' = D(\alpha, f)$ *and* $Q(\alpha, f) \cap L' = D(\alpha, L)$.
$\dot{Q}(\alpha, f) \cap M' = \dot{D}(\alpha, f)$ *and* $\dot{Q}(\alpha, f) \cap L' = \dot{D}(\alpha, L)$.

(c) $Q(\alpha, f) \cap Q(\beta, f) = Q(\alpha\cdot\beta, f)$, *where* $\alpha\cdot\beta$ *is the simplex of L spanned by* α *and* β, *if there is one,* $\alpha\cdot\beta = \varnothing$ *otherwise, and* $Q(\varnothing, f) = \varnothing$.

(d) $C^* = \sum \{Q(\alpha, f) \mid \alpha < L\}$.

(e) $\dot{Q}(\alpha, f) = \sum \{Q(\beta, f) \mid \alpha < \dot{\beta}\}$.

(f) $Q(\alpha, f) = b(\alpha)[\dot{Q}(\alpha, f) + D(\alpha, f)]$.

LEMMA (7.8). *If* $\alpha^i < L$, *then* $Q(\alpha^i, f) \simeq B^{n-i+1}$.

PROOF. Let $A^i = b(\alpha^0) \cdots b(\alpha^i)$ be a maximal simplex of $(\alpha^i)'$. As a partition of C_f, C^* is an $(n+1)$-manifold, and since $A^i \subset |\partial C_f|$, we see that $A^i < \partial C^*$. Hence $\mathrm{Lk}(A^i, C^*) \simeq B^{n-i}$. But $\mathrm{Lk}(A^i, C^*) = \dot{Q}(\alpha^i, f) + D(\alpha^i, f)$. Therefore $Q(\alpha^i, f) = b(\alpha^i)\mathrm{Lk}(A^i, C^*) \simeq B^{n-i+1}$, q.e.d.

LEMMA (7.9). (1) $\partial Q(\alpha, f) = \dot{Q}(\alpha, f) + Q(\alpha, f \mid \partial M) + D(\alpha, f) + D(\alpha, L)$
(2) $\dot{Q}(\alpha, f) \cap Q(\alpha, f \mid \partial M) = \dot{Q}(\alpha, f \mid \partial M)$.

PROOF. If A is a simplex of $D(\alpha, f)$, then every principal simplex of $\mathrm{Lk}(A, Q(\alpha, f))$ contains $b(\alpha)$. Thus $\mathrm{Lk}(A, Q(\alpha, f))$, being a ball or sphere by (7.8), and a cone over $b(\alpha)$, is a ball.

If $A = b(\beta) \cdots b(\gamma) < D(\alpha, L)$, then

$$\mathrm{Lk}(A, Q(\alpha, f)) \cap D(\alpha, f) = D(\gamma, f) \searrow 0 .$$

Since this is true for all $A < D(\alpha, L)$, it follows from (3.2) that

$$\mathrm{Lk}(A, Q(\alpha, f)) \searrow 0 .$$

Hence this link is a ball.

If $A = b(\beta) \cdots b(\gamma)b(\sigma) \cdots b(\tau) < Q(\alpha, f)$ meets both $D(\alpha, f)$ and $D(\alpha, L)$ then, by our usual method of analyzing links, we see that

$$\mathrm{Lk}(A, Q(\alpha, f)) = D(\alpha, \dot{\beta})(\text{sphere})X(\text{sphere})\dot{D}(\tau, M) ,$$

where X consists of all simplexes which fit between $b(\gamma)$ and $b(\sigma)$. That is

$$X = \{b(\gamma\delta_1) \cdots b(\gamma\delta_r)b(\mu) \cdots b(\nu) \mid \varnothing \neq \delta_i < \mathrm{Lk}(\gamma, L), \gamma\delta_r < f(\mu), \nu < \dot{\sigma}\} + D(\gamma, f \mid \dot{\sigma}) + \dot{D}(\gamma, f(\sigma)) .$$

We claim that X is a sphere. This implies that $\mathrm{Lk}(A, Q(\alpha, f))$ is a ball if and only if $\alpha \neq \beta$ or $\tau < \partial M$; that is, if and only if

$$A < \dot{Q}(\alpha, f) + Q(\alpha, f \mid \partial M) ,$$

and so completes the proof of (1). The proof of (2) is straight-forward.

To see that X is a sphere, notice that $\mathrm{Lk}(\gamma b(\sigma), C_{f|\sigma})$ is a sphere by (7.2). But we have

$$\begin{aligned} &\mathrm{Lk}(\gamma b(\sigma), C_{f|\sigma}) \\ &\quad = \{\delta b(\mu) \cdots b(\nu) \mid \delta < \mathrm{Lk}(\gamma, L), \gamma\delta < f(\mu), \nu < \dot{\sigma}\} \\ &\qquad + D(\gamma, f \mid \dot{\sigma}) + \mathrm{Lk}(\gamma, f(\sigma)) \\ &[\mathrm{Lk}(\gamma b(\sigma), C_{f|\sigma})]^* \\ &\quad = \{b(\delta_1) \cdots b(\delta_r)b(\mu) \cdots b(\nu) \mid \varnothing \neq \delta_i < \mathrm{Lk}(\gamma, L), \gamma\delta_r < f(\mu), \nu < \dot{\sigma}\} \\ &\qquad + D(\gamma, f \mid \dot{\sigma}) + \mathrm{Lk}(\gamma, f(\sigma))' . \end{aligned}$$

The correspondence which takes

$$b(\gamma\delta_1) \cdots b(\gamma\delta_r)b(\mu) \cdots b(\nu) \quad \text{to} \quad b(\delta_1) \cdots b(\delta_r)b(\mu) \cdots b(\nu)$$

is an isomorphism of X onto the sphere $[\mathrm{Lk}(\gamma b(\sigma), C_{f|\sigma})]^*$, q.e.d.

Let F_1 be the family of piecewise linear balls, each a subpolyhedron of $|M' \times I|$, consisting of the elements $D(\alpha, f) \times I$, $D(\alpha, f) \times i$ $(i = 0, 1)$, $D(\alpha, f \mid \partial M) \times I$, and $D(\alpha, f \mid \partial M) \times i$ $(i = 0, 1)$, where α ranges over L.

Let F_2 be the family of balls, each a subcomplex of C^*, consisting of the elements $Q(\alpha, f)$, $D(\alpha, f)$, $D(\alpha, L)$, $Q(\alpha, f \mid \partial M)$, $D(\alpha, f \mid \partial M)$, and $D(\alpha, \partial L)$, where α ranges over L.

LEMMA (7.10). *F_2 is a polyhedral cell complex covering $|C_f|$. Each cell is a ball in $|C_f|$. The intersection of two cells in F_2 is a cell in F_2 which lies in the boundary of each. Moreover, the boundary of each i-cell in F_2 is the union of cells in F_2 of dimension less than i. A similar statement holds for F_1 and $|M' \times I|$.*

PROOF. This follows from (7.7)–(7.9) and (5.6).

The following assertion should be obvious.

LEMMA (7.11). *If $\psi: F_1 \to F_2$ is defined by*

$$\begin{aligned} \psi[D(\alpha, f) \times 0] &= D(\alpha, f) \\ \psi[D(\alpha, f) \times 1] &= D(\alpha, L) \\ \psi[D(\alpha, f) \times I] &= Q(\alpha, f) \\ \psi[D(\alpha, f \mid \partial M) \times 0] &= D(\alpha, f \mid \partial M) \\ \psi[D(\alpha, f \mid \partial M) \times 1] &= D(\alpha, \partial L) \\ \psi[D(\alpha, f \mid \partial M) \times I] &= Q(\alpha, f \mid \partial M) , \end{aligned}$$

then ψ is an isomorphism of cell complexes.

Let us identify $M' \times 0 = M' < C_f$ and consider f as a mapping of $M' \times 1$ onto L', where $K' = K' \times 1 < M' \times 1$. With these conventions the conditions on h in the statement of (B_n) make sense.

PROOF OF THE ASSERTION (B_n). Let F_j^i be the set of all cells of F_j of dimension less than or equal to i $(j = 1, 2)$. We construct the required homeomorphism $h: |F_1| \to |F_2|$ inductively on the i-skeleton of F_1.

Let $h_0: |F_1^0| \to |F_2^0|$ by $h_0(c^0) = \psi(c^0)$ for each 0-cell $c^0 \in F_1^0$. Note that $h_0 \mid F_1^0 \cap (M' \times 0) = 1$ and $h_0(c^0) = f(c^0)$ if $c^0 \in (K' \times 1) \cap F_1^0$ (i.e., if c^0 is the barycenter of an n-simplex of K or an $(n-1)$-simplex of $K \cap \partial M$).

If $\alpha < f(K)$ is an i-simplex, then $f^{-1}(\alpha)$ is also an i-simplex since $f^{-1}f(K) = K$ and $f \mid K$ is an isomorphism. Denote $\bar{\alpha} = f^{-1}(\alpha)$. Then

$$D(\alpha, f) = b(\bar{\alpha})\dot{D}(\alpha, f) ; \qquad D(\alpha, L) = b(\alpha)\dot{D}(\alpha, L) .$$

If $h: \dot{D}(\alpha, f) \to \dot{D}(\alpha, L)$ is a homeomorphism then, by *the cone over h*, we mean the homeomorphism $h_*: D(\alpha, f) \to D(\alpha, L)$ defined by

$$h_*\big(tb(\bar{\alpha}) + (1 - t)x\big) = tb(\alpha) + (1 - t)h(x) .$$

Suppose now that homeomorphisms $h_j\colon |F_1^j| \to |F_2^j|$ have been defined for $0 \leqq j < i$ such that

(a) $h_j(c^j) = \psi(c^j)$ for each $c^j \in F_1^j$.

(b) $h_j \,|\, |F_1^j| \cap (M' \times 0) = 1$.

(c) If $\alpha < f(K)$ and $D(\alpha, f) = D(\alpha, f) \times 1 < F_1^j$, then

$$h_j\colon D(\alpha, f) \longrightarrow D(\alpha, L)$$

is the cone over $h_{j-1}\colon \dot{D}(\alpha, f) \to \dot{D}(\alpha, L)$. Similarly $h_j\colon D(\alpha, f \,|\, \partial M) \to D(\alpha, \partial L)$ is the cone over $h_{j-1}\colon \dot{D}(\alpha, f \,|\, \partial M) \to \dot{D}(\alpha, \partial L)$, if $\alpha < f(\partial M) \cap f(K)$ and $D(\alpha, f \,|\, \partial M) < F_1^j$.

(d) $h_j = h_k \,|\, |F_1^j|$ if $j < k$.

Let c^i be an i-cell of F_1. Notice that $h_{i-1}(\partial c^i) = \psi(\partial c^i) = \partial\psi(c^i)$, by condition (a), (7.9), (5.6), and definition of ψ. Define $h_i\colon |F_1^i| \to |F_2^i|$ as follows:

(i) If $c^i = D(\alpha, f) \times 0 < (M' \times 0)$ then $\psi(c^i) = c^i$ and $h_{i-1} \,|\, \partial c^i = 1$. Define $h_i \,|\, c^i = 1$. Similarly if $c^i = D(\alpha, f \,|\, \partial M) \times 0$.

(ii) (In this case the duals mentioned are subcomplexes of $M' \times 1$.) If $c^i = D(\alpha, f \,|\, \partial M)$ where $\alpha < f(K) \cap \partial L$, then $\partial c^i = \dot{D}(\alpha, f \,|\, \partial M)$ by (5.6) and $b(\bar{\alpha})$ is an interior point of c^i. Define $h_i \,|\, c^i$ as the cone over $h_{i-1} \,|\, \dot{D}(\alpha, f \,|\, \partial M)$. We follow the same procedure if $c^i = D(\alpha, f)$ where $\alpha < f(K)$ and $\alpha \not< \partial L$. Finally, if $c^i = D(\alpha, f)$ where $\alpha < (\partial L) \cap f(K)$, then $c^i = b(\bar{\alpha})\dot{D}(\alpha, f)$ and $\partial c^i = D(\alpha, f \,|\, \partial M) + \dot{D}(\alpha, f) = b(\bar{\alpha})\dot{D}(\alpha, f \,|\, \partial M) + \dot{D}(\alpha, f)$. Again we define $h_i \,|\, c^i$ as the cone over $h_{i-1} \,|\, \dot{D}(\alpha, f)$. We must check that $h_i \,|\, \partial c^i = h_{i-1} \,|\, \partial c^i$. This is certainly true on $\dot{D}(\alpha, f)$. Moreover $h_i \,|\, D(\alpha, f \,|\, \partial M)$ is the cone over $h_{i-1} \,|\, \dot{D}(\alpha, f \,|\, \partial M)$, and by inductive assumption (c), $h_{i-1} \,|\, D(\alpha, f \,|\, \partial M)$ is the cone over its restriction to $\dot{D}(\alpha, f \,|\, \partial M)$.

(iii) If c^i is an i-cell of F_1 not covered in cases (i) or (ii), then c^i and $\psi(c^i)$ are i-dimensional balls and h_{i-1} is a homeomorphism taking ∂c^i onto $\partial\psi(c^i)$. Choose $h_i \,|\, c^i$ to be any extension of $h_{i-1} \,|\, \partial c^i$ to a homeomorphism of c^i onto $\psi(c^i)$.

Now h_i is a well-defined one-one function on $|F_1^i|$ because the intersection of distinct i-cells is contained in F_1^{i-1} and the images of their interiors are the interiors of disjoint i-cells of F_2. Also h_i is onto because $\psi(F_1^i) = (F_2^i)$. Thus h_i, being one-one and onto, is a homeomorphism because it is a homeomorphism on each cell. It is clear that h_i satisfies (a)–(d).

Let $h = h_{n+1}$. By (a) $h\colon (M' \times I, M' \times 0, M' \times 1) \to (C_f, M', L)$. By (b), $h \,|\, (M' \times 0) = 1$. From (c) and (d), an easy inductive argument shows that $h \,|\, |K| = f \,|\, |K|$, q.e.d.

8. The assumption that C_f is a manifold

THEOREM (8.1). *Assume M is an n-manifold, and $f\colon M \to L$ is a simplicial mapping. If C_f is an $(n+1)$-manifold with $L < \partial C_f$, then f is dual-collapsible. If further, L is an n-manifold, then f is transversely cellular.*

PROOF. To prove the first assertion, suppose that $\alpha^i < L$, and it is known that $D(\alpha^j, f) \simeq B^{n-j}$ for each $j > i$, $\alpha^j < L$. Since $\dim L \leqq n$ and

$$\dim \operatorname{Lk}(\alpha^i, C_f) = (n - i) ,$$

we see that $\operatorname{Lk}(\alpha^i, C_f) \cap M' = D(\alpha^i, f) \neq \emptyset$. Thus by (5.6), $D(\alpha^i, f)$ is an $(n-i)$-manifold. But $L < \partial C_f$ so $\operatorname{Lk}(\alpha^i, C_f) \simeq B^{n-i}$. We show that

$$\operatorname{Lk}(\alpha^i, C_f) \searrow D(\alpha^i, f) .$$

It then follows from the uniqueness of regular neighborhoods that

$$D(\alpha^i, f) \simeq B^{n-i} .$$

By Example 2 of § 3, $\operatorname{Lk}(\alpha^i, C_f)$ is a reduced join from $\operatorname{Lk}(\alpha^i, L)$ to $D(\alpha^i, f)$. Suppose $\emptyset \neq \beta^j < \operatorname{Lk}(\alpha^i, L)$. Then

$$\begin{aligned} \operatorname{Lk}\big(\beta^j, \operatorname{Lk}(\alpha^i, C_f)\big) \cap D(\alpha^i, f) &= \operatorname{Lk}(\alpha^i\beta^j, C_f) \cap M' \\ &= D(\alpha^i\beta^j, f) \simeq B^{n-i-j-1} \searrow 0 , \end{aligned}$$

using the induction hypothesis to get the combinatorial equivalence. By (3.1) $\operatorname{Lk}(\alpha^i, C_f) \searrow D(\alpha^i, f)$. This completes the proof of the first assertion.

Suppose now that L is an n-manifold and $\partial M \neq \emptyset$. We first show that $f(\partial M) = \partial L$. f is a contractible mapping since, as we have already seen, $D(\alpha, f)$ is a ball for each $\alpha < L$. $f \mid \dot{D}(\alpha, f)$ is a contractible mapping for each $\alpha < L$ because $\dot{D}(\alpha, f) = f^{-1}\dot{D}(\alpha, L)$. But contractible mappings preserve homotopy type (by [13] or by (11.1) ahead), whence $\dot{D}(\alpha, f)$ and $\dot{D}(\alpha, L)$ have the same homotopy type for each $\alpha < L$. If $\alpha < \partial L$, $\dot{D}(\alpha, L)$ is a ball, so $\dot{D}(\alpha, f)$ is not a sphere. However $\partial D(\alpha, f) = \dot{D}(\alpha, f) + D(\alpha, f \mid \partial M)$. Therefore $D(\alpha, f \mid \partial M) \neq \emptyset$ and $\alpha < f(\partial M)$. Conversely, if $\alpha < f(\partial M)$, $D(\alpha, f \mid \partial M) \neq \emptyset$, and this implies that $\partial D(\alpha, f) \neq \dot{D}(\alpha, f)$. Then $\dot{D}(\alpha, L)$ is not a homotopy sphere. Thus $\dot{D}(\alpha, L)$ is a ball, and so $\alpha < \partial L$. This proves that $f(\partial M) = \partial L$.

$\partial C_f = L + C_{f|\partial M} + M'$, as we see from (7.5). Moreover

$$L \cap C_{f|\partial M} = f(\partial M) = \partial L .$$

Therefore $C_{f|\partial M}$ is the closure of the complement of $M' + L$ in ∂C_f. It is thus an n-manifold with boundary $\partial M' + \partial L = \partial M' + f(\partial M)$. By the first part of this theorem, it follows that $D(\alpha, f \mid \partial M)$ is a ball for each $\alpha < \partial L$. Therefore f is transversely cellular, q.e.d.

COROLLARY (8.2). *If M_1 and M_2 are n-manifolds with boundary and there exists a dual-collapsible mapping $f: M_1 \to M_2$, then M_1 and M_2 are homeomorphic.*

PROOF. C_f is an $(n+1)$-manifold and $M_2 < \partial C_f$, by (7.1) and (7.4). Therefore, by (8.1), the assumption that M_2 is a manifold implies that f is transversely cellular. Hence (7.1) implies that $M_1 \simeq M_2$, q.e.d.

9. The structure of simplicial mapping cylinders

In this section we derive some basic properties of simplicial mapping cylinders. Many (perhaps all) of these facts have been part of the folklore, but a unified listing with proofs has not been given. Since we will need these results in the next section, the requirements of logic and exposition dictate that we present them here.

Mappings of manifolds. We state

PROPOSITION (9.1). *If f is a simplicial mapping of the n-simplex σ onto the simplex τ, then $C_f \simeq B^{n+1}$.*

PROPOSITION (9.2). *If M is an n-manifold, $f: M \to L$ is a simplicial mapping and A is a j-simplex of C_f meeting M', then* $\mathrm{Lk}(A, M)$ *is an $(n-j)$-ball if $A < M'$ or $A < C_{f|\partial M}$, and* $\mathrm{Lk}(A, M)$ *is an $(n-j)$-sphere otherwise.*

Proposition (9.1) is an immediate consequence of (7.1). Proposition (9.2) is just a restatement of (7.2) and (7.3). E. C. Zeeman has informed the author that these results were known to M. H. A. Newman ten years before the author was born. Professor Newman's proof of (9.1) is given in [16, Ch. 7].

Mappings of complexes. We state

PROPOSITION (9.3). *If $f: K \to L$ is a simplicial mapping then*

$$[N(K'', C'_f), K'', \dot{N}(K'', C'_f)] \simeq [K \times I, K \times 0, K \times 1] .$$

PROOF. Suppose σ is a simplex of K. By (9.1), $C_{f|\sigma} \simeq \sigma \times I$. Then it is easy to see (or one can invoke (10.4)) that there is a homeomorphism of $N(\sigma'', C'_{f|\sigma})$ onto $\sigma \times I$ which takes σ'' onto $\sigma \times 0$ and $\dot{N}(\sigma'', C'_{f|\sigma})$ onto $\sigma \times 1$. The cell complex consisting of the cells σ'', $N(\sigma'', C'_{f|\sigma})$ and $\dot{N}(\sigma'', C_{f|\sigma})$, with σ ranging over K, is isomorphic to the cell complex $K \times I$. Using this isomorphism one builds, as in the proof of (7.1), the desired homeomorphism, q.e.d.

COROLLARY (9.4). *The simplicial mapping cylinder W_f defined by Whitehead is homeomorphic to our mapping cylinder C_f.* (See § 4.)

PROPOSITION (9.5). *If $f: K \to L$ is a simplicial mapping and if K_* and*

L_* are partitions of K and L such that $f = f_*\colon K_* \to L_*$ is simplicial, then $(C_f, K', L) \simeq (C_{f_*}, K_*', L_*)$.

PROOF. If $\sigma^i = \sigma < K$, let $\sigma_* < K_*$ be the subcomplex of K_* underlying σ. Then $f \mid \sigma_*$ is a collapsible mapping so $C_{f|\sigma_*}$ is an $(i + 1)$-ball whose boundary is the union of σ_*', $f(\sigma_*)$ and the balls $C_{f|\tau_*}$ $(\tau < \sigma)$, using (7.1) and (7.5). Thus C_{f_*} is a cell complex whose typical cells are the balls σ_*', $C_{f|\sigma_*} f(\sigma_*)$ and $\alpha_*(\alpha < L)$. Similarly C_f is a cell complex with typical cells σ, $C_{f|\sigma}$, $f(\sigma)$ and $\alpha(\alpha < L)$. We construct the desired homeomorphism as before, q.e.d.

Mapping cylinders and stellar neighborhoods. Suppose that L is a full subcomplex of K. We define two functions, $g\colon N(L, K) \to [0, 1]$ and $T\colon g^{-1}(1/2) \to L$.

$g\colon N(L, K) \to [0, 1]$ is the unique simplicial mapping such that $g(L) = 0$, and $g(\dot{N}(L, K)) = 1$. If $\sigma\tau < K$, $\sigma < L$, and $\tau < \dot{N}(L, K)$, then $(g \mid \sigma\tau)^{-1}(1/2)$ is a convex cell which is homeomorphic to $\sigma \times \tau$, and we denote it by $\sigma \times \tau$. $(g \mid \sigma\tau)^{-1}[0, 1/2]$ is a convex cell of one higher dimension and we denote it by $D(\sigma\tau)$.

If $z \in g^{-1}(1/2)$, then z is uniquely expressible in the form $z = (1/2)x + (1/2)y$ where carrier $x = \sigma < L$, carrier $y = \tau < \dot{N}(L, K)$ and carrier $z = \sigma\tau$. We define $T((1/2)x + (1/2)y) = x$. Notice that $T \mid \sigma \times \tau$ is linear, $T(\sigma \times \tau) = \sigma$, and $(T \mid \sigma \times \tau)^{-1}(x) \simeq \tau$ for each $x \in \sigma$.

T is piecewise linear. In fact, if we triangulate the convex cell complex $g^{-1}(1/2)$ without adding any new vertices, then $T\colon g^{-1}(1/2) \to L$ is simplicial. (Such a triangulation is possible by the argument for [16, Lem. 1].) If $f\colon L \to L_1$ is simplicial, then let C_{fT} denote the simplicial mapping cylinder with respect to any partitions of $g^{-1}(1/2)$ and L_1 such that fT is simplicial.

In this context we can now state

PROPOSITION (9.6). *If $f\colon L \to L_1$ is a simplicial mapping and F is the standard extension of f to $N(L, B(K/L))$, where the barycenters of simplexes of $K - L$ have been chosen so that $g^{-1}[0, 1/2] = |N(L, B(K/L))|$, then there is a homeomorphism*

$$h\colon (Fg^{-1}[0, 1/2] + L_1, Fg^{-1}(1/2), L_1) \longrightarrow (C_{fT}, g^{-1}(1/2), L_1)\,.$$

PROOF. For any convex i-cell $\sigma \times \tau \subset g^{-1}(1/2)$, it is clear that $fT \mid (\sigma \times \tau)$ is a collapsible mapping. Hence $C_{fT|(\sigma\times\tau)}$ is an $(i + 1)$-ball with the i-ball $\sigma \times \tau$ and $f(\sigma)$ in its boundary. Thus we may view C_{fT} as a cell complex, the typical cells of which are the balls $C_{fT|\sigma\times\tau}$, $\sigma \times \tau$, $f(\sigma)$, and the simplexes of L_1.

On the other hand, $F \mid D(\sigma\tau) = F \mid N(\sigma, B(\sigma\tau/\sigma))$ is transversely cellular since σ is well situated in $B(\sigma\tau/\sigma)$ and in $B(\partial(\sigma\tau)/\sigma)$. Hence $F(D(\sigma\tau))$ is an

$(i+1)$-ball with $\sigma \times \tau$ and $f(\sigma)$ in its boundary. Thus $Fg^{-1}[0, 1/2] + L_1$ may be viewed as a cell complex made up of cells of the form $(\sigma \times \tau) \subset g^{-1}(1/2)$, $F(D(\sigma\tau))$, $f(\sigma)$ and the simplexes of L_1.

The natural correspondence between these cell complexes leads to the desired homeomorphism, q.e.d.

Taking $f = 1_L$, we get

COROLLARY (9.7). *If L is a full subcomplex of K then*

$$\big(N(L', K'), \dot{N}(L', K'), L'\big) \simeq \big(C_T, \dot{N}(L', K'), L\big) .$$

Relationship between the simplicial and topological cylinders. In [15, § 10], Whitehead proved the following.

PROPOSITION (9.8). *If $f\colon K \to L$ is a simplicial mapping and M_f is the topological mapping cylinder of f, then there is a topological homeomorphism $h\colon (M_f, K, L) \to (C_f, K, L)$ such that $h\big(M_{f|\sigma}, \sigma, f(\sigma)\big) = \big(C_{f|\sigma}, \sigma, f(\sigma)\big)$ for each $\sigma < K$, and such that $h \mid (K \cup L) = 1$.*

10. Three applications

THEOREM (10.1). *Suppose that X is a polyhedral n-manifold, Y is a polyhedron, and $f\colon X \to Y$. If f is transversely cellular with respect to one pair of triangulations of X, Y, then it is transversely cellular with respect to any other pair for which it is simplicial.*

PROOF. Assume that M_i and L_i are triangulations of X and Y $(i = 1, 2)$, and that $f = f_1\colon M_1 \to L_1$ and $f = f_2\colon M_2 \to L_2$ are simplicial. If f_1 is transversely cellular, then f is a compact mapping and

$$(C_{f_1}, M_1', L_1) \simeq (M_1 \times I, M_1 \times 0, M_1 \times 1) .$$

Since f is compact and the complexes are locally finite, there are common partitions M of M_1, M_2 and L of L_1, L_2 such that $f\colon M \to L$ is simplicial. (This can be proved by modifying the arguments of [16, Lems. 1–6].) Therefore, using (9.5),

$$(C_{f_1}, M_1', L_1) \simeq (C_f, M', L) \simeq (C_{f_2}, M_2', L_2) .$$

This shows that C_{f_2} is an $(n+1)$-manifold and L_2 is an n-manifold in its boundary. By (8.1), f_2 is transversely cellular, q.e.d.

If $A \subset X$ and $f\colon A \to Y$, we let X/f denote the decomposition space of X, the elements of which are the sets $\{x\}$ $(x \in X - A)$ and $f^{-1}f(a)$ $(a \in A)$.

THEOREM (10.2). *If L is a full subcomplex of K, $f\colon L \to L_1$ is a simplicial surjection, $X = |K|$, $\pi\colon X \to X/f$ is the quotient mapping, F is the standard extension of f to $B(K/L)$, and $B(K/L) = N\big(L, B(K/L)\big) + P$, then there is a* *topological homeomorphism* $h: F(X) \to X/f$ *such that* $hF(x) = \pi(x)$ *for all* $x \in |L + P|$. *Moreover,* h *can be extended to a topological homeomorphism* $H: (C_F, X, F(X)) \to (M_\pi, X, X/f)$.

If we identify $F(X)$ with X/f by the homeomorphism h, then the conclusion of (10.2) can be stated in the following more useful form.

PROPOSITION (10.2)′. *The quotient space* X/f *is triangulable. There is a simplicial mapping* $F: X \to X/f$ *such that* F *agrees with* π *on* $|P + L|$, *and the only new inverse sets introduced by* F *are convex cells. Moreover,* $(C_F, X, X/f) \approx (M_\pi, X, X/f)$ *by a topological homeomorphism which is the identity on* $X \cup X/f$.

REMARK. If X is compact and $\varepsilon > 0$, we can, by taking fine enough triangulations, make F an ε-approximation to π.

PROOF OF (10.2). Define $g: N(L, K) \to [0, 1]$, $T: g^{-1}(1/2) \to |L|$, $\sigma \times \tau$, and $D(\sigma\tau)$ as in the previous section, and assume that $g^{-1}[0, 1/2] = |N(L, B(K/L))|$. Let "$\approx$" denote topological homeomorphism. If $\sigma < L$, $\tau < \dot{N}(L, K)$, and $\sigma\tau < K$, then we claim that $\pi(D(\sigma\tau)) = D(\sigma\tau)/f \mid \sigma$ is a topological $(i+1)$-ball, where $i = \dim(\sigma \times \tau)$. For

$$D(\sigma\tau)/f \mid \sigma \approx (\sigma \times \tau \times I)/fT = M_{fT|\sigma\times\tau} .$$

The claim follows because $M_{fT|\sigma\times\tau} \approx C_{fT|\sigma\times\tau}$ by (9.8), and in the proof of (9.6) it was demonstrated that $C_{fT|\sigma\times\tau} \simeq B^{i+1}$.

Now (see [15, §8]) the quotient topology on $\pi g^{-1}[0, 1/2]$ is precisely the weak topology on $\pi g^{-1}[0, 1/2]$ considered as a CW-complex with closed cells of the form $\pi(\sigma \times \tau)$, $\pi D(\sigma\tau)$, and $\pi(\sigma)$. Each of these is a topological ball whose boundary is the union of cells of lower dimension. Similarly $Fg^{-1}[0, 1/2]$ is a cell complex with closed cells of the form $\sigma \times \tau = F(\sigma \times \tau)$, $FD(\sigma\tau)$, and $F(\sigma)$. It was shown (9.6) that $FD(\sigma\tau) \simeq B^{i+1}$, so each of these closed cells is a topological ball. We can now build a topological homeomorphism

$$h': Fg^{-1}[0, 1/2] \longrightarrow \pi g^{-1}[0, 1/2]$$

inductively on the skeletons, using the natural isomorphism between these complexes. If at the i^{th} stage we choose the homeomorphism h'_i on cells of the form $\sigma \times \tau$ or $F(\sigma)$ to satisfy $h'_i F(x) = \pi(x)$, then h' will satisfy this condition on $|\dot{N}(L, B(K/L)) + L|$. We then extend h' trivially to the desired homeomorphism $h: F(X) \to X/f$.

The same argument will show that h has an extension to

$$H: (C_F, X, F(X)) \longrightarrow (M_\pi, X, X/f)$$

once we prove that $C_{F|D(\sigma\tau)}$ and $M_{\pi|D(\sigma\tau)}$ are topological $(i + 2)$-balls. The first of these certainly is, since $F \mid D(\sigma\tau)$ is transversely cellular. The second is a topological ball because $M_{\pi|D(\sigma,\tau)} = D(\sigma\tau) \times I/(f \mid \sigma \times 1)$. With respect to an appropriate triangulation this is just the topological quotient space of the $(i+2)$-ball $D(\sigma\tau) \times I$ induced by the simplicial mapping of the simplex $\sigma \times 1$ which is well situated in the ball and in its boundary. By the first part of the theorem, this topological quotient space is topologically homeomorphic to the image of $D(\sigma\tau) \times I$ under the standard extension of f. This standard extension is transversely cellular, so the image is a ball, q.e.d.

A good example in this realm is the figure-eight decomposition of S^3 given in [3]. Here $K = S^3$, L is the union of two disks which meet in a common radius, and $f: L \longrightarrow [0, 1]$ is the simplicial mapping which takes the boundary of one disk to 0 and the boundary of the other disk to 1. Bing proves that $S^3/f \approx S^3$. We sketch a proof of the following corollary to (10.2), which is both interesting and instructive.

COROLLARY (10.3). *If* S^3 *is considered as the unit sphere in* S^4, *with* $L < S^3 < S^4$, *then* $S^4/f \not\approx S^4$.

SKETCH OF PROOF. Let Q be a collar of S^3 in S^4. Then Q/f is topologically homeomorphic to two copies of M_π, sewn along $\pi(S^3)$. Let $F: S^3 \longrightarrow S^3/f$ be as in (10.2)′. Q/f is topologically homeomorphic to two copies of C_F sewn along $F(S^3)$. Let $\alpha^0 < \alpha^1 = F(L)$. Then $D(\alpha^1, F)$ is the regular neighborhood of the figure-eight $F^{-1}b(\alpha^1)$ in the boundary of the solid torus $D(\alpha^0, F)$. Thus $D(\alpha^1, F)$ is a 2-dimensional torus with a hole in it. Arguing as in the proofs of (7.4) and (8.1), $\mathrm{Lk}(\alpha^1, C_F)$ is a 2-manifold which collapses to $D(\alpha^1, F)$. Thus $\mathrm{Lk}(\alpha^1, C_F)$ is a torus with a hole in it, the boundary of which is the 1-sphere $\mathrm{Lk}(\alpha^1, F(S^3))$. Then $\mathrm{Lk}(\alpha^1, 2C_F)$ is a 2-sphere with two handles. Therefore $2C_F$ is not a topological manifold, whence S^4/f is not a topological manifold, q.e.d.

THEOREM (10.4). (Join-cobordism theorem). *If* M_0 *and* M_1 *are* n-*manifolds,* $V < M_0M_1$ *is an* $(n+1)$-*manifold and* $M_i < \partial V$ $(i = 0, 1)$, *then there is a homeomorphism*

$$h: (V, M_0, M_1) \longrightarrow (M_0 \times I, M_0 \times 0, M_0 \times 1) .$$

PROOF. Let $g: V \longrightarrow [0, 1]$ be the simplicial mapping such that $g(M_0) = 0$, and $g(M_1) = 1$. Let V' be a first barycentric subdivision of V chosen so that

$$g^{-1}[0, 1/2] = N(M_0', V') = N_0 ,$$
$$g^{-1}[1/2, 1] = N(M_1', V') = N_1 ,$$

and $g^{-1}(1/2) = \dot{N}_0 = \dot{N}_1$. Let $T: g^{-1}(1/2) \longrightarrow M_0$ be defined as before.

By (9.7), there is a homeomorphism $h'\colon (N_0, \dot{N}_0, M_0') \to (C_T, \dot{N}_0, M_0)$. But (3.4) asserts that N_0 is an $(n+1)$-manifold and $\dot{N}_0$ is an n-manifold. Therefore C_T and $\dot{N}_0$ have these properties. Thus by (8.1), T is transversely cellular. Hence there is a homeomorphism of $(C_T, \dot{N}_0, M_0)$ onto $(M_0 \times I, M_0 \times 1, M_0 \times 0)$. Composing this with h' we get a homeomorphism

$$h_0\colon (N_0, \dot{N}_0, M_0') \longrightarrow (M_0 \times I, M_0 \times 1, M_0 \times 0) .$$

Similarly there is a homeomorphism

$$h_1\colon (N_1, \dot{N}_1, M_1') \longrightarrow (M_1 \times I, M_1 \times 0, M_1 \times 1) .$$

Then $h\colon V \to M_0 \times [0, 2]$ is a homeomorphism if we define it by:

(i) $h \mid N_0 = h_0$

(ii) $h(h_1^{-1}(x, t)) = h_0 h_1^{-1}(x, 0) + (0, t)$ for all $(x, t) \in M_1 \times I$, where $M_0 \subset R^q$, $M_0 \times [0, 2] \subset R^q \times R^1$ and $(0, t) \in 0 \times R^1$, q.e.d.

COROLLARY (10.5). *If M^n is a submanifold of ∂V^{n+1}, then*

$$\bigl(N(M'', V''), M'', \dot{N}(M'', V'')\bigr) \simeq (M \times I, M \times 0, M \times 1) .$$

11. Contractible mappings

We prove two results about contractible mappings.

THEOREM (11.1). *If K and L are finite complexes and f is a contractible simplicial mapping of K onto L, then f is a simple homotopy equivalence.*

THEOREM (11.2). *Let P_j and Q_j be the statements:*

P_j: *If M^j is a closed combinatorial j-manifold with the same homotopy type as S^j, then $M^j \simeq S^j$.*

Q_j: *If M_1 and M_2 are combinatorial j-manifolds without boundary, and if there exists a compact contractible simplicial mapping of M_1 onto M_2, then $M_1 \simeq M_2$.*

Then P_j is true for all $j \leqq n \Longleftrightarrow \mathrm{Q}_j$ is true for all $j \leqq n$.

REMARK. There are examples [6], [8] of pairs of closed manifolds M_1, M_2 which are of the same simple homotopy type, but which are not piecewise linearly homeomorphic. It is conceivable, by (11.1), that there is a contractible simplicial mapping $f\colon M_1 \to M_2$. If this is so, then (11.2) implies that the Poincaré conjecture is false in some dimension, and indeed (as the proof will show) there is a dual cell such that either it or its boundary is a counterexample. Looked at the other way, if P_j is true for all j, then no simple homotopy equivalence $f\colon M_1 \to M_2$ is a piecewise-linear contractible mapping. Theorem (11.1) already shows that no homotopy equivalence between the lens spaces $L_{7,1}$ and $L_{7,2}$ is both piecewise-linear and contractible.

PROOF OF (11.1). It suffices to show that $|K|$ is a deformation retract of M_f and $\tau(M_f, |K|) = 0$, where M_f is the topological mapping cylinder of f with the natural CW-structure, and $\tau(X, Y)$ denotes the torsion of the CW-pair (X, Y). Using (9.8) and the fact that torsion is invariant under subdivision, we may equivalently demonstrate this for the pair $(|C^*|, |K'|)$. Here C^* is the complex defined in § 7 which is gotten by barycentrically subdividing the base of C_f.

For each $\alpha < L$ let $Q(\alpha, f)$ and $\dot{Q}(\alpha, f)$ be defined as in § 7. $Q(\alpha, f)$ is a "cell" stretching from $D(\alpha, f)$ to $D(\alpha, L)$. It is not a topological ball in general, but $Q(\alpha, f) = b(\alpha)[\dot{Q}(\alpha, f) + D(\alpha, f)]$ is a cone, and we use these "cells" as the simply connected blocks across which we deform $|C^*|$ back onto $|K'|$. Define $\operatorname{Int} Q(\alpha, f) = |Q(\alpha, f)| - |\dot{Q}(\alpha, f) + D(\alpha, f)|$. Notice that properties (b)–(f) of (7.7) are true in this context.

We claim, for each $\alpha < L$, that

(1) $\dot{Q}(\alpha, f) + D(\alpha, f)$ deformation retracts onto $D(\alpha, f)$.

(2) $\dot{Q}(\alpha, f) + D(\alpha, f)$ is contractible.

(3) $Q(\alpha, f)$ deformation retracts onto $\dot{Q}(\alpha, f) + D(\alpha, f)$.

$D(\alpha, f)$ is contractible by hypothesis and (5.5). Thus (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (3) because a compact contractible polyhedron is always a deformation retract of the cone over it.

Suppose that $\alpha = \alpha^i$ is an i-simplex, and (1)–(3) are known for each simplex of L of dimension greater than i. If α^i is principal then $\dot{Q}(\alpha, f) = \emptyset$ so (1) holds trivially. If α^i is not principal, choose a maximal "cell" of $\dot{Q}(\alpha^i, f)$. This "cell" is of the form $Q(\alpha^{i+1}, f)$ where $\alpha^i < \alpha^{i+1} < L$. By induction hypothesis this deformation retracts onto $\dot{Q}(\alpha^{i+1}, f) + D(\alpha^{i+1}, f)$. This deformation extends to a deformation of $\dot{Q}(\alpha^i, f) + D(\alpha^i, f)$ onto

$$|\dot{Q}(\alpha^i, f) + D(\alpha^i, f)| - \operatorname{Int} Q(\alpha^{i+1}, f)$$

because $\operatorname{Int} Q(\alpha^{i+1}, f)$ does not meet any other "cells" in $\dot{Q}(\alpha^i, f) + D(\alpha^i, f)$. We repeat this process for each $(i + 1)$-simplex of L containing α^i. Then we deformation retract, in turn each "cell" $Q(\alpha^{i+2}, f)$ onto $\dot{Q}(\alpha^{i+2}, f) + D(\alpha^{i+2}, f)$ $(\alpha^i < \alpha^{i+2} < L)$. Continuing in this manner we obtain, in a finite number of steps, a deformation retraction of $\dot{Q}(\alpha^i, f) + D(\alpha^i, f)$ onto $D(\alpha^i, f)$. Thus (1), (2) and (3) are true for all $\alpha < L$.

Enumerate the simplexes of L, $\alpha_1, \alpha_2, \cdots, \alpha_r$, so that $\dim \alpha_i \leqq \dim \alpha_{i+1}$. Let $X_i = |C^*| - \bigcup_{j \leqq i} \operatorname{Int} Q(\alpha_j, f)$ where $X_0 = |C^*|$, and $X_r = |K'|$. By (3) above, X_{i+1} is a deformation retract of X_i. Therefore $|K'|$ is a deformation retract of $|C^*|$. Moreover $X_i - X_{i+1} = \operatorname{Int} Q(\alpha_{i+1}, f)$ is an open cone and so is simply connected. Thus by [9, Lems. 7.1 and 7.2],

$$\tau(|C^*|, |K'|) = \sum_{i=0}^{r-1} \tau(X_i, X_{i+1}) = 0 , \qquad \text{q.e.d.}$$

PROOF OF (11.2). We first show that Q_j implies P_j. Let Σ be a homotopy j-sphere with principal simplex σ. Let $X \subset \operatorname{Int}(\Sigma - \operatorname{Int}\sigma)$ be a spine of $\Sigma - \operatorname{Int}\sigma$. Then (having taken appropriate triangulations) the standard extension F, to Σ, of the map which takes X to a point, is a contractible simplicial mapping. Since $F(\Sigma) \simeq S^j$ it follows from Q_j that $\Sigma \simeq S^j$.

We now show that P_j ($j \leqq n$) implies Q_n. For convenience we introduce the assertion P'_j which is well known to be equivalent to P_j.

P'_j: Every compact contractible j-manifold with sphere boundary is a j-ball.

Let $f: M_1^n \to M_2^n$ be a compact contractible simplicial surjection. If $\alpha^i < M_2^n$, then $D(\alpha^i, f)$ is a compact contractible $(n-i)$-manifold with boundary $\dot{D}(\alpha^i, f)$. Moreover $\dot{D}(\alpha^i, f) = f^{-1}\dot{D}(\alpha^i, M_2^n)$. Thus $f \mid \dot{D}(\alpha^i, f)$ is a contractible mapping; so by (11.1), $\dot{D}(\alpha^i, f)$ is a homotopy $(n-i-1)$-sphere. By P_{n-i-1}, it follows that $\dot{D}(\alpha^i, f)$ is a sphere. Hence P'_{n-i} implies that $D(\alpha^i, f) \simeq B^{n-i}$. Therefore f is transversely cellular, and $M_1^n \simeq M_2^n$, q.e.d.

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